A note on analytic functions in the unit circle. By Mr R. E. A. C. PALEY, Trinity College, and A. ZYGMUND.

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1. Let

$$f(z)=u(r,\theta)+iv(r,\theta) \qquad (z=re^{i\theta}) \tag{1}$$

be a function regular for $|z|<1$. We say that u belongs to the class L^p $(p>0)$ if

$$\int_0^{2\pi} |u(r,\theta)|^p d\theta = O(1).$$

It has been proved by M. Riesz that, for $p>1$, if $u(r,\theta)$ belongs to L^p, so does $v(r,\theta)$. Littlewood and later Hardy and Littlewood have shown that for $0<p<1$ the theorem is no longer true: there exists* an $f(z)$ such that $u(r,\theta)$ belongs to every $L^{1-\epsilon}$ and $v(r,\theta)$ belongs to no L^ϵ $(0<\epsilon<1)$. The proof was based on the theorem (due to F. Riesz) that if, for an $\epsilon>0$, we have

$$\int_0^{2\pi} |f(re^{i\theta})|^\epsilon d\theta = O(1), \tag{2}$$

then $\lim_{r\to 1} f(re^{i\theta})$ exists for almost every θ.

We prove the following:

THEOREM 1. *Given two arbitrary functions*

$$\chi(t) \quad \text{and} \quad \psi(t) \qquad (0\leqslant t<\infty),$$

non-negative, measurable, bounded in any finite interval and subject to the conditions

$$\chi(t)=o(t), \tag{3}$$

$$\psi(t)\to\infty \qquad (t\to\infty), \tag{4}$$

there exists an $f(z)$, regular for $|z|<1$ and such that, for $z\to 1$,

$$\int_0^{2\pi} \chi(|u(r,\theta)|)\, d\theta = O(1), \tag{5}$$

$$\int_0^{2\pi} \psi(|v(r,\theta)|)\, d\theta \neq O(1). \tag{6}$$

We use for the proof the same class of functions that Hardy and Littlewood used in their paper referred to, namely

$$f(z)=\Sigma a_n \frac{z^{\lambda_n}}{1-z^{2\lambda_n}}, \tag{7}$$

* J. E. Littlewood, "On inequalities in the theory of functions", *Proc. London Math. Soc.* (2), 23 (1924), 481–519 (516–517) (no proof given), G. H. Hardy and J. E. Littlewood, "Some properties of conjugate functions", *Journal für Math.*, 167 (1932), 405–423 (416 ff.).

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where $\lambda_0 < \lambda_1 < \lambda_2 < \dots$ are positive integers. The proof of (5) for general χ is substantially the same as in the case* of $\chi(t) \equiv t^{1-\epsilon}$. To prove (6) we must proceed otherwise, for although it is known (Ostrowski, R. and F. Nevanlinna) that the condition

$$\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta = O(1) \tag{8}$$

involves the existence of $\lim f(re^{i\theta})$ as r tends to 1 for almost every θ, it may be shown that the logarithm in (8) cannot be replaced by anything tending more slowly to $+\infty$ (see Theorem 2 below). A useful tool here appears to be the series

$$f_t(z) = \Sigma \frac{z^{\lambda_n}}{1 - z^{2\lambda_n}} \phi_n(t) \qquad (0 \leqslant t \leqslant 1), \tag{9}$$

where $\phi_0(t), \phi_1(t), \dots$ are Rademacher's well-known functions

$$\phi_0(t) = \operatorname{sign} \sin 2\pi t \quad (0 \leqslant t \leqslant 1), \qquad \phi_m(t) = \phi_0(2^m t).$$

The functions (9) enable us to prove also the result we have just stated. More precisely, we prove the following theorems:

THEOREM 2. *Given any function $\chi(t)$ $(0 \leqslant u < \infty)$, non-negative, measurable, bounded in every finite interval and satisfying* (3), *there exists a function $f(z)$, regular for $|z| < 1$, such that*

$$\int_0^{2\pi} \chi(\log^+ |f(re^{i\theta})|)\, d\theta = O(1), \tag{9a}$$

although $\lim_{r \to 1} f(re^{i\theta})$ exists for almost no θ.

THEOREM 3. *Given any function $\omega(r)$ $(0 \leqslant r < \infty)$, real and tending to $+\infty$ with r, there exists an $f(z)$, regular for $|z| < 1$, having no radial limit for almost every θ, and such that*

$$\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta = O(\omega(r)).$$

2. LEMMA α. *Let $\chi(t)$ $(0 \leqslant t < \infty)$ be an increasing function, continuous, vanishing for $t = 0$ and satisfying the condition*

$$\chi(t) = o(t) \quad \text{for} \quad t \to \infty.$$

Let

$$P(r, \theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$

Then the expression

$$\int_0^{2\pi} \chi(P(r, \theta))\, d\theta \tag{10}$$

tends to 0 for $r \to 1$.

* Hardy and Littlewood, *loc. cit.*

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In fact, suppose that $\chi(t) \leqslant \epsilon t$ for $t \geqslant t_0 = t_0(\epsilon)$. Let E_r be the set of θ such that $P(r, \theta) \geqslant t_0$, then

$$\int_0^{2\pi} \chi(P(r,\theta))\, d\theta = \int_{E_r} + \int_{CE_r} < \epsilon \int_0^{2\pi} P(r,\theta)\, d\theta + \int_0^{2\pi} g_r(\theta)\, d\theta,$$

where $g_r(\theta)$ is equal to $\chi(P(r,\theta))$ in CE_r and to 0 in E_r. The first integral on the right is equal to $2\pi\epsilon$ and the second tends to zero since the functions $g_r(\theta)$ are uniformly bounded and tend to zero for $\theta \neq 0, 2\pi$.

LEMMA β. *Let $\chi(t)$ be any concave function satisfying the conditions of the preceding lemma. Then, if the coefficients a_n in (7) are bounded (e.g. if $|a_n| \leqslant 1$) and if the exponents λ_n increase sufficiently rapidly, we have*

$$\int_0^{2\pi} \chi(|\Re f(e^{i\theta})|)\, dt = O(1). \tag{11}$$

The function χ being concave, we have, for every $x_1, x_2 \geqslant 0$, the inequality $\chi(x_1 + x_2) \leqslant \chi(x_1) + \chi(x_2)$. Taking into account that, for $z = re^{i\theta}$ $(0 \leqslant r < 1)$,

$$\Re \frac{z}{1-z^2}\Big| = \left|\frac{r(1-r^2)\cos\theta}{1-2r^2\cos 2\theta + r^4}\right| \leqslant r\,\frac{(1-r^4)}{1-2r^2\cos 2\theta + r^4} = rP(r^2, 2\theta),$$

we have

$$|\Re f(z)| \leqslant \sum_{n=1}^{\infty} r^{\lambda_n} P(r^{2\lambda_n}, 2\lambda_n\theta) = \sum_{n=1}^{N} + \sum_{n=N+1}^{\infty} = K_r + L_r,$$

where $N = N(r)$ is the greatest integer satisfying the inequality $2^{\lambda_N} \geqslant 1/2$. Suppose that $\lambda_{k+1} \geqslant 2\lambda_k$ $(k = 1, 2, \ldots)$. Then

$$L_r = \sum_{n=N+1}^{\infty} O(r^{\lambda_n}) = O(r^{\lambda_{N+1}})(1 + r^{\lambda_{N+2}-\lambda_{N+1}} + r^{\lambda_{N+3}-\lambda_{N+1}} + \ldots)$$

$$= O(1).(1 + r^{\lambda_{N+1}} + r^{2\lambda_{N+1}} + \ldots) = O(1).(1 + 2^{-1} + 2^{-2} + \ldots) = O(1).$$

Since

$$\chi(|\Re f|) \leqslant \chi(K_r + L_r) \leqslant \chi(K_r) + \chi(L_r)$$

it remains to be shown that

$$\int_0^{2\pi} \chi(K_r)\, d\theta = O(1).$$

Denoting by $\epsilon(1-r)$ the expression (10), we see that the preceding integral does not exceed

$$\int_0^{2\pi} \chi\left(\sum_{n=1}^{N} P(r^{2\lambda_n}, 2\lambda_n\theta)\right) d\theta \leqslant \sum_{n=1}^{N} \int_0^{2\pi} \chi(P(r^{2\lambda_n}, 2\lambda_n\theta)) \tag{12}$$

$$= \sum_1^N \epsilon(1 - r^{2\lambda_n}).$$

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Let ξ_k be a sequence of numbers such that $\epsilon(t) \leqslant 2^{-k}$ for $0 \leqslant t \leqslant \xi_k$. Taking into account that $|2^{-x}-1| \leqslant x\,(x \geqslant 0)$ we have two series of inequalities,

$$r^{\lambda_N} \geqslant 1/2, \quad r^{\lambda_{N-1}} \geqslant (1/2)^{\lambda_{N-1}/\lambda_N}, \quad r^{\lambda_{N-2}} \geqslant (1/2)^{\lambda_{N-2}/\lambda_N}, \ldots,$$

$$1-r^{2\lambda_N} < 2(1-r^{\lambda_N}) \leqslant 2.1/2, \quad 1-r^{2\lambda_{N-1}} < 2(1-2^{-\lambda_{N-1}/\lambda_N}) \leqslant 2\lambda_{N-1}/\lambda_N,$$

$$1-r^{2\lambda_{N-2}} < 2\lambda_{N-2}/\lambda_N, \ \ldots.$$

If we suppose that

$$2\lambda_1/\lambda_2 < \xi_1, \quad 2\lambda_2/\lambda_3 < \xi_2, \ \ldots,$$

then, for $n < N$, we have

$$2\lambda_n/\lambda_N \leqslant 2\lambda_n/\lambda_{n+1} < \xi_n,$$

hence $\epsilon(1-r^{2\lambda_n}) \leqslant 2^{-n}$ and the expression (12) is $O(1)$.

LEMMA γ. *Let $a_1, a_2, \ldots$ be a sequence of complex numbers for which $A^2 = \Sigma |a_k|^2 < \infty$, and so the series $\Sigma a_k \phi_k(t)$ (ϕ_k denoting Rademacher's functions) converges almost everywhere to a sum $S(t)$. Then*

$$|S(t)| \geqslant \tfrac{1}{2}A$$

in a set E of t, $m(E) > 1/64$. In particular, if $\psi(u)$ ($\psi \geqslant 0$, $0 \leqslant u < \infty$) increases with u, we have

$$\int_0^1 \psi(|S(t)|)\,dt \geqslant \tfrac{1}{64}\psi(\tfrac{1}{2}A).$$

We have already* proved the lemma in the case where the numbers a_n are real. The extension to the complex case is immediate. The number 1/64 can be improved but it is of no consequence.

LEMMA δ. *Let $\psi(u)$ ($\psi \geqslant 0$, $0 \leqslant u < \infty$) be an arbitrary increasing function tending to $+\infty$ with u, and let f_t be given by the formula (9) where the integers λ_n are increasing. Then for almost every t we have*

$$\int_0^{2\pi} \psi(|f_t(re^{i\theta})|)\,d\theta \neq O(1), \tag{13}$$

for $r \to 1$.

Let us fix an arbitrary sequence $r_1 < r_2 < \ldots < r_n < \ldots \to 1$. We put

$$S(r, \theta, t) = \sum_{k=1}^{\infty} \frac{z^{\lambda_k}}{1-z^{2\lambda_k}}\,\phi_k(t), \tag{14}$$

$$A^2(r, \theta) = \sum_{k=1}^{\infty} \left|\frac{z^{\lambda_k}}{1-z^{2\lambda_k}}\right|^2,$$

$$A(r) = \operatorname*{Min}_{0 \leqslant \theta < 2\pi} A(r, \theta), \quad A_n = A(r_n).$$

* R. E. A. C. Paley and A. Zygmund, "On some series of functions, (3)", *Proc. Camb. Phil. Soc.*, 28 (1932), 190–205, Lemma 19 (192).

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It is obvious that $A(r) \to \infty$. Let E_n be the set of points t, θ of the rectangle $(R)\, 0 \leqslant t \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$, such that

$$|S(r_n, \theta, t)| \geqslant \tfrac{1}{2} A(r_n, \theta) \geqslant \tfrac{1}{2} A_n.$$

Since the linear measure of the intersection of E_n with a line $\theta = \theta_0$ is at least $1/64$, we deduce that $m(E_n) \geqslant \pi/32$ and hence $m(R - E_n) \leqslant 63\pi/32$. It follows that the measure of the set t_0, such that the intersection of $R - E_n$ with $t = t_0$ has its measure $\geqslant 127\pi/64$, does not exceed $126/127$. Consequently, there exists a set F_n of t, $m(F_n) \geqslant \alpha = 1/127$, such that if $t_0 \subset F_n$, the measure of the set of points of E_n for which $t = t_0$ is not less than $\beta = \pi/64$. Let $F = \limsup F_n$, $m(F) \geqslant \alpha$. It is obvious that if $t \subset F$, the integral (13), for infinitely many values of r tending to 1, is not less than

$$\beta \psi [\tfrac{1}{2} A(r)] \to \infty.$$

To prove that (13) is true for almost every t (which, by the way, is not essential for our purposes) we may suppose that $(a)\ \psi(u) \leqslant \sqrt{u}$ for all u sufficiently large, $(b)\ \psi(u)$ is concave (otherwise we can find a function $\psi_1(u) \leqslant \psi(u)$ $(u \geqslant u_0)$, concave, tending to $+\infty$ with u and it is sufficient to prove Lemma δ for ψ_1). Then, denoting by $S_n(r, \theta, t)$ the sum of the first n terms of (14) we have, by concavity,

$$\psi(|S - S_n|) \geqslant \psi(|S|) - \psi(|S_n|).$$

In virtue of (a) the integral of $\psi(|S_n|)$ with respect to θ is bounded. Hence the integrals of $\psi(|S - S_n|)$ and $\psi(|S|)$ are bounded or unbounded together. It follows that the set of t for which (13) is true has periods 2^{-n} $(n = 1, 2, \ldots)$ and so is either of measure 0 or 1.

3. Now Theorem 1 follows easily from Lemmas β and δ. In fact, given the conditions (3) we can find two positive concave functions $\chi_1(u)$, $\psi_1(u)$ such that

$$\chi_1(u) \geqslant \chi(u), \quad \chi_1(u) = o(u)\ (u \to \infty);$$
$$\psi_1(u) \leqslant \psi(u)\ \ (u \geqslant u_0), \quad \psi_1(u) \to \infty.$$

Without loss of generality we may suppose that $\chi_1 \geqslant \psi_1$. We choose a sequence of λ_n increasing so rapidly that (11) (with χ_1 instead of χ) is satisfied. Then we can find a t_0 such that (13) (with $t = t_0$ and ψ_1 instead of ψ) is true. Let $f_{t_0} = u + iv$. By concavity,

$$\int_0^{2\pi} \psi_1(|f_{t_0}|)\, d\theta \leqslant \int_0^{2\pi} \psi_1(|u|)\, d\theta + \int_0^{2\pi} \psi_1(|v|)\, d\theta$$
$$\leqslant \int_0^{2\pi} \chi_1(|u|)\, d\theta + \int_0^{2\pi} \psi_1(|v|)\, d\theta.$$

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The last integral but one is $O(1)$, hence it follows from (13) (with ψ_1 instead of ψ) that the last integral is unbounded. Consequently (5) and (6) are satisfied.

To prove Theorem 2 we consider an arbitrary function $f_{t_0}(z)$, t_0 belonging to F (see the proof of Lemma 8). Since $t_0 \subset F$ there exists a sequence of values of r tending to 1, for which

$$|f_{t_0}(re^{i\theta})| \geqslant A(r)/2 \qquad (A(r) \to \infty)$$

in sets H_r of θ of measure $\geqslant \beta = \pi/64$. Let $H = H(t_0)$ $(m(H) > 0)$ denote the set of θ such that (a) $\theta/2\pi$ is irrational, (b) $f_{t_0}(re^{i\theta})$ does not tend radially to a limit. Let us suppose that the ratios

$$\lambda_{n+1}/\lambda_n \qquad (n = 0, 1, 2, \ldots)$$

are integers. Then H has periods $2\pi/\lambda_n$ $(n = 0, 1, \ldots)$. Consequently $m(H)$ is either 0 or 2π, i.e. $m(H) = 2\pi$. In other words, we have proved that, if $t_0 \subset F$ $(m(F) > 0)$, then

$$\limsup_{r \to 1} |f_{t_0}(re^{i\theta})| = \infty \tag{15}$$

for almost every θ. Since the set of t_0 having that property is of measure 0 or 1, we find that (15) is true for almost every t_0 and almost every θ. Let us suppose that there exists a t' such that

$$\lim_{r \to 1} \Re f_{t'}(re^{i\theta}) = -\infty \tag{16}$$

in a set $K = K(t')$ of θ, $m(K) > 0$. The last inequality implies $m(K) = 2\pi$. We say that the set of such values of t' is (if not empty) of measure 0. Otherwise it would be of measure 1 and we should have (16) for almost every t' and θ. Putting $t'' = 1 - t'$ and taking into account that $\phi_n(1-t) = -\phi_n(t)$, we should have

$$\lim_{r \to 1} \Re f_{t''}(re^{i\theta}) = -\lim_{r \to 1} \Re f_{t'}(re^{i\theta}) = +\infty$$

for almost every t'' and θ, which contradicts our hypothesis. Consequently, if t_0 belongs to a set T $(m(T) = 1)$, we have (15) for almost every θ and

$$\lim_{r \to 1} \Re f_{t_0}(re^{i\theta}) = -\infty$$

for almost no θ. Let $f(z) = \exp f_{t_0}(z)$, $t_0 \subset T$. Then $\exp f_{t_0}(re^{i\theta})$ has a radial limit for almost no θ. On the other hand, if $f_{t_0} = u + iv$, we have

$$\overset{+}{\log} |f(z)| = \overset{+}{u}(r, \theta) \leqslant |u(r, \theta)|$$

and the condition $(9a)$ is satisfied if λ_n increases sufficiently rapidly.

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Theorem 3 is rather trivial. In fact, it is known* that

$$f(re^{i\theta}) = \Sigma a_k z^{2^k} \qquad (z = re^{i\theta})$$

has no limit for almost every θ, provided that $\Sigma |a_k|^2 = \infty$. We may take $\Sigma |a_k|^2$ diverging so slowly that

$$\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \Sigma |a_k|^2 r^{2^{k+1}} \leqslant \omega(r) \qquad (r_0 \leqslant r < 1).$$

Since

$$\int_0^{2\pi} \overset{+}{\log} |f| \, d\theta \leqslant \int_0^{2\pi} |f|^2 \, d\theta + O(1),$$

Theorem 3 follows at once.

* A. Zygmund, "On the convergence of lacunary trigonometric series", *Fundamenta Math.*, 16 (1930), 90–107.

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