Bull. Aust. Math. Soc. **112** (2025), 418–429
doi:10.1017/S0004972725000267

MONOTONIC COLLATZ SUBSEQUENCES WITH TERMS CONGRUENT MODULO A FIXED POWER OF TWO

W. E. LONGSTAFF

(Received 16 November 2024; accepted 28 January 2025; first published online 16 May 2025)

Abstract

A class of sequences called L-sequences is introduced, each one being a subsequence of a Collatz sequence. Every ordered pair (v, w) of positive integers determines an odd positive integer P such that there exists an L-sequence of length n for every positive integer n, each term of which is congruent to P modulo 2^{v+w+1}. The smallest possible initial term of such a sequence is described. If $3^v > 2^{v+w}$ the L-sequence is increasing. Otherwise, it is decreasing, except if it is the constant sequence P. A central role is played by Bezout's identity.

2020 *Mathematics subject classification*: primary 11B75.

Keywords and phrases: Collatz, congruence, sequences.

1. Introduction

Two integers x, y are *congruent modulo* m, where $m \in \mathbb{Z}^+$, if $x - y$ is divisible by m. This relation is denoted by $x \equiv y \pmod{m}$. For fixed m, this is an equivalence relation on $\mathbb{Z}$. The set of integers congruent to x, that is, the equivalence class containing x, is $\{y \in \mathbb{Z} : y = x + ms, \text{ for some } s \in \mathbb{Z}\}$. Each integer x is congruent to a unique integer between 0 and $m - 1$. This unique integer will be denoted by $[x]_m$, and will be called x *modulo* m. Then $x = [x]_m + mt$, for some integer t. It is easily verified that $[x + y]_m \equiv [x]_m + [y]_m \pmod{m}$ and $[x \cdot y]_m \equiv [x]_m \cdot [y]_m \pmod{m}$, for all integers x, y.

If x, y are coprime integers, there exist integers a, b such that $xa - yb = 1$. This follows from the well-known Bezout's identity. We will take $x = 2^u$ and $y = 3^v$, or vice versa, for positive integers u, v. If (a, b) is a solution of $2^u x - 3^v y = 1$, that is, $2^u a - 3^v b = 1$, the *general solution*, which is the set of all solutions of $2^u x - 3^v y = 1$, is $\{(x, y) : x = a + 3^v t, y = b + 2^u t, t \in \mathbb{Z}\}$. If either of a, b is positive then so is the other. If (a, b) is a solution, then a is not divisible by 3 and b is odd. Also, if $a > 0$, the least solution (x, y) with $x > 0$ is $x = [a]_{3^v}$. Similar comments apply to the equation $3^v x - 2^u y = 1$.

© The Author(s), 2025. Published by Cambridge University Press on behalf of Australian Mathematical Publishing Association Inc.

https://doi.org/10.1017/S0004972725000267 Published online by Cambridge University Press

DEFINITION 1.1. If z is a nonzero integer, the odd part of z, denoted by $O(z)$, is defined to be the largest odd positive integer dividing z.

REMARK 1.2. For any nonzero integer y and positive integer v, $O(2^v y) = O(y) \leq |y|$.

DEFINITION 1.3. The *Collatz map* is the map $C : \mathbb{Z} \to \mathbb{Z}$ defined by

$$C(x) = \begin{cases} 3x+1 & \text{if } x \text{ is odd} \\ x/2 & \text{if } x \text{ is even.} \end{cases}$$

A *Collatz sequence* is a sequence of integers in which each term S is obtained from its predecessor T by applying the Collatz map, that is, $S = C(T)$. So the sequence with intitial term T is $T, C(T), C^{(2)}(T), C^{(3)}(T), \ldots$. The Collatz conjecture is that, starting at any positive integer T, there exists $n \in \mathbb{N}$ such that $C^{(n)}(T) = 1$. This conjecture arose in 1937 in the work of L. Collatz [3] and has yet to be resolved. Recent important results are [1, 4].

Any odd positive integer M can be written uniquely as $M = 2^m x - 1$ where $m, x \in \mathbb{Z}^+$. Applying the Collatz rule mentioned above gives the next term as $2^m \cdot 3x - 2$ and the next as $2^{m-1} \cdot 3x - 1$. So, after two applications of the rule, a two has been replaced by a three. After $2(m-1)$ applications of the rule, $m-1$ twos have been replaced by threes and we get $2 \cdot 3^{m-1} - 1$, and a further two applications gives the even number $3^m x - 1$, where all the twos have been replaced with threes. Replacing a two with a three gives a larger-valued term, so the sequence is increasing up to this term. The next odd term will be $O(3^m x - 1)$.The latter is less than its predecessor and equals $(3^m x - 1)/2^q$ for some positive integer q.

We will consider sequences each beginning at a positive odd integer M expressed in the form $M = 2^m x - 1$, where $m, x \in \mathbb{Z}^+$ and x is odd. We define the map

$$L : \{M \in \mathbb{Z}^+ : M \text{ odd}\} \to \{M \in \mathbb{Z}^+ : M \text{ odd}\} \quad \text{by} \quad L(2^m x - 1) = O(3^m x - 1).$$

Of course, $O(3^m x - 1) = (3^m x - 1)/2^q$ for some positive integer q. The sequences

$$T, L(T), L^{(2)}(T), L^{(3)}(T), \ldots,$$

where T is an odd positive integer, are the sequences under consideration in this paper. For convenience we will call them *L-sequences*.

Slightly more detail of the sequence $T, L(T), L^{(2)}(T), L^{(3)}(T), \ldots$ with initial term $T = 2^{m_1} x_1 - 1$, comes from

$$2^{m_1} x_1 - 1 \mapsto 2^{m_2} x_2 - 1 \mapsto 2^{m_3} x_3 - 1 \mapsto 2^{m_4} x_4 - 1 \mapsto \ldots,$$

where x_k is an odd positive integer defined by $L(2^{m_k} x_k - 1) = O(3^{m_k} x_k - 1) = (3^{m_k} x_k - 1)/2^{q_k} = 2^{m_{k+1}} x_{k+1} - 1$, for all $k \geq 1$. *Throughout the paper the notation $x \mapsto y$ will be reserved for the L-map, that is, it will mean that x and y are positive odd integers and $L(x) = y$.*

https://doi.org/10.1017/S0004972725000267 Published online by Cambridge University Press

EXAMPLE 1.4. (i) $3\,165 \mapsto 1\,187 \mapsto 167 \mapsto 283 \mapsto 319$ since

$$\begin{aligned} 3\,165 &= 2 \cdot 1\,583 - 1 \quad \text{and} \quad 3 \cdot 1\,583 - 1 = 2^2 \cdot 1\,187, \\ 1\,187 &= 2^2 \cdot 297 - 1 \quad \text{and} \quad 3^2 \cdot 297 - 1 = 2^4 \cdot 167, \\ 167 &= 2^3 \cdot 21 - 1 \quad \text{and} \quad 3^3 \cdot 21 - 1 = 2 \cdot 283, \\ 283 &= 2^2 \cdot 71 - 1 \quad \text{and} \quad 3^2 \cdot 71 - 1 = 2 \cdot 319. \end{aligned}$$

(ii) $331\,247 \mapsto 838\,471 \mapsto 1\,414\,921 \mapsto 1\,061\,191 \mapsto 1\,790\,761$ since

$$\begin{aligned} 331\,247 &= 2^4 \cdot 20\,703 - 1 \quad \text{and} \quad 3^4 \cdot 20\,703 - 1 = 2 \cdot 838\,471, \\ 838\,471 &= 2^3 \cdot 104\,809 - 1 \quad \text{and} \quad 3^3 \cdot 104\,809 - 1 = 2 \cdot 1\,414\,921, \\ 1\,414\,921 &= 2 \cdot 707\,461 - 1 \quad \text{and} \quad 3 \cdot 707\,461 - 1 = 2 \cdot 1\,061\,191, \\ 1\,061\,191 &= 2^3 \cdot 132\,649 - 1 \quad \text{and} \quad 3^3 \cdot 132\,649 - 1 = 2 \cdot 1\,790\,761. \end{aligned}$$

DEFINITION 1.5. For $u, P \in \mathbb{Z}^+$ with P odd satisfying $1 \le P \le 2^{u+1} - 1$ define

$$C^+(u, P) = \{2^{u+1}N + P : N \in \mathbb{N}\}.$$

Below we consider L-sequences all of whose terms belong to a class $C^+(u, P)$ for some fixed u and P.

2. Main theorem

Before proving our main theorem we consider the action of the L map $L(2^m x - 1) = O(3^m x - 1)$, where $m, x \in \mathbb{Z}^+$ and x is odd, on the classes $C^+(u, P)$. Proposition 2.1 below shows that, given a class $C^+(u, P)$, there always exists an element T of it such that $L(T)$ does not belong to it. Proposition 2.2 shows that all of the elements of $C^+(u, P)$ can have this property, so that there are no L-sequences in $C^+(u, P)$ of length more than one. Proposition 2.4 shows that there is a $C^+(u, P)$ containing L-sequences of arbitrary length. Our main theorem will show that there are many such classes.

PROPOSITION 2.1. *Let $u, P \in \mathbb{Z}^+$ with P odd satisfying $1 \le P \le 2^{u+1} - 1$. There exists $T \in C^+(u, P)$ such that $L(T) \notin C^+(u, P)$.*

PROOF. If $L(P) \notin C^+(u, P)$ the statement is proved. Otherwise, let $P = 2^v d - 1$ with $v, d \in \mathbb{Z}^+$ and d odd. Also, let $3^v d - 1 = 2^r f$, where $r, f \in \mathbb{Z}^+$ and f is odd. Then $P \mapsto f$ since $P = 2^v d - 1 \mapsto O(3^v d - 1) = f$. Hence $f = 2^{u+1}M + P$ for some $M \in \mathbb{N}$. Consider

$$2^{u+1} \cdot (2^{v+r-1}) + P = 2^v\{2^{u+r} + d\} - 1 \mapsto O(3^v \cdot 2^{u+r} + 2^r f) = 3^v \cdot 2^u + f.$$

If $3^v \cdot 2^u + f \in C^+(u, P)$ then $3^v \cdot 2^u + f = 2^{u+1}K + P$, for some $K \in \mathbb{N}$. Then

$$3^v \cdot 2^u + f = 3^v \cdot 2^u + 2^{u+1}M + P = 2^{u+1}K + P,$$

so $2^{u+1}K - 3^v \cdot 2^u = 2^{u+1}M$ and hence $2K - 3^v = 2M$, which is a contradiction. Thus $T = 2^{u+1} \cdot (2^{v+r-1}) + P$ satisfies $L(T) \notin C^+(u, P)$. □

https://doi.org/10.1017/S0004972725000267 Published online by Cambridge University Press

PROPOSITION 2.2. *Let $u, P \in \mathbb{Z}^+$ with P odd satisfying $1 \le P \le 2^{u+1} - 1$. If $P = 2^v d - 1$ with $v, d \ge 1$ and d odd, and $3^v d - 1 = 2^r f$ with $r, f \ge 1$ and f odd, then there are no $T \in C^+(u, P)$ such that $L(T) \in C^+(u, P)$ if $u - v > r$ and $(f - P)/2$ is odd.*

PROOF. Let $u - v > r$ and $(f - P)/2$ be odd. Then, for any $N \in \mathbb{N}$,

$$T = 2^{u+1}N + P = 2^v\{2^{u-v+1}N + d\} - 1 \mapsto O(3^v \cdot 2^{u-v+1} + 3^v \cdot d - 1)$$
$$= O(3^v \cdot 2^{u-v+1} + 2^r f) = O(3^v \cdot 2^{u-v-r+1} + f) = 3^v \cdot 2^{u-v-r+1} + f.$$

Now $3^v \cdot 2^{u-v-r+1} + f \in C^+(u, P)$ if and only if there exists $M \in \mathbb{N}$ for which we have $3^v \cdot 2^{u-v-r+1} + f = 2^{u+1}M + P$, that is, if and only if $2^u M - 3^v \cdot 2^{u-v-r}N = (f - P)/2$. Since $u - v > r$ and $(f - P)/2$ is odd, no such M can exist. □

EXAMPLE 2.3. If $u \ge 4$ there is no $N \in \mathbb{N}$ such that $L(2^{u+1}N + 11) = 2^{u+1}M + 11$, for some $M \in \mathbb{N}$. Here $v = 2, d = 3, w = 1, r = 1, f = 13$. Briefly, $2^{u+1}N + 11 = 2^2\{2^{u-1}N + 3\} - 1 \mapsto O(3^2 \cdot 2^{u-1}N + 26) = 3^2 \cdot 2^{u-2}N + 13 = 2^{u+1}M + 11$ if and only if $2^u M - 3^2 \cdot 2^{u-3}N = 1$.

PROPOSITION 2.4. *For every $n \in \mathbb{Z}^+, n \ge 2$, let $T_1^{(n)} = 4 \cdot 8^{n+1} - 5$. Then $T_1^{(n)}$ is the first term of an L-sequence in $C^+(4, 27)$ with n terms.*

PROOF. For $2 \le k \le n$ define $T_k^{(n)}$ by $T_k^{(n)} = 4 \cdot 3^{2(k-1)} \cdot 8^{n-k+2} - 5$. We show that $L(T_k^{(n)}) = T_{k+1}^{(n)}$, for $1 \le k \le n - 1$. This is true if $k = 1$ since $T_1^{(n)} = 4 \cdot 8^{n+1} - 5 = 4\{8^{n+1} - 1\} - 1 \mapsto O(3^2 \cdot 8^{n+1} - 10) = 4 \cdot 3^2 \cdot 8^n - 5$. Let $2 \le k \le n - 1$. We have $T_k^{(n)} = 4 \cdot 3^{2(k-1)} \cdot 8^{n-k+2} - 5 = 4\{3^{2(k-1)} \cdot 8^{n-k+2} - 1\} - 1 \mapsto O(3^{2k} \cdot 8^{n-k+2} - 10) = 4 \cdot 3^{2k} \cdot 8^{n-k+1} - 5$. Since $T_k^{(n)} = 32(3^{2(k-1)} \cdot 8^{n-k+1} - 1) + 27$, for $1 \le k \le n$, $T_1^{(n)}$ is the first term of an L-sequence in $C^+(4, 27)$ with n terms. □

DEFINITION 2.5. For $p \in \mathbb{N}$ and $U \in \mathbb{Z}, U \ne 1$, define $Z_p(U) = (U^p - 1)/(U - 1)$, taking $0^0 = 1$ so that $Z_0(0) = 0$.

REMARK 2.6. Note that, for $p \ge 1$, $Z_p(U) = 1 + U + U^2 + \cdots + U^{p-1}$, so $Z_{p+1}(U) = Z_p(U) + U^p$ and $Z_p(U) = UZ_{p-1}(U) + 1$.

LEMMA 2.7. *For all $k \in \mathbb{N}$, if $a, b, u, v \in \mathbb{Z}^+$ with $2^u a - 3^v b = 1$ and $Y = 3^v(a - b)$, then $3^{v(k+1)}b(a - b)^k + 3^v a Z_k(Y) = 2^u a Z_{k+1}(Y) - 1$.*

PROOF. Notice first that $3^v a Z_k(Y) = 2^u a Z_k(Y) + Y^k - 1$. This follows from

$$(2^u a - 3^v a)Z_k(Y) = (2^u a - 3^v b + 3^v(b - a))Z_k(Y) = (1 - Y)Z_k(Y) = 1 - Y^k.$$

Then

$$3^{v(k+1)}b(a - b)^k + 3^v a Z_k(Y) = 3^v b Y^k + 2^u a Z_k(Y) + Y^k - 1$$
$$= (3^v b - 2^u a)Y^k + 2^u a(Y^k + Z_k(Y)) + Y^k - 1$$
$$= -Y^k + 2^u a Z_{k+1}(Y) + Y^k - 1 = 2^u a Z_{k+1}(Y) - 1.$$ □

https://doi.org/10.1017/S0004972725000267 Published online by Cambridge University Press

THEOREM 2.8 (Main Theorem). *Let $v, w \in \mathbb{Z}^+$ and let (f, d) be the solution to $3^v d - 2^w f = 1$ which has f as the smallest odd positive integer. Let $P = 2^v d - 1$ and $u = v + w$. Let (a, b) be the least positive solution of $2^u a - 3^v b = 1$ and let (A, B) be the least positive solution of $2^u A - 3^v B = C$ where $C = (f - P)/2$. Let $E = b(A - B)$ and $F = a(A - B)$.*

(i) *The integer d is odd and $1 \le P < 2^{u+1} - 1$.*
(ii) *$L(2^{u+1}N + P) = 2 \cdot 3^v N + f$, that is, $2^{u+1}N + P \mapsto 2 \cdot 3^v N + f$, for every $N \in \mathbb{N}$.*
(iii) *For $n \in \mathbb{Z}^+$, $n \ge 2$, define $T_1^{(n)}$ by*

$$T_1^{(n)} = 2^{u+1}B + P + E\,2^{2u+1}Z_{n-2}(X),$$

where $X = 2^u(a - b)$. Let $t \in \mathbb{Z}$ be such that $S_1^{(n;t)} = T_1^{(n)} + 2^{nu+1}t$ is positive. Then $\{S_k^{(n;t)} : 1 \le k \le n\}$ is an L-sequence in $C^+(u, P)$ where

$$\begin{aligned} S_k^{(n;t)} &= 2^{u+1}A + P + E2^{u+1} \cdot 3^{v(k-1)}(a - b)^{k-2}Z_{n-k}(X) \\ &\quad + F2^{u+1} \cdot 3^v Z_{k-2}(Y) + 3^{(k-1)v} \cdot 2^{(n-k+1)u+1}t, \end{aligned}$$

for $2 \le k \le n$, where $Y = 3^v(a - b)$.
(iv) *We have $C > 0$ if and only if $3^v > 2^u$ if and only if $L(2^{u+1}N + P) > 2^{u+1}N + P$, for every $N \in \mathbb{N}$. Consequently, $\{S_k^{(n;t)} : 1 \le k \le n\}$ is an increasing sequence if $3^v > 2^u$.*
(v) *If $C \ne 0$ we have $C < 0$ if and only if $3^v < 2^u$ if and only if $L(2^{u+1}N + P) < 2^{u+1}N + P$, for every $N \in \mathbb{N}$. Consequently, $\{S_k^{(n;t)} : 1 \le k \le n\}$ is a decreasing sequence if $3^v < 2^u$ and $C \ne 0$.*
(vi) (a) *With $T_n^{(n)}$ defined by $T_n^{(n)} = 2^{u+1}A + P + F\,2^{u+1} \cdot 3^v Z_{n-2}(Y)$, every L-sequence in $C^+(u, P)$ with $n \ge 2$ terms has initial term $T_1^{(n)} + 2^{nu+1}s$, and final term $T_n^{(n)} + 2^{u+1} \cdot 3^{(n-1)v}s$ for some $s \in \mathbb{Z}$.*
(b) *The least initial term of an L-sequence in $C^+(u, P)$ with n terms is $[T_1^{(n)}]_{2^{nu+1}}$.*

PROOF. (i) Clearly d must be odd. Let d_0, f_0 be the smallest positive solution of $3^v d_0 - 2^w f_0 = 1$. Then $f = f_0$ if f_0 is odd and $f_0 + 3^v$ otherwise. Correspondingly, $d = d_0$ or $d_0 + 2^w$. But $d_0 < 2^w$ so $d < 2^{w+1}$ and $P = 2^v d - 1 < 2^{u+1} - 1$ since $u = v + w$.

(ii) Let $N \in \mathbb{N}$. Then $2^{u+1}N + P = 2^{u+1}N + 2^v d - 1 = 2^v\{2^{w+1}N + d\} - 1 \mapsto O(3^v \cdot 2^{w+1}N + 3^v d - 1) = O(3^v \cdot 2^{w+1}N + 2^w f) = 3^v \cdot 2N + f$.

(iii) Clearly $S_k^{(n;t)}$ is congruent to P modulo 2^{u+1}, for $1 \le k \le n$. Since $L(T) \ge 1$, for every $T \in \mathbb{Z}^+$ it only remains to show that $L(S_k^{(n;t)}) = S_{k+1}^{(n;t)}$, for $1 \le k \le n$. We have

$$\begin{aligned} S_1^{(n;t)} &= 2^{u+1}B + P + E2^{2u+1}Z_{n-2}(X) + 2^{nu+1}t \\ &= 2^v\{2^{w+1}B + E2^{u+w+1}Z_{n-2}(X) + 2^{(n-1)u+w+1}t + d\} - 1 \\ &\mapsto O(3^v \cdot 2^{w+1}B + E2^{u+w+1} \cdot 3^v Z_{n-2}(X) + 3^v \cdot 2^{(n-1)u+w+1}t + 2^w f) \end{aligned}$$

https://doi.org/10.1017/S0004972725000267 Published online by Cambridge University Press

$$= 3^v \cdot 2B + E2^{u+1} \cdot 3^v Z_{n-2}(X) + 3^v \cdot 2^{(n-1)u+1} t + f$$
$$= 2^{u+1} A + P + E2^{u+1} \cdot 3^v Z_{n-2}(X) + 3^v \cdot 2^{(n-1)u+1} t,$$

since $2^u A - 3^v B = (f - P)/2$. Thus $L(S_1^{(n;t)}) = S_2^{(n;t)}$. Let $2 \le k \le n$. Then

$$\begin{aligned} S_k^{(n;t)} &= 2^{u+1} A + P + E2^{u+1} \cdot 3^{v(k-1)} (a-b)^{k-2} Z_{n-k}(X) \\ &\quad + F2^{u+1} \cdot 3^v Z_{k-2}(Y) + 3^{(k-1)v} \cdot 2^{(n-k+1)u+1} t \\ &= E2^{u+1} \cdot 3^{v(k-1)} (a-b)^{k-2} Z_{n-k}(X) + F2^{u+1} \cdot 3^v Z_{k-2}(Y) \\ &\quad + 2^{u+1} A + 3^{(k-1)v} \cdot 2^{(n-k+1)u+1} t + 2^v d - 1 \\ &= 2^v \{E2^{w+1} \cdot 3^{v(k-1)} (a-b)^{k-2} Z_{n-k}(X) + F2^{w+1} \cdot 3^v Z_{k-2}(Y) \\ &\quad + 2^{w+1} A + 3^{(k-1)v} \cdot 2^{(n-k)u+w+1} t + d\} - 1 \\ &\mapsto O(E2^{w+1} \cdot 3^{vk} (a-b)^{k-2} Z_{n-k}(X) + F2^{w+1} \cdot 3^{2v} Z_{k-2}(Y) \\ &\quad + 2^{w+1} \cdot 3^v A + 3^{kv} \cdot 2^{(n-k)u+w+1} t + 2^w f) \\ &= E2 \cdot 3^{vk} (a-b)^{k-2} Z_{n-k}(X) + F2 \cdot 3^{2v} Z_{k-2}(Y) + 2 \cdot 3^v A \\ &\quad + 3^{kv} \cdot 2^{(n-k)u+1} t + f = L(S_k^{(n;t)}). \end{aligned}$$

Since $Z_p(U) = UZ_{p-1} + 1$,

$$\begin{aligned} L(S_k^{(n;t)}) &= E2 \cdot 3^{vk} (a-b)^{k-2} X Z_{n-k-1}(X) + E2 \cdot 3^{vk} (a-b)^{k-2} \\ &\quad + F2 \cdot 3^{2v} Z_{k-2}(Y) + 2 \cdot 3^v A + 3^{kv} \cdot 2^{(n-k)u+1} t + f. \end{aligned}$$

Now, using Lemma 2.7,

$$\begin{aligned} E2 \cdot 3^{vk} (a-b)^{k-2} + F2 \cdot 3^{2v} Z_{k-2}(Y) &= 2(A-B) 3^v \{3^{v(k-1)} (a-b)^{k-2} b + a 3^v Z_{k-2}(Y)\} \\ &= 2(A-B) 3^v (a 2^u Z_{k-1}(Y) - 1). \end{aligned}$$

So

$$\begin{aligned} &E2 \cdot 3^{vk} (a-b)^{k-2} + F2 \cdot 3^{2v} Z_{k-2}(V) + 2 \cdot 3^v A + f \\ &\quad = 2(A-B) 3^v \{a 2^u Z_{k-1}(Y) - 1\} + 2 \cdot 3^v A + f \\ &\quad = 2^{u+1} F 3^v Z_{k-1}(Y) - 2(A-B) 3^v + 2 \cdot 3^v A + f \\ &\quad = 2^{u+1} F 3^v Z_{k-1}(Y) + 2 \cdot 3^v B + f = 2^{u+1} F 3^v Z_{k-1}(Y) + 2^{u+1} A + P. \end{aligned}$$

Since $X = 2^u(a-b)$ this gives the value of $L(S_k^{(n;t)})$ as

$$\begin{aligned} &E2 \cdot 3^{vk} (a-b)^{k-2} X Z_{n-k-1}(X) + 2^{u+1} F 3^v Z_{k-1}(Y) + 2^{u+1} A + P + 3^{kv} \cdot 2^{(n-k)u+1} t \\ &\quad = E2^{u+1} \cdot 3^{vk} (a-b)^{k-1} Z_{n-k-1}(X) + 2^{u+1} F 3^v Z_{k-1}(Y) + 2^{u+1} A + P + 3^{kv} \cdot 2^{(n-k)u+1} t, \end{aligned}$$

that is, $S_{k+1}^{(n;t)}$. Thus $\{S_k^{(n;t)} : 1 \le k \le n\}$ is an L-sequence in $C^+(u, P)$.

(iv) and (v). We have $2C = f - P$ giving the equations $2^{w+1} C = 2^w f - 2^w (2^v d - 1) = 3^v d - 1 - 2^u d + 2^w = (3^v - 2^u) d + 2^w - 1$ so $2^{w+1} C = (3^v - 2^u) d + 2^w - 1$. Rearranging the latter gives a second equation $(3^v - 2^u) d = 2^w (2C - 1) + 1$.

https://doi.org/10.1017/S0004972725000267 Published online by Cambridge University Press

From the first equation $3^v > 2^u$ implies $C > 0$. From the second equation $C > 0$ implies $3^v > 2^u$.

Suppose that $C \neq 0$. The first equation shows that $C < 0$ implies $3^v < 2^u$. The second equation shows that $3^v < 2^u$ implies $2C < 1$ which implies $C < 0$.

By (i), for every $n \in \mathbb{N}$, $L(2^{u+1}N + P) - (2^{u+1}N + P) = 2N(3^v - 2^u) + 2C$. The desired results follow.

(vi)(a) We prove the result by induction on n.

Let $W_1 = 2^{u+1}N_1 + P$ and $W_2 = 2^{u+1}N_2 + P$ for some $N_1, N_2 \in \mathbb{N}$ with $L(W_1) = W_2$. By (i), $L(W_1) = 2 \cdot 3^v N_1 + f$ so $2^{u+1}N_2 + P = 2 \cdot 3^v N_1 + f$. This gives $2^u N_2 - 3^v N_1 = C$. Since $2^u A - 3^v B = C$ it follows that $N_1 = B + 2^u s, N_2 = A + 3^v s$, for some $s \in \mathbb{Z}$. Hence $W_1 = 2^{u+1}B + P + 2^{2u+1}s$ and $W_2 = 2^{u+1}A + P + 2^{u+1} \cdot 3^v s$. Since $T_1^{(2)} = 2^{u+1}B + P$ and $T_2^{(2)} = 2^{u+1}A + P$ the result is proved for $n = 2$.

Let $n \geq 2$ and suppose that the result is true for n. Let $\{W_k\}_1^{n+1}$ be an L-sequence in $C^+(u, P)$. By the induction assumption, since $\{W_k\}_1^n$ and $\{W_k\}_2^{n+1}$ are L-sequences in $C^+(u, P)$, $W_1 = T_1^{(n)} + 2^{nu+1}p$ for some $p \in \mathbb{Z}$ and $W_2 = T_1^{(n)} + 2^{nu+1}q$, for some $q \in \mathbb{Z}$. Also $W_{n+1} = T_n^{(n)} + 2^{u+1} \cdot 3^{(n-1)v}q$. Additionally, since $W_2 = L(W_1)$ we have $2^u N_2 - 3^v N_1 = C$ where $W_1 = 2^{u+1}N_1 + P$ and $W_2 = 2^{u+1}N_2 + P$. Let $T_1^{(n)} = 2^{u+1}N_1^{(n)} + P$. Then $N_1 = N_1^{(n)} + 2^{(n-1)u}p$, $N_2 = N_1^{(n)} + 2^{(n-1)u}q$ and

$$2^u\{N_1^{(n)} + 2^{(n-1)u}q\} - 3^v\{N_1^{(n)} + 2^{(n-1)u}p\} = C.$$

This gives

$$2^{nu}q - 2^{(n-1)u} \cdot 3^v p = C - (2^u - 3^v)N_1^{(n)}. \tag{2.1}$$

Now $N_1^{(n)} = B + b(A - B)2^u Z_{n-2}(X)$ and $C = 2^u A - 3^v B$, so

$$\begin{aligned} C - (2^u - 3^v)N_1^{(n)} &= 2^u A - 3^v B - 2^u B + 3^v B - (2^u - 3^v)b(A - B)2^u Z_{n-2}(X) \\ &= 2^u(A - B)\{1 - b(2^u - 3^v)Z_{n-2}(X)\} \\ &= 2^u(A - B)\{1 - (2^u b - 2^u a + 2^u a - 3^v b)Z_{n-2}(X)\} \\ &= 2^u(A - B)\{1 - (1 - 2^u(a - b))Z_{n-2}(X)\} \\ &= 2^u(A - B)\{1 - (1 - X)Z_{n-2}(X)\} = 2^u(A - B)X^{n-2} \\ &= 2^{(n-1)u}(A - B)(a - b)^{n-2}. \end{aligned}$$

Substituting this in (2.1) gives

$$2^u q - 3^v p = (A - B)(a - b)^{n-2}.$$

Since $2^u a - 3^v b = 1$ this implies that $p = b(A - B)(a - b)^{n-2} + 2^u s$ and $q = a(A - B)(a - b)^{n-2} + 3^v s$ for some $s \in \mathbb{Z}$. Substituting these values for p, q in the equations for W_1 and W_{n+1} gives

$$\begin{aligned} W_1 &= T_1^{(n)} + 2^{nu+1}\{b(A - B)(a - b)^{n-2} + 2^u s\} \\ &= \{T_1^{(n)} + 2^{nu+1}b(A - B)(a - b)^{n-2}\} + 2^{(n+1)u+1}s = T_1^{(n+1)} + 2^{(n+1)u+1}s \end{aligned}$$

https://doi.org/10.1017/S0004972725000267 Published online by Cambridge University Press

and

$$W_{n+1} = T_n^{(n)} + 2^{u+1} \cdot 3^{(n-1)v}\{a(A-B)(a-b)^{n-2} + 3^v s\}$$
$$= \{T_n^{(n)} + 2^{u+1} \cdot 3^{(n-1)v} a(A-B)(a-b)^{n-2}\} + 2^{u+1} \cdot 3^{nv} = T_{n+1}^{(n+1)} + 2^{u+1} \cdot 3^{nv} s,$$

where we have used the facts

$$T_1^{(n+1)} - T_1^{(n)} = b(A-B)2^{2u+1}\{Z_{n-1}(X) - Z_{n-2}(X)\}$$
$$= b(A-B)2^{2u+1}X^{n-2} = b(A-B)2^{nu+1}(a-b)^{n-2}$$

and

$$T_{n+1}^{(n+1)} - T_n^{(n)} = a(A-B)2^{u+1} \cdot 3^v\{Z_{n-1}(Y) - Z_{n-2}(Y)\}$$
$$= a(A-B)2^{u+1} \cdot 3^v Y^{n-2} = a(A-B)2^{u+1} \cdot 3^{(n-1)v}(a-b)^{n-2}.$$

This completes the proof by induction.

(vi)(b) Let $t \in \mathbb{Z}$ and suppose that $W_1 = T_1^{(n)} + 2^{nu+1}t \in C^+(u,P)$. Let $\{W_k\}_1^n$ be the L-sequence with initial term W_1. Let $T_k^{(n)} = 2^{u+1}N_k^{(n)} + P$ for $1 \le k \le n$. We prove by induction that, for $1 \le k \le n$,

$$W_k = 2^{u+1}N_k^{(n)} + 2^{(n-k+1)u+1} \cdot 3^{(k-1)v}t + P.$$

The result is true for $k = 1$. Let $1 \le k \le n-1$ and suppose that the result is true for k. Then

$$W_k = 2^{u+1}N_k^{(n)} + 2^{(n-k+1)u+1} \cdot 3^{(k-1)v}t + P = 2^v\{2^{w+1}N_k^{(n)} + 2^{(n-k)u+w+1} \cdot 3^{(k-1)v}t + d\} - 1$$
$$\mapsto O(2^{w+1} \cdot 3^v N_k^{(n)} + 2^{(n-k)u+w+1} \cdot 3^{kv}t + 2^w f) = 2 \cdot 3^v N_k^{(n)} + 2^{(n-k)u+1} \cdot 3^{kv}t + f.$$

But $T_{k+1}^{(n)} = L(T_k^{(n)}) = 2 \cdot 3^v N_k^{(n)} + f$, so

$$W_{k+1} = T_{k+1}^{(n)} + 2^{(n-k)u+1} \cdot 3^{kv}t = 2^{u+1}N_{k+1}^{(n)} + 2^{(n-k)u+1} \cdot 3^{kv}t + P.$$

This completes the induction proof. Clearly, $W_k \in C^+(u,P)$ for $1 \le k \le n$.

By (iii) and (vi)(a), a positive odd integer T is the initial term of an L-sequence with $n \ge 2$ terms in $C^+(u,P)$ if and only if $T \equiv T_1^{(n)} \pmod{2^{nu+1}}$. The least such T is $[T_1^{(n)}]_{2^{nu+1}}$. □

REMARK 2.9. The following remarks are presented in the context of the hypotheses of Theorem 2.8.

If $a = b$, the formula $T_1^{(n)} = 2^{u+1}B + P + E\,2^{2u+1}Z_{n-2}(X)$ in the theorem becomes

$$T_1^{(n)} = 2^{u+1}B + P + b(A-B)2^{2u+1}Z_{n-2}(0) = 2^{u+1}B + P + b(A-B)2^{2u+1}.$$

This is independent of n for $n \ge 2$. An explanation for this is as follows. If $a = b$ then $(2^u - 3^v)b = 1$. Thus $b = 1$ and $2^u - 3^v = 1$ so $u > 1$. By the truth of the Catalan conjecture [2] we must then have $v = 1$, so $u = 2$ and $w = 1$. Also $3^v d - 2^w f = 1$ becomes $3d - 2f = 1$ so $f = d = 1$. Then $P = 1$ and $C = 0$. Since $2^u A - 3^v B = C$, $A = 3$ and $B = 4$, so $A - B = -1$. Substituting these values into the formula for $T_1^{(n)}$

https://doi.org/10.1017/S0004972725000267 Published online by Cambridge University Press

we get $T_1^{(n)} = 2^3 \cdot 4 + 1 - 2^5 = 1$. Of course $L(1) = 1$ so the L-sequence that we get in $C^+(2, 1)$ is the constant sequence 1.

If $A = B$, the formula $T_1^{(n)} = 2^{u+1}B + P + E\,2^{2u+1}$ becomes $T_1^{(n)} = 2^{u+1}B + P$. Again this is independent of n for $n \geq 2$. If $A = B$, then $(2^u - 3^v)B = C$. Since $2C = f - P$,

$$2^{w+1}(2^u - 3^v)B = 2^w(f - P) = 3^v d - 1 - 2^u d + 2^w.$$

Rearranging this gives

$$(2^u - 3^v)\{2^{w+1}B + d\} = 2^w - 1.$$

Since $2^w - 1 > 0$, $2^u - 3^v \geq 1$ so $B = 0$ since $2^{w+1} > 2^w$. Then $C = 0$ so $f = P$. Since we always have $L(P) = f$ we deduce that $T_1^{(n)} = P$ and the L-sequence that we get in $C^+(u, P)$ is the constant sequence P. (If $(2^{v+w} - 3^v)d = 2^w - 1$ has only the trivial solution $v = w = d = 1$ then $P = 1$. The author has not been able to prove this.)

EXAMPLE 2.10. Again, the examples are based on the hypotheses of Theorem 2.8.

(1) Suppose $v = w = 1$. Clearly $d = f = 1$ and $P = 1, C = (f - p)/2 = 0$. Also, $u = 2$ and $a = b = 1$ since $2^2a - 3b = 1$. By the case $a = b$ of Remark 2.9, the L-sequence that we get in $C^+(2, 1)$ is the constant sequence 1.

(2) Suppose $v = 2, w = 3$. Since $3^2d - 2^3f = 1$, once again $d = f = 1$. This time $P = 3, u = 5$ and $C = -1$. Since $2^5a - 3^2b = 1$, we have $a = 2, b = 7$ so $X = 2^5(-5) = -160$. Since $2^5A - 3^2B = -1$, we have $A = 7, B = 25$. Then

$$T_1^{(n)} = 2^6 \cdot 25 + 3 + 7(-18)2^{11}Z_{n-2}(-160).$$

For example, taking $n = 4$ gives $T_1^{(4)} = 41\,031\,235$ and we get the L-sequence

$$41\,031\,235 \mapsto 11\,540\,035 \mapsto 3\,245\,635 \mapsto 912\,835$$

in $C^+(5, 3)$. (Here $C < 0$ and $3^v - 2^u = -23 < 0$ so we expect this sequence to be decreasing.) Since $[T_1^{(n)}]_{2^{nu+1}} = [41\,031\,235]_{2\,097\,152} = 1\,185\,347$, the least initial term of an L-sequence in $C^+(5, 3)$ with four terms is 1 185 347. In fact,

$$1\,185\,347 \mapsto 333\,379 \mapsto 93\,763 \mapsto 26\,371.$$

(3) Suppose $v = 4, w = 2$. Since $3^4d - 2^2f = 1$, we have $d = 5$ and $f = 101$ (recall that f must be odd). This time, $P = 79, u = 6$ and $C = 11$. Since $2^6a - 3^4b = 1$, we have $a = 19, b = 15$, so $X = 2^6(4) = 2^8$. Since $2^6A - 3^4B = 11$, we have $A = 47, B = 37$. Then

$$T_1^{(n)} = 2^7 \cdot 37 + 79 + 15(10)2^{13}Z_{n-2}(2^8).$$

For example, taking $n = 4$ gives $T_1^{(4)} = 315\,806\,415$ and we get the L-sequence

$$315\,806\,415 \mapsto 399\,692\,495 \mapsto 505\,860\,815 \mapsto 640\,230\,095$$

https://doi.org/10.1017/S0004972725000267 Published online by Cambridge University Press

TABLE 1. Classes $C^+(u, P)$ for Theorem 2.8 with $1 \le v, w \le 4$.

r	v_r	w_r	d_r	f_r	P_r	u_r	C_r	a_r	b_r	A_r	B_r	incr./decr.
(1)	1	1	1	1	1	2	0	1	1	3	4	neither
(2)	1	2	7	5	13	3	−4	2	5	1	4	decreasing
(3)	1	3	3	1	5	4	−2	1	5	1	6	decreasing
(4)	1	4	27	5	53	5	−24	2	21	3	40	decreasing
(5)	2	1	3	13	11	3	1	8	7	8	7	increasing
(6)	2	2	5	11	19	4	−4	4	7	2	4	decreasing
(7)	2	3	1	1	3	5	−1	2	7	7	25	decreasing
(8)	2	4	9	5	35	6	−15	1	7	3	23	decreasing
(9)	3	1	1	13	7	4	3	22	13	12	7	increasing
(10)	3	2	7	47	55	5	−4	11	13	10	12	decreasing
(11)	3	3	11	37	87	6	−25	19	45	11	27	decreasing
(12)	3	4	3	5	23	7	−9	23	109	9	43	decreasing
(13)	4	1	3	121	47	5	37	38	15	29	11	increasing
(14)	4	2	5	101	79	6	11	19	15	47	37	increasing
(15)	4	3	9	91	143	7	−52	50	79	73	116	decreasing
(16)	4	4	1	5	15	8	−5	25	79	37	117	decreasing

in $C^+(6, 79)$. (Here $3^v - 2^u = 17 > 0$ so we expect this sequence to be increasing.) Since $[T_1^{(n)}]_{2^{mu+1}} = [315\,806\,415]_{33\,554\,432} = 13\,816\,527$, the least initial term of an L-sequence in $C^+(6, 79)$ with four terms is 13 816 527. In fact,

$$13\,816\,527 \mapsto 17\,486\,543 \mapsto 22\,131\,407 \mapsto 28\,010\,063.$$

Table 1 shows all of the classes $C^+(u, P)$ satisfying the hypotheses of our main theorem for values of the parameters v and w satisfying $1 \le v, w \le 4$. It also shows whether an increasing or decreasing L-sequence is obtained by applying the theorem.

3. Increasing sequences

Table 1 indicates that examples of increasing L-sequences are harder to find. These are the type of sequences that may be useful in developing techniques to solve the Collatz problem. Table 2 gives more examples of increasing L-sequences derived from Theorem 2.8(iii).

We conclude by considering increasing L-sequences in the classes of the form $C^+(4, P)$ for P an odd integer satisfying $1 \le P \le 31$. The following list of data (which continues on the next page) will be useful. For every $N \in \mathbb{N}$,

$$32N + 1 = 2\{16N + 1\} - 1 \mapsto 24N + 1 \ (< 32N + 1 \text{ if } N > 0),$$
$$32N + 3 = 4\{8N + 1\} - 1 \mapsto O(9N + 1) \le 9N + 1 < 32N + 3,$$
$$32N + 5 = 2\{16N + 3\} - 1 \mapsto 6N + 1 < 32N + 5,$$

https://doi.org/10.1017/S0004972725000267 Published online by Cambridge University Press

TABLE 2. Increasing L-sequences.

r	v_r	w_r	d_r	f_r	P_r	u_r	C_r	a_r	b_r	A_r	B_r
(17)	5	1	1	121	31	6	45	19	5	126	33
(18)	5	2	7	425	223	7	101	131	69	109	57
(19)	6	1	3	1 093	191	7	451	131	23	32	5
(20)	6	2	5	911	319	8	296	430	151	434	152
(21)	6	3	1	91	63	9	14	215	151	94	66
(22)	7	1	1	1 093	127	8	483	1 888	221	2 112	247
(23)	7	2	7	3 827	895	9	1 466	944	221	1 720	402
(24)	8	1	3	9 841	767	9	4 537	5 318	415	2 969	231
(25)	7	3	11	3 007	1 407	10	800	472	221	1 436	672

$$32N + 9 = 2\{16N + 5\} - 1 \mapsto 24N + 7 < 32N + 9,$$
$$32N + 13 = 2\{16N + 7\} - 1 \mapsto 12N + 5 < 32N + 13,$$
$$32N + 17 = 2\{16N + 9\} - 1 \mapsto 24N + 13 < 32N + 17,$$
$$32N + 19 = 4\{8N + 5\} - 1 \mapsto 18N + 11 < 32N + 19,$$
$$32N + 21 = 2\{16N + 11\} - 1 \mapsto O(3N + 2) \leq 3N + 2 < 32N + 21,$$
$$32N + 23 = 8\{4N + 3\} - 1 \mapsto O(27N + 20) \leq 27N + 20 < 32N + 23,$$
$$32N + 25 = 2\{16N + 13\} - 1 \mapsto 24N + 19 < 32N + 25,$$
$$32N + 29 = 2\{16N + 15\} - 1 \mapsto 12N + 11 < 32N + 29,$$

$$32N + 7 = 8\{4N + 1\} - 1 \mapsto 54N + 13 > 32N + 7,$$
$$32N + 11 = 4\{8N + 3\} - 1 \mapsto 36N + 13 > 32N + 11,$$
$$32N + 27 = 4\{8N + 7\} - 1 \mapsto 36N + 31 > 32N + 27,$$

$$32N + 15 = 16\{2N + 1\} - 1 \mapsto O(81N + 40),$$
$$32N + 31 = 32\{N + 1\} - 1 \mapsto ?$$

For the first eleven cases in the above list there can be no increasing L-sequences in $C^+(4, P)$ since we have $L(T) < T$, for every $T \neq 1$.

For $P \in \{7, 15, 31\}$ increasing L-sequences in $C^+(4, P)$ exist, directly or indirectly, by (iii) of our main theorem. Proof of this for the case $P = 7$ follows directly from (iii) of our main theorem (see row (9) of Table 1). For the case $P = 15$, proof follows indirectly from (iii) of our main theorem, once it is observed that $C^+(5, 47) \subseteq C^+(4, 15)$ and $C^+(5, 47)$ occurs as row (10) of Table 1. For the case $P = 31$, proof follows indirectly from (iii) of our main theorem. Note that each of $C^+(7, 191), C^+(9, 63), C^+(8, 127), C^+(9, 895), C^+(10, 1\,407)$ is a subset of $C^+(4, 31)$, since 191, 63, 127, 895, 1 407 are all congruent to 31 modulo 32. These classes occur as rows (19), (21), (22), (23), (25), respectively, of Table 2.

https://doi.org/10.1017/S0004972725000267 Published online by Cambridge University Press

The case $P = 27$ is established by Proposition 2.4. Note that from the data listed above, $L(T) > T$ for all $T \in C^+(4, 27)$ so this L-sequence is increasing.

The case $P = 11$ is extreme. We observed in Example 2.3 that $L(T) \notin C^+(4, 11)$ for every $T \in C^+(4, 11)$ (so the longest L-sequence in $C^+(4, 11)$ has length one).

REMARK 3.1. With the hypotheses and notation of the main theorem, notice that

$$T_1^{(n+1)} - T_1^{(n)} = b(A - B)2^{2u+1}\{Z_{n-1}(X) - Z_{n-2}(X)\} = b(A - B)(a - b)^{n-2}2^{nu+1}.$$

The examples of increasing L-sequences that we obtain from (iv) of our main theorem arise when $3^v > 2^u$. In this case $T_1^{(n)}$ increases with n. For if $3^v > 2^u$ we must have $a - b > 0$ and $A - B > 0$. Indeed, if $a < b$ then $1 = 2^u a - 3^v b < (2^u - 3^v)b < 0$. Similarly, if $A < B$ then $C = 2^u A - 3^v B < (2^u - 3^v)B < 0$, a contradiction.

For example, taking $P = 1\,407$ and $u = 10$ (so $v = 7, w = 3$) gives

$$T_1^{(n+1)} = T_1^{(n)} + 168\,844(251)^{n-2} \cdot 2^{10n+1}.$$

Using this iteration, $T_1^{(4)} = 1\,599\,633\,425\,791$. Since this is less that 2^{41}, it follows from (vi)(b) of our main theorem that the L-sequence with four terms, each term congruent to $1\,407$ modulo 2^{11} with least initial term has initial term $1\,599\,633\,425\,791 \approx 1.599 \times 10^{12}$. It is

$$1\,599\,633\,425\,791 \mapsto 3\,416\,404\,591\,999 \mapsto 7\,296\,559\,416\,703 \mapsto 15\,583\,569\,769\,855.$$

References

[1] J. C. Lagarias, *The Ultimate Challenge: The* $3x + 1$ *Problem* (American Mathematical Society, Providence, RI, 2010).

[2] P. Mihailescu, *Reflection, Bernoulli Numbers and the Proof of Catalan's Conjecture* (Georg-August-Universitat, Gottingen, 2005).

[3] J. J. O'Connor. and E. F. Robertson, *Lothar Collatz* (St Andrews University, School of Mathematics and Statistics, St Andrews, 2006).

[4] T. Tao, 'Almost all orbits of the Collatz map attain almost bounded values', *Forum Math. Pi* **10** (2022), e12.

W. E. LONGSTAFF, 702/106 Pacific Parade, Bilinga,
Queensland 4225, Australia
e-mail: welongstaff10@gmail.com

https://doi.org/10.1017/S0004972725000267 Published online by Cambridge University Press