SPINOR IDENTITIES

By R. SHAW

Received 24 May 1954

1. *Introduction.* Using a method similar to one of Pauli's (1), identities will be constructed involving four spinor fields. However, we do not use c-number spinors, but instead we assume the anti-commutativity of different spinor fields. The main result is the expression of the Wigner-Critchfield interaction (2) in terms of the usual invariants of β-decay, this result being proved for a general representation of the Dirac γ-matrices.

2. *Notation.* γ-matrices will be used satisfying

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu = 2g_{\mu\nu} \quad (\mu,\nu = 0,1,2,3).$$

$g_{\mu\nu}$ is the metric tensor of special relativity ($g_{00} = -1$, $g_{ii} = 1$). There exists a matrix B (see Pauli (1)) such that

$$B\gamma_\mu B^{-1} = -\gamma_\mu^T \quad \text{and} \quad B^T = -B. \tag{1}$$

B^T here denotes the transpose of the matrix B. (In the Majorana representation of the γ's, with γ_0, γ_5 anti-symmetric and γ_1, γ_2, γ_3 symmetric, we find B can be taken as γ_0.)

We put $B' = B\gamma^5$, $\Lambda^\mu = B\gamma^\mu$, $\Lambda'^\mu = B\gamma^\mu\gamma^5$, $\Lambda^{\mu\nu} = \frac{1}{2}B(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)$

and form the usual covariant quantities

$$\Omega_{12} = i\psi^1B\psi^2, \quad \Omega'_{12} = i\psi^1B'\psi^2, \quad S'^\mu_{12} = i\psi^1\Lambda'^\mu\psi^2,$$
$$S^\mu_{12} = i\psi^1\Lambda^\mu\psi^2, \quad M^{\mu\nu}_{12} = i\psi^1\Lambda^{\mu\nu}\psi^2,$$

where ψ^1 and ψ^2 are two different spinors.

Note that, using (1), B, B', Λ'^μ are anti-symmetric, and $\Lambda^\mu, \Lambda^{\mu\nu}$ are symmetric. Hence (since ψ^1 and ψ^2 anti-commute),

$$\Omega_{12} = \Omega_{21}, \quad \Omega'_{12} = \Omega'_{21}, \quad S'^\mu_{12} = S'^\mu_{21}, \quad S^\mu_{12} = -S^\mu_{21}, \quad M^{\mu\nu}_{12} = -M^{\mu\nu}_{21}. \tag{2}$$

3. Let $\{\gamma^A\}$ denote the set of sixteen matrices

$$I, \quad \gamma^\mu, \quad \gamma^5, \quad \gamma^\mu\gamma^5, \quad \tfrac{1}{2}(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu).$$

We now apply the orthogonality relations to the representation by the matrices $\{\pm\gamma^A\}$ of a group of order 32. We obtain

$$\frac{1}{32}\sum_{A=1}^{16}\{\gamma^A_{ij}(\gamma^A)^{-1}_{kl} + (-\gamma^A)_{ij}(-\gamma^A)^{-1}_{kl}\} = \tfrac{1}{4}\delta_{il}\delta_{kj},$$

or

$$\sum_{A=1}^{16}\gamma^A_{ij}(\gamma^A)^{-1}_{kl} = 4\delta_{il}\delta_{kj}. \tag{3}$$

In order to obtain relations between Lorentz covariant quantities, we multiply (3) by $B_{i'i}B_{k'k}$, sum over i and k and then drop the primes on i' and k'. We find

$$\sum_{A=1}^{16}[B\gamma^A]_{ij}[B(\gamma^A)^{-1}]_{kl} = 4B_{il}B_{kj}. \tag{4}$$

https://doi.org/10.1017/S0305004100030097 Published online by Cambridge University Press

Now the inverses of

$$I, \quad \gamma^0, \quad \gamma^i, \quad \gamma^0\gamma^i, \quad \gamma^i\gamma^j, \quad \gamma^0\gamma^5, \quad \gamma^i\gamma^5, \quad \gamma^5 \quad (i,j = 1,2,3)$$

are respectively

$$I, \quad -\gamma^0, \quad \gamma^i, \quad \gamma^0\gamma^i, \quad -\gamma^i\gamma^j, \quad -\gamma^0\gamma^5, \quad \gamma^i\gamma^5, \quad -\gamma^5.$$

These changes in sign allow us to write (4) in terms of Lorentz invariant products. We can also split (4) up into two equations, one symmetric, the other anti-symmetric in i and j. We obtain

$$\Lambda^{\mu}_{ij}\Lambda^{\mu}_{kl} - \Lambda^{\mu\nu}_{ij}\Lambda^{\mu\nu}_{kl} = 2(B_{il}B_{kj} + B_{ik}B_{lj}) \tag{5}$$

and

$$B_{ij}B_{kl} - B'_{ij}B'_{kl} + \Lambda'^{\mu}_{ij}\Lambda'^{\mu}_{kl} = 2(B_{il}B_{kj} - B_{ik}B_{lj}). \tag{6}$$

For convenience we have used the notation $a^\mu b^\mu = -a^0b^0 + a^ib^i$.

We obtain a further separation of the various covariant quantities by multiplying (5) and (6) by $\gamma^5_{kk'}\gamma^5_{ll'}$. Noting

$$\gamma^T_5 B\gamma_5 = -B, \quad \gamma^T_5 B'\gamma_5 = -B', \quad \gamma^T_5 \Lambda'^{\mu}\gamma_5 = \Lambda'^{\mu},$$

$$\gamma^T_5 \Lambda^{\mu}\gamma_5 = \Lambda^{\mu}, \quad \gamma^T_5 \Lambda^{\mu\nu}\gamma_5 = -\Lambda^{\mu\nu},$$

we arrive at the following four equations:

$$\Lambda^{\mu}_{ij}\Lambda^{\mu}_{kl} = B_{il}B_{kj} + B_{ik}B_{lj} + B'_{il}B'_{kj} + B'_{ik}B'_{lj}, \tag{7}$$

$$\Lambda^{\mu\nu}_{ij}\Lambda^{\mu\nu}_{kl} = B_{il}B_{kj} + B_{ik}B_{lj} - B'_{il}B'_{kj} - B'_{ik}B'_{lj}, \tag{8}$$

$$\Lambda'^{\mu}_{ij}\Lambda'^{\mu}_{kl} = B_{il}B_{kj} - B_{ik}B_{lj} + B'_{il}B'_{kj} - B'_{ik}B'_{lj}, \tag{9}$$

$$B_{ij}B_{kl} + B_{ik}B_{lj} + B_{il}B_{jk} = B'_{ij}B'_{kl} + B'_{ik}B'_{lj} + B'_{il}B'_{jk}. \tag{10}$$

On multiplying equations (7)–(10) by $\psi^1_i\psi^2_j\psi^3_k\psi^4_l$, identities are obtained involving Lorentz invariant quantities. Thus equation (7) gives the relation

$$S^{\mu}_{12}S^{\mu}_{34} = -\Omega_{14}\Omega_{23} + \Omega_{13}\Omega_{24} - \Omega'_{14}\Omega'_{23} + \Omega'_{13}\Omega'_{24},$$

from which we obtain

$$S^{\mu}_{12}S^{\mu}_{34} + S^{\mu}_{13}S^{\mu}_{42} + S^{\mu}_{14}S^{\mu}_{23} = 0. \tag{11}$$

Similarly, using (8), we obtain

$$M^{\mu\nu}_{12}M^{\mu\nu}_{34} + M^{\mu\nu}_{13}M^{\mu\nu}_{42} + M^{\mu\nu}_{14}M^{\mu\nu}_{23} = 0. \tag{12}$$

The left-hand sides of both (11) and (12) are seen from (2) to be anti-symmetric in the exchange of any pair of spinors, and no other such quantities can be formed which are Lorentz invariant (from (2) only the anti-symmetric S^μ and $M^{\mu\nu}$ can be used). Hence no non-vanishing interaction exists which is anti-symmetric in four different spinors.

From (10) immediately, and from (9) after an intervening step, we obtain

$$\begin{aligned}\Omega_{12}\Omega_{34} + \Omega_{13}\Omega_{42} + \Omega_{14}\Omega_{23} &= \Omega'_{12}\Omega'_{34} + \Omega'_{13}\Omega'_{42} + \Omega'_{14}\Omega'_{23} \\ &= -\tfrac{1}{4}(\Lambda'^{\mu}_{12}\Lambda'^{\mu}_{34} + \Lambda'^{\mu}_{13}\Lambda'^{\mu}_{42} + \Lambda'^{\mu}_{14}\Lambda'^{\mu}_{23}) \\ &= W, \quad \text{say.}\end{aligned}$$

From the anti-symmetry of (10) in i, j, k, l, we see that

$$W = c\epsilon_{ijkl}\psi^1_i\psi^2_j\psi^3_k\psi^4_l.$$

https://doi.org/10.1017/S0305004100030097 Published online by Cambridge University Press

If B is normalized so that $|B| = 1$, then $c = \pm 1$. Hence W is the Wigner-Critchfield interaction (2), completely symmetric in the four spinors. Now (6) can be written

$$-B_{ij}B_{kl} - B'_{ij}B'_{kl} + \Lambda'^{\mu}_{ij}\Lambda'^{\mu}_{kl} = -2(B_{ij}B_{kl} + B_{ik}B_{lj} + B_{il}B_{jk}).$$

Hence
$$-2W = \Omega_{12}\Omega_{34} + \Omega'_{12}\Omega'_{34} - S'^{\mu}_{12}S'^{\mu}_{34}.$$

Thus W has been expressed in terms of the usual invariants. In the usual notation

$$-2W = S + P - A.$$

The author wishes to thank Dr Salam for his interest in the work. Thanks are also due to the Department of Scientific and Industrial Research for a maintenance grant.

REFERENCES

(1) Pauli, W. *Ann. Inst. Poincaré*, 6 (1936), 109.
(2) Wigner, E. and Critchfield, C. L. *Phys. Rev.* 63 (1943), 417.

Trinity College
Cambridge

https://doi.org/10.1017/S0305004100030097 Published online by Cambridge University Press