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ASYMPTOTIC BEHAVIOR OF THE WEAK APPROXIMATION TO A CLASS OF GAUSSIAN PROCESSES

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Abstract

We study, under mild conditions, the weak approximation constructed from a standard Poisson process for a class of Gaussian processes, and establish its sample path moderate deviations. The techniques consist of a good asymptotic exponential approximation in moderate deviations, the Besov–Lèvy modulus embedding, and an exponential martingale technique. Moreover, our results are applied to the weak approximations associated with the moving average of Brownian motion, fractional Brownian motion, and an Ornstein–Uhlenbeck process.

Keywords: Exponential martingale; Gaussian process; moderate deviations; Poisson process; weak approximation

2010 Mathematics Subject Classification: Primary 60F10
Secondary 60F17; 60B10

1. Introduction and main result

1.1. Introduction

Let $N = \{N(t), t \geq 0\}$ be a standard Poisson process. Stroock [18] constructed a family of continuous processes $X_\epsilon = \{X_\epsilon(t), 0 \leq t \leq 1\}$ which converge weakly to a standard Brownian motion $W = \{W(t), 0 \leq t \leq 1\}$ in the space C ([0, 1]) with the uniform topology. Here, the processes $\{X_\epsilon(t), 0 \leq t \leq 1\}$ are defined as

$$X_\epsilon(t) = \frac{1}{\epsilon} \int_0^t (-1)^{N(\epsilon^{-2}r)}\, \mathrm{d}r, \qquad 0 \leq t \leq 1.$$

Notice that a wide class of Gaussian processes admit an integral representation in law of the form

$$Y(t) = \int_0^1 K(t, r)\, \mathrm{d}W(r), \tag{1}$$

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where K is the deterministic kernel. By choosing K appropriately we can get some well-known processes. For instance, Y is the Ornstein–Uhlenbeck process when $K(t, r) = Ce^{\lambda(r-t)}$. Moreover, if $K(t, r) = K^H(t, r)$ is defined as

$$K^H(t,r) = \begin{cases} c_H r^{\frac{1}{2}-H} \mathbf{1}_{[0,t]}(r) \int_r^t (u-r)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} \, \mathrm{d}u, & \text{if } H > \frac{1}{2}, \\ \tilde{c}_H \mathbf{1}_{[0,t]}(r) \left((t/r)^{H-\frac{1}{2}} (t-r)^{H-\frac{1}{2}} - \left(H-\frac{1}{2}\right) r^{\frac{1}{2}-H} \int_r^t (u-r)^{H-\frac{1}{2}} u^{H-\frac{3}{2}} \mathrm{d}u \right), & \\ & \text{if } H < \frac{1}{2}, \end{cases} \tag{2}$$

where

$$c_H = \left(\frac{H(2H-1)}{\beta\left(2-2H, H-\frac{1}{2}\right)} \right)^{1/2}, \qquad \tilde{c}_H = \left(\frac{2H}{(1-2H)\beta\left(1-2H, H+\frac{1}{2}\right)} \right)^{1/2},$$

then the process Y is a fractional Brownian motion with Hurst index H [16, Chapter 5]. To approximate the process Y, inspired by the result in [18], Delgado and Jolis [9] introduced the process by replacing $\mathrm{d}W(r)$ by $\mathrm{d}X_\epsilon(r) = \frac{1}{\epsilon}(-1)^{N(\epsilon^{-2}r)} \, \mathrm{d}r$ in (1), i.e.

$$Y_\epsilon(t) = \frac{1}{\epsilon} \int_0^1 K(t, r)(-1)^{N(\epsilon^{-2}r)} \, \mathrm{d}r, \qquad 0 \le t \le 1.$$

They proved that Y_ϵ converges weakly to Y in $C([0, 1])$ with the uniform topology. There are also many other generalizations of [18]. To be explicit, [3] studied the weak approximation of the Brownian sheet from a Poisson process in the plane, while [15] applied this method to approximate fractional Brownian motion. As to the approximation of stochastic (partial) differential equations, see [4, 5, 12]. For more details on this topic, see [2, 19] and the references therein.

In this paper, under mild conditions, we will study the sample path moderate deviation for the process Y_ϵ in $C([0, 1])$ with the uniform topology, refining the already known result in [9]. This may be useful for estimating deviation domains of suprema or infema of sample paths, which is of particular interest in statistics. Generally speaking, a moderate deviation fulfills the gap between a weak convergence result and large deviation. More precisely, consider the estimation of $\mathbb{P}(\lambda_\epsilon Y_\epsilon(\cdot) \in A)$, where A is a given domain of deviations and λ_ϵ denotes the scale of deviation. When $\lambda_\epsilon \equiv 1$, this is exactly the estimation of a weak convergence result. When $\lambda_\epsilon \equiv \epsilon$, this corresponds to a large deviation. And when λ_ϵ lies between 1 and ϵ, i.e. $\lambda_\epsilon \to 0$ and $\epsilon/\lambda_\epsilon \to 0$ as $\epsilon \to 0$, this is a so-called moderate deviation.

1.2. Main result

Throughout this paper, the deterministic kernel K:$[0, 1] \times [0, 1] \to \mathbb{R}$ satisfies the following conditions.

Condition 1.1. *K is measurable and $K(0, r) = 0$ for any $r \in [0, 1]$.*

Condition 1.2. *There exists a strictly increasing and continuous function G:$[0, 1] \to \mathbb{R}$ and a constant $\alpha > 0$ such that, for any $0 \le s < t \le 1$,*

$$\int_0^1 (K(t, r) - K(s, r))^2 \, dr \le (G(t) - G(s))^\alpha.$$

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Theorem 1.1. *Let* $\{\lambda_\epsilon, \epsilon > 0\}$ *be a family of positive numbers such that*

$$\lim_{\epsilon\to 0} \lambda_\epsilon = 0, \qquad \lim_{\epsilon\to 0} \frac{\epsilon}{\lambda_\epsilon} = 0.$$

Under Conditions 1.1 and 1.2, the family $\{\lambda_\epsilon Y_\epsilon, \epsilon > 0\}$ *satisfies the sample path moderate deviation in* C *([0, 1]) equipped with the uniform topology with speed* λ_ϵ^2 *and, for any* $f \in C([0, 1])$*, the rate function* I *defined by*

$$I(f) = \inf\left\{\frac{1}{2}\int_0^1 \phi^2(r)\,\mathrm{d}r : f(t) = \int_0^1 K(t, r)\phi(r)\,\mathrm{d}r,\ t \in [0, 1],\ \phi \in L^2([0, 1])\right\}. \tag{3}$$

Explicitly, for any Borel-measurable set $A \subseteq C([0, 1])$*,*

$$-\inf_{f\in A^\circ} I(f) \le \limsup_{\epsilon\to 0} \lambda_\epsilon^2 \log \mathbb{P}\left(\lambda_\epsilon Y_\epsilon(\cdot) \in A\right) \le -\inf_{f\in \overline{A}} I(f).$$

It should be mentioned that for any fixed $t \in [0, 1]$, $K(t, \cdot)$ just belongs to $L^2([0, 1])$ and is not necessarily differentiable or even continuous and bounded, and so integration by parts formulae may not be available. To overcome this limitation, we will use the good asymptotic exponential approximation in moderate deviations [10], Besov–Lèvy modulus embedding [14], and stochastic analysis techniques.

The remainder of this paper is organized as follows. In Section 2 we give an explicit characterization of the rate function $I(\cdot)$. Moreover, the sample path moderate deviations for the weak approximation associated with the moving average of Brownian motion, fractional Brownian motion, and an Ornstein–Uhlenbeck process will be considered. Finally, we will state the proof of Theorem 1.1 in Section 3.

2. Further study of the rate function and applications

2.1. Characterization of rate function

For practical applications, we need to further study the rate function $I(\cdot)$ defined by (3). In this subsection, some properties and characterization of I will be established.

Set $\mathrm{Dom}(I) = \{f : I(f) < +\infty\}$ and define the following mapping $\mathbb{S}$ on $L^2([0, 1])$:

$$\mathbb{S}(\phi)(t) = \int_0^1 K(t, r)\phi(r)\,\mathrm{d}r, \qquad \phi \in L^2([0, 1]), \quad t \in [0, 1]. \tag{4}$$

Obviously, $\mathbb{S}(L^2([0, 1])) := \left\{\mathbb{S}(\phi) : \phi \in L^2([0, 1])\right\}$ is the reproducing kernel Hilbert space associated with Gaussian process Y defined by (1).

Proposition 2.1. *Under Conditions 1.1 and 1.2, we have the following results.*

(i) $\mathrm{Dom}(I) = \mathbb{S}(L^2([0, 1]))$.

(ii) *For any* $f \in \mathrm{Dom}(I)$*, there exists an element* $\phi \in L^2([0, 1])$ *such that, for any* $t \in [0, 1]$*,* $f(t) = \int_0^1 K(t, r)\phi(r)\,\mathrm{d}r$ *and* $I(f) = \frac{1}{2}\int_0^1 \phi^2(r)\,\mathrm{d}r$*. Furthermore,* ϕ *is unique with respect to the Lebesgue measure* $\mathcal{L}(\cdot)$.

Proof. By the definition of $I(\cdot)$ in (3), (i) is obvious. Then, we turn to prove (ii). In fact, for any $f \in \mathrm{Dom}(I)$, there exists a sequence of $\phi_n \in L^2([0, 1])$ such that

$$f(t) = \int_0^1 K(t, r)\phi_n(r)\,\mathrm{d}r,\ t \in [0, 1], \quad \text{and} \quad \lim_{n\to+\infty} \frac{1}{2}\int_0^1 \phi_n^2(r)\,\mathrm{d}r = I(f).$$

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Then, there exists a subsequence of ϕ_{n_i} converging weakly to an element $\phi \in L^2([0,1])$, which implies that

$$f(t) = \int_0^1 K(t, r)\phi(r)\,\mathrm{d}r, \qquad t \in [0,1].$$

Note that for any $\psi \in L^2([0,1])$,

$$\begin{aligned}\left|\int_0^1 \phi(r)\psi(r)\,\mathrm{d}r\right| &= \lim_{n_i\to\infty}\left|\int_0^1 \phi_{n_i}(r)\psi(r)\,\mathrm{d}r\right| \\ &\le \left(\int_0^1 \psi^2(r)\,\mathrm{d}r\right)^{1/2}\left(\lim_{n_i\to+\infty}\int_0^1 \phi_{n_i}^2(r)\,\mathrm{d}r\right)^{1/2},\end{aligned}$$

which implies that

$$\frac{1}{2}\int_0^1 \phi^2(r)\,\mathrm{d}r \le \frac{1}{2}\lim_{n_i\to+\infty}\int_0^1 \phi_{n_i}^2(r)\,\mathrm{d}r = I(f).$$

Moreover, by the definition of $I(f)$, we also have

$$\frac{1}{2}\int_0^1 \phi^2(r)\,\mathrm{d}r \ge I(f).$$

Therefore, $I(f) = \frac{1}{2}\int_0^1 \phi^2(r)\,\mathrm{d}r$.

Finally, we show the uniqueness of ϕ. Suppose there exists some $\psi \in L^2([0,1])$ such that the infimum is obtained, and $\psi \neq \phi$ in $L^2([0,1])$. Then there exists a measurable set $A \subset [0,1]$ such that, for any $r \in A$, $\psi(r) \neq \phi(r)$ and $\mathcal{L}(A) \neq 0$. Define $\xi := \frac{1}{2}(\psi + \phi)$; then,

$$f(t) = \int_0^1 K(t, r)\xi(r)\,\mathrm{d}r \quad \text{and} \quad \int_0^1 \xi^2(r)\,\mathrm{d}r = \int_A \xi^2(r)\,\mathrm{d}r + \int_{A^c} \xi^2(r)\,\mathrm{d}r.$$

By the strict convexity of x^2,

$$\begin{aligned}\int_A \xi^2(r)\,\mathrm{d}r &< \frac{1}{2}\left(\int_A \phi^2(r)\,\mathrm{d}r + \int_A \psi^2(r)\,\mathrm{d}r\right), \\ \int_{A^c} \xi^2(r)\,\mathrm{d}r &\le \frac{1}{2}\left(\int_{A^c} \phi^2(r)\,\mathrm{d}r + \int_{A^c} \psi^2(r)\,\mathrm{d}r\right).\end{aligned}$$

Therefore, $\frac{1}{2}\int_0^1 \xi^2(r)\,\mathrm{d}r < I(f)$, which contradicts the definition of $I(f)$. □

Now, we give an equivalent expression of the rate function $I(\cdot)$.

Proposition 2.2. *Assume that $\mathbb{S}$ is an injection on $L^2([0,1])$. Under Conditions 1.1 and 1.2, we have*

$$I(f) = \begin{cases} \frac{1}{2}\left\|\mathbb{S}^{-1}f\right\|^2_{L^2([0,1])}, & \text{if } f \in \mathbb{S}(L^2([0,1])), \\ +\infty, & \text{otherwise.}\end{cases}$$

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I is also the rate function of the probability measures $\{\mu_\epsilon, \epsilon > 0\}$ on $C([0, 1])$ with speed ϵ, where

$$\mu_\epsilon(A) := \mathbb{P}\left(\sqrt{\epsilon}\int_0^1 K(\cdot, r)\,\mathrm{d}W(r) \in A\right), \qquad A \subseteq C([0, 1]),$$

and $\{W(r): r \in [0, 1]\}$ is a Brownian motion in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Proof. By the definition of I and $\mathbb{S}$, the first result is easily obtained if $\mathbb{S}$ is an injection.

Now we turn to the second result. For any finite sign measure ν on $[0, 1]$, by stochastic Fubini's theorem [20, Theorem 4.18] we have

$$\int_0^1\int_0^1 K(t, r)\,\mathrm{d}W(r)\nu(\mathrm{d}t) = \int_0^1\int_0^1 K(t, r)\nu(\mathrm{d}t)\,\mathrm{d}W(r),$$

which implies that $\int_0^1\int_0^1 K(t, r)\,\mathrm{d}W(r)\nu(\mathrm{d}t) \sim N\big(0, \int_0^1\big(\int_0^1 K(t, r)\nu(\mathrm{d}t)\big)^2\,\mathrm{d}r\big)$. By [11, Theorem 3.4.12] with the Cameron–Martin space $H = \{h; h = \int_0^t \dot{h}(r)\,\mathrm{d}r; \dot{h} \in L^2([0, 1])\}$, $E = C([0, 1])$, and $S: H \to E$ defined by $S(h) = \int_0^1 K(t, r)\dot{h}(r)\,\mathrm{d}r$, it is sufficient to prove that

$$S^*(\nu)(r) = \int_0^r\int_0^1 K(t, r)\nu(\mathrm{d}t)\,\mathrm{d}r, \qquad r \in [0, 1]$$

in H, where S^* is the adjoint map to S.

Indeed, by the definition of S, we have, for any $h \in H$,

$$\langle S^*(\nu), h\rangle = \langle \nu, S(h)\rangle = \int_0^1\left(\int_0^1 K(t, r)\dot{h}(r)\,\mathrm{d}r\right)\nu(\mathrm{d}t) = \int_0^1\left(\int_0^1 K(t, r)\nu(\mathrm{d}t)\right)\dot{h}(r)\,\mathrm{d}r.$$

Therefore, $S^*(\nu)(r) = \int_0^r\int_0^1 K(t, r)\nu(\mathrm{d}t)\,\mathrm{d}r$ for $r \in [0, 1]$, and the desired result is proved.

Note that the operator $\mathbb{S}$ may not be injective, or not easy to verify. However, when the kernel K has some product structure, the rate function has the following explicit formulation.

Proposition 2.3. *Suppose that $K(t, r) = \xi(t)\eta(r)\mathbf{1}_{[0,t]}(r)$, $t, r \in [0, 1]$. Here, ξ and η are measurable functions satisfying $\xi \in C([0, 1])$, $\eta \in L^2([0, 1])$, and $\mathcal{L}(\{\xi = 0\}) = 0$. Under Conditions 1.1 and 1.2, we have*

$$I(f) = \begin{cases} \frac{1}{2}\int_0^1 \left(\left(\frac{f}{\xi}\right)'(r)\right)^2 \frac{\mathbf{1}_{\{\eta\neq 0\}}(r)}{\eta^2(r)}\,\mathrm{d}r, & \text{if } \frac{f}{\xi} \text{ is absolutely continuous and} \\ & \left(\frac{f}{\xi}\right)'\frac{\mathbf{1}_{\{\eta\neq 0\}}}{\eta} \in L^2([0, 1]), \\ +\infty, & \text{otherwise.} \end{cases}$$

Proof. If $f \in \mathrm{Dom}(I)$ then, for any choice $\phi \in L^2([0, 1])$ such that

$$f(t) = \xi(t)\int_0^t \eta(r)\phi(r)\,\mathrm{d}r, \qquad t \in [0, 1],$$

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the fact that $\mathcal{L}(\{\xi = 0\}) = 0$ implies that $\frac{f}{\xi}(t) = \int_0^t \eta(r)\phi(r)\,\mathrm{d}r$ is absolutely continuous and, for almost all $t \in [0, 1]$, $\left(\frac{f}{\xi}\right)'(t) = \eta(t)\phi(t)$. That is, on $\{t{:}\eta(t) \neq 0\}$, $\phi(t) = \frac{1}{\eta(t)}\left(\frac{f}{\xi}\right)'(t)$.

Define $\psi(t) = \frac{\mathbf{1}_{\eta\neq 0}(t)}{\eta(t)}\left(\frac{f}{\xi}\right)'(t)$ for almost all $t \in [0, 1]$. Then $\psi(t) = \phi(t)\mathbf{1}_{\{\eta\neq 0\}}(t)$ for almost all $t \in [0, 1]$, and

$$f(t) = \xi(t)\int_0^t \eta(r)\psi(r)\,\mathrm{d}r, \qquad t \in [0, 1].$$

Obviously, $\int_0^1 \phi^2(r)\,\mathrm{d}r \geq \int_0^1 \phi^2(r)\mathbf{1}_{\{\eta(r)\neq 0\}}\,\mathrm{d}r = \int_0^1 \psi^2(r)\,\mathrm{d}r$, which completes the proof of this proposition.

2.2. Applications

In this subsection we apply our results in Theorem 1.1 and Propositions 2.2 and 2.3 to the weak approximation of some Gaussian processes.

2.2.1 *Moving average of Brownian motion* Let the kernel $K(t, r) = \mathbf{1}_{[0,t]}(r)(2\pi)^{1/2}(t - r)^{H-1/2}$, with $0 < H < 1/2$. It is well known that the corresponding mapping $\mathbb{S}$ defined by (4) is an injection and $\mathbb{S}^{-1}f$ is known as the solution to Abel's equation [17, (2.1), p. 29]. By [17, Theorem 2.1], $f \in \mathbb{S}(L^2([0, 1]))$ implies that $\left\{\int_0^t \frac{f(r)}{(t-r)^{H+\frac{1}{2}}}\,\mathrm{d}r,\ t \in [0, 1]\right\}$ is absolutely continuous, $\lim_{t\to 0+}\int_0^t \frac{f(r)}{(t-r)^{H+\frac{1}{2}}}\,\mathrm{d}r = 0$. Furthermore, if $f \in \mathbb{S}(L^2([0, 1]))$, then $f(t) = \int_0^1 K(t, r)\phi(r)\,\mathrm{d}r$, where $\phi \in L^2([0, 1])$ [17, (2.6), p. 30] can be written as

$$\phi(t) = -\frac{1}{\sqrt{2\pi}\Gamma\left(\frac{1}{2} + H\right)\Gamma\left(\frac{1}{2} - H\right)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_t^1 \frac{f(r)\,\mathrm{d}r}{(t - r)^{H+\frac{1}{2}}}\right). \tag{5}$$

It is also known that Conditions 1.1 and 1.2 both hold with $G(t) = Ct$, $C > 0$, $\alpha = 2H$ [9].

Now, let $Y_\epsilon(t) = \frac{(2\pi)^{1/2}}{\epsilon}\int_0^t (t - r)^{H-1/2}(-1)^{N(\epsilon^{-2}r)}\,\mathrm{d}r$, $0 \leq t \leq 1$. By using Theorem 1.1 and Proposition 2.2, we have the following result.

Corollary 2.1. *The family* $\{\lambda_\epsilon Y_\epsilon, \epsilon > 0\}$ *satisfies the sample path moderate deviation with speed* λ_ϵ^2 *and rate function I defined by*

$$I(f) = \begin{cases} \frac{1}{2}\int_0^1 \phi^2(r)\,\mathrm{d}r, & \text{if } \int_0^t \frac{f(r)}{(t-r)^{H+\frac{1}{2}}}\,\mathrm{d}r \text{ is absolutely continuous and} \\ & \lim_{t\to 0+}\int_0^t \frac{f(r)}{(t-r)^{H+\frac{1}{2}}}\,\mathrm{d}r = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Here, $\phi \in L^2([0, 1])$ *is given by (5).*

2.2.2. *Fractional Brownian motion* For $K^H(t, r)$ given by (2), $Y(t) := \int_0^1 K^H(t, r)\,\mathrm{d}W(r)$ is a fractional Brownian motion with Hurst index H. Applying [8, Theorem 2.1], the mapping $\mathbb{S}$ defined by (4) is an injection. Moreover, Conditions 1.1 and 1.2 both hold with $G(t) = Ct$, $C > 0$, $\alpha = 2H$ [9].

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Let $Y_\epsilon(t) = \frac{1}{\epsilon}\int_0^1 K^H(t,r)(-1)^{N(\epsilon^{-2}r)}\,\mathrm{d}r$, $0 \le t \le 1$. Together with Theorem 1.1 and Corollary 2.2, the following result holds.

Corollary 2.2. *The family* $\{\lambda_\epsilon Y_\epsilon, \epsilon > 0\}$ *satisfies the sample path moderate deviation with speed* λ_ϵ^2 *and rate function I given by*

$$I(f) = \begin{cases} \frac{1}{2}\left\|\mathbb{S}^{-1}f\right\|^2_{L^2([0,1])}, & \text{if } f \in \mathbb{S}(L^2([0,1])), \\ +\infty, & \text{otherwise.} \end{cases}$$

2.2.3. *Ornstein–Uhlenbeck process* For the kernel $K(t,r) = C\mathrm{e}^{\lambda(r-t)}\mathbf{1}_{(0,t]}(r)$ with $\lambda > 0$, Conditions 1.1 and 1.2 both hold with $G(t) = Ct$, $C > 0$ [9]. In particular, $Y_t := \int_0^1 C\mathrm{e}^{\lambda(r-t)}\,\mathrm{d}W(r)$ satisfies $\mathrm{d}Y_t = -\lambda Y_t + C\,\mathrm{d}W(t)$. Hence, this process is the well-known Ornstein–Uhlenbeck process.

Let $Y_\epsilon(t) = \frac{1}{\epsilon}\int_0^t C\mathrm{e}^{\lambda(r-t)}(-1)^{N(\epsilon^{-2}r)}\,\mathrm{d}r$, $0 \le t \le 1$. By Theorem 1.1 and Corollary 2.3, we conclude this section with the following corollary.

Corollary 2.3. $\{\lambda_\epsilon Y_\epsilon, \epsilon > 0\}$ *satisfies the sample path moderate deviation with speed* λ_ϵ^2 *and rate function I defined by*

$$I(f) = \begin{cases} \frac{1}{2C^2}\int_0^1 (f'(r) + \lambda f(r))^2\,\mathrm{d}r, & \text{if } f \text{ is absolutely continuous and} \\ & \int_0^1 |f'(r)|^2\,\mathrm{d}r < +\infty, \\ +\infty, & \text{otherwise.} \end{cases}$$

3. Proof of Theorem 1.1

To get Theorem 1.1, by [13, Theorem 4.28] or [10, Theorems 4.2.4 and 4.6.1], we only need to prove the moderate deviations of finite-dimensional distributions $\mathbb{P}\left(\lambda_\epsilon\left(Y_\epsilon(t_1), \ldots, Y_\epsilon(t_k)\right)^\top \in \cdot\right)$, $0 < t_1 < \cdots < t_k \le 1$, $k \ge 1$, and the following exponential tightness: for any $L > 0$, there exists a compact set $K_L \subset C([0,1])$ such that $\limsup_{\epsilon\to 0}\lambda_\epsilon^2 \log \mathbb{P}\left(\lambda_\epsilon Y_\epsilon \in K_L^c\right) \le -L$. Here, $x^\top$ denotes the transpose of $x \in \mathbb{R}^k$, $k \in \mathbb{N}$.

We first provide some notation to be used later. Let $N(\mathrm{d}t, \mathrm{d}x)$ be a Poisson random measure with the intensity measure $\mathrm{d}t \otimes \mathrm{d}x$ on $(\mathbb{R}_+ \times [0,1], \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}([0,1]))$. For any measure μ on $(\mathbb{R}_+ \times [0,1], \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}([0,1]))$, define $\mu(t,1) = \mu([0,t] \times [0,1])$. Obviously, $\{N(t,1) := N([0,t] \times [0,1]), t \ge 0\}$ is a version of the standard Poisson process. Then, the processes X_ϵ and Y_ϵ can be written as

$$X_\epsilon(t) = \int_0^t \theta_\epsilon(s)\,\mathrm{d}s, \qquad Y_\epsilon(t) = \int_0^1 K(t,r)\theta_\epsilon(r)\,\mathrm{d}r, \qquad \theta_\epsilon(t) = \epsilon^{-1}(-1)^{N^{\epsilon^{-2}}(t,1)},$$

where $N^{\epsilon^{-2}}$ is a Poisson random measure with intensity density $\epsilon^{-2}\,\mathrm{d}t \otimes \mathrm{d}x$.

3.1. High-order moments

For any $f \in L^2([0,1])$, we extend the result in [9, Lemma 2], and then give a sharper estimation for the high-order moments of $\int_0^1 f(r)\theta_\epsilon(r)\,\mathrm{d}r$.

Lemma 3.1. *For any* $f \in L^2([0,1])$ *and even number* $m \in \mathbb{N}$,

$$\left(\mathbb{E}\left(\int_0^1 f(r)\theta_\epsilon(r)\,\mathrm{d}r\right)^m\right)^{1/m} \le \mathrm{e}^{-\frac{1}{2}}\left(\frac{\mathrm{e}}{\sqrt{\pi}}\right)^{\frac{1}{m}}\sqrt{m}\left(\int_0^1 f^2(r)\,\mathrm{d}r\right)^{1/2}.$$

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Proof. We have

$$\mathbb{E}\left(\int_0^1 f(r)\theta_\epsilon(r)\,\mathrm{d}r\right)^m$$
$$= \frac{m!}{\epsilon^m}\int_{[0,1]^m} \mathbf{1}_{\{r_1<\cdots<r_m\}} f(r_1)\cdots f(r_m)\exp\left\{-\frac{2}{\epsilon^2}\sum_{i=1}^{m/2}(r_{2i}-r_{2i-1})\right\}\mathrm{d}r_1\cdots\mathrm{d}r_m$$
$$= \frac{m!}{\epsilon^{m-2}}\int_{[0,1]^{m-2}} \mathbf{1}_{\{r_3<\cdots<r_m\}} G_1(r_3) f(r_3)\cdots f(r_m)\times\exp\left\{-\frac{2}{\epsilon^2}\sum_{i=3}^{m/2}(r_{2i}-r_{2i-1})\right\}\mathrm{d}r_3\cdots\mathrm{d}r_m,$$

where the first equality is shown in [9, Lemma 2] and, for any $r_3 < r_4$,

$$\begin{aligned}
G_1(r_3) &:= \int_{[0,1]^2} \mathbf{1}_{\{r_1<r_2<r_3\}}\frac{f(r_1)}{\epsilon}\cdot\frac{f(r_2)}{\epsilon}\exp\left\{-\frac{2}{\epsilon^2}(r_2-r_1)\right\}\mathrm{d}r_1\,\mathrm{d}r_2\\
&\le \left(\int_{[0,1]^2} \mathbf{1}_{\{r_1<r_2<r_3\}}\frac{f^2(r_1)}{\epsilon^2}\exp\left\{-\frac{2}{\epsilon^2}(r_2-r_1)\right\}\mathrm{d}r_1\,\mathrm{d}r_2\right)^{1/2}\\
&\times\left(\int_{[0,1]^2} \mathbf{1}_{\{r_1<r_2<r_3\}}\frac{f^2(r_2)}{\epsilon^2}\exp\left\{-\frac{2}{\epsilon^2}(r_2-r_1)\right\}\mathrm{d}r_1\,\mathrm{d}r_2\right)^{1/2}\\
&\le \frac{1}{2}\left(\int_{r_1<r_3} f^2(r_1)\,\mathrm{d}r_1\right)^{1/2}\cdot\left(\int_{r_2<r_3} f^2(r_2)\,\mathrm{d}r_2\right)^{1/2}\\
&\le \frac{1}{2}\left(\int_{r<r_3} f^2(r)\,\mathrm{d}r\right)^{1/2}\cdot\left(\int_{r<r_4} f^2(r)\,\mathrm{d}r\right)^{1/2}.
\end{aligned}$$

Substituting the above estimation into the original equation,

$$\begin{aligned}
\mathbb{E}\left(\int_0^1 f(r)\theta_\epsilon(r)\,\mathrm{d}r\right)^m =& \frac{m!}{\epsilon^{m-4}}\int_{[0,1]^{m-4}} \mathbf{1}_{\{r_5<\cdots<r_m\}} G_2(r_5) f(r_5)\cdots f(r_m)\\
&\times\exp\left\{-\frac{2}{\epsilon^2}\sum_{i=5}^{m/2}(r_{2i}-r_{2i-1})\right\}\mathrm{d}r_5\cdots\mathrm{d}r_m,
\end{aligned}$$

where, for any $r_5 < r_6$,

$$\begin{aligned}
G_2(r_5) &:= \int_{[0,1]^2} \mathbf{1}_{\{r_3<r_4<r_5\}} G_1(r_3)\frac{f(r_3)}{\epsilon}\cdot\frac{f(r_4)}{\epsilon}\exp\left\{-\frac{2}{\epsilon^2}(r_4-r_3)\right\}\mathrm{d}r_3\,\mathrm{d}r_4\\
&\le \frac{1}{2}\int_{[0,1]^2} \mathbf{1}_{\{r_3<r_4<r_5\}}\left(\int_{r<r_3} f^2(r)\,\mathrm{d}r\right)^{1/2}\frac{f(r_3)}{\epsilon}\exp\left\{-\frac{1}{\epsilon^2}(r_4-r_3)\right\}\\
&\times\left(\int_{r<r_4} f^2(r)\,\mathrm{d}r\right)^{1/2}\frac{f(r_4)}{\epsilon}\exp\left\{-\frac{1}{\epsilon^2}(r_4-r_3)\right\}\mathrm{d}r_3\,\mathrm{d}r_4
\end{aligned}$$

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$$\le \frac{1}{2^2}\left(\int_{r_3<r_5}\left(\int_{r<r_3} f^2(r)\,\mathrm{d}r\right) f^2(r_3)\,\mathrm{d}r_3\right)^{1/2}$$
$$\times \left(\int_{r_4<r_5}\left(\int_{r<r_4} f^2(r)\,\mathrm{d}r\right) f^2(r_4)\,\mathrm{d}r_4\right)^{1/2}$$
$$\le \frac{1}{2^2}\left(\frac{\left(\int_{r<r_5} f^2(r)\,\mathrm{d}r\right)^2}{2}\right)^{1/2}\left(\frac{\left(\int_{r<r_6} f^2(r)\,\mathrm{d}r\right)^2}{2}\right)^{1/2}.$$

Substituting the above into the previous equation again, by the same argument and induction, we have

$$\mathbb{E}\left(\int_0^1 f(r)\theta_\epsilon(r)\,\mathrm{d}r\right)^m \le \frac{m!}{2^{m/2}}\cdot\frac{\left(\int_0^1 f^2(r)\,\mathrm{d}r\right)^{m/2}}{(m/2)!}.$$

For any positive integer n, we have the following Stirling formula:

$$\sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n \le n! \le \mathrm{e}\sqrt{n}\left(\frac{n}{\mathrm{e}}\right)^n. \tag{6}$$

Therefore, $\frac{m!}{(m/2)!} \le \frac{\mathrm{e}}{\sqrt{\pi}}\cdot 2^{\frac{m}{2}}\cdot\exp\left\{\frac{m\log m}{2}-\frac{m}{2}\right\}$. Then,

$$\mathbb{E}\left(\int_0^1 f(r)\theta_\epsilon(r)\,\mathrm{d}r\right)^m \le \frac{\mathrm{e}}{\sqrt{\pi}}\exp\left\{\frac{m}{2}\log m-\frac{m}{2}\right\}\left(\int_0^1 f^2(r)\,\mathrm{d}r\right)^{m/2}. \tag{7}$$

Therefore, $\left(\mathbb{E}\left(\int_0^1 f(r)\theta_\epsilon(r)\,\mathrm{d}r\right)^m\right)^{1/m} \le \mathrm{e}^{-\frac{1}{2}}\left(\frac{\mathrm{e}}{\sqrt{\pi}}\right)^{\frac{1}{m}}\sqrt{m}\left(\int_0^1 f^2(r)\,\mathrm{d}r\right)^{1/2}$. □

As an application of Lemma 3.1 we can obtain the following result, which plays a crucial role in our analysis.

Corollary 3.1. *For any* $0 \le \eta < \frac{1}{2}$,

$$\sup_{f\in L^2([0,1]),\,\|f\|_{L^2}\neq 0}\mathbb{E}\exp\left\{\frac{\eta\left|\int_0^1 f(r)\theta_\epsilon(r)\,\mathrm{d}r\right|^2}{\int_0^1 f^2(r)\,\mathrm{d}r}\right\} < \infty.$$

Proof. Using (7) and the Stirling formula (6), for any $0 \le \eta < \frac{1}{2}$,

$$\mathbb{E}\exp\left\{\frac{\eta\left|\int_0^1 f(r)\theta_\epsilon(r)\,\mathrm{d}r\right|^2}{\int_0^1 f^2(r)\,\mathrm{d}r}\right\} = \sum_{n=0}^\infty \frac{1}{n!}\frac{\eta^n\mathbb{E}\left|\int_0^1 f(r)\theta_\epsilon(r)\,\mathrm{d}r\right|^{2n}}{\left(\int_0^1 f^2(r)\,\mathrm{d}r\right)^n}$$
$$\le \frac{\mathrm{e}}{\sqrt{\pi}}\left(1+\sum_{n=1}^\infty\frac{(2\eta)^n}{\sqrt{2\pi n}}\right) < +\infty,$$

which completes the proof of this corollary.

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3.2. Finite-dimensional moderate deviations

In this subsection, for fixed $0 < t_1 < \cdots < t_k \le 1$, we will study the moderate deviations for $\lambda_\epsilon \big(Y_\epsilon(t_1), \cdots, Y_\epsilon(t_k)\big)^\top$. Since $K(t_i, \cdot) \in L^2([0, 1])$, there exists a sequence of finite variational functions (step functions, for example) $\{K_m(t, \cdot), m \in \mathbb{N}\}$ such that

$$\lim_{m\to\infty} \max_{1\le i\le k} \int_0^1 \big|K(t_i, r) - K_m(t_i, r)\big|^2 \,\mathrm{d}r = 0.$$

Define

$$Y_\epsilon(t_1, \ldots, t_k) = \left(\int_0^1 K(t_1, r)\theta_\epsilon(r)\,\mathrm{d}r, \ldots, \int_0^1 K(t_k, r)\theta_\epsilon(r)\,\mathrm{d}r\right)^\top,$$

$$Y_{m,\epsilon}(t_1, \ldots, t_k) = \left(\int_0^1 K_m(t_1, r)\theta_\epsilon(r)\,\mathrm{d}r, \ldots, \int_0^1 K_m(t_k, r)\theta_\epsilon(r)\,\mathrm{d}r\right)^\top.,$$

We will show that $Y_{m,\epsilon}(t_1, \ldots, t_k)$ is a good approximation of $Y_\epsilon(t_1, \ldots, t_k)$ by the following lemma.

Lemma 3.2. *For any $\delta > 0$,*

$$\limsup_{m\to\infty} \limsup_{\epsilon\to 0} \lambda_\epsilon^2 \log \mathbb{P}\big(\lambda_\epsilon \big\|Y_{m,\epsilon}(t_1, \ldots, t_k) - Y_\epsilon(t_1, \ldots, t_k)\big\| > \delta\big) = -\infty,$$

where $\|\cdot\|$ denotes the Euclidean norm.

Proof. Without loss of generality, we may assume that for $1 \le i \le k$, $\int_0^1 \big|K(t_i, r) - K_m(t_i, r)\big|^2 \,\mathrm{d}r \ne 0$. Otherwise, $\int_0^1 K(t_i, r)\theta_\epsilon(r)\,\mathrm{d}r = \int_0^1 K_m(t_i, r)\theta_\epsilon(r)\,\mathrm{d}r$ $\mathbb{P}$-almost everywhere (a.e.) corresponds to the trivial case.

For $0 < \eta < \frac{1}{2}$, by Corollary 3.1, it follows that

$$\begin{aligned}
&\mathbb{P}\big(\lambda_\epsilon \|Y_{m,\epsilon}(t_1, \ldots, t_k) - Y_\epsilon(t_1, \ldots, t_k)\| > \delta\big) \\
&\quad\le k \max_{1\le i\le k} \mathbb{P}\left(\left|\int_0^1 K(t_i, r)\theta_\epsilon(r)\,\mathrm{d}r - \int_0^1 K_m(t_i, r)\theta_\epsilon(r)\,\mathrm{d}r\right| > (k\lambda_\epsilon)^{-1}\delta\right) \\
&\quad\le k \max_{1\le i\le k} \mathbb{E}\exp\left\{\frac{\eta\big|\int_0^1 (K(t_i, r) - K_m(t_i, r))\theta_\epsilon(r)\,\mathrm{d}r\big|^2}{\int_0^1 \big|K(t_i, r) - K_m(t_i, r)\big|^2\,\mathrm{d}r}\right\} \\
&\qquad\times \exp\left\{-\frac{\eta k^{-2}\lambda_\epsilon^{-2}\delta^2}{\max_{1\le i\le k}\int_0^1 \big|K(t_i, r) - K_m(t_i, r)\big|^2\,\mathrm{d}r}\right\} \\
&\quad\le C\exp\left\{-\frac{\eta k^{-2}\lambda_\epsilon^{-2}\delta^2}{\max_{1\le i\le k}\int_0^1 \big|K(t_i, r) - K_m(t_i, r)\big|^2\,\mathrm{d}r}\right\}.
\end{aligned}$$

Consequently, we can get

$$\begin{aligned}
&\limsup_{m\to\infty} \limsup_{\epsilon\to 0} \lambda_\epsilon^2 \log \mathbb{P}\big(\lambda_\epsilon \big\|Y_{m,\epsilon}(t_1, \ldots, t_k) - Y_\epsilon(t_1, \ldots, t_k)\big\| > \delta\big) \\
&\qquad\le \limsup_{m\to\infty} -\frac{\eta k^{-2}\delta^2}{\max_{1\le i\le k}\int_0^1 \big|K(t_i, r) - K_m(t_i, r)\big|^2\,\mathrm{d}r} = -\infty,
\end{aligned}$$

which completes the proof of this lemma. □

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By [10, Theorem 4.2.16] and Lemma 3.2, to get the moderate deviations for $\lambda_\epsilon Y_\epsilon(t_1, \ldots, t_k)$ it is sufficient to study those of $\lambda_\epsilon Y_{m,\epsilon}(t_1, \ldots, t_k)$. The exponential martingale method will be applied here.

Lemma 3.3.

(i) *For any* $t \in [0, 1]$,

$$Y_{m,\epsilon}(t) = -\epsilon^2 \int_0^1 \int_0^1 K_m(t, r)\theta_\epsilon(r-)\widetilde{N}^{\epsilon^{-2}}(\mathrm{d}r, \mathrm{d}x) - \frac{\epsilon^2}{2}\int_0^1 K_m(t, r)\,\mathrm{d}\theta_\epsilon(r),$$

where $\widetilde{N}^{\epsilon^{-2}}(\mathrm{d}r, \mathrm{d}x) = N^{\epsilon^{-2}}(\mathrm{d}r, \mathrm{d}x) - \epsilon^{-2}\,\mathrm{d}r \otimes \mathrm{d}x$ *is the compensated Poisson random measure of* $N^{\epsilon^{-2}}(\mathrm{d}r, \mathrm{d}x)$.

(ii) *The family* $\{\lambda_\epsilon Y_{m,\epsilon}(t_1, \ldots, t_k),\ \epsilon > 0\}$ *satisfies the moderate deviations with speed* λ_ϵ^2 *and rate function* $J_{m;t_1,\ldots,t_k}(x) = \frac{1}{2}x^\top \Sigma^{-1}_{m;t_1,\ldots,t_k}x$, $x \in \mathbb{R}^k$, *where*

$$\Sigma_{m;t_1,\ldots,t_k} = (\sigma_{ij}(m))_{k\times k}, \qquad \sigma_{ij}(m) = \int_0^1 K_m(t_i, r)K_m(t_j, r)\,\mathrm{d}r.$$

Proof: First, by Itô's formula [1, Theorem 4.4.7],

$$\begin{aligned}\theta_\epsilon(t) &= \epsilon^{-1} - 2\int_0^t \int_0^1 \theta_\epsilon(s-)N^{\epsilon^{-2}}(\mathrm{d}s, \mathrm{d}x)\\ &= \epsilon^{-1} - 2\int_0^t \int_0^1 \theta_\epsilon(s-)\widetilde{N}^{\epsilon^{-2}}(\mathrm{d}s, \mathrm{d}x) - 2\epsilon^{-2}X_\epsilon(t), .\end{aligned}$$

where $X_\epsilon(t) = \frac{1}{\epsilon}\int_0^t (-1)^{N(\epsilon^{-2}r)}\,\mathrm{d}r$. Then,

$$X_\epsilon(t) = \frac{\epsilon}{2} - \epsilon^2 \int_0^t \int_0^1 \theta_\epsilon(s-)\widetilde{N}^{\epsilon^{-2}}(\mathrm{d}s, \mathrm{d}x) - \frac{\epsilon^2}{2}\theta_\epsilon(t),$$

which implies

$$Y_{m,\epsilon}(t) = \int_0^1 K_m(t, r)\,\mathrm{d}X_\epsilon(r) = -\epsilon^2 \int_0^1\int_0^1 K_m(t, r)\theta_\epsilon(r-)\widetilde{N}^{\epsilon^{-2}}(\mathrm{d}r, \mathrm{d}x) - \frac{\epsilon^2}{2}\int_0^1 K_m(t, r)\,\mathrm{d}\theta_\epsilon(r).$$

Second, letting $\widetilde{Y}_{m,\epsilon}(t) = -\epsilon^2 \int_0^1\int_0^1 K_m(t, r)\theta_\epsilon(r-)\widetilde{N}^{\epsilon^{-2}}(\mathrm{d}r, \mathrm{d}x)$, $t \in [0, 1]$, we now prove that the two families $\{\lambda_\epsilon Y_{m,\epsilon}(t_1, \ldots, t_k),\ \epsilon > 0\}$ and $\{\lambda_\epsilon \widetilde{Y}_{m,\epsilon}(t_1, \ldots, t_k),\ \epsilon > 0\}$ are exponentially equivalent. In fact,

$$\begin{aligned}\lambda_\epsilon \left\| Y_{m,\epsilon}(t_1, \ldots, t_k) - \widetilde{Y}_{m,\epsilon}(t_1, \ldots, t_k)\right\| &\le \max_{1\le i\le k}\left|\frac{\lambda_\epsilon\epsilon^2}{2}\int_0^1 K_m(t_i, r)\,\mathrm{d}\theta_\epsilon(r)\right|\\ &= \frac{\lambda_\epsilon\epsilon}{2}\max_{1\le i\le k}\left|\int_0^1 K_m(t_i, r)\,\mathrm{d}(-1)^{N^{\epsilon^{-2}}(r,1)}\right|\\ &= \lambda_\epsilon\epsilon\max_{1\le i\le k}\left|\sum_{j\le N^{\epsilon^{-2}}(1,1)}(-1)^{k-1}K_m(t_i, T_{\epsilon,j})\right|\\ &\le \lambda_\epsilon\epsilon\max_{1\le i\le k}\|K_m(t_i, \cdot)\|_{\mathrm{var}},\end{aligned}$$

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where $T_{\epsilon,j} \le 1$ is the jth event time in the interval [0,1], and $\|K_m(t,\cdot)\|_{\text{var}}$ is the total variation of $K_m(t,\cdot)$ in [0,1]. Therefore, for any $\delta > 0$,

$$\lim_{\epsilon\to 0} \lambda_\epsilon^2 \log \mathbb{P}\big(\lambda_\epsilon \big\| Y_{m,\epsilon}(t_1,\dots,t_k) - \widetilde{Y}_{m,\epsilon}(t_1,\dots,t_k)\big\| > \delta\big) = -\infty.$$

Finally, we study the moderate deviations for the family $\big\{\lambda_\epsilon \widetilde{Y}_{m,\epsilon}(t_1,\dots,t_k),\ \epsilon > 0\big\}$, where $\widetilde{Y}_{m,\epsilon}(t_1,\dots,t_k) = \big(\widetilde{Y}_{m,\epsilon}(t_1),\dots,\widetilde{Y}_{m,\epsilon}(t_k)\big)^\top$. Indeed, for any $\iota \in \mathbb{R}^k$, we can construct the following exponential martingale:

$$\mathcal{E}_\iota(t) = \exp\left\{\int_0^t\int_0^1 W_\epsilon(\iota,r)\widetilde{N}^{\epsilon^{-2}}(\mathrm{d}r,\mathrm{d}x) - \epsilon^{-2}\int_0^t\int_0^1 \left(\mathrm{e}^{W_\epsilon(\iota,r)} - 1 - W_\epsilon(\iota,r)\right)\mathrm{d}r\,\mathrm{d}x\right\}, \tag{8}$$

where

$$W_\epsilon(\iota,r) = -\epsilon^2\theta_\epsilon(r-,1)\iota^\top K_m(t_1,\dots,t_k;r),$$
$$K_m(t_1,\dots,t_k;r) = (K_m(t_1,r),\dots,K_m(t_k,r))^\top.$$

Note that $W_\epsilon(\iota,r)$ is a bounded function; by [7, Lemma 2.3] with $\varphi(r) = \exp(W_\epsilon(\iota,r))$, we have $\mathbb{E}\mathcal{E}_\iota(1) = 1$ for any ι. From Taylor's formula, there exists a random variable ξ with values in [0,1] such that

$$\begin{aligned}
&\mathrm{e}^{W_\epsilon(\lambda_\epsilon^{-1}\iota,r)} - 1 - W_\epsilon(\lambda_\epsilon^{-1}\iota,r)\\
&\quad= \frac{W_\epsilon^2(\lambda_\epsilon^{-1}\iota,r)}{2}\mathrm{e}^{\xi W_\epsilon(\lambda_\epsilon^{-1}\iota,r)} = \frac{W_\epsilon^2(\lambda_\epsilon^{-1}\iota,r)}{2}\left(1+\epsilon\lambda_\epsilon^{-1}O(1)\right)\\
&\quad= \frac{1}{2}\epsilon^2\lambda_\epsilon^{-2}\iota^\top K_m(t_1,\dots,t_k;r)\big(K_m(t_1,\dots,t_k;r)\big)^\top\iota\left(1+\epsilon\lambda_\epsilon^{-1}O(1)\right),
\end{aligned} \tag{9}$$

where $|O(1)| \le M$ for some M independent of ϵ. By $\mathbb{E}\mathcal{E}_{\lambda_\epsilon^{-1}\iota}(1) = 1$, (8) and (9) imply that

$$\begin{aligned}
&\mathbb{E}\exp\left\{\lambda_\epsilon^{-1}\iota^\top\widetilde{Y}_{m,\epsilon}(t_1,\dots,t_k)\right\} = \mathbb{E}\exp\left\{\int_0^t\int_0^1 W_\epsilon(\lambda_\epsilon^{-1}\iota,r)\widetilde{N}^{\epsilon^{-2}}(\mathrm{d}r,\mathrm{d}x)\right\}\\
&= \mathbb{E}\exp\left\{\int_0^t\int_0^1 W_\epsilon(\lambda_\epsilon^{-1}\iota,r)\widetilde{N}^{\epsilon^{-2}}(\mathrm{d}r,\mathrm{d}x) - \epsilon^{-2}\int_0^t\int_0^1\left(\mathrm{e}^{W_\epsilon(\iota,r)} - 1 - W_\epsilon(\iota,r)\right)\mathrm{d}r\,\mathrm{d}x\right.\\
&\qquad\left. + \frac{1}{2}\lambda_\epsilon^{-2}\iota^\top K_m(t_1,\dots,t_k;r)\big(K_m(t_1,\dots,t_k;r)\big)^\top\iota\left(1+\epsilon\lambda_\epsilon^{-1}O(1)\right)\right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}\iota^\top K_m(t_1,\dots,t_k;r)\big(K_m(t_1,\dots,t_k;r)\big)^\top\iota\left(1-\epsilon\lambda_\epsilon^{-1}M\right)\\
&\qquad\le \lambda_\epsilon^2\log\mathbb{E}\exp\left\{\lambda_\epsilon^{-1}\iota^\top\widetilde{Y}_{m,\epsilon}(t_1,\dots,t_k)\right\}\\
&\qquad\le \frac{1}{2}\iota^\top K_m(t_1,\dots,t_k;r)\big(K_m(t_1,\dots,t_k;r)\big)^\top\iota\left(1+\epsilon\lambda_\epsilon^{-1}M\right),
\end{aligned}$$

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implying that

$$\lim_{\epsilon\to 0}\lambda_\epsilon^2 \log \mathbb{E}\exp\left\{\lambda_\epsilon^{-1}\iota^\top \widetilde{Y}_{m,\epsilon}(t_1,\ldots,t_k)\right\}$$
$$= \frac{1}{2}\iota^\top \left(\int_0^1 K_m(t_1,\ldots,t_k;r)\big(K_m(t_1,\ldots,t_k;r)\big)^\top \mathrm{d}r\right)\iota.$$

By the Gärtner–Ellis theorem [10, Theorem 2.3.6]. the family $\left\{\lambda_\epsilon \widetilde{Y}_{m,\epsilon}(t_1,\ldots,t_k),\ \epsilon>0\right\}$ satisfies the moderate deviations with speed λ_ϵ^{-2} and rate function

$$J_{m;t_1,\ldots,t_k}(x) = \sup_{\iota\in\mathbb{R}^k}\left\{\iota^\top x - \frac{1}{2}\iota^\top\left(\int_0^1 K_m(t_1,\ldots,t_k;r)\big(K_m(t_1,\ldots,t_k;r)\big)^\top \mathrm{d}r\right)\iota\right\}$$
$$= \frac{1}{2}x^\top \Sigma^{-1}_{m;t_1,\ldots,t_k}x, \qquad x\in\mathbb{R}^k.$$

We end this subsection with the moderate deviations of $\left\{\lambda_\epsilon Y_\epsilon(t_1,\ldots,t_k),\ \epsilon>0\right\}$.

Proposition 3.1. *The family* $\left\{\lambda_\epsilon Y_\epsilon(t_1,\ldots,t_k),\ \epsilon>0\right\}$ *satisfies the weak moderate deviations with speed* λ_ϵ^2 *and rate function* $J_{t_1,\ldots,t_k}(x)=\frac{1}{2}x^\top\Sigma^{-1}_{t_1,\ldots,t_k}x$, $x\in\mathbb{R}^k$, *where* $\Sigma_{t_1,\ldots,t_k}=(\sigma_{ij})_{k\times k}$, $\sigma_{ij}=\int_0^1 K(t_i,r)K(t_j,r)\,\mathrm{d}r$.

Proof. By [10, Theorem 4.2.16] and Lemma 3.3, the family $\left\{\lambda_\epsilon Y_\epsilon(t_1,\ldots,t_k),\ \epsilon>0\right\}$ satisfies the moderate deviations with speed λ_ϵ^2 and rate function

$$\widetilde{J}_{t_1,\ldots,t_k}(x) = \sup_{\delta>0}\liminf_{m\to\infty}\inf_{y\in B_x(\delta)} J_{m;t_1,\ldots,t_k}(y), \tag{10}$$

where $B_x(\delta)=\left\{y:\|x-y\|\le\delta\right\}$, and $J_{m;t_1,\ldots,t_k}$ is defined in Lemma 3.3. As $m\to\infty$,

$$\sigma_{ij}(m)=\int_0^1 K_m(t_i,r)K_m(t_j,r)\,\mathrm{d}r \to \int_0^1 K(t_i,r)K(t_j,r)\,\mathrm{d}r=\sigma_{ij}, \qquad i,j=1,\ldots,k,$$

which implies that $\widetilde{J}_{t_1,\ldots,t_k}(x)=J_{t_1,\ldots,t_k}(x)$ by classical argument and the definition of (10). □

3.3. Exponential tightness and proof of Theorem 1.1

For the exponential tightness of $\{\lambda_\epsilon Y_\epsilon,\ \epsilon>0\}$, we utilize Besov–Lèvy modulus embedding as illustrated by [14, Theorem A.19].

Proposition 3.2. *The family* $\{\lambda_\epsilon Y_\epsilon,\ \epsilon>0\}$ *is exponentially tight in C([0,1]) equipped with the uniform topology.*

Proof. Since G is strictly increasing and continuous, then G^{-1}, its inverse function, is also strictly increasing and continuous. Define $\mathbb{Y}_\epsilon(t)=Y_\epsilon\big(G^{-1}(t)\big)$, $t\in[0,1]$. Note that

$$Y_\epsilon\big(G^{-1}(t)\big)-Y_\epsilon\big(G^{-1}(s)\big)=\int_0^1 (K(G^{-1}(t),r)-K(G^{-1}(s),r))\theta_\epsilon(r)\,\mathrm{d}r.$$

Without of loss of generality, we may assume that $\int_0^1 (K(G^{-1}(t),r)-K(G^{-1}(s),r))^2\,\mathrm{d}r\neq 0$, otherwise $Y_\epsilon\big(G^{-1}(t)\big)-Y_\epsilon\big(G^{-1}(s)\big)=0$, $\mathbb{P}$-a.e., corresponds to the trivial case.

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On the other hand, for $t \neq s$, $d(t, s) := |t-s|^{\alpha/2}$,

$$\frac{|\mathbb{Y}_\epsilon(t) - \mathbb{Y}_\epsilon(s)|^2}{d(t,s)} = \frac{|\mathbb{Y}_\epsilon(t) - \mathbb{Y}_\epsilon(s)|^2}{\int_0^1 (K(G^{-1}(t), r) - K(G^{-1}(s), r))^2 \, \mathrm{d}r} = \frac{\int_0^1 (K(G^{-1}(t), r) - K(G^{-1}(s), r))^2 \, \mathrm{d}r}{d(t,s)}.$$

Therefore, Corollary 3.1 and Condition 1.2 imply that

$$1 + \sup_{s,t\in[0,1]} \mathbb{E}\psi\left(\frac{|\mathbb{Y}_\epsilon(t) - \mathbb{Y}_\epsilon(s)|}{d(t,s)}\right) = \sup_{t,s\in[0,1]} \mathbb{E}\exp\left\{\frac{\eta|\mathbb{Y}_\epsilon(t) - \mathbb{Y}_\epsilon(s)|^2}{|t-s|^\alpha}\right\} < \infty,$$

where $d(t,s) = |t-s|^{\alpha/2}$ and $\psi(x) = \mathrm{e}^{\eta x^2} - 1$, $0 < \alpha < 2$, $0 < \eta < \frac{1}{2}$. Using [14, Theorem A.19], there exists $c = c(\alpha)$ such that

$$\mathbb{E}\exp\left\{c\eta \sup_{t,s\in[0,1]} \frac{|\mathbb{Y}_\epsilon(t) - \mathbb{Y}_\epsilon(s)|^2}{|t-s|^{\alpha'}}\right\} < \infty,$$

where $0 < \alpha' < \alpha$. Hence, for any $L > 0$, it follows that

$$\begin{aligned}
&\limsup_{\delta\to 0} \limsup_{\epsilon\to 0} \lambda_\epsilon^2 \log \mathbb{P}\left\{\sup_{|t-s|<\delta} |\lambda_\epsilon \mathbb{Y}_\epsilon(t) - \lambda_\epsilon \mathbb{Y}_\epsilon(s)| > L\right\} \\
&\le \limsup_{\delta\to 0} \limsup_{\epsilon\to 0} \lambda_\epsilon^2 \left(-\frac{c\eta L^2}{\lambda_\epsilon^2 \delta^{\alpha'}} + \log \mathbb{E}\exp\left\{c\eta \sup_{t,s\in[0,1]} \frac{|\mathbb{Y}_\epsilon(t) - \mathbb{Y}_\epsilon(s)|^2}{|t-s|^{\alpha'}}\right\}\right) \\
&= -\infty,
\end{aligned}$$

which implies the exponential tightness of $\{\lambda_\epsilon \mathbb{Y}_\epsilon, \epsilon > 0\}$; i.e. for any $L > 0$, there exists a compact set $\mathcal{K}_L \subset C([0,1])$ such that $\limsup_{\epsilon\to 0} \lambda_\epsilon^2 \log \mathbb{P}(\lambda_\epsilon \mathbb{Y}_\epsilon \in \mathcal{K}_L^c) \le -L$.

Since G is strictly increasing and continuous, $\tilde{\mathcal{K}}_L := \{f : f = g \circ G \text{ for some } g \in \mathcal{K}_L\}$ is also a compact set in $C([0,1])$ and $h \in \tilde{\mathcal{K}}_L$ if and only if $h \circ G^{-1} \in \mathcal{K}_L$. Therefore, $\limsup_{\epsilon\to 0} \lambda_\epsilon^2 \log \mathbb{P}(\lambda_\epsilon Y_\epsilon \in \tilde{\mathcal{K}}_L^c) \le \limsup_{\epsilon\to 0} \lambda_\epsilon^2 \log \mathbb{P}(\lambda_\epsilon \mathbb{Y}_\epsilon \in \mathcal{K}_L^c) \le -L$, which implies the desired result. □

Proof of Theorem 1.1. By Propositions 3.1 and 3.2, the family $\{\lambda_\epsilon Y_\epsilon, \epsilon > 0\}$ satisfies the sample path moderate deviation in $C([0,1])$ with uniform topology, with speed λ_ϵ^2 and rate function $\widetilde{I}(\cdot)$ defined by

$$\widetilde{I}(f) = \sup_{k\in\mathbb{N}} \sup_{0\le t_1\le\cdots\le t_k} J_{t_1,\ldots,t_k}(f(t_1), \ldots, f(t_k)), \qquad f \in C([0,1]),$$

where $J_{t_1,\ldots,t_k}(\cdot)$ is defined in Proposition 3.1.

To end the proof of this theorem, we only need to show $\widetilde{I}(f) = I(f)$ for $f \in C([0,1])$, where $I(\cdot)$ is defined by (3). In fact, by [11, Theorem 3.4.5] or the weak convergence approach introduced in [6, 7], $\{\lambda_\epsilon \int_0^1 K(t,r)\,\mathrm{d}W_r, t \in [0,1]\}$ satisfies the sample path moderate deviations with speed λ_ϵ^2 and rate function $I(\cdot)$. Moreover, it is noted that the rate function for the finite-dimensional vector $\{\lambda_\epsilon(\int_0^1 K(t_1,r)\mathrm{d}B_r, \ldots, \int_0^1 K(t_k,r)\mathrm{d}B_r), \epsilon > 0\}$ is just $J_{t_1,\ldots,t_k}(\cdot)$. Therefore, by the Dawson–Gärtner projective limits theorem and exponential tightness [10, Theorems 4.6.1, 4.2.4], we have $\widetilde{I}(f) = I(f)$. □

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