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Cohomology of dominated diffeomorphism-valued cocycles over hyperbolic systems

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Abstract. We prove a rigidity theorem for dominated Hölder cocycles with values on diffeomorphism groups of a compact manifold over hyperbolic homeomorphisms. More precisely, we show that if two such cocycles have equal periodic data, then they are cohomologous.

1. *Introduction*

Given a homeomorphism $f : M \circlearrowleft$ and a topological group G, a *G-cocycle over f* is a continuous map $\alpha : \mathbb{Z} \times M \to G$ satisfying

$$\alpha^{(m+n)}(x) = \alpha^{(m)}(f^n(x))\alpha^{(n)}(x) \quad \text{for all } m, n \in \mathbb{Z} \text{ for all } x \in M. \tag{1}$$

Two cocycles α and β over f are said to be *cohomologous* whenever there exists a continuous map $P : M \to G$, usually called a *transfer map*, such that

$$\alpha^{(n)}(x) = P(f^n(x))\beta^{(n)}(x)P(x)^{-1} \quad \text{for all } n \in \mathbb{Z} \text{ for all } x \in M.$$

As can be easily verified, the cohomology relation is an equivalence one over the space of (continuous) cocycles.

In the particular case that β is the constant function equal to e_G (where e_G denotes the identity element of G), we say that α is a *coboundary*.

Many important questions in dynamical systems can be reduced to the problem of determining whether certain cocycles are cohomologous. So, it is not surprising that this equivalence relation has been extensively studied in recent decades (see for instance [**Kat01**] for a survey).

As had been already noticed in [**Koc13**], in general the C^0-category is not the right one to study cohomology of dynamical systems and, depending on the dynamical properties

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of the system, different degrees of regularity are required to guarantee a rather reasonable description of cohomology classes.

When f is a hyperbolic homeomorphism (see §2 for precise definitions), it seems that Hölder regularity (for cocycles) is enough to study its cohomology. In fact, the seminal works of Livšic [**Liv71**, **Liv72**] claim that, when G admits a compatible bi-invariant complete distance, the *periodic data* of f completely characterizes Hölder-continuous G-coboundaries. More precisely, a Hölder G-cocycle α over f is a coboundary if and only if

$$\alpha^{(n)}(p) = e_G \quad \text{for all } p \in \mathrm{Fix}(f^n).$$

This result has been considerably extended to more general groups and to higher regularity by several different authors [**dlLMM86**, **NT96**, **NP01**, **dlLW10**, **Kal11**, **KP13**], and such results are usually called *Livšic-type theorems* (see also [**KN11**] and references therein).

More generally, inspired by these results, it would be natural to ask whether two Hölder G-cocycles α and β over a hyperbolic system f satisfying the so-called *periodic orbit condition*, *POC* for short,

$$\alpha^{(n)}(p) = \beta^{(n)}(p) \quad \text{for all } n \in \mathbb{Z} \text{ for all } p \in \mathrm{Fix}(f^n), \tag{POC}$$

are indeed cohomologous.

When G is an Abelian group, the space of G-cocycles is naturally an Abelian group itself, and coboundaries form a subgroup. Hence, under the hypothesis of commutativity, since it can be easily verified that two cocycles are cohomologous if and only if their difference is a coboundary, the problems of characterization of coboundaries and cohomology classes are completely analogous.

However, when G is non-Abelian, the second problem is much more complicated than the first one.

In the present paper we study the characterization of cohomology classes for cocycles with values in the group $G = \mathrm{Diff}^r(N)$ (i.e. the group of C^r-diffeomorphisms of a compact manifold N) and satisfying a domination condition. The domination condition is used to construct invariant holonomies, which play a key role in our arguments.

This generalizes the previous result independently obtained by the first author [**Bac13**] and Sadovskaya [**Sad13**] for linear cocycles (i.e. where $G = \mathrm{GL}_d(\mathbb{R})$) satisfying a fiber bunching condition. Similar results have been previously obtained by Parry [**Par99**], for cocycles admitting a bi-invariant distance, and by Schmidt [**Sch99**], for cocycles satisfying a 'bounded distortion condition', which is much stronger than our domination assumption.

These paper is organized as follows: in §1.1 we present the precise statement of the main result. In §2 we give some definitions and preliminary results. Section 2.3 is devoted to describe a distance in $\mathrm{Diff}^r(N)$ and some auxiliary results. In §3 we build the main tool used in our proof: invariant holonomies. In §§4 and 5 we construct, in some sense, explicitly the transfer map, while §6 is devoted to improving the regularity of these transfer maps.

1.1. *Main results.* The main result of this work is the following one (see §2 for precise definitions).

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THEOREM 1.1. *Let $f : M \circlearrowleft$ be a Lipschitz continuous transitive hyperbolic homeomorphism on a compact metric space (M, d), N a compact smooth manifold and $\alpha, \beta : \mathbb{Z} \times M \to \mathrm{Diff}^r(N)$ two $(2r-1)$-dominated ν-Hölder cocycles over f satisfying the* periodic data condition:

$$\alpha^{(n)}(p) = \beta^{(n)}(p) \quad \text{for all } n \in Z \text{ for all } p \in \mathrm{Fix}(f^n). \tag{2}$$

Then there exists a ν-Hölder continuous map $P : M \to \mathrm{Diff}^{r-4}(N)$ such that

$$\alpha^{(n)}(x) = P(f^n(x)) \circ \beta^{(n)}(x) \circ P(x)^{-1} \quad \text{for all } x \in M \text{ for all } n \in \mathbb{Z}. \tag{3}$$

Moreover, $P(M) \subset \mathrm{Diff}^{r-1}(N)$ and, if f is a C^{r-1} Anosov diffeomorphism and the cocycles α and β are C^{r-1}, then P is $C^{r-1-\varepsilon}$ for any small $\varepsilon > 0$.

At this point we would like to remark that by $P : M \to \mathrm{Diff}^{r-4}(N)$ being a ν-Hölder continuous map we mean that P is ν-Hölder with respect to the distance function d_{r-4} given by (4) below and the cocycle $\alpha : \mathbb{Z} \times M \to \mathrm{Diff}^r(N)$ being C^{r-1} means that the induced map $\alpha^{(1)} : M \times N \to N$ is C^{r-1}.

The strategy of proving this theorem is closely related to the proof of an analogous rigidity theorem for linear cocycles, proved by the first author in [**Bac13**]. However, here we have additional difficulties due to the fact that the group of diffeomorphisms is infinite dimensional.

The $(2r-1)$-domination condition is used to guarantee the existence of invariant holonomies with good regularity. Using these invariant holonomies, we 'explicitly' construct the transfer map on a dense subset, where we prove that it is ν-Hölder. The last step consists in extending P to the whole space M and to proving that it has, at least in some cases, a better regularity.

As a consequence of our methods, we also obtain the following.

THEOREM 1.2. *Let M, N and f be as in Theorem 1.1 and $\alpha, \beta : \mathbb{Z} \times M \to \mathrm{Diff}^r(N)$ be two $(2r-1)$-dominated ν-Hölder cocycles over f. Let us assume that there exists $P : M \to \mathrm{Diff}^1(N)$ ν-Hölder such that*

$$\alpha^{(1)}(x) = P(f(x)) \circ \beta^{(1)}(x) \circ P(x)^{-1} \quad \text{for all } x \in M$$

and assume also that there exists $x_0 \in M$ such that $P(x_0) \in \mathrm{Diff}^{r-1}(N)$. Then $P(M) \subset \mathrm{Diff}^{r-1}(N)$. Moreover, if f is a C^{r-1} Anosov diffeomorphism and the cocycles α and β are C^{r-1}, then P is $C^{r-1-\varepsilon}$ for any $\varepsilon > 0$.

2. *Preliminaries and notation*

2.1. *Hyperbolic homeomorphisms.* Let (M, d) be a compact metric space and $f : M \circlearrowleft$ be a homeomorphism. Given any $x \in M$ and $\varepsilon > 0$, define the *local stable* and *unstable sets* by

$$W^s_\varepsilon(x, f) := \{y \in M : d(f^n(x), f^n(y)) \le \varepsilon \text{ for all } n \ge 0\},$$
$$W^u_\varepsilon(x, f) := \{y \in M : d(f^n(x), f^n(y)) \le \varepsilon \text{ for all } n \le 0\},$$

respectively. Where there is no risk of ambiguity, we just write $W^s_\varepsilon(x)$ instead of $W^s_\varepsilon(x, f)$, and the same holds for the local unstable set.

Following [**AV10**], we introduce the following definition.

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Definition 2.1. A homeomorphism $f : M \circlearrowleft$ is said to be *hyperbolic with local product structure* (or just *hyperbolic* for short) whenever there exist constants $C_1, \varepsilon, \tau > 0$ and $\lambda \in (0, 1)$ such that the following conditions are satisfied:

(h1) $d(f^n(y_1), f^n(y_2)) \leq C_1\lambda^n d(y_1, y_2)$, for all $x \in M$, for all $y_1, y_2 \in W^s_\varepsilon(x)$, for all $n \geq 0$;

(h2) $d(f^{-n}(y_1), f^{-n}(y_2)) \leq C_1\lambda^n d(y_1, y_2)$, for all $x \in M$, for all $y_1, y_2 \in W^u_\varepsilon(x)$, for all $n \geq 0$;

(h3) if $d(x, y) \leq \tau$, then $W^s_\varepsilon(x)$ and $W^u_\varepsilon(y)$ intersect in a unique point, which is denoted by $[x, y]$ and depends continuously on x and y.

For such homeomorphisms, one can define the *stable* and *unstable sets* by

$$W^s(x, f) := \bigcup_{n\geq 0} f^{-n}(W^s_\varepsilon(f^n(x))) \quad \text{and} \quad W^u(x, f) := \bigcup_{n\geq 0} f^n(W^u_\varepsilon(f^{-n}(x))),$$

respectively.

Notice that shifts of finite type and basic sets of Axiom A diffeomorphisms are particular examples of hyperbolic homeomorphisms with local product structure (see for instance [**Mañ87**, Ch. IV, §9] for details).

2.2. *Cocycles and cohomology.* Given any homeomorphism $f : M \circlearrowleft$ and a topological group G, a *G-cocycle* over f is a continuous map $\alpha : \mathbb{Z} \times M \to G$ such that

$$\alpha^{(m+n)}(x) = \alpha^{(m)}(f^n(x))\alpha^{(n)}(x) \quad \text{for all } m, n \in \mathbb{Z} \text{ for all } x \in M.$$

On the other hand, if H is another topological group such that $G \subset H$ just as groups (i.e. their topologies are not necessarily related), then two G-cocycles α and β over f are said to be *H-cohomologous* whenever there exists a continuous map $P : M \to H$ such that

$$\alpha^{(n)}(x) = P(f^n(x))\beta^{(n)}(x)P(x)^{-1} \quad \text{for all } n \in \mathbb{Z} \text{ for all } x \in M.$$

2.3. *Diffeomorphism groups and their topologies.* In this subsection we recall some concepts about C^r-topologies on groups of diffeomorphisms and, to do that, we mainly follow [**dlLW10**, §5.2]. From now on, N will denote a compact smooth Riemannian manifold. We write $\mathrm{Diff}^r(N)$ for the group of C^r-diffeomorphisms of N.

It is well known that $\mathrm{Diff}^r(N)$ has a Banach manifold structure modeled on the space of C^r vector fields with local charts given by composition with the exponential map.

The topology induced by this Banach manifold structure coincides with the one induced by the *C^r-distance* d_r, which is defined as follows.

Let us consider a smooth curve $p : \mathbb{R} \to \mathrm{Diff}^r(N)$. Then, for any $t \in \mathbb{R}$ and $y \in N$, there exists neighborhoods V of t and U of $0 \in U \subset T_yN$ such that for any $s \in V$ the local representative $\tilde{p}(s)_y : U \subset T_yN \to T_{p(t)(y)}$ is defined by

$$p(s)(\exp_y v) = \exp_{p(t)(y)}(\tilde{p}(s)_y(v)).$$

Using the standard identification of the tangent space to a linear space with the linear space itself, we obtain

$$D^n\tilde{p}(s)_y : T_yN^{\otimes n} \to T_{p(t)(y)}N.$$

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Now, since this is a curve in a fixed linear space, we can differentiate it with respect to s to get

$$\frac{d}{dt}D_y^n p(t) := \frac{d}{ds}D^n \tilde{p}(s)_y \Big|_{s=t}(0).$$

Now consider a path $p:[0,1]\to \mathrm{Diff}^r(N)$ such that $(d/dt)D^n p_t$ is piecewise continuous in t for each $0\le n\le r$. Such p will be called a piecewise C^1 path in $\mathrm{Diff}^r(N)$. We define the length of p by

$$l_r(p) = \max_{0\le n\le r}\max_{y\in N}\int_0^1 \left\| \frac{d}{dt}D_y^n p_t \right\| dt,$$

and its partial length is given by

$$l_r(p;s) := \max_{0\le n\le r}\max_{y\in N}\int_0^s \left\| \frac{d}{dt}D_y^n p_t \right\| dt.$$

This length structure on $\mathrm{Diff}^r(N)$ allows us to define a metric d_r as follows: given $h, g\in \mathrm{Diff}^r(N)$, we define

$$d_r(h,g) := \inf_{p\in\mathcal{P}} \max\{l_r(p), l_r(p^{-1})\} \tag{4}$$

where

$$\mathcal{P} := \{p\in C^1_{\mathrm{pw}}([0,1],\mathrm{Diff}^r(N)) : p(0)=h,\ p(1)=g\}.$$

As was already mentioned above, this metric d_r induces the usual C^r-uniform topology on $\mathrm{Diff}^r(N)$, which coincides with the one induced by the Banach manifold structure.

Finishing this subsection, we recall four lemmas from [**dlLW10**, Lemmas 5.1–5.3 and 5.5] that we will use in forthcoming sections.

LEMMA 2.2. *For every C^1-path $p:\mathbb{R}\to\mathrm{Diff}^r(N)$, every $1\le k\le r$ and all $s>0$,*

$$\|D^k p_s\| \le e^{\kappa l_0(p;s)}(l_k(p;s) + \|D^k p_0\|),$$

where $\|\cdot\|$ denotes the operator norm for linear maps from $(TN)^{\otimes k}$ to TN, and κ is a real constant depending on the Riemannian metric of N. In particular,

$$\|Dp_s\|_{r-1} \le e^{\kappa l_0(p;s)}(l_r(p;s) + \|Dp_0\|_{r-1}).$$

LEMMA 2.3. *Let $p\in C^1([0,1],\mathrm{Diff}^{r-1}(N))$ and $h\in\mathrm{Diff}^r(N)$. Then there exists a constant $C>0$ depending just on r such that*

$$l_{r-1}(h\circ p_s) \le C\|Dh\|_{r-1}\Big(1+\max_{s\in[0,1]}\|Dp_s\|_{r-2}\Big)^{r-1} l_{r-1}(p_s) \quad \text{for all } s\in[0,1].$$

On the other hand, if $p\in C^1([0,1],\mathrm{Diff}^r(N))$ and $h\in\mathrm{Diff}^r(N)$, then

$$l_r(p_s\circ h) \le C\max_{k_1,\dots,k_r}\|D^1h\|^{k_1}\cdots\|D^r h\|^{k_r} l_r(p_s) \quad \text{for all } s\in[0,1],$$

where the maximum is taken over all $k_1,\dots,k_r\ge 0$ such that

$$k_1+2k_2+\cdots+rk_r \le r.$$

Crudely, this may be estimated by

$$l_r(p_s\circ h) \le C(1+\|Dh\|_{r-1})^r l_r(p_s) \quad \text{for all } s\in[0,1].$$

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LEMMA 2.4. *Let $C > 0$ and $r \in \mathbb{N}$ be arbitrary. Suppose that $h \in \mathrm{Diff}^r(N)$ and $g_1, g_2 \in \mathrm{Diff}^{r-1}(N)$. Then there exists a constant $C' = C'(C, r, N) > 0$ depending only on C, r and the manifold N such that*

$$d_{r-1}(h \circ g_1, h \circ g_2) < C' d_{r-1}(g_1, g_2),$$
$$d_{r-1}(g_1 \circ h, g_2 \circ h) < C' d_{r-1}(g_1, g_2)$$

whenever

$$d_r(h, \mathrm{Id}) < C, \quad d_{r-1}(g_1, \mathrm{Id}) < C \quad \text{and} \quad d_{r-1}(g_2, \mathrm{Id}) < C.$$

LEMMA 2.5. *Given a Lipschitz continuous cocycle $\alpha : \mathbb{Z} \times M \to \mathrm{Diff}^r(N)$ over a homeomorphism $f : M \circlearrowleft$, let us define*

$$\rho_0 := \max_{w \in M} d_0(\alpha^{(1)}(w), \mathrm{Id}_N)$$

and

$$\rho_1 := \max_{w \in M} \max\{\|D\alpha^{(1)}(w)\|, \|D\alpha^{(1)}(w)^{-1}\|\}.$$

Then we have the following estimates for $m \le r$:

$$d_0(\alpha^{(n)}(w), \mathrm{Id}_N) \le \rho_0 |n| \quad \text{and} \quad \|D^m \alpha^{(n)}(w)\| \le C \rho_1^{m|n|}$$

for all $(n, w) \in \mathbb{Z} \times M$.

2.4. *Hölder cocycles and domination.* Let us consider a hyperbolic homeomorphism $f : M \circlearrowleft$ on a compact metric space (M, d) and let $\alpha : \mathbb{Z} \times M \to \mathrm{Diff}^r(N)$ be a cocycle over f, where N denotes a smooth compact manifold. Given $\nu > 0$, we say that α is a *ν-Hölder cocycle* when there exists a constant $C_2 > 0$ such that

$$d_r(\alpha^{(1)}(x), \alpha^{(1)}(y)) \le C_2 d(x, y)^\nu \quad \text{for all } x, y \in M. \tag{5}$$

On the other hand, let us define $\rho = \rho(\alpha) > 0$ by

$$\rho := \max_{z \in N} \max_{x \in M} \{\|D(\alpha^{(1)}(x))_y\|, \|D(\alpha^{(1)}(x)^{-1})_y\|\} \tag{6}$$

and let $\lambda > 0$ be the constant associated to f given by Definition 2.1. Then, given a real number $t > 0$, we will say that the ν-Hölder cocycle α is *t-dominated* whenever

$$\rho^t \lambda^\nu < 1.$$

In what follows, for simplicity of the presentation we will assume that $\nu = 1$; the general case is entirely analogous.

3. *Constructing invariant holonomies*

In this section we start the proof of Theorem 1.1. The first step of the proof is Proposition 3.1, where we show the existence of invariant holonomies for $(2r - 1)$-dominated cocycles. This result is mainly inspired by [**Via08**, Proposition 2.5] and [**dlLW10**, Theorem 5.8].

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PROPOSITION 3.1. *Let $f : M \circlearrowleft$ be a hyperbolic homeomorphism on a compact metric space (M, d) and $\alpha : M \to \mathrm{Diff}^r(N)$ be a $(2r-1)$-dominated 1-Hölder cocycle over f. Then there exists a constant $C_4 > 0$ such that, for any $x \in M$ and any $y, z \in W^s(x, f)$, the limit*

$$H^{s,\alpha}_{yz} := \lim_{n\to+\infty} \alpha^{(n)}(z)^{-1} \circ \alpha^{(n)}(y)$$

exists in the metric space $(\mathrm{Diff}^{r-1}(N), d_{r-1})$ and

$$d_{r-1}(H^{s,\alpha}_{yz}, \mathrm{Id}_N) \le C_4 d(y, z)$$

whenever $y, z \in W^s_\varepsilon(x, f)$, where the constant $\varepsilon > 0$ associated to f is given by Definition 2.1.

On the other hand, if $y, z \in W^u(x, f)$, we can analogously define

$$H^{u,\alpha}_{yz} := \lim_{n\to\infty} \alpha^{(-n)}(z)^{-1} \circ \alpha^{(-n)}(y),$$

and the very same Hölder estimates hold for these maps when $y, z \in W^u_\varepsilon(x, f)$.

Finally, for every $x \in M$ and $\sigma \in \{s, u\}$,

$$H^{\sigma,\alpha}_{yz} = H^{\sigma,\alpha}_{xz} \circ H^{\sigma,\alpha}_{yx}$$

and

$$H^{\sigma,\alpha}_{f^n(y)f^n(z)} \circ \alpha^{(n)}(y) = \alpha^{(n)}(z) \circ H^{\sigma,\alpha}_{yz}$$

for every $y, z \in W^\sigma(x, f)$ and every $n \in \mathbb{Z}$.

Definition 3.2. The maps $H^{s,\alpha}$ and $H^{u,\alpha}$ given by Proposition 3.1 are called *stable* and *unstable holonomies*, respectively.

Remark 3.3. Assuming that α is just 1-dominated in Proposition 3.1, it can be rather easily shown that the invariant holonomies exist, but in general they are only C^0.

Proof of Proposition 3.1. After taking forward iterates if necessary, we can assume that $y, z \in W^s_\varepsilon(x)$. Let us define $\gamma_n(y, z) := \alpha^{(n)}(z)^{-1} \circ \alpha^{(n)}(y)$. Then let us estimate the d_{r-1}-distance between $\gamma_{n+1}(y, z)$ and $\gamma_n(y, z)$. First of all, observe that

$$\gamma_{n+1}(y, z) = \alpha^{(n)}(z)^{-1} \circ \alpha^{(1)}(f^n(z))^{-1} \circ \alpha^{(1)}(f^n(y)) \circ \alpha^{(n)}(y).$$

Then let us consider a continuous path $p : [0, 1] \to \mathrm{Diff}^r(N)$ connecting the diffeomorphism $\alpha^{(1)}(f^n(z))^{-1} \circ \alpha^{(1)}(f^n(y))$ to Id_N. Since, by compactness of M, $d_{r-1}(\alpha^{(1)}(w_1)^{-1} \circ \alpha^{(1)}(w_2), \mathrm{Id}_N)$ is uniformly bounded for any $w_1, w_2 \in M$, we may assume that the d_{r-1}-lengths $l_{r-1}(p_s)$ and $l_{r-1}(p_s^{-1})$ are also uniformly bounded, independently of x, y, z and n.

In what follows, we will use the letter C to denote a positive constant that may differ in each step. By Lemma 2.3, we know that

$$l_{r-1}(\alpha^{(n)}(z)^{-1} \circ p_s) \le C \|D\alpha^{(n)}(z)^{-1}\|_{r-1} \Big(1 + \max_{s\in[0,1]} \|Dp_s\|_{r-2}\Big)^{r-1} l_{r-1}(p_s)$$

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for every $s \in [0, 1]$ and, by Lemmas 2.2 and 2.5, we know that the right-hand side is less than or equal to $C\rho^{rn} l_{r-1}(p_s)$, where C is independent of n. Now, by Lemma 2.3, it easily follows that

$$\begin{aligned} & l_{r-1}(\alpha^{(n)}(z)^{-1} \circ p_s \circ \alpha^{(n)}(y)) \\ & \quad \le C \max_{k_1,\dots,k_{r-1}} \|D^1\alpha^{(n)}(y)\|^{k_1} \cdots \|D^{r-1}\alpha^{(n)}(y)\|^{k_{r-1}} l_{r-1}(\alpha^{(n)}(z)^{-1} \circ p_s) \\ & \quad \le C\rho^{rn} \max_{k_1,\dots,k_{r-1}} \|D^1\alpha^{(n)}(y)\|^{k_1} \cdots \|D^{r-1}\alpha^{(n)}(y)\|^{k_{r-1}} l_{r-1}(p_s) \end{aligned}$$

for every $s \in [0, 1]$.

Then, applying Lemma 2.5, we get that

$$l_{r-1}(\alpha^{(n)}(z)^{-1} \circ p_s \circ \alpha^{(n)}(y)) \le C\rho^{rn}\rho^{(r-1)n} l_{r-1}(p_s).$$

On the other hand, by symmetry the very same estimate holds for the inverse and, thus,

$$d_{r-1}(\gamma_{n+1}(y, z), \gamma_n(y, z)) \le C\rho^{(2r-1)n} d_{r-1}(\alpha^{(1)}(f^n(z))^{-1} \circ \alpha^{(1)}(f^n(y)), \mathrm{Id}_N).$$

Now, as M is compact, there is a constant $C > 0$ such that $d_r(\alpha^{(1)}(w), \mathrm{Id}_N) \le C$ for all $w \in M$ and, consequently, by Lemma 2.4, there exists $K > 1$ such that

$$d_{r-1}(\alpha^{(1)}(w_1)^{-1} \circ \alpha^{(1)}(w_2), \mathrm{Id}_N) \le K d_{r-1}(\alpha^{(1)}(w_1), \alpha^{(1)}(w_2))$$

for every $w_1, w_2 \in M$. Thus,

$$\begin{aligned} d_{r-1}(\gamma_{n+1}(y, z), \gamma_n(y, z)) &\le C\rho^{(2r-1)n} d_{r-1}(\alpha^{(1)}(f^n(z))^{-1} \circ \alpha^{(1)}(f^n(y)), \mathrm{Id}_N) \\ &\le CK\rho^{(2r-1)n} d_{r-1}(\alpha^{(1)}(f^n(z)), \alpha^{(1)}(f^n(y))), \end{aligned}$$

where the last line of the estimate, by the Hölder condition on α, is less than or equal to $CKC_2\rho^{(2r-1)n} d(f^n(z), f^n(y))$, where C_2 is the positive constant given by the Hölder condition (5). Then, invoking our assumption that $y, z \in W^s_\varepsilon(x)$, we conclude that

$$\begin{aligned} d_{r-1}(\gamma_{n+1}(y, z), \gamma_n(y, z)) &\le CKC_2\rho^{(2r-1)n} d(f^n(z), f^n(y)) \\ &\le CKC_2\rho^{(2r-1)n}\lambda^n d(z, y). \end{aligned}$$

Since $\rho^{2r-1}\lambda < 1$, this proves that the sequence is Cauchy in the metric space $(\mathrm{Diff}^r(N), d_{r-1})$ and, hence, the limit $H^{s,\alpha}_{y,z}$ exists and satisfies

$$d_{r-1}(H^{s,\alpha}_{y,z}, \mathrm{Id}) \le C_4 d(y, z),$$

with $C_4 := \sum_{n=0}^{\infty} CKC_2(\rho^{(2r-1)}\lambda)^n$. □

4. *Constructing a transfer map assuming* $\mathrm{Fix}(f) \neq \emptyset$

In this section we continue with the proof of Theorem 1.1, using the invariant holonomies constructed in §3 to define the transfer map P that solves the cohomological equation (3) in the case where f exhibits a fixed point.

So, we will assume that there exists $x \in M$ such that $f(x) = x$. For such a point, we write $W(x) := W^s(x) \cap W^u(x)$. We start by defining $P : W(x) \to \mathrm{Diff}^{r-1}(N)$ by

$$P(y) = H^{s,\alpha}_{xy} \circ (H^{s,\beta}_{xy})^{-1} = H^{s,\alpha}_{xy} \circ H^{s,\beta}_{yx},$$

where $H^{\sigma,\alpha}$ and $H^{\sigma,\beta}$ are the holonomy maps given by Proposition 3.1.

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Note that P satisfies the following relation:

$$\alpha^{(n)}(y) = P(f^n(y)) \circ \beta^{(n)}(y) \circ P(y)^{-1} \quad \text{for all } y \in W(x) \text{ for all } n \in \mathbb{N}. \tag{7}$$

Indeed,

$$\begin{aligned} P(f(y)) &= H^{s,\alpha}_{xf(y)} \circ H^{s,\beta}_{f(y)x} = H^{s,\alpha}_{f(x)f(y)} \circ H^{s,\beta}_{f(y)f(x)} \\ &= \alpha^{(1)}(y) \circ H^{s,\alpha}_{xy} \circ \alpha^{(1)}(x)^{-1} \circ \beta^{(1)}(x) \circ H^{s,\beta}_{yx} \circ \beta^{(1)}(y)^{-1} \\ &= \alpha^{(1)}(y) \circ H^{s,\alpha}_{xy} \circ H^{s,\beta}_{yx} \circ \beta^{(1)}(y)^{-1} \\ &= \alpha^{(1)}(y) \circ P(y) \circ \beta^{(1)}(y)^{-1}. \end{aligned}$$

Combining this equation with the cocycle property (1), we can easily get (7). Observe that we have strongly used the existence of a fixed point, the periodic orbit condition (POC) and Proposition 3.1.

Up to now we have our transfer map P defined on a dense subset of M, where it satisfies the desired property (7). Then we will show that P can be extended to the whole space M. In order to do this, we will prove that P is Lipschitz on $W(x)$ and the following lemma plays a key role in proving that.

LEMMA 4.1. *If α and β are 1-dominated, then*

$$P(y) = H^{s,\alpha}_{xy} \circ H^{s,\beta}_{yx} = H^{u,\alpha}_{xy} \circ H^{u,\beta}_{yx} \quad \textit{for all } y \in W(x).$$

The following classical result (see for instance [**KH96**, Corollary 6.4.17]) will be used in the proof.

LEMMA 4.2. (Anosov closing lemma) *Let f and $\lambda > 0$ be as in Definition 2.1. Then, given any $\theta > \lambda$, there exist $C_5 > 0$ and $\varepsilon_0 > 0$ such that for any $z \in M$ and every $n \in \mathbb{Z}$ satisfying $d(f^n(z), z) < \varepsilon_0$, there exists a periodic point $p \in \mathrm{Fix}(f^n)$ such that*

$$d(f^j(z),\, f^j(p)) \le C_5 \theta^{\min\{j, n-j\}} d(f^n(z), z) \quad \textit{for } j = 0, 1, \ldots, n.$$

Proof of Lemma 4.1. Let us consider the distance function $\tilde{d}$ on $\mathrm{Diff}^r(N)$ given by

$$\tilde{d}(g, h) = \sup_{y \in N} d(g(y), h(y)) \quad \text{for all } g, h \in \mathrm{Diff}^r(N).$$

Observe that the distance $\tilde{d}$ exhibits the following properties:

- it is right invariant, i.e. $\tilde{d}(g \circ u, h \circ u) = \tilde{d}(g, h)$ for every $g, h, u \in \mathrm{Diff}^r(N)$;
- $\tilde{d}(u \circ g, u \circ h) \le (\mathrm{Lip}\, u)\tilde{d}(g, h)$ for any $g, h, u \in \mathrm{Diff}^r(N)$ and where $\mathrm{Lip}\, u$ denotes the Lipschitz constant of u given by $\mathrm{Lip}\, u := \sup_{x \neq y} d(u(x), u(y))/d(x, y)$. Notice that $\mathrm{Lip}\, u \le \|Du\|$.

Let $\lambda > 0$ be the hyperbolic constant associated to f given by Definition 2.1. Let us fix $\theta \in (\lambda, 1)$ such that $\rho\theta < 1$, where $\rho(\alpha)$ and $\rho(\beta)$ are given by (6) and $\rho := \max\{\rho(\alpha), \rho(\beta)\} > 0$. Let $C_5 > 0$ and $\varepsilon_0 > 0$ be the constants given by Lemma 4.2.

Fix an arbitrary point $y \in W(x)$. We begin by noticing that, since $y \in W(x)$, there exist $C_6 > 0$ and $n_0 \in \mathbb{N}$ such that

$$d(f^{-n}(y),\, f^n(y)) \le C_6 \lambda^{n-n_0} \quad \text{for all } n \ge n_0.$$

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In fact, this easily follows from the fact that, as $y \in W(x) = W^s(x) \cap W^u(x)$, there exists $n_0 \in \mathbb{N}$ such that $f^{n_0}(y) \in W^s_\varepsilon(x)$ and $f^{-n_0}(y) \in W^u_\varepsilon(x)$ and the exponential convergence towards x in $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$.

Now let $n_1 \geq n_0$ be such that, for all $n \geq n_1$, $d(f^n(y), f^{-n}(y)) < \varepsilon_0$. Thus, by Lemma 4.2, for every $n \geq n_1$ there exists a periodic point $p_n \in \mathrm{Fix}(f^{2n})$ such that

$$d(f^j(f^{-n}(p_n)), f^j(f^{-n}(y))) \leq C_5\theta^{\min\{j,2n-j\}}d(f^{-n}(y), f^n(y))$$

for each $j = 0, 1, \ldots, 2n$. Using the periodic orbit condition (POC) and noticing that $f^{2n}(f^{-n}(p_n)) = f^{-n}(p_n)$, we get

$$\alpha^{(2n)}(f^{-n}(p_n)) = \beta^{(2n)}(f^{-n}(p_n)),$$

which can be rewritten as

$$\alpha^{(n)}(p_n) \circ \alpha^{(n)}(f^{-n}(p_n)) = \beta^{(n)}(p_n) \circ \beta^{(n)}(f^{-n}(p_n))$$

or, equivalently,

$$\alpha^{(n)}(f^{-n}(p_n)) \circ \beta^{(n)}(f^{-n}(p_n))^{-1} = \alpha^{(n)}(p_n)^{-1} \circ \beta^{(n)}(p_n).$$

Hence, since

$$\begin{aligned}\alpha^{(n)}(f^{-n}(p_n)) &= \alpha^{(1)}(f^{-1}(p_n)) \circ \cdots \circ \alpha^{(1)}(f^{-n}(p_n)) \\ &= (\alpha^{(1)}(f^{-n}(p_n))^{-1} \circ \cdots \circ \alpha^{(1)}(f^{-1}(p_n))^{-1})^{-1} = \alpha^{(-n)}(p_n)^{-1}\end{aligned}$$

and analogously

$$\beta^{(n)}(f^{-n}(p_n))^{-1} = \beta^{(-n)}(p_n),$$

it follows that

$$\alpha^{(-n)}(p_n)^{-1} \circ \beta^{(-n)}(p_n) = \alpha^{(n)}(p_n)^{-1} \circ \beta^{(n)}(p_n). \tag{8}$$

Now we claim that

$$\tilde{d}(\alpha^{(n)}(y)^{-1} \circ \beta^{(n)}(y), \alpha^{(n)}(p_n)^{-1} \circ \beta^{(n)}(p_n)) \to 0 \tag{9}$$

and

$$\tilde{d}(\alpha^{(-n)}(y)^{-1} \circ \beta^{(-n)}(y), \alpha^{(-n)}(p_n)^{-1} \circ \beta^{(-n)}(p_n)) \to 0 \tag{10}$$

when $n \to +\infty$.

Thus, it follows from (8) and our claim that

$$\tilde{d}(\alpha^{(n)}(y)^{-1} \circ \beta^{(n)}(y), \alpha^{(-n)}(y)^{-1} \circ \beta^{(-n)}(y)) \to 0$$

when $n \to +\infty$. Observing that

$$\begin{aligned}&\tilde{d}(\alpha^{(n)}(y)^{-1} \circ \beta^{(n)}(y), H^{s,\alpha}_{xy} \circ H^{s,\beta}_{yx}) \\ &\quad = \tilde{d}(\alpha^{(n)}(y)^{-1} \circ \alpha^{(n)}(x) \circ \beta^{(n)}(x)^{-1} \circ \beta^{(n)}(y), H^{s,\alpha}_{xy} \circ H^{s,\beta}_{yx}) \to 0\end{aligned}$$

as $n \to +\infty$, and

$$\begin{aligned}&\tilde{d}(\alpha^{(-n)}(y)^{-1} \circ \beta^{(-n)}(y), H^{u,\alpha}_{xy} \circ H^{u,\beta}_{yx}) \\ &\quad = \tilde{d}(\alpha^{(-n)}(y)^{-1} \circ \alpha^{(-n)}(x) \circ \beta^{(-n)}(x)^{-1}\beta^{(-n)}(y), H^{u,\alpha}_{xy} \circ H^{u,\beta}_{yx}) \to 0\end{aligned}$$

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as $n \to +\infty$, we would get

$$P(y) = H^{s,\alpha}_{xy} \circ H^{s,\beta}_{yx} = H^{u,\alpha}_{xy} \circ H^{u,\beta}_{yx},$$

as desired.

So, in order to complete the proof, it remains to prove our claim. In fact, we shall only show (9). The other case (10) is completely analogous.

As before, in what follows we will use C as a generic notation for positive constants that may differ at each step.

We begin by noticing that

$$\begin{aligned}
&\tilde{d}(\alpha^{(n)}(y) \circ \alpha^{(n)}(p_n)^{-1}, \mathrm{Id}) \\
&\quad \le \sum_{j=0}^{n-1} \tilde{d}(\alpha^{(n-j)}(f^j(y)) \circ \alpha^{(n-j)}(f^j(p_n))^{-1}, \\
&\qquad\qquad \alpha^{(n-j-1)}(f^{j+1}(y)) \circ \alpha^{(n-j-1)}(f^{j+1}(p_n))^{-1}) \\
&\quad = \sum_{j=0}^{n-1} \tilde{d}(\alpha^{(n-j)}(f^j(y)) \circ \alpha^{(1)}(f^j(p_n))^{-1} \circ \alpha^{(n-j-1)}(f^{j+1}(p_n))^{-1}, \\
&\qquad\qquad \alpha^{(n-j-1)}(f^{j+1}(y)) \circ \alpha^{(n-j-1)}(f^{j+1}(p_n))^{-1}) \\
&\quad = \sum_{j=0}^{n-1} \tilde{d}(\alpha^{(n-j)}(f^j(y)) \circ \alpha^{(1)}(f^j(p_n))^{-1}, \alpha^{(n-j-1)}(f^{j+1}(y))) \\
&\quad \le \sum_{j=0}^{n-1} \mathrm{Lip}(\alpha^{(n-j-1)}(f^{j+1}(y)))\tilde{d}(\alpha^{(1)}(f^j(y)) \circ \alpha^{(1)}(f^j(p_n))^{-1}, \mathrm{Id}). \qquad (11)
\end{aligned}$$

On the other hand, observe that

$$\begin{aligned}
&\tilde{d}(\alpha^{(1)}(f^j(y)) \circ \alpha^{(1)}(f^j(p_n))^{-1}, \mathrm{Id}) \\
&\quad = \tilde{d}(\alpha^{(1)}(f^j(y)), \alpha^{(1)}(f^j(p_n))) \\
&\quad \le C d(f^j(y),\ f^j(p_n)) = C d(f^{n+j}(f^{-n}(y)),\ f^{n+j}(f^{-n}(p_n))) \\
&\quad \le C\theta^{\min\{n+j,2n-(n+j)\}} d(f^{-n}(y),\ f^n(y)) = C\theta^{n-j} d(f^{-n}(y),\ f^n(y)) \qquad (12)
\end{aligned}$$

for $j \in \{0, 1, \ldots, n-1\}$. Thus, combining (11) and (12), and recalling that $\|D_N\alpha(k, z)\| \le \rho^k$ for all $z \in M$, $\rho\theta < 1$ and $d(f^{-n}(y),\ f^n(y)) \le C_6\lambda^{n-n_0}$, we get

$$\begin{aligned}
&\tilde{d}(\alpha^{(n)}(y) \circ \alpha^{(n)}(p_n)^{-1}, \mathrm{Id}) \\
&\quad \le \sum_{j=0}^{n-1} \|D\alpha^{(n-j-1)}(f^{j+1}(y))\| C\theta^{n-j} d(f^{-n}(y),\ f^n(y)) \\
&\quad \le C \sum_{j=0}^{n-1} \rho^{n-j}\theta^{n-j} d(f^{-n}(y),\ f^n(y)) \le C d(f^{-n}(y),\ f^n(y)) \sum_{j=0}^{\infty} (\rho \cdot \theta)^{n-j} \\
&\quad \le C d(f^{-n}(y),\ f^n(y)) \le C\lambda^{n-n_0}.
\end{aligned}$$

Analogously, we can prove that

$$\tilde{d}(\beta^{(n)}(p_n) \circ \beta^{(n)}(y)^{-1}, \mathrm{Id}) \le C\lambda^{n-n_0}.$$

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Now observe that, since $0 < \rho\lambda < 1$, it follows that

$$\begin{aligned}
&\tilde{d}(\alpha^{(n)}(y)^{-1} \circ \beta^{(n)}(y), \alpha^{(n)}(p_n)^{-1} \circ \beta^{(n)}(p_n)) \\
&\quad \le \mathrm{Lip}(\alpha^{(n)}(p_n))\tilde{d}(\alpha^{(n)}(p_n) \circ \alpha^{(n)}(y)^{-1} \circ \beta^{(n)}(y), \beta^{(n)}(p_n)) \\
&\quad = \mathrm{Lip}(\alpha^{(n)}(p_n))\tilde{d}(\alpha^{(n)}(p_n) \circ \alpha^{(n)}(y)^{-1}, \beta^{(n)}(p_n) \circ \beta^{(n)}(y)^{-1}) \\
&\quad \le \|D\alpha^{(n)}(p_n)\|(\tilde{d}(\alpha^{(n)}(p_n) \circ \alpha^{(n)}(y)^{-1}, \mathrm{Id}) + \tilde{d}(\mathrm{Id}, \beta^{(n)}(p_n) \circ \beta^{(n)}(y)^{-1})) \\
&\quad \le 2C\rho^n\lambda^{n-n_0} = 2C\lambda^{-n_0}(\rho\lambda)^n \to 0 \qquad (13)
\end{aligned}$$

as $n \to +\infty$, and claim (9) is proved. □

Then we can finally prove that P can be continuously extended to the whole space M.

LEMMA 4.3. *The map $P : W(x) \to \mathrm{Diff}^{r-1}(N)$ is Lipschitz with respect to the C^{r-4}-distance.*

Proof. Let y denote an arbitrary point in $W(x)$ and let us define $C_1(y) := d_{r-2}(P(y), \mathrm{Id}) + 1$.

Then let us show that there exists a constant $C_2(y) > 0$ such that

$$d_{r-2}(H^{s,\alpha}_{xz} \circ H^{s,\beta}_{yx}, \mathrm{Id}) < C_2(y) \quad \text{for all } z \in W^s_\varepsilon(y), \qquad (14)$$

where $C_2(y)$ depends on ε, the constant C_4 given by Proposition 3.1, r, the manifold N and $C_1(y)$. Indeed, defining

$$C_2(y) := C'(C_4\varepsilon, r, N)C_1(y) + C_4\varepsilon,$$

where $C'(\cdot\,,\cdot\,,\cdot)$ is the constant given by Lemma 2.4, we get

$$\begin{aligned}
&d_{r-2}(H^{s,\alpha}_{xz} \circ H^{s,\beta}_{yx}, \mathrm{Id}) \\
&\quad = d_{r-2}(H^{s,\alpha}_{xz} \circ H^{s,\alpha}_{yx} \circ H^{s,\alpha}_{xy} \circ H^{s,\beta}_{yx}, \mathrm{Id}) \\
&\quad = d_{r-2}(H^{s,\alpha}_{xz} \circ H^{s,\alpha}_{yx} \circ P(y), \mathrm{Id}) = d_{r-2}(H^{s,\alpha}_{yz} \circ P(y), \mathrm{Id}) \\
&\quad \le d_{r-2}(H^{s,\alpha}_{yz} \circ P(y), H^{s,\alpha}_{yz}) + d_{r-2}(H^{s,\alpha}_{yz}, \mathrm{Id}) \\
&\quad \le C'(C_4\varepsilon, r, N)d_{r-2}(P(y), \mathrm{Id}) + C_4\varepsilon < C_2(y).
\end{aligned}$$

Then, for each $z \in W^s_\varepsilon(y)$,

$$\begin{aligned}
&d_{r-3}(P(y), P(z)) \\
&\quad = d_{r-3}(H^{s,\alpha}_{xy} \circ H^{s,\beta}_{yx}, H^{s,\alpha}_{xz} \circ H^{s,\beta}_{zx}) \\
&\quad \le d_{r-3}(H^{s,\alpha}_{xy} \circ H^{s,\beta}_{yx}, H^{s,\alpha}_{xz} \circ H^{s,\beta}_{yx}) + d_{r-3}(H^{s,\alpha}_{xz} \circ H^{s,\beta}_{yx}, H^{s,\alpha}_{xz} \circ H^{s,\beta}_{zx}) \\
&\quad = d_{r-3}(H^{s,\alpha}_{xy} \circ H^{s,\alpha}_{zx} \circ H^{s,\alpha}_{xz} \circ H^{s,\beta}_{yx}, H^{s,\alpha}_{xz} \circ H^{s,\beta}_{yx}) \\
&\qquad + d_{r-3}(H^{s,\alpha}_{xz} \circ H^{s,\beta}_{yx}, H^{s,\alpha}_{xz} \circ H^{s,\beta}_{yx} \circ H^{s,\beta}_{xy} \circ H^{s,\beta}_{zx}) \\
&\quad \le C'(C_2(y), r, N)(d_{r-3}(H^{s,\alpha}_{xy} \circ H^{s,\alpha}_{zx}, \mathrm{Id}) + d_{r-3}(\mathrm{Id}, H^{s,\beta}_{xy} \circ H^{s,\beta}_{zx})) \\
&\quad \le C_3(y)d(y, z), \qquad (15)
\end{aligned}$$

where $C_3(y) := 2C_4C'(C_2(y), r, N)$ and the last estimate is a consequence of Proposition 3.1.

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Analogously, invoking Lemma 4.1, we can show that for every $y \in W(x)$,

$$d_{r-3}(P(y),\, P(z)) \le C_3(y)d(y,\, z) \quad \text{for all } z \in W^u_\varepsilon(y). \tag{16}$$

Now, given $y, z \in W(x)$ with $d(y, z) < \tau$, where τ is the constant given by Definition 2.1, we can consider $w = [y, z] = W^s_\varepsilon(y) \cap W^u_\varepsilon(z)$. Then, as $w \in W^s_\varepsilon(y)$ and $w \in W^u_\varepsilon(z)$, by (15) and (16) it follows that

$$\begin{aligned} d_{r-3}(P(y),\, P(z)) &\le d_{r-3}(P(y),\, P(w)) + d_{r-3}(P(w),\, P(z)) \\ &\le C_3(y)d(y,\, w) + C_3(w)d(w,\, z). \end{aligned}$$

From the proof, we get that $C_3(w)$ depends on $C_4\varepsilon$, r, N and $C_1(w)$. Now note that

$$d_{r-3}(P(w), \mathrm{Id}) \le d_{r-3}(P(w),\, P(y)) + d_{r-3}(P(y), \mathrm{Id}) \le C_3(y)\varepsilon + C_1(y).$$

In other words, we can estimate $d_{r-3}(P(w), \mathrm{Id})$ from above by some constant that depends on y but does not depend on w.

Thus, repeating the previous arguments and replacing r by $r - 1$, we can show that for every $y \in W(x)$ there exists $C_y > 0$ such that

$$d_{r-4}(P(y),\, P(z)) \le C_y d(y,\, w) + C_y d(w,\, z)$$

for every $z \in W(x)$ such that $d(y, z) < \tau$ and $w = [y, z]$.

Then, invoking the fact that there exists $D > 0$ such that

$$d(y,\, w) + d(w,\, z) \le Dd(y,\, z),$$

for every y, z and w as above, it follows that there exists $K_y = K_y(r, N, C_y, f) > 0$ such that

$$d_{r-4}(P(y),\, P(z)) \le K_y d(y,\, z)$$

whenever $y, z \in W(x)$ and $d(y, z) < \tau$. Now, as $W(x)$ is dense in M and M is compact, it follows that there exists a constant $K > 0$ such that, for all $y, z \in W(x)$ with $d(y, z) < \tau$, we have that

$$d_{r-4}(P(y),\, P(z)) \le Kd(y,\, z),$$

i.e. P is Lipschitz with respect to the C^{r-4} topology. □

Therefore, we can extend $P : W(x) \to \mathrm{Diff}^{r-4}(N)$ to the closure of $W(x)$, that is, the whole space M. By continuity, such extension clearly satisfies (3) and we complete the proof of Theorem 1.1 in the case where f exhibits a fixed point.

5. *Constructing a transfer map in the general case*

In this section we conclude the proof of Theorem 1.1, eliminating the assumption of the existence of a fixed point for f that we used in §4.

We shall use the following notation:

$$\begin{aligned} C(\alpha^{(n)}, \beta^{(n)}) := \{P : M \to \mathrm{Diff}^{r-4}(N) : P \text{ is Lipschitz and} \\ \alpha^{(n)}(x) = P(f^n(x)) \circ \beta^{(n)}(x) \circ P(x)^{-1} \text{ for all } x \in M\}. \end{aligned}$$

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To deal with the general case, let $x_0 \in M$ be a periodic point and n_0 its period. Consider now the new cocycles $\tilde{\alpha}, \tilde{\beta} : \mathbb{Z} \times M \to \mathrm{Diff}^r(N)$ over $F = f^{n_0}$ given by

$$\tilde{\alpha}^{(1)}(x) := \alpha^{(n_0)}(x) = \alpha^{(1)}(f^{n_0-1}(x)) \circ \cdots \circ \alpha^{(1)}(x)$$

and

$$\tilde{\beta}^{(1)}(x) = \beta^{(n_0)}(x) = \beta^{(1)}(f^{n_0-1}(x)) \circ \cdots \circ \beta^{(1)}(x).$$

It is easy to see that $\tilde{\alpha}$ and $\tilde{\beta}$ are $(2r-1)$-dominated over F, since f is Lipschitz, they are Lipschitz themselves and $F(x_0) = x_0$. Thus, applying results of §4 to these objects, we get a Lipschitz map $P : M \to \mathrm{Diff}^{r-4}(N)$ satisfying

$$\tilde{\alpha}^{(n)}(y) = P(F^n(y)) \circ \tilde{\beta}^{(n)}(y) \circ P(y)^{-1}$$

for all $y \in M$ and $n \in \mathbb{N}$.

Rewriting this in terms of the original cocycles, we get

$$\alpha^{(n\cdot n_0)}(y) = P(f^{n\cdot n_0}(y)) \circ \beta^{(n\cdot n_0)}(y) \circ P(y)^{-1}$$

for every $y \in M$ and $n \in \mathbb{N}$. In other words, $P \in C(\alpha^{(n_0)}, \beta^{(n_0)})$ and $P(x_0) = \mathrm{Id}$.

What we are going to do in the rest of this section consists in showing that, in fact, this P is a transfer map for α and β, i.e. $P \in C(\alpha^{(1)}, \beta^{(1)})$.

To prove that, the main step of our argument is the following.

LEMMA 5.1. *Let $P \in C(\alpha^{(n)}, \beta^{(n)})$ and $x \in M$ be an arbitrary point. Then*

$$P(z) = H^{s,\alpha}_{xz} \circ P(x) \circ H^{s,\beta}_{zx} \quad \textit{for all } z \in W^s(x).$$

An analogous result holds for points on the unstable set $W^u(x)$.

Proof. Since $P \in C(\alpha^{(n)}, \beta^{(n)})$, we have that

$$\alpha^{(n)}(y) = P(f^n(y)) \circ \beta^{(n)}(y) \circ P(y)^{-1} \quad \text{for all } y \in M,$$

which can be restated as

$$\alpha^{(nk)}(y) = P(f^{nk}(y)) \circ \beta^{(nk)}(y) \circ P(y)^{-1} \tag{17}$$

for all $k \in \mathbb{N}$ and $y \in M$.

Let $z \in W^s_\varepsilon(x)$ be an arbitrary point. Observe initially that, by (17), we have that

$$\begin{aligned}
&\alpha^{(nk)}(z)^{-1} \circ P(f^{nk}(x)) \circ \beta^{(nk)}(z) \\
&\quad = \alpha^{(nk)}(z)^{-1} \circ \alpha^{(nk)}(x) \circ P(x) \circ \beta^{(nk)}(x)^{-1} \circ \beta^{(nk)}(z) \to H^{s,\alpha}_{xz} \circ P(x) \circ H^{s,\beta}_{zx}
\end{aligned}$$

as k goes to infinity. Now, considering the distance $\tilde{d}$ defined in the proof of Lemma 4.1, we have again by (17) that

$$\begin{aligned}
&\tilde{d}(P(z), \alpha^{(nk)}(z)^{-1} \circ P(f^{nk}(x)) \circ \beta^{(nk)}(z)) \\
&\quad = \tilde{d}(\alpha^{(nk)}(z)^{-1} \circ P(f^{nk}(z)) \circ \beta^{(n\cdot k)}(z), \alpha^{(nk)}(z)^{-1} \circ P(f^{nk}(x)) \circ \beta^{(nk)}(z)) \\
&\quad = \tilde{d}(\alpha^{(nk)}(z)^{-1} \circ P(f^{nk}(z)), \alpha^{(nk)}(z)^{-1} \circ P(f^{nk}(x))) \\
&\quad \le \rho^{nk} \tilde{d}(P(f^{nk}(z)), P(f^{nk}(x))) \le C\rho^{nk} d(f^{n\cdot k}(z), f^{n\cdot k}(x)) \\
&\quad \le C\rho^{nk}\lambda^{nk} d(z, x) = C(\rho\lambda)^{nk} d(z, x) \to 0
\end{aligned}$$

when k goes to infinity, because $\rho\lambda < 1$.

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Therefore, since

$$\alpha^{(n\cdot k)}(z)^{-1} \circ P(f^{n\cdot k}(x)) \circ \beta^{(n\cdot k)}(z) \longrightarrow H^{s,\alpha}_{xz} \circ P(x) \circ H^{s,\beta}_{zx} \quad \text{as } k \to \infty,$$

it follows that

$$P(z) = H^{s,\alpha}_{xz} \circ P(x) \circ H^{s,\beta}_{zx},$$

as desired. The case when $z \in W^s(x)$ follows easily from the previous one. □

Remark 5.2. The previous lemma holds with α and β being just 1-dominated. Another superfluous hypothesis is that $P(M) \subset \mathrm{Diff}^{r-4}(N)$. In fact, the same result holds if $P(M) \subset \mathrm{Diff}^{s}(N)$ for any $s \geq 1$ and $P : M \to \mathrm{Homeo}(N)$ is Lipschitz with respect to the distance $\tilde{d}$. This will be used in the last section.

Let $x \in M$ be a periodic point and assume that its period is n_0. Consider $P \in C(\alpha^{(n_0)}, \beta^{(n_0)})$ such that $P(x) = \mathrm{Id}$, which we know to exist by the previous comments, and $y \in W^s(x) \cap W^u(f^{n_0-1}(x))$, which in particular is such that $f(y) \in W^u(x)$. By Lemma 5.1, we know that

$$P|_{W^s(x)} : W^s(x) \to \mathrm{Diff}^{r-1}(N)$$

is given by $P(z) = H^{s,\alpha}_{xz} \circ H^{s,\beta}_{zx}$ and

$$P|_{W^u(x)} : W^u(x) \to \mathrm{Diff}^{r-1}(N)$$

is given by $P(w) = H^{u,\alpha}_{xw} \circ H^{u,\beta}_{wx}$ and consequently

$$P(y) = H^{u,\alpha}_{xy} \circ H^{u,\beta}_{yx} \quad \text{and} \quad P(f(y)) = H^{u,\alpha}_{xf(y)} \circ H^{u,\beta}_{f(y)x}.$$

We claim now that

$$\alpha^{(1)}(y) = P(f(y)) \circ \beta^{(1)}(y) \circ P(y)^{-1},$$

that is,

$$P(y) = \alpha^{(1)}(y)^{-1} \circ P(f(y)) \circ \beta^{(1)}(y).$$

The proof of this claim is similar to the proof of Lemma 4.1, so we will just indicate how to proceed.

Observe initially that

$$P(y) = H^{u,\alpha}_{xy} \circ H^{u,\beta}_{yx} = \lim_{n\to\infty} \alpha^{(n\cdot n_0)}(y)^{-1} \circ \alpha^{(n\cdot n_0)}(x) \circ \beta^{(n\cdot n_0)}(x)^{-1} \beta^{(n\cdot n_0)}(y),$$

which, by the periodic orbit condition (POC), is equal to $\lim_{n\to\infty} \alpha^{(n\cdot n_0)}(y)^{-1} \circ \beta^{(n\cdot n_0)}(y)$. Analogously,

$$\begin{aligned} P(f(y)) &= H^{u,\alpha}_{xf(y)} \circ H^{u,\beta}_{f(y)x} \\ &= \lim_{n\to\infty} \alpha^{(-n\cdot n_0)}(f(y))^{-1} \circ \alpha^{(-n\cdot n_0)}(x) \circ \beta^{(-n\cdot n_0)}(x)^{-1} \circ \beta^{(-n\cdot n_0)}(f(y)) \\ &= \lim_{n\to\infty} \alpha^{(-n\cdot n_0)}(f(y))^{-1} \circ \beta^{(-n\cdot n_0)}(f(y)) \end{aligned}$$

and, so,

$$\alpha^{(1)}(y)^{-1} \circ P(f(y)) \circ \beta^{(1)}(y) = \lim_{n\to\infty} \alpha^{(-n\cdot n_0+1)}(y)^{-1} \circ \beta^{(-n\cdot n_0+1)}(y).$$

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Thus, what we have to prove is that

$$\lim_{n\to\infty} \alpha^{(n\cdot n_0)}(y)^{-1} \circ \beta^{(n\cdot n_0)}(y) = \lim_{n\to\infty} \alpha^{(-n\cdot n_0+1)}(y)^{-1} \circ \beta^{(-n\cdot n_0+1)}(y).$$

As already mentioned, we will proceed analogously to what we did in the proof of Lemma 4.1. Fix C_5 and $\varepsilon_0 > 0$ such that the Anosov closing lemma 4.2 holds for $\theta \in (\lambda, 1) > 0$ such that $\rho \cdot \theta < 1$, where $\rho = \rho(\alpha, \beta) > 0$ is defined in §2.4, and consider the distance $\tilde{d}$ on $\mathrm{Diff}^r(N)$ as defined in the proof of Lemma 4.1. As $y \in W^s(x)$, $f(y) \in W^u(x)$ and $f^{n_0}(x) = x$, we can find $C_7 > 0$ and $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$ we have

$$d(f^{-n\cdot n_0+1}(y),\ f^{n\cdot n_0}(y)) \leq C_7\lambda^{(n-n_2)\cdot n_0}.$$

Fix $n_3 \geq n_2$ such that for all $n \geq n_3$ we have that $d(f^{-n\cdot n_0+1}(y),\ f^{n\cdot n_0}(y)) < \varepsilon_0$ and, thus, by the Anosov closing lemma 4.2, for all $n \geq n_3$ there exists a periodic point $p_n \in M$ with $f^{2n\cdot n_0-1}(p_n) = p_n$ and such that

$$d(f^j(f^{-n\cdot n_0+1}(p_n)),\ f^j(f^{-n\cdot n_0+1}(y))) \leq C_5\theta^{\min\{j,2n\cdot n_0-1-j\}}d(f^{-n\cdot n_0+1}(y),\ f^{n\cdot n_0}(y))$$

for all $j = 0, 1, \ldots, (2n \cdot n_0 - 1)$. Using the periodic orbit condition (POC) and $f^{2n\cdot n_0-1}(f^{-n\cdot n_0+1}(p_n)) = f^{-n\cdot n_0+1}(p_n)$, we get that

$$\alpha^{(2n\cdot n_0-1)}(f^{-n\cdot n_0+1}(p_n)) = \beta^{(2n\cdot n_0-1)}(f^{-n\cdot n_0+1}(p_n)),$$

which can be rewritten as

$$\alpha^{(n\cdot n_0)}(p_n) \circ \alpha^{(n\cdot n_0-1)}(f^{-n\cdot n_0+1}(p_n)) = \beta^{(n\cdot n_0)}(p_n) \circ \beta^{(n\cdot n_0-1)}(f^{-n\cdot n_0+1}(p_n))$$

or, equivalently, as

$$\alpha^{(n\cdot n_0)}(p_n)^{-1} \circ \beta^{(n\cdot n_0)}(p_n) = \alpha^{(n\cdot n_0-1)}(f^{-n\cdot n_0+1}(p_n)) \circ \beta^{(n\cdot n_0-1)}(f^{-n\cdot n_0+1}(p_n))^{-1}.$$

Observing now that

$$\begin{aligned}&\alpha^{(n\cdot n_0-1)}(f^{-n\cdot n_0+1}(p_n))\\ &\quad = \alpha^{(1)}(f^{-1}(p_n)) \circ \cdots \circ \alpha^{(1)}(f^{-n\cdot n_0+1}(p_n)) = \alpha^{(-n\cdot n_0+1)}(p_n)^{-1}\end{aligned}$$

and

$$\beta^{(n\cdot n_0-1)}(f^{-n\cdot n_0+1}(p_n))^{-1} = \beta^{(-n\cdot n_0+1)}(p_n),$$

we get that

$$\alpha^{(n\cdot n_0)}(p_n)^{-1} \circ \beta^{(n\cdot n_0)}(p_n) = \alpha^{(-n\cdot n_0+1)}(p_n)^{-1} \circ \beta^{(-n\cdot n_0+1)}(p_n). \tag{18}$$

Following the same lines as in the proof of the claim in Lemma 4.1, we get that both

$$\tilde{d}(\alpha^{(n\cdot n_0)}(p_n)^{-1} \circ \beta^{(n\cdot n_0)}(p_n),\ \alpha^{(n\cdot n_0)}(y)^{-1} \circ \beta^{(n\cdot n_0)}(y))$$

and

$$\tilde{d}(\alpha^{(-n\cdot n_0+1)}(p_n)^{-1} \circ \beta^{(-n\cdot n_0+1)}(p_n),\ \alpha^{(-n\cdot n_0+1)}(y)^{-1} \circ \beta^{(-n\cdot n_0+1)}(y))$$

go to zero when n goes to infinity. Thus, using this fact and (18), we get that

$$\lim_{n\to\infty} \alpha^{(n\cdot n_0)}(y)^{-1} \circ \beta^{(n\cdot n_0)}(y) = \lim_{n\to\infty} \alpha^{(-n\cdot n_0+1)}(y)^{-1} \circ \beta^{(-n\cdot n_0+1)}(y)$$

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and consequently that

$$\alpha^{(1)}(y) = P(f(y)) \circ \beta^{(1)}(y) \circ P(y)^{-1},$$

proving our claim.

Summarizing, we have obtained a map $P \in C(\alpha^{(n_0)}, \beta^{(n_0)})$ with $P(x) = \mathrm{Id}$, where $x \in \mathrm{Fix}(f^{n_0})$, and such that, for every $y \in W^s(x) \cap W^u(f^{n_0-1}(x))$, it satisfies

$$\alpha^{(1)}(y) = P(f(y)) \circ \beta^{(1)}(y) \circ P(y)^{-1}.$$

Thus, as all the objects involved are uniformly continuous and $W^s(x) \cap W^u(f^{n_0-1}(x))$ is dense in M, it follows that

$$\alpha^{(1)}(y) = P(f(y)) \circ \beta^{(1)}(y) \circ P(y)^{-1}$$

for all $y \in M$, that is, $P \in C(\alpha^{(1)}, \beta^{(1)})$, which completes the proof of the existence part of Theorem 1.1.

Another way to conclude the proof of the existence part is combining the fact that there exists $P \in C(\alpha^{(n_0)}, \beta^{(n_0)})$ such that for some $y \in M$ it satisfies

$$\alpha^{(1)}(y) = P(f(y)) \circ \beta^{(1)}(y) \circ P(y)^{-1}$$

with the next lemma, which is simple and interesting by itself.

LEMMA 5.3. *Let $P \in C(\alpha^{(n_0)}, \beta^{(n_0)})$. If there exist some $y \in M$ such that $\alpha^{(1)}(y) = P(f(y)) \circ \beta^{(1)}(y) \circ P(y)^{-1}$, then $P \in C(\alpha^{(1)}, \beta^{(1)})$.*

Proof. By Lemma 5.1, we know that

$$P|_{W^s(y)} : W^s(y) \to \mathrm{Diff}^{r-4}(N)$$

is given by $P(z) = H^{s,\alpha}_{yz} \circ P(y) \circ H^{s,\beta}_{zy}$. Thus,

$$\begin{aligned}\alpha^{(1)}(z) \circ P(z) \circ \beta^{(1)}(z)^{-1} &= \alpha^{(1)}(z) \circ H^{s,\alpha}_{yz} \circ P(y) \circ H^{s,\beta}_{zy} \circ \beta^{(1)}(z)^{-1} \\ &= H^{s,\alpha}_{f(y)f(z)} \circ \alpha^{(1)}(y) \circ P(y) \circ \beta^{(1)}(y)^{-1} \circ H^{s,\beta}_{f(z)f(y)} \\ &= H^{s,\alpha}_{f(y)f(z)} \circ P(f(y)) \circ H^{s,\beta}_{f(z)f(y)},\end{aligned}$$

which again, by Lemma 5.1, is equal to $P(f(z))$. So,

$$\alpha^{(1)}(z) = P(f(z)) \circ \beta^{(1)}(z) \circ P(z)^{-1}$$

for all $z \in W^s(y)$. Now, as $W^s(y)$ is dense in M and P is uniformly continuous, it follows that

$$\alpha^{(1)}(z) = P(f(z)) \circ \beta^{(1)}(z) \circ P(z)^{-1}$$

for all $z \in M$, as we want. □

Remark 5.4. We would like to stress again that, even though we are working with $\nu = 1$, all results are valid for any $\nu > 0$ and their proofs are analogous. Obviously, in the case $\nu \in (0, 1)$ our constructions above will produce a ν-Hölder map $P : M \to \mathrm{Diff}^{r-4}(N)$ instead of a Lipschitz one.

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6. *Improving regularity*

The existence part of Theorem 1.1 gives us a Lipschitz map $P : M \to \mathrm{Diff}^{r-4}(N)$ such that

$$\alpha^{(n)}(x) = P(f^n(x)) \circ \beta^{(n)}(x) \circ P(x)^{-1}$$

for all $x \in M$ and $n \in \mathbb{N}$. In this section we are going to show that the image of this P is contained in $\mathrm{Diff}^{r-1}(N)$ and also that when f, α and β exhibit a higher regularity, so does P.

Let $\tilde{d}$ be the distance in $\mathrm{Homeo}(N)$ as defined in the proof of Lemma 4.1.

THEOREM 6.1. *Let M, N and f be as in Theorem 1.1 and $\alpha, \beta : \mathbb{Z} \times M \to \mathrm{Diff}^r(N)$ be two $(2r-1)$-dominated Lipschitz cocycles over f. Let us assume that there exists $P : M \to \mathrm{Diff}^1(N)$, a Lipschitz continuous map with respect to the distance $\tilde{d}$, such that*

$$\alpha^{(1)}(x) = P(f(x)) \circ \beta^{(1)}(x) \circ P(x)^{-1} \quad \textit{for all } x \in M.$$

Assume also that there exists $x \in M$ such that $P(x) \in \mathrm{Diff}^{r-1}(N)$. Then $P(M) \subset \mathrm{Diff}^{r-1}(N)$. Moreover, if f is a C^{r-1} Anosov diffeomorphism and the cocycles α and β are C^{r-1}, then P is $C^{r-1-\varepsilon}$ for any small $\varepsilon > 0$.

Recall that the cocycle $\alpha : \mathbb{Z} \times M \to \mathrm{Diff}^r(N)$ being C^{r-1} means that the induced map $\alpha^{(1)} : M \times N \to N$ is C^{r-1}.

Proof. Note that by Lemma 5.1, Remark 5.2 and Proposition 3.1, we have automatically that $P(W^s(x) \cup W^u(x)) \subset \mathrm{Diff}^{r-1}(N)$, since $P(x) \in \mathrm{Diff}^{r-1}(N)$ and $H^{s,\alpha}_{xy}$, $H^{u,\alpha}_{xz}$, $H^{s,\beta}_{xy}$ and $H^{u,\beta}_{xz} \in \mathrm{Diff}^{r-1}(N)$ for any $y \in W^s(x)$ and $z \in W^u(x)$. Applying again this argument, we get that $P(W^s(y) \cup W^u(y)) \subset \mathrm{Diff}^{r-1}(N)$ for any $y \in W^s(x) \cup W^u(x)$. Now, as any transitive hyperbolic homeomorphism is accessible, that is, any two points can be connected by a path which is a concatenation of subpaths, where each of which lies entirely on a single stable or unstable leaf, it follows by the previous arguments that $P(M) \subset \mathrm{Diff}^{r-1}(N)$, proving the first assertion.

Assume now that f is a C^{r-1} Anosov diffeomorphism and that α and β are C^{r-1} cocycles. Let us consider the skew product $F_\alpha : M \times N \to M \times N$ given by

$$F_\alpha(x, \xi) = (f(x), \alpha^{(1)}(x)(\xi)), \quad (x, \xi) \in M \times N,$$

which is a C^{r-1} partially hyperbolic diffeomorphism since the cocycle α is $(2r-1)$-dominated. We know that the distributions E^s and E^u are integrable and the corresponding foliations $\hat{W}^s$ (strong stable) and $\hat{W}^u$ (strong unstable) respectively, are C^{r-1} foliations, that is, the leaves $\hat{W}^s(x, \xi)$ and $\hat{W}^u(x, \xi)$ are C^{r-1} and depend continuously on the point $(x, \xi) \in M \times N$ in the C^{r-1} topology. Moreover, we can observe that they are graphs over $W^s(x)$ and $W^u(x)$, respectively. More precisely,

$$\hat{W}^s(x, \xi) = \{(y, H^{s,\alpha}_{xy}(\xi));\ y \in W^s(x)\} \quad \text{and} \quad \hat{W}^u(x, \xi) = \{(y, H^{s,\alpha}_{xy}(\xi));\ y \in W^u(x)\}.$$

This follows from [**HPS77**, Theorem 5.5]. Now, since $\hat{W}^s(x, \xi)$ and $\hat{W}^u(x, \xi)$ are C^{r-1} submanifolds, it follows that $H^{s,\alpha}_x : W^s(x) \times \{\xi\} \to N$ and $H^{u,\alpha}_x : W^u(x) \times \{\xi\} \to N$ are C^{r-1}.

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Analogously, $H_x^{s,\beta} : W^s(x) \times \{\xi\} \to N$ and $H_x^{u,\beta} : W^u(x) \times \{\xi\} \to N$ are C^{r-1}.

Combining this with the fact that for every $x \in M$, $y \in W^s(x)$ and $z \in W^u(x)$ we have $H_{x,y}^{s,\alpha}$, $H_{x,z}^{u,\alpha}$, $H_{x,y}^{s,\beta}$, $H_{x,z}^{u,\beta} \in \mathrm{Diff}^{r-1}(N)$ (see Proposition 3.1) and with Lemma 5.1 and Remark 5.2, it follows that $P|_{W^s(x)}$ and $P|_{W^u(x)}$ are C^{r-1} for every $x \in M$. Then, applying Journé's theorem (see [**Jou88**]), we get that P is $C^{r-1-\varepsilon}$ for any small $\varepsilon > 0$, as we want. □

To complete the proof of Theorem 1.1, we have just to combine what we have done so far: in §§4 and 5, given a periodic point $x \in M$, we have constructed 'explicitly' a Lipschitz continuous map $P : M \to \mathrm{Diff}^{r-4}(N)$ such that $P(x) = \mathrm{Id}$ and

$$\alpha^{(n)}(x) = P(f^n(x)) \circ \beta^{(n)}(x) \circ P(x)^{-1}$$

for all $x \in M$ and $n \in \mathbb{Z}$. Therefore, by Theorem 6.1, $P(M) \subset \mathrm{Diff}^{r-1}(N)$ and if f is a C^{r-1} Anosov diffeomorphism and α and β are C^{r-1}, then P is $C^{r-1-\varepsilon}$ for any small $\varepsilon > 0$, which completes the proof of Theorem 1.1.

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