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A UNIFIED APPROACH TO CONTINUOUS, MEASURABLE SELECTIONS, AND SELECTIONS FOR HYPERSPACES

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Abstract. In this paper we provide a unified approach, based on methods of descriptive set theory, for proving some classical selection theorems. Among them is the zero-dimensional Michael selection theorem, the Kuratowski–Ryll-Nardzewski selection theorem, as well as a known selection theorem for hyperspaces.

§1. *Introduction.* There are three classical results in the theory of selection theorems that we would like to focus on. The first one is due to Michael [22].

ZERO-DIMENSIONAL MICHAEL SELECTION THEOREM. *Let X be a paracompact Hausdorff space with* $\dim(X) = 0$*, Y a completely metrizable space and $\Phi : X \to 2^Y \setminus \{\emptyset\}$ a lower semicontinuous multifunction such that $\Phi(x)$ is a closed subset of Y for all $x \in X$. Then there exists a continuous $\phi : X \to Y$ such that $\phi(x) \in \Phi(x)$ for all $x \in X$.*

The second theorem is due to Kuratowski and Ryll-Nardzewski [17], and we can think of it as the measurable version of Michael's zero-dimensional theorem.

KURATOWSKI–RYLL-NARDZEWSKI SELECTION THEOREM. *Let (X, Σ) be a measurable space, Y a Polish space and $\Phi : X \to 2^Y \setminus \{\emptyset\}$ a multifunction such that $\{x \in X : \Phi(x) \cap U \neq \emptyset\}$ belongs to Σ for each $U \subseteq Y$ open and $\Phi(x)$ is a closed subset of Y for all $x \in X$. Then there exists a Σ-measurable $\phi : X \to Y$ such that $\phi(x) \in \Phi(x)$ for all $x \in X$.*

The third theorem (for the purpose of this paper it will be called the selection theorem for hyperspaces), which is a solution to the so-called hyperspace selection problem, was proved independently by Choban [4] and Engelking, Heath and Michael [9].

SELECTION THEOREM FOR HYPERSPACES. *Let X be a completely metrizable space with* $\dim(X) = 0$ *and $\mathcal{F}(X)$ the family of all non-empty closed subsets of X equipped with the Vietoris topology. Then there exists a continuous $\phi : \mathcal{F}(X) \to X$ such that $\phi(A) \in A$ for all $A \in \mathcal{F}(X)$.*

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This result, as Michael [20] first observed, yields a selection theorem for continuous multifunctions (for more details see §6.3).

So, roughly speaking, these three selection theorems cover three basic types of "measurability" in multifunctions: lower semicontinuous, measurable, and continuous multifunctions. The purpose of this paper is to provide a unified approach, based on methods of descriptive set theory, that gives new proofs of those three theorems.

There have been other unified approaches: Maitra and Rao [19] and Mägerl [18] proved general theorems that include as special cases the zero-dimensional Michael selection theorem and the Kuratowski–Ryll-Nardzewski selection theorem. They also share the same strategy with the original proofs of Michael and Kuratowski–Ryll-Nardzewski; that is, the selection is defined as the uniform limit of an appropriate sequence of functions.

In our approach we follow a different strategy. There is a standard technique in descriptive set theory by which we can embed every compact metrizable zero-dimensional space in the Cantor space $\{0, 1\}^\omega$, or every Polish zero-dimensional space in the Baire space ω^ω (see, for example, [16, Theorems 7.4 and 7.7]). Arvanitakis [3], combining elements of this technique with ideas from the study of regular averaging and regular extension operators (see [2, 6, 7, 13, 14]), gave a simultaneous selection theorem, which could be seen in the context of convex-valued selection theorems (see also [27, 29]). In this paper, using the aforementioned language and ideas of descriptive set theory (Kechris [16]), we adjust the approach of Arvanitakis [3] in order to provide another unified method of constructing continuous and measurable selections.

Let us outline our approach. Assume for simplicity that space Y is the Cantor space $\{0, 1\}^\omega$. Let also $\{V_t : t \in \{0, 1\}^{<\omega}\}$ be the usual clopen base for the topology of $\{0, 1\}^\omega$. There are two well-known useful facts concerning the structure of $\{V_t : t \in \{0, 1\}^{<\omega}\}$. The first is that

$$V_\emptyset = Y \quad \text{and} \quad V_t = \bigcup \{V_s : s \text{ is an immediate successor of } t\} \quad \text{for all } t \in \{0, 1\}^{<\omega}.$$

The second is that for each $b \in \{0, 1\}^\omega$ we have

$$\bigcap_{n \in \omega} V_{b|n} = \{b\} \quad \text{and} \quad \{V_{b|n} : n \in \omega\} \text{ is a neighborhood base at } b.$$

Assume now that X is a non-empty set, Σ a family of subsets of X and $\Phi : X \to 2^Y \setminus \{\emptyset\}$ a multifunction such that $\{x \in X : \Phi(x) \cap U \neq \emptyset\} \in \Sigma$ for each $U \subseteq Y$ open and $\Phi(x)$ is a closed subset of Y for all $x \in X$. If we set

$$U_t = \{x \in X : \Phi(x) \cap V_t \neq \emptyset\},$$

it is easy to see that $U_t \in \Sigma$ for all $t \in \{0, 1\}^{<\omega}$ and, moreover,

$$U_\emptyset = X \quad \text{and} \quad U_t = \bigcup \{U_s : s \text{ is an immediate successor of } t\} \quad \text{for all } t \in \{0, 1\}^{<\omega}.$$

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In our method the existence of a Σ-measurable selection (a map $\phi : X \to Y$ such that $\phi(x) \in \Phi(x)$ for all $x \in X$ and $\phi^{-1}(V) \in \Sigma$ for all $V \subseteq Y$ open) relies on the ability of constructing a family $\{G_t : t \in \{0, 1\}^{<\omega}\}$ such that for all t:

(i) $G_t \in \Sigma$;
(ii) $G_\emptyset = X$ and $\{G_s : s \text{ successor of } t\}$ is a pairwise disjoint cover of G_t;
(iii) $G_t \subseteq U_t$.

In this situation, (ii) gives that for each $x \in X$ there is a unique $b_x \in \{0, 1\}^\omega$ such that $x \in \bigcap_{n\in\omega} G_{b_x|n}$. Then $\phi : X \to Y$ defined by $\{\phi(x)\} = \bigcap_{n\in\omega} V_{b_x|n} = \{b_x\}$ is a selection of Φ, since from (iii) we have $\Phi(x) \cap V_{b_x|n} \neq \emptyset$ for all $n \in \omega$. Moreover, if Σ is closed under countable unions, then the fact that $\{V_{b|n} : n \in \omega\}$ is a neighborhood base at b for each $b \in \{0, 1\}^\omega$ implies that ϕ is Σ-measurable.

This approach is presented in a general form in Theorem 4.2. With this theorem we give new proofs of the zero-dimensional Michael selection theorem (§6.1) and the Kuratowski–Ryll-Nardzewski selection theorem (§6.2). Along with these proofs we give new characterizations for the existence of continuous and measurable selections. The core of Theorem 4.2 is also used to obtain an elementary proof of the selection theorem for hyperspaces (§6.3).

Finally, we would like to mention that, to the best of our knowledge, our approach shares some similar ideas with [**4**, **10**, **23**] in the context of multivalued selection theorems.

§2. *Preliminaries and notation.*

2.1. *Some topological terminology.* We begin by fixing some topological terminology; for reference see [**8**, **28**]. Let X be a topological space and $A \subseteq X$. The closure of A in X will be denoted by $\overline{A}$ or $\mathrm{cl}_X(A)$, whereas the interior of A in X will be denoted by $\mathrm{int}_X(A)$.

Let $\{W_i\}_{i\in I}$ be a cover of X. We say that $\{G_i\}_{i\in I}$ is a precise refinement of $\{W_i\}_{i\in I}$ (or that $\{W_i\}_{i\in I}$ has a precise refinement $\{G_i\}_{i\in I}$), if $\{G_i\}_{i\in I}$ is a cover of X and $G_i \subseteq W_i$ for all $i \in I$; $\{G_i\}_{i\in I}$ is a disjoint precise refinement of $\{W_i\}_{i\in I}$ if, in addition, $\{G_i\}_{i\in I}$ is a pairwise disjoint family.

X will be called zero-dimensional, if there exists a base for the topology of X consisting of clopen sets; $\dim(X) = 0$ concerns the covering dimension of a normal space.

2.2. *Multifunctions.* Secondly we fix the notation and terminology about multifunctions; many of them can be found in [**1**, **5**, **15**, **25**].

Let X, Y be non-empty sets. By 2^Y we denote the set of all subsets of Y and by $2^Y \setminus \{\emptyset\}$ the set of all non-empty subsets of Y. A function $\Phi : X \to 2^Y$ will be called multifunction.

If Y is equipped with a topology, we denote by $\mathcal{F}(Y)$ the set of all non-empty closed subsets of Y and by $\mathcal{K}(Y)$ the set of all non-empty compact subsets of Y. A multifunction $\Phi : X \to \mathcal{F}(Y)$ will be called closed-valued; if $\mathcal{K}(Y)$ is instead of $\mathcal{F}(Y)$, then the multifunction will be called compact-valued.

We say that $\phi : X \to Y$ is a selection of Φ, if $\phi(x) \in \Phi(x)$ for all $x \in X$. In the case where there exists such a ϕ, we also say that Φ admits

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a single-valued selection (we often use the term "single-valued" to make a distinction from selections that are multifunctions).

Let W be a subset of Y. The lower inverse image of W under Φ is defined as

$$\Phi^l(W) = \{x \in X : \Phi(x) \cap W \neq \emptyset\},$$

whereas the upper inverse image of W under Φ is defined as

$$\Phi^u(W) = \{x \in X : \Phi(x) \subseteq W\}.$$

If X and Y are equipped with a topology, we say that Φ is lower semicontinuous (and we use the abbreviation "lsc") if $\Phi^l(U)$ is open in X for every $U \subseteq Y$ open. Respectively, Φ is upper semicontinuous (and we use the abbreviation "usc") if $\Phi^u(U)$ is open in X for every $U \subseteq Y$ open, equivalently, if $\Phi^l(F)$ is closed in X for every $F \subseteq Y$ closed. Finally, Φ is continuous if and only if it is both lsc and usc.

More generally, let $\Sigma \subseteq 2^X$ and Y be a topological space. We say that Φ is Σ-measurable if $\Phi^l(U) \in \Sigma$ for every $U \subseteq Y$ open. Similarly, a function $\phi : X \to Y$ is Σ-measurable if $\phi^{-1}(U) \in \Sigma$ for every $U \subseteq Y$ open. Obviously, semicontinuous multifunctions and continuous selections are special cases of measurable multifunctions and measurable selections respectively, so the term "measurable" will often be used in general.

2.3. *Trees.* Trees are the basic tool in our method of constructing single-valued selections. In this subsection we mainly follow [**16**] for the necessary terminology.

Let A be a non-empty set. We set $A^{<\omega} = \bigcup_{n<\omega} A^n$ and $A^{\leqslant\omega} = \bigcup_{n\leqslant\omega} A^n$ and we follow the convention $A^0 = \{\emptyset\}$. We define a partial order $\sqsubseteq$ on $A^{\leqslant\omega}$ as follows: $t \sqsubseteq s$ if and only if there exists $n \leqslant \omega$ so that $t = s|n$, where $s|n = (s(0), \ldots, s(n-1))$ denotes the restriction of the function s on the domain $n = \{0, 1, \ldots, n-1\}$; we also make the convention that $s|0 = \emptyset$ for all $s \in A^{\leqslant\omega}$. The corresponding strict partial order is denoted by $\sqsubset$. For $t, s \in A^{<\omega}$ with $t \sqsubseteq s$, we say that t is a predecessor of s or s is a successor of t. The height of $t \in A^{\leqslant\omega}$, denoted by $|t|$, is defined as the $n \leqslant \omega$ such that $t \in A^n$.

Let T be a non-empty subset of $A^{<\omega}$. We say that T is a tree (on A of ω height), if it is closed under predecessors. Since a tree T is by definition non-empty and closed under predecessors, we have $\emptyset \sqsubseteq t$ for all $t \in A^{<\omega}$, and thus, $\emptyset \in T$; in the present setting T is always a rooted tree. Moreover, we say that T' is a subtree of T, if $T' \subseteq T$ and T' is a tree (on A).

For $n \leqslant \omega$ the n-level of T is defined as $T^n = \{t \in T : |t| = n\}$. The height of T, denoted by $ht(T)$, is defined as the least ordinal $n \leqslant \omega$ such that $T^n = \emptyset$.

For each $t \in T$ we define the set of all immediate successors of t as

$$\mathrm{succ}(t) = \{s \in T : t \sqsubset s \text{ and } |s| = |t| + 1\}.$$

If k is a cardinal, we say that T is k-branching if the cardinality of $\mathrm{succ}(t)$, denoted by $|\mathrm{succ}(t)|$, is strictly less than k for all $t \in T$; an $\aleph_1$-branching tree

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will be called countably branching; an $\aleph_0$-branching tree will be called finitely branching.

We say that T is pruned, if $\mathrm{succ}(t) \neq \emptyset$ for all $t \in T$; it is easy to observe that a pruned tree T has height equal to ω.

Finally, the body of a pruned tree T is defined as

$$[T] = \{b \in A^\omega : b|n \in T \text{ for all } n \in \omega\},$$

which is obviously non-empty.

2.4. *Topology on the bodies of trees.* Finally, we are going to briefly recall the usual topology on the bodies of pruned trees.

First of all, if T is a pruned tree on A, for each $t \in T$ we define

$$V_t = \{b \in [T] : t \sqsubset b\}.$$

We assume that A is equipped with the discrete topology and A^ω with the product topology. $[T]$ is equipped with the relative topology induced by A^ω and $\{V_t : t \in T\}$ is a base for $[T]$ consisting of clopen sets.

Evidently $[T]$ is completely metrizable; $[T]$ is Polish in the case where T is countably branching. It is easy to see that $\dim([T]) = 0$ (thus, $[T]$ is also zero-dimensional). Moreover, it is known that a non-empty $F \subseteq A^\omega$ is closed if and only if there exists a pruned tree T on A such that $F = [T]$; a non-empty $K \subseteq A^\omega$ is compact if and only if there exists a finitely branching pruned tree T on A such that $K = [T]$. For more details we refer to [**16**, Chs. I-2 and I-4].

§3. *Tree schemes.* Considering the notions of Cantor and Lusin schemes in descriptive set theory (see [**16**, Definitions 6.1 and 7.5]), we give the following definitions.

Definition 3.1. A tree scheme on a non-empty set X is a family $(W_t)_{t\in T}$ of subsets of X such that:

(i) T is a pruned tree;
(ii) $W_\emptyset = X$ and $W_t = \bigcup\{W_s : s \in \mathrm{succ}(t)\}$ for all $t \in T$.

Definition 3.2. Let $(W_t)_{t\in T}$ be a tree scheme on a non-empty set X.

(i) If $\Sigma \subseteq 2^X$ and $(W_t)_{t\in T} \subseteq \Sigma$, then we say that $(W_t)_{t\in T}$ is a Σ tree scheme.
In the case where X is a topological space, we use the following alternative terminology: if Σ is the family of open subsets of X, we say that $(W_t)_{t\in T}$ is an open tree scheme; if Σ is the family of closed subsets of X, we say that $(W_t)_{t\in T}$ is a closed tree scheme; if Σ is the family of clopen subsets of X, we say that $(W_t)_{t\in T}$ is a clopen tree scheme.
If $\mathcal{A} \subseteq 2^\Sigma$ and $\{W_s : s \in \mathrm{succ}(t)\} \in \mathcal{A}$ for all $t \in T$, then we say that $(W_t)_{t\in T}$ is an $\mathcal{A}$ tree scheme. For example, $\mathcal{A}$ could stand for the collection of point-finite families of open subsets of X.
(ii) $(W_t)_{t\in T}$ is said to be disjoint, if $\{W_s : s \in \mathrm{succ}(t)\}$ is a pairwise disjoint family for all $t \in T$.

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(iii) $(W_t)_{t\in T}$ is said to be point-finite, if $\{W_s : s \in \mathrm{succ}(t)\}$ is point-finite in W_t for each $t \in T$.

(iv) We say that $(G_t)_{t\in T}$ is a precise refinement of $(W_t)_{t\in T}$ (or that $(W_t)_{t\in T}$ has a precise refinement $(G_t)_{t\in T}$), if $(G_t)_{t\in T}$ is a tree scheme on X and $\{G_t : |t| = n\}$ is a precise refinement of $\{W_t : |t| = n\}$ for each $n \in \omega$.

(v) If X is a topological space, then $(W_t)_{t\in T}$ is said to be locally finite, if $\{W_s : s \in \mathrm{succ}(t)\}$ is locally finite in the subspace W_t for each $t \in T$.

In the next lemma we prove some basic properties concerning tree schemes, which reveal their convenient structure; roughly speaking, a tree scheme decomposes the space in a "uniform" way.

LEMMA 3.3. *Let $(W_t)_{t\in T}$ be a tree scheme on a non-empty set X. Then the following conditions are satisfied.*

(i) *$W_s \subseteq W_t$ for all $t \sqsubseteq s$.*

(ii) *For every $x \in X$ there exists $b_x \in [T]$ such that $x \in \bigcap_{n\in\omega} W_{b_x|n}$. Moreover, if $(W_t)_{t\in T}$ is disjoint, then b_x is unique.*

(iii) *$\{W_t : |t| = n\}$ is a cover of X for all $n \in \omega$. Moreover, if $(W_t)_{t\in T}$ is disjoint, then $\{W_t : |t| = n\}$ is a pairwise disjoint cover of X for all $n \in \omega$.*

(iv) *$(W_t)_{t\in T}$ is point-finite if and only if $\{W_t : |t| = n\}$ is point-finite in X for all $n \in \omega$.*

(v) *If X is a topological space and $(W_t)_{t\in T}$ is closed, then $(W_t)_{t\in T}$ is locally finite if and only if $\{W_t : |t| = n\}$ is locally finite in X for all $n \in \omega$.*

Proof. (i) We are going to proceed by induction. Obviously $W_s \subseteq W_t$ for all $t \sqsubset s$ with $|s| = |t|$. By definition of the tree scheme we have that $W_s \subseteq W_t$ for all $t \sqsubset s$ with $|s| = |t| + 1$. Suppose that for $n \in \omega$ we have $W_s \subseteq W_t$ for all $t \sqsubset s$ with $|s| = |t| + n$. Let $t \sqsubset s$ with $|s| = |t| + n + 1$. There is $r \in T$ such that $t \sqsubset r \sqsubset s$, $|r| = |t| + n$ and $|s| = |r| + 1$, and thus, $W_r \subseteq W_t$ and $W_s \subseteq W_r$, and this completes the inductive step.

(ii) Let $x \in X$. We set $t_x^0 = \emptyset$. Suppose that there is $t_x^n \in T^n$ with $x \in W_{t_x^n}$. There is $s \in \mathrm{succ}(t_x^n)$ such that $x \in W_s$. We set $t_x^{n+1} = s$. Thus, we can recursively define an increasing (with respect to $\sqsubset$) sequence $\{t_x^n\}_{n\in\omega}$ such that $|t_x^n| = n$ and $x \in W_{t_x^n}$ for all $n \in \omega$. We conclude the result by defining $b_x = \bigcup_{n\in\omega} t_x^n$.

Now let us examine the uniqueness of b_x in the case where $(W_t)_{t\in T}$ is disjoint. Suppose that there exist $b_x \neq b_x' \in [T]$ such that $x \in \bigcap_{n\in\omega} W_{b_x|n}$ and $x \in \bigcap_{n\in\omega} W_{b_x'|n}$. There is a minimum $n \in \omega$ such that $b_x|n \neq b_x'|n$, so $W_{b_x|n} \cap W_{b_x'|n} = \emptyset$, which is a contradiction.

(iii) Follows easily from (ii).

(iv) Let $(W_t)_{t\in T}$ be point-finite; the inverse direction is obvious. We are going to proceed by induction. Firstly, $\{W_t : |t| = 0\}$ is point-finite in X. Suppose that $\{W_t : |t| = n\}$ is point-finite in X for some $n \in \omega$. If x belongs to infinitely many elements of $\{W_t : |t| = n + 1\}$, then there should be $t \in T^n$ such that $x \in W_t$ and $x \in W_s$ for infinitely many $s \in \mathrm{succ}(t)$, which is a contradiction. Thus, $\{W_t : |t| = n + 1\}$ is point-finite in X.

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(v) Let $(W_t)_{t\in T}$ be a closed tree scheme on X. Assume that $(W_t)_{t\in T}$ is locally finite; the inverse direction is obvious. First of all, $\{W_t : |t| = 0\}$ is locally finite in X. We assume that $\{W_t : |t| = n\}$ is locally finite in X and we will show that $\{W_t : |t| = n + 1\}$ is locally finite in X. Let $x \in X$. By the inductive hypothesis there exists U_x, an open neighborhood of x, such that the set,

$$T^n_{U_x} = \{t \in T : U_x \cap W_t \neq \emptyset \text{ and } |t| = n\},$$

is finite. Let also

$$T^n_x = \{t \in T : x \in W_t \text{ and } |t| = n\},$$

which is finite as a subset of $T^n_{U_x}$. We observe that

$$U_x \setminus \bigcup\{W_t : t \in T^n_{U_x} \setminus T^n_x\}$$

is an open (since $\{W_t : t \in T^n_{U_x} \setminus T^n_x\}$ is a finite family of closed subsets) neighborhood of x which intersects W_t with $|t| = n$ if and only if $t \in T^n_x$. $\{W_s : s \in \mathrm{succ}(t)\}$ is locally finite in the subspace W_t for all $t \in T$, so for each $t \in T^n_x$ there exists an open (with respect to the relative topology of W_t) neighborhood $U_t(x)$ of x, such that the set,

$$T^{n+1}_{U_t(x)} \cap \mathrm{succ}(t) = \{s \in \mathrm{succ}(t) : U_t(x) \cap W_s \neq \emptyset\},$$

is finite. Since $U_t(x)$ is open in W_t, there exists an open (with respect to the topology of X) neighborhood $V_t(x)$ of x such that $U_t(x) = V_t(x) \cap W_t$. Observe that the set

$$T^{n+1}_{V_t(x)} \cap \mathrm{succ}(t) = \{s \in \mathrm{succ}(t) : V_t(x) \cap W_s \neq \emptyset\}$$

is equal to

$$T^{n+1}_{U_t(x)} \cap \mathrm{succ}(t) = \{s \in \mathrm{succ}(t) : U_t(x) \cap W_s \neq \emptyset\}.$$

Thus, the set

$$\left(U_x \setminus \bigcup\{W_t : t \in T^n_{U_x} \setminus T^n_x\}\right) \cap \left(\bigcap\{V_t(x) : t \in T^n_x\}\right),$$

is an open (with respect to the topology of X) neighborhood of x that intersects finitely many elements of $\{W_t : |t| = n + 1\}$. □

Definition 3.4. We say that a tree scheme $(W_t)_{t\in T}$ on a topological space X is complete, if the following are satisfied:

(i) for each $b \in [T]$ such that $W_{b|n} \neq \emptyset$ for all $n \in \omega$ there exists $x_b \in X$ with $\bigcap_{n\in\omega} W_{b|n} = \{x_b\}$;
(ii) if $\{x\} = \bigcap_{n\in\omega} W_{b|n}$, then $\{W_{b|n} : n \in \omega\}$ is a neighborhood base at x.

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Remark 3.5. Let $(W_t)_{t\in T}$ be a complete open tree scheme on X. By Lemma 3.3(ii) for each $x \in X$ there is $b_x \in [T]$ such that $x \in \bigcap_{n\in\omega} W_{b_x|n}$. Now since $(W_t)_{t\in T}$ is complete, we have $\{x\} = \bigcap_{n\in\omega} W_{b_x|n}$. Therefore, for each $x \in X$ there is $b_x \in [T]$ such that $\{x\} = \bigcap_{n\in\omega} W_{b_x|n}$ and $\{W_{b_x|n} : n \in \omega\}$ is a neighborhood base at x.

The next proposition asserts that we can obtain complete tree schemes on completely metrizable spaces with "nice" properties. These results are somehow well-known in descriptive set theory (see, for example, [**16**, Theorem 6.2]), but here are stated in a form that fits our definitions and would be useful for our purposes. First, let us prove a useful lemma.

LEMMA 3.6. *Let Y be a metrizable space and $(U_t)_{t\in T}$ an open tree scheme on Y. Then $(U_t)_{t\in T}$ has a closed precise refinement $(F_t)_{t\in T}$, which in turn has an open precise refinement $(W_t)_{t\in T}$ and, moreover, $\{F_t : |t| = n\}$ and $\{W_t : |t| = n\}$ are both locally finite in Y for all $n < \omega$.*

Proof. We are going to define the desirable tree schemes recursively. Set $F_\emptyset = Y$ and suppose that F_t is closed in Y and $F_t \subseteq U_t$. Since F_t is paracompact Hausdorff, as a subspace of a metrizable space, and $\{F_t \cap U_s : s \in \mathrm{succ}(t)\}$ is an open cover of F_t in the relative topology, it has a locally finite open precise refinement $\{D_s : s \in \mathrm{succ}(t)\}$ such that $\mathrm{cl}_{F_t}(D_s) \subseteq F_t \cap U_s$ for all $s \in \mathrm{succ}(t)$. We define $F_s = \mathrm{cl}_{F_t}(D_s)$, which is closed in Y, and we observe that

$$F_t = \bigcup\{\mathrm{int}_{F_t}(F_s) : s \in \mathrm{succ}(t)\} = \bigcup\{F_s : s \in \mathrm{succ}(t)\}.$$

What is more, since $\{D_s : s \in \mathrm{succ}(t)\}$ is locally finite in F_t, the same holds for $\{F_s : s \in \mathrm{succ}(t)\}$ and $\{\mathrm{int}_{F_t}(F_s) : s \in \mathrm{succ}(t)\}$. The recursively constructed locally finite closed tree scheme $(F_t)_{t\in T}$ is a precise refinement of $(U_t)_{t\in T}$; by Lemma 3.3(v), it follows that $\{F_t : |t| = n\}$ is locally finite in Y for all $n < \omega$.

Now set $W_\emptyset = Y$ and suppose that W_t is open in Y and $W_t \subseteq F_t$. Observe that

$$W_t \subseteq \mathrm{int}_X(F_t) \subseteq \bigcup\{\mathrm{int}_{F_t}(F_s) : s \in \mathrm{succ}(t)\}$$

and

$$\mathrm{int}_X(F_t) \cap \mathrm{int}_{F_t}(F_s) \subseteq \mathrm{int}_X(F_s).$$

Hence, $\{W_t \cap \mathrm{int}_X(F_s) : s \in \mathrm{succ}(t)\}$ is an open cover of W_t in the relative topology. Following the same argument as above, there is a locally finite open precise refinement $\{W_s : s \in \mathrm{succ}(t)\}$. Thus, the recursively constructed open tree scheme $(W_t)_{t\in T}$ is a precise refinement of $(F_t)_{t\in T}$, which implies that $\{W_t : |t| = n\}$ is locally finite in Y for all $n < \omega$. □

PROPOSITION 3.7. *Let Y be a completely metrizable space.*

(i) *There exists a complete open tree scheme $(W_t)_{t\in T}$ on Y such that $\{W_t : |t| = n\}$ is locally finite in Y for all $n < \omega$.*

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(ii) *If Y is separable, then T in (i) could be countably branching.*
(iii) *If $\dim(Y) = 0$, then there exists a complete clopen disjoint tree scheme $(W_t)_{t\in T}$ on Y.*

Proof. (i) First we are going to construct a complete open tree scheme on Y. Let d be a complete compatible bounded metric on Y and $\{U_i : i \in I\}$ a base for the topology of Y consisted of non-empty sets. For each $i \in I$ we define

$$I(i) = \{j \in I : \overline{U_j} \subseteq U_i \text{ and } \operatorname{diam}_d(\overline{U_j}) \leqslant \tfrac{1}{2}\operatorname{diam}_d(U_i)\}$$

and we observe that $U_i = \bigcup_{j\in I(i)} U_j = \bigcup_{j\in I(i)} \overline{U_j}$. Then we define the pruned tree $T \subseteq I^{<\omega}$ as follows:

$$t = (i_1, \dots, i_n) \in T \;\Leftrightarrow\; i_{k+1} \in I(i_k) \quad \text{for all } k \in \{1, \dots, n-1\}$$

and

$$\operatorname{diam}_d(\overline{U_{i_1}}) \leqslant \tfrac{1}{2}\operatorname{diam}_d(Y).$$

We set $U_\emptyset = Y$ and $U_t = U_{i_n}$ for every $t = (i_1, \dots, i_n) \in T$. We will show that $(U_t)_{t\in T}$ is a complete open tree scheme on Y. Let $t = (i_1, \dots, i_n) \in T$. If $s \in \operatorname{succ}(t)$, then, according to the definition of T, there exists $i_{n+1} \in I(i_n)$ so that $s = t^\frown i_{n+1} = (i_1, \dots, i_n, i_{n+1})$, which implies that $\overline{U_s} = \overline{U_{i_{n+1}}} \subseteq U_{i_n} = U_t$. Thus,

$$\bigcup \{U_s : s \in \operatorname{succ}(t)\} \subseteq U_t. \tag{1}$$

On the other hand, let $y \in U_t$, where $t = (i_1, \dots, i_n) \in T$. There is $\varepsilon > 0$ such that $2\varepsilon < \frac{1}{2}\operatorname{diam}_d(U_{i_n})$ and $\overline{B_d(y, \varepsilon)} \subseteq U_t = U_{i_n}$. Since $\{U_i : i \in I\}$ is a base for the metric topology, there exists $j \in I$ such that $y \in U_j \subseteq B_d(y, \varepsilon)$. Hence, $\overline{U_j} \subseteq U_{i_n}$ and $\operatorname{diam}_d(U_j) \leqslant 2\varepsilon < \frac{1}{2}\operatorname{diam}_d(U_{i_n})$, which implies $j \in I(i_n)$. So $y \in U_s$, where $s = (i_1, \dots, i_n, j) \in \operatorname{succ}(t)$, and thus,

$$U_t \subseteq \bigcup \{U_s : s \in \operatorname{succ}(t)\}. \tag{2}$$

From (1) and (2) we have

$$U_t = \bigcup \{U_s : s \in \operatorname{succ}(t)\} \quad \text{for all } t \in T.$$

So $(U_t)_{t\in T}$ is an open tree scheme. It remains to show that $(U_t)_{t\in T}$ is complete. Let $b \in [T]$. We observe that $\bigcap_{n\in\omega} U_{b|n} = \bigcap_{n\in\omega} \overline{U_{b|n}}$ and that $\{\overline{U_{b|n}} : n \in \omega\}$ is a decreasing sequence of non-empty closed subsets with vanishing diameter; therefore, since d is complete, it follows that there is $y_b \in Y$ so that $\bigcap_{n\in\omega} U_{b|n} = \{y_b\}$. Moreover, $\{U_{b|n} : n \in \omega\}$ is a neighborhood base at y_b, since, as we observed above, $\operatorname{diam}(U_{b|n}) \to 0$.

Now, by applying Lemma 3.6, $(U_t)_{t\in T}$ has an open precise refinement $(W_t)_{t\in T}$ such that $\{W_t : |t| = n\}$ is locally finite in Y for all $n < \omega$. It is easy to see that $(W_t)_{t\in T}$ is also complete.

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(ii) If Y is separable, then the initial base $\{U_i : i \in I\}$ in the proof of (i) can be chosen to be countable, and thus, T in (i) would be countably branching.

(iii) From (i) we get a complete open tree scheme $(W_t)_{t\in T}$ on Y such that $\{W_t : |t| = n\}$ is locally finite in Y for all $n < \omega$. We are going to define recursively a clopen disjoint precise refinement of $(W_t)_{t\in T}$. We set $G_\emptyset = X$. Let $G_t \subseteq W_t$ be clopen and observe that $G_t = \bigcup_{s\in \mathrm{succ}(t)}(G_t \cap W_s)$. Since G_t is normal with $\dim(G_t) = 0$ and $\{G_t \cap W_s : s \in \mathrm{succ}(t)\}$ is locally finite in G_t, there exists a clopen precise refinement $\{U_s : s \in \mathrm{succ}(t)\}$. We consider a strict well-order $<$ on $\mathrm{succ}(t)$ and we define $G_s = U_s \setminus \bigcup_{s'<s} U_{s'}$, which is clopen since $\{U_s : s \in \mathrm{succ}(t)\}$ is locally finite in G_t. So the recursively defined $(G_t)_{t\in T}$ is a clopen disjoint tree scheme, and it is easy to see that, as a precise refinement of $(W_t)_{t\in T}$, is also complete. □

§4. *A general selection theorem.* In Theorem 4.2 we present the main idea of this paper: a unified method of constructing continuous and measurable single-valued selections. Proposition 4.3 provides measurable multifunctions with range a body of a tree that are either closed-valued or compact-valued. Such multifunctions will be used in §6, where we characterize the structure of the domain in terms of tree schemes. Let us first begin with a simple lemma. Both the lemma and Proposition 4.3 are essentially based on [**3**, Proposition 2.2, Lemma 2.4].

LEMMA 4.1.

(i) *If X, Y are non-empty sets, $(W_t)_{t\in T}$ a tree scheme on Y and $\Phi : X \to 2^Y \setminus \{\emptyset\}$ a multifunction, then $(\Phi^l(W_t))_{t\in T}$ is a tree scheme on X.*

(ii) *If, in addition to (i), $\Sigma \subseteq 2^X$, Y is a topological space, Φ is Σ-measurable and $(W_t)_{t\in T}$ is open, then $(\Phi^l(W_t))_{t\in T}$ is a Σ tree scheme on X.*

(iii) *If, in addition to (ii), Σ is a topology, Φ is continuous and $(W_t)_{t\in T}$ is clopen, then $(\Phi^l(W_t))_{t\in T}$ is a clopen tree scheme on X.*

Proof. For (i) it suffices to observe that $\Phi^l(Y) = X$ and that the lower inverse image under Φ preserves arbitrary unions; (ii) and (iii) follow immediately from the definition of Σ-measurable or continuous multifunction respectively. □

THEOREM 4.2. *Let X be a non-empty set, $\Sigma \subseteq 2^X$, Y a topological space and $\Phi : X \to \mathcal{F}(Y)$ a multifunction.*

(i) *If there exists a complete open tree scheme $(W_t)_{t\in T}$ on Y such that $(\Phi^l(W_t))_{t\in T}$ has a disjoint precise refinement $(G_t)_{t\in T}$, then Φ admits a single-valued selection ϕ.*

(ii) *If Σ is a topology on X and $(G_t)_{t\in T}$ in (i) is, in addition, a Σ tree scheme, then ϕ in (i) is continuous.*

(iii) *If Σ is closed under countable unions and $(G_t)_{t\in T}$ in (i) is, in addition, a Σ tree scheme where T is countably branching, then ϕ in (i) is Σ-measurable.*

Proof. (i) Let $(W_t)_{t\in T}$ be a complete open tree scheme on Y and $(G_t)_{t\in T}$ a disjoint precise refinement of $(\Phi^l(W_t))_{t\in T}$. By Lemma 3.3(iii), for each $x \in X$

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there exists a unique $b_x \in [T]$ such that $x \in \bigcap_{n\in\omega} G_{b_x|n}$. Notice that $W_{b_x|n} \neq \emptyset$ for all $n \in \omega$ and $(W_t)_{t\in T}$ is complete, therefore we can define $\phi : X \to Y$ by $\{\phi(x)\} = \bigcap_{n\in\omega} W_{b_x|n}$. We will show that ϕ is a selection of Φ. Let $x \in X$. Since Φ is closed-valued, it suffices to show that $\phi(x) \in \overline{\Phi(x)}$. But this follows immediately from the fact that $\{W_{b_x|n} : n \in \omega\}$ is a neighborhood base at $\phi(x)$ and $\Phi(x) \cap W_{b_x|n} \neq \emptyset$ for all $n \in \omega$.

(ii) Let W be an open subset of Y. For each $x \in \phi^{-1}(W)$ there exists $n_x \in \omega$ such that $\phi(x) \in W_{b_x|n_x} \subseteq W$. Then for each $x \in \phi^{-1}(W)$ we have that $x \in G_{b_x|n_x} \subseteq \phi^{-1}(W_{b_x|n_x}) \subseteq \phi^{-1}(W)$, and thus, $\phi^{-1}(W) = \bigcup_{x\in\phi^{-1}(W)} G_{b_x|n_x}$, which is an open subset of Y since each $G_{b_x|n_x}$ is open.

(iii) As we proved in (ii) for each $W \subseteq Y$ open we have $\phi^{-1}(W) = \bigcup_{x\in\phi^{-1}(W)} G_{b_x|n_x}$. Moreover, Σ is closed under countable unions, $\{b_x|n_x : x \in \phi^{-1}(W)\}$ is countable (T is countably branching) and each $G_{b_x|n_x}$ belongs to Σ, and thus, $\phi^{-1}(W) \in \Sigma$. □

PROPOSITION 4.3. *Let X be a non-empty set, $\Sigma \subseteq 2^X$, $(U_t)_{t\in T}$ a Σ tree scheme on X and $\Phi : X \to 2^{[T]} \setminus \{\emptyset\}$ defined by $\Phi(x) = \{b \in [T] : x \in \bigcap_{n\in\omega} U_{b|n}\}$.*

(i) *If Σ is a topology, then Φ is lsc.*
(ii) *If T is countably branching and Σ is closed under countable unions, then Φ is Σ-measurable.*
(iii) *Φ is closed-valued.*
(iv) *If $(U_t)_{t\in T}$ is point-finite, then Φ is compact-valued.*
(v) *If Σ is a topology and $(U_t)_{t\in T}$ is locally finite and clopen, then Φ is continuous.*

Proof. (i) It suffices to observe that $\{V_t : t \in T\}$ is a base for the topology of $[T]$, the lower inverse image under Φ preserves arbitrary unions and finally $\Phi^l(V_t) = U_t \in \Sigma$ for all $t \in T$.

(ii) Let V be an open subset of $[T]$ and $S \subseteq T$ such that $V = \bigcup_{t\in S} V_t$. Obviously T is countable and $\Phi^l(V) = \bigcup_{t\in S} \Phi^l(V_t)$, and thus, $\Phi^l(V) \in \Sigma$.

(iii) Let $x \in X$. It is easy to see that $\Phi(x) = [T_x]$, where $T_x = \{t \in T : x \in U_t\}$ is a pruned subtree of T. In other words, $\Phi(x)$ is a body of a pruned subtree of T, and thus, is a closed subset of $[T]$.

(iv) If, moreover, $(U_t)_{t\in T}$ is point-finite, then $T_x = \{t \in T : x \in U_t\}$ is a pruned finitely branching subtree of T, and thus, $\Phi(x) = [T_x]$ is compact.

(v) If $(U_t)_{t\in T}$ is locally finite and clopen, then, due to (i), it remains to show that Φ is upper semicontinuous. Let V be an open subset of $[T]$ and $x \in \Phi^u(V)$. For every $b \in \Phi(x)$ there exists $n_b \in \omega$ such that $V_{b|n_b} \subseteq V$, so $\{V_{b|n_b}\}_{b\in\Phi(x)}$ is an open cover of $\Phi(x)$. From (iv) we have that $\Phi(x)$ is compact, and hence, there is a subcover $\{V_{t_i}\}_{i=1}^k$. We can assume that $|t_i| = n_0$ for some $n_0 \in \omega$ and for all $i = 1, \ldots, k$. Since $(U_t)_{t\in T}$ is locally finite and clopen, by virtue of Lemma 3.3(v) there exists an open neighborhood U of x such that the set

$$T_U^{n_0} = \{t \in T : |t| = n_0 \text{ and } U \cap U_t \neq \emptyset\}$$

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is finite. Let

$$T_x^{n_0} = \{t \in T : |t| = n_0 \text{ and } x \in U_t\}$$

and

$$G = U \setminus \bigcup \{U_t : t \in T_U^{n_0} \setminus T_x^{n_0}\}.$$

First we observe that G is an open neighborhood of x. Finally, by the definition of Φ, we observe that $G \subseteq \Phi^u(V)$, since we have that $T_x^{n_0} \subseteq \{t_1, \ldots, t_k\}$ and $\Phi(x) \subseteq \bigcup_{i=1}^k V_{t_i}$. □

§5. *Disjoint refinements of tree schemes.* As we figured out in the previous section (Theorem 4.2), disjoint precise refinements of tree schemes are of particular interest, since, through them, we achieve the construction of measurable single-valued selections.

Maitra and Rao [**19**], in their approach, defined a weaker version of Kuratowski's reduction principle as follows: if k is a cardinal and $\mathcal{F}$ a family of subsets of X, then $\mathcal{F}$ satisfies the k-weak reduction principle, if whenever $\{A_\alpha : \alpha < \beta\} \subseteq \mathcal{F}$, $\beta < k$ and $X = \bigcup_{\alpha<\beta} A_\alpha$, there exists a disjoint precise refinement $\{B_\alpha : \alpha < \beta\} \subseteq \mathcal{F}$.

Here we define a hereditary version of this weak reduction principle and we prove that it is the essential factor in order to construct disjoint precise refinements of tree schemes which belong to a certain class.

Recall that if Γ is a class of sets and X a set, then we denote by $\Gamma(X)$ the collection of subsets of X which belong to Γ. For example, $\Gamma(X)$ could stand for a topology on X. We also use the following notation: $\sim\Gamma(X) = \{X \setminus A : A \in \Gamma(X)\}$, $\Delta(X) = \sim\Gamma(X) \cap \Gamma(X)$ and $\Gamma(Y) = \Gamma(X)|Y = \{A \cap Y : A \in \Gamma(X)\}$ for any $Y \subseteq X$ (for reference see [**16**, p. 167]).

The above notation can be extended as follows: if $\mathcal{A}$ is a class of families of sets, then we denote by $\mathcal{A}(X)$ the collection of families of subsets of X which belong to $\mathcal{A}$ and by $\mathcal{A}(\Gamma(X))$ the collection of subfamilies of $\Gamma(X)$ which belong in $\mathcal{A}(X)$. For example, $\mathcal{A}(X)$ could be the collection of families of subsets of X which are locally finite in X and $\mathcal{A}(\Gamma(X))$ the collection of families of open subsets of X which are locally finite in X.

Definition 5.1. Let Γ be a class of sets, $\mathcal{A}$ a class of families of sets, X a set and k a cardinal.

(i) We say that $\Gamma(X)$ satisfies the k-weak reduction principle, if each $\{A_i\}_{i\in I} \subseteq \Gamma(X)$ such that $|I| < k$ and $X = \bigcup_{i\in I} A_i$, has a disjoint precise refinement $\{B_i\}_{i\in I} \subseteq \Gamma(X)$.

If $\Gamma(X)$ satisfies the k-weak reduction principle for any cardinal k, then we say that $\Gamma(X)$ satisfies the weak reduction principle.

(ii) We say that $\Gamma(X)$ satisfies the k-weak reduction principle $\Delta(X)$-hereditary, if $\Gamma(Y)$ satisfies the k-weak reduction principle for each $Y \in \Delta(X)$.

If $\Gamma(X)$ satisfies the k-weak reduction principle $\Delta(X)$-hereditary for any cardinal k, then we say that $\Gamma(X)$ satisfies the weak reduction principle $\Delta(X)$-hereditary.

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(iii) We say that $\mathcal{A}(\Gamma(X))$ satisfies the k-weak reduction principle, if each $\{A_i\}_{i\in I} \in \mathcal{A}(\Gamma(X))$ such, that $|I| < k$ and $X = \bigcup_{i\in I} A_i$, has a disjoint precise refinement $\{B_i\}_{i\in I} \subseteq \Gamma(X)$.

If $\mathcal{A}(\Gamma(X))$ satisfies the k-weak reduction principle for any cardinal k, then we say that $\mathcal{A}(\Gamma(X))$ satisfies the weak reduction principle.

(iv) We say that $\mathcal{A}(\Gamma(X))$ satisfies the k-weak reduction principle $\Delta(X)$-hereditary, if $\mathcal{A}(\Gamma(Y))$ satisfies the k-weak reduction principle for each $Y \in \Delta(X)$.

If $\mathcal{A}(\Gamma(X))$ satisfies the k-weak reduction principle $\Delta(X)$-hereditary for any cardinal k, then we say that $\mathcal{A}(\Gamma(X))$ satisfies the weak reduction principle $\Delta(X)$-hereditary.

PROPOSITION 5.2. *Let k be a cardinal, Γ a class of sets and X a set such that:*

(1) $X \in \Gamma(X)$*;*
(2) $\Gamma(X)$ *is closed under finite intersections.*

Then the following are equivalent:

(i) $\Gamma(X)$ *satisfies the weak reduction principle $\Delta(X)$-hereditary (k-weak reduction principle $\Delta(X)$-hereditary);*
(ii) *every $\Gamma(X)$ tree scheme $(U_t)_{t\in T}$ on X (T is k-branching) has a $\Gamma(X)$ disjoint precise refinement.*

The statement of this proposition includes two cases. The one stated inside the parentheses can be proved exactly in the same way, so its proof is omitted.

Proof. (i) $\Rightarrow$ (ii) Let $(U_t)_{t\in T}$ be a $\Gamma(X)$ tree scheme on X. Set $G_\emptyset = X$ and suppose that $G_t \subseteq U_t$ and $G_t \in \Delta(X)$. Since $\{U_s : s \in \mathrm{succ}(t)\} \subseteq \Gamma(X)$, we observe that $\{G_t \cap U_s : s \in \mathrm{succ}(t)\} \subseteq \Gamma(G_t)$ covers G_t, so it has a disjoint precise refinement $\{G_s : s \in \mathrm{succ}(t)\} \subseteq \Gamma(G_t)$. From (1) and (2) we conclude that the recursively defined disjoint precise refinement $(G_t)_{t\in T}$ belongs to $\Gamma(X)$.

(ii) $\Rightarrow$ (i) Let $Y \in \Delta(X)$. We have to show that $\Gamma(Y)$ satisfies the weak reduction principle. Let $\{A_i\}_{i\in I} \subseteq \Gamma(Y)$ such that $Y = \bigcup_{i\in I} A_i$. From (2) we have that $\{A_i\}_{i\in I} \subseteq \Gamma(X)$. Set $U_\emptyset = X$. Consider that the immediate successors of $\emptyset$ are exactly t_1, t_2 and set $U_{t_1} = Y$ and $U_{t_2} = X \setminus Y$. Then consider that the immediate successors of t_1 is the set I and for every $i \in \mathrm{succ}(t_1)$ set $U_i = A_i$. Finally, consider that $U_t = U_{t_2}$ for any $t \in \mathrm{succ}(t_2)$, and $U_t = U_i$ for any $t \in \mathrm{succ}(i)$ and $i \in I$. It is easy to see that in this way we can define a pruned tree T (on some set) and a $\Gamma(X)$ tree scheme $(U_t)_{t\in T}$, which, by hypothesis, has a $\Gamma(X)$ disjoint precise refinement $(G_t)_{t\in T}$. To complete the proof it suffices to observe that $\{G_i\}_{i\in I} \subseteq \Gamma(X)$ is a disjoint precise refinement of $\{A_i\}_{i\in I}$. □

PROPOSITION 5.3. *Let k be a cardinal, Γ a class of sets, $\mathcal{A}$ a class of families of sets and X a set such that:*

(1) $\{Y\} \in \mathcal{A}(\Gamma(Y))$ *for any $Y \subseteq X$;*
(2) $\Gamma(X)$ *is closed under finite intersections;*

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(3) *if* $Y \subseteq X$ *and* $\{A_i\}_{i\in I} \in \mathcal{A}(\Gamma(X))$, *then* $\{A_i \cap Y\}_{i\in I} \in \mathcal{A}(\Gamma(Y))$;
(4) *if* $Y \in \Delta(X)$ *and* $\{A_i\}_{i\in I} \in \mathcal{A}(\Gamma(Y))$, *then* $\{A_i\}_{i\in I} \in \mathcal{A}(\Gamma(X))$;
(5) *if* $\{A_i\}_{i\in I} \subseteq \Gamma(X)$ *and it is pairwise disjoint, then* $\{A_i\}_{i\in I} \in \mathcal{A}(\Gamma(X))$;
(6) *if* $\{A_i\}_{i\in I} \in \mathcal{A}(\Gamma(X))$, *then* $\{A_i\}_{i\in J} \in \mathcal{A}(\Gamma(X))$ *for each* $J \subseteq I$.

Then the following are equivalent:

(i) $\mathcal{A}(\Gamma(X))$ *satisfies the weak reduction principle* $\Delta(X)$*-hereditary (k-weak reduction principle* $\Delta(X)$*-hereditary);*
(ii) *every* $\mathcal{A}(\Gamma(X))$ *tree scheme* $(U_t)_{t\in T}$ *on* X *(*T *is k-branching) has a* $\Gamma(X)$ *disjoint precise refinement.*

As in the previous proposition, the statement again includes two cases. The one stated inside the parentheses can be proved exactly in the same way, so its proof is omitted.

Proof. (i) $\Rightarrow$ (ii) Let $(U_t)_{t\in T}$ be a $\mathcal{A}(\Gamma(X))$ tree scheme on X. Set $G_\emptyset = X$ and suppose that $G_t \subseteq U_t$ and $G_t \in \Delta(X)$. Since $\{U_s : s \in \mathrm{succ}(t)\} \in \mathcal{A}(\Gamma(X))$, by applying (3) we observe that $\{G_t \cap U_s : s \in \mathrm{succ}(t)\}$ belongs to $\mathcal{A}(\Gamma(G_t))$ and, moreover, covers G_t, so it has a disjoint precise refinement $\{G_s : s \in \mathrm{succ}(t)\} \in \Gamma(G_t)$. From (2) we have that the recursively defined disjoint precise refinement $(G_t)_{t\in T}$ is a $\Gamma(X)$ tree scheme.

(ii) $\Rightarrow$ (i) Let $Y \in \Delta(X)$. We have to show that $\mathcal{A}(\Gamma(Y))$ satisfies the weak reduction principle. Let $\{A_i\}_{i\in I} \in \mathcal{A}(\Gamma(Y))$ such that $Y = \bigcup_{i\in I} A_i$. From (4) we have that $\{A_i\}_{i\in I} \in \mathcal{A}(\Gamma(X))$. Set $U_\emptyset = X$. Consider that the immediate successors of $\emptyset$ are exactly t_1, t_2 and set $U_{t_1} = Y$ and $U_{t_2} = X \setminus Y$. From (5) we have that $\{U_s : s \in \mathrm{succ}(\emptyset)\} \in \mathcal{A}(\Gamma(X))$. Then consider that the immediate successors of t_1 is the set I and for every $i \in \mathrm{succ}(t_1)$ set $U_i = A_i$. Finally, consider that $U_t = U_{t_2}$ for any $t \in \mathrm{succ}(t_2)$, and $U_t = U_i$ for any $t \in \mathrm{succ}(i)$ and $i \in I$. It is easy to see ((6) should be used) that in this way we can define a pruned tree T (on some set) and a $\mathcal{A}(\Gamma(X))$ tree scheme $(U_t)_{t\in T}$, which, by hypothesis, has a $\Gamma(X)$ disjoint precise refinement $(G_t)_{t\in T}$. To complete the proof it suffices to observe that $\{G_i\}_{i\in I} \subseteq \Gamma(X)$ is a disjoint precise refinement of $\{A_i\}_{i\in I}$. □

§6. *Single-valued selection theorems.*

6.1. *Single-valued continuous selections of lsc multifunctions and characterizations.* Michael [**21**] showed that the existence of continuous selections characterize the topological structure of the domain space, such as paracompactness. Along the same lines, Maitra and Rao [**19**] showed that the existence of measurable selections characterize the domain space through the weak reduction principle. As we have already seen (Theorem 4.2), our method of constructing continuous selections is based on the existence of disjoint precise refinements of tree schemes. So it is natural to expect that the existence of continuous selections can also characterize the domain in terms of tree schemes. Considering the results of the previous section and the characterizations in [**19**], it is more than expected that continuous selections also characterize the domain through the hereditary version of the weak reduction principle we defined in the previous section.

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In this subsection the main result we are interested in is the zero-dimensional Michael selection theorem. Here we give a new proof of this theorem and of some other known selection theorems that can be found in [18, 19, 22, 24]. In fact, thinking similarly to [19], we characterize the existence of continuous selections in three different ways: in terms of tree schemes, of the hereditary version of the weak reduction principle, and of the topological structure of the domain space.

THEOREM 6.1. *Let (X, τ) be a topological space. The following are equivalent.*

(i) *If Y is a completely metrizable space and $\Phi : X \to \mathcal{F}(Y)$ a lsc multifunction, then Φ admits a continuous single-valued selection.*
(ii) *Every open tree scheme $(U_t)_{t\in T}$ on X has an open disjoint precise refinement.*
(iii) *τ satisfies the weak reduction principle $(\sim\tau \cap \tau)$-hereditary.*
(iv) *(X, τ) is paracompact Hausdorff with $\dim(X) = 0$.*

Proof. (i) $\Rightarrow$ (ii) Let $(U_t)_{t\in T}$ be an open tree scheme on X. We define $\Phi : X \to 2^{[T]} \setminus \{\emptyset\}$ by

$$\Phi(x) = \left\{ b \in [T] : x \in \bigcap_{n\in\omega} U_{b|n} \right\}.$$

Φ is lsc and closed-valued from Proposition 4.3 and, moreover, $[T]$ is completely metrizable, and thus, Φ admits a continuous selection $\phi : X \to [T]$. We set $G_t = \phi^{-1}(V_t)$ and we observe that $(G_t)_{t\in T}$ is an open disjoint precise refinement of $(U_t)_{t\in T}$.

(ii) $\Rightarrow$ (i) Let Y be a completely metrizable space and $\Phi : X \to \mathcal{F}(Y)$ a lsc multifunction. From Proposition 3.7 there exists a complete open tree scheme $(W_t)_{t\in T}$ on Y. Due to Theorem 4.2, it suffices to show that $(\Phi^l(W_t))_{t\in T}$ has an open disjoint precise refinement. By Lemma 4.1, $(\Phi^l(W_t))_{t\in T}$ is an open tree scheme on X, so, by hypothesis, it has an open disjoint precise refinement.

(ii) $\Leftrightarrow$ (iii) Follows immediately from Proposition 5.2 for $\Gamma(X) = \tau$.

(iii) $\Rightarrow$ (iv) It is immediate.

(iv) $\Rightarrow$ (iii) Let G be a clopen subset of X and $\{U_i\}_{i\in I}$ an open family of subsets of G (in the relative topology) such that $G = \bigcup_{i\in I} U_i$. We observe that G, as a closed subspace of X, is paracompact Hausdorff with $\dim(G) = 0$, and thus, there exists an open disjoint precise refinement $\{G_i\}_{i\in I}$. Since $G \in (\sim\tau \cap \tau)$, it follows that $\{G_i\}_{i\in I} \subseteq (\sim\tau \cap \tau)$. □

THEOREM 6.2. *If (X, τ) is Lindelöf zero-dimensional, Y completely metrizable, and $\Phi : X \to \mathcal{F}(Y)$ lsc multifunction, then Φ admits a continuous single-valued selection.*

Proof. Due to Theorem 6.1 it suffices to show that τ satisfies the weak reduction principle $(\sim\tau \cap \tau)$-hereditary. Let G be a clopen subset of X and

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$\{U_i\}_{i\in I}$ an open cover of G in the relative topology. G is obviously a Lindelöf zero-dimensional subspace. Let $\mathcal{V}$ be the family of all clopen subsets of G such that each is contained in some U_i. Since G has a basis of clopen sets, then $\mathcal{V}$ is an open cover of G. Assume that $\{W_n\}_{n\in\omega}$ is a countable subcover of $\mathcal{V}$ and define $G_n = W_n \setminus \bigcup_{m<n} W_m$. Then $\{G_n\}_{n\in\omega}$ is a clopen disjoint refinement of $\{U_i\}_{i\in I}$, so $\{G_n\}_{n\in\omega}$ is also locally finite in G. For every $n \in \omega$ we select a unique $i_n \in I$ such that $G_n \subseteq U_{i_n}$ and we define $G_i = \bigcup\{G_n : i_n = i\}$ for all $i \in I$. Since $\{G_n\}_{n\in\omega}$ is locally finite in G, then G_i is clopen for all $i \in I$. Hence, $\{G_i\}_{i\in I}$ is a disjoint clopen precise refinement of $\{U_i\}_{i\in I}$. □

THEOREM 6.3. *Let (X, τ) be a topological space. The following are equivalent.*

(i) *If Y is a Polish space and $\Phi : X \to \mathcal{F}(Y)$ a lsc multifunction, then Φ admits a continuous single-valued selection.*
(ii) *Every open tree scheme $(U_t)_{t\in T}$ on X, where T is countably branching, has an open disjoint precise refinement.*
(iii) *τ satisfies the $\aleph_1$-weak reduction principle ($\sim\tau \cap \tau$)-hereditary.*
(iv) *(X, τ) is countably paracompact normal with* $\dim(X) = 0$.

Proof. (i) ⇒ (ii) The only difference with the proof of (i) ⇒ (ii) of Theorem 6.1 is that $[T]$ is also separable.

(ii) ⇒ (i) Similarly, the only difference with the proof of (ii) ⇒ (i) of Theorem 6.1 is that the tree scheme $(W_t)_{t\in T}$ on Y could be such that T is countably branching (Proposition 3.7).

(ii) ⇔ (iii) Follows immediately from Proposition 5.2 for $\Gamma(X) = \tau$ and $k = \aleph_1$.

(iii) ⇒ (iv) It is immediate.

(iv) ⇒ (iii) Let G be a clopen subset of X and $\{U_n\}_{n\in\omega}$ a countable open cover of G in the relative topology. We observe that G, as a closed subspace of X, is countably paracompact normal with $\dim(G) = 0$, and thus, we can construct an open disjoint precise refinement $\{G_n\}_{n\in\omega}$. Since $G \in (\sim\tau \cap \tau)$, it follows that $\{G_n\}_{n\in\omega} \subseteq (\sim\tau \cap \tau)$. □

LEMMA 6.4. *Let X be a non-empty set, Y a topological space, $\Phi : X \to \mathcal{K}(Y)$ a multifunction and $\{W_i\}_{i\in I}$ a family of open subsets of Y that is locally finite in Y. Then $\{\Phi^l(W_i)\}_{i\in I}$ is point-finite in X.*

Proof. Let $x \in X$. For every $y \in \Phi(x)$ there exists an open neighborhood U_y of y that intersects finitely many W_i. $\{U_y\}_{y\in\Phi(x)}$ is an open cover of $\Phi(x)$, and thus, due to compactness, there is a finite subcover $\mathcal{U}$. Let $V = \bigcup \mathcal{U}$. It is easy to see that V intersects finitely many W_i, and hence, the same applies to $\Phi(x)$, which implies that x belongs to finitely many $\Phi^l(W_i)$. □

THEOREM 6.5. *Let (X, τ) be a topological space and for each $Z \subseteq X$ let τ_Z be the relative topology on Z and $\mathcal{A}(\tau_Z)$ the collection of families of open subsets of Z which are point-finite in Z. The following are equivalent.*

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(i) *If Y is a Polish space and $\Phi : X \to \mathcal{K}(Y)$ a lsc multifunction, then Φ admits a continuous single-valued selection.*
(ii) *Every point-finite open tree scheme $(U_t)_{t\in T}$ on X, where T is countable branching, has an open disjoint precise refinement.*
(iii) *$\mathcal{A}(\tau)$ satisfies the $\aleph_1$-weak reduction principle ($\sim\tau \cap \tau$)-hereditary.*
(iv) *(X, τ) is normal with* $\dim(X) = 0$.

Proof. (i) $\Rightarrow$ (ii) Let $(U_t)_{t\in T}$ be a point-finite open tree scheme on X, where T is countable branching. We define $\Phi : X \to 2^{[T]} \setminus \{\emptyset\}$ by

$$\Phi(x) = \left\{ b \in [T] : x \in \bigcap_{n\in\omega} U_{b|n} \right\}.$$

Φ is compact-valued lsc from Proposition 4.3 and $[T]$ is Polish, and thus, Φ admits a continuous selection $\phi : X \to [T]$. We set $G_t = \phi^{-1}(V_t)$ and we observe that $(G_t)_{t\in T}$ is an open disjoint precise refinement of $(U_t)_{t\in T}$.

(ii) $\Rightarrow$ (i) Let Y be a Polish space and $\Phi : X \to \mathcal{K}(Y)$ a lsc multifunction. From Proposition 3.7 there exists a complete open tree scheme $(W_t)_{t\in T}$ on Y, where T is countable branching, such that $\{W_t : |t| = n\}$ is locally finite in Y for all $n \in \omega$. Since Φ is compact-valued, from the above lemma and Lemma 4.1, $(\Phi^l(W_t))_{t\in T}$ is a point-finite open tree scheme on X, so, by hypothesis, it has an open disjoint precise refinement. Thus, by virtue of Theorem 4.2, Φ admits a continuous selection.

(ii) $\Leftrightarrow$ (iii) Follows immediately from Proposition 5.3.

(iii) $\Rightarrow$ (iv) It is immediate.

(iv) $\Rightarrow$ (iii) Let G be a clopen subset of X and $\{U_i\}_{i\in I} \in \mathcal{A}(\tau_G)$ with $|I| \leqslant \aleph_0$ such that $G = \bigcup_{i\in I} U_i$. G is normal as a closed subspace of X and $\{U_i\}_{i\in I}$ is a point-finite open cover of G in the relative topology, and thus, there exists an open precise refinement $\{V_i\}_{i\in I}$ such that $\mathrm{cl}_G(V_i) \subseteq U_i$ for all $i \in I$. Moreover, $\dim(G) = 0$, since G is a closed subspace, so there exists a family $\{W_i\}_{i\in I}$ of clopen subsets of G (which are also clopen in X) such that $\mathrm{cl}_G V_i \subseteq W_i \subseteq U_i$ for all $i \in I$. Since $|I| \leqslant \aleph_0$, we can assume a strict well-order $<$ on I which is isomorphic with an ordinal less or equal to ω. Then for all $i \in I$ we define $G_i = W_i \setminus \bigcup_{j<i} W_j$ and we observe that $\{G_i\}_{i\in I}$ is a clopen disjoint precise refinement of $\{U_i\}_{i\in I}$. □

6.2. *Single-valued measurable selections of measurable multifunctions and characterizations.* The main result of this subsection is the Kuratowski–Ryll-Nardzewski selection theorem. Here, on the same wavelength with [**19**], we characterize the existence of measurable single-valued selections in terms of tree schemes and of the hereditary version of the weak reduction principle, and then we provide a new proof of the Kuratowski–Ryll-Nardzewski selection theorem and of some known variations of this theorem; all of them can be found in [**17**].

THEOREM 6.6. *Let X be a non-empty set and Σ a non-empty family of subsets of X that is closed under countable unions, finite intersections and $X \in \Sigma$. The following are equivalent.*

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(i) *If Y is a Polish space and $\Phi : X \to \mathcal{F}(Y)$ a Σ-measurable multifunction, then Φ admits a Σ-measurable single-valued selection.*

(ii) *Every Σ tree scheme $(U_t)_{t\in T}$ on X, where T is countably branching, has a Σ disjoint precise refinement.*

(iii) *Σ satisfies the $\aleph_1$-weak reduction principle ($\sim\Sigma \cap \Sigma$)-hereditary.*

Proof. (i) $\Rightarrow$ (ii) Let $(U_t)_{t\in T}$ be a Σ tree scheme on X such that T is countably branching. We define $\Phi : X \to 2^{[T]} \setminus \{\emptyset\}$ by

$$\Phi(x) = \left\{ b \in [T] : x \in \bigcap_{n\in\omega} U_{b|n} \right\}.$$

Φ is Σ-measurable and closed-valued from Proposition 4.3. Moreover, $[T]$ is Polish, and thus, Φ admits a Σ-measurable selection $\phi : X \to [T]$. We set $G_t = \phi^{-1}(V_t)$ and we observe that $(G_t)_{t\in T}$ is a Σ disjoint precise refinement of $(U_t)_{t\in T}$.

(ii) $\Rightarrow$ (i) Let Y be a Polish space and $\Phi : X \to \mathcal{F}(Y)$ a Σ-measurable multifunction. From Proposition 3.7 there exists a complete open tree scheme $(W_t)_{t\in T}$ on Y such that T is countably branching. By Lemma 4.1, $(\Phi^l(W_t))_{t\in T}$ is a Σ tree scheme on X; therefore, it has a Σ disjoint precise refinement. The requested selection, as usual, is a consequence of Theorem 4.2.

(ii) $\Leftrightarrow$ (iii) Follows immediately from Proposition 5.2 for $\Gamma(X) = \Sigma$ and $k = \aleph_1$. □

THEOREM 6.7. *If (X, Σ) is a measurable space, Y a Polish space and $\Phi : X \to \mathcal{F}(Y)$ a Σ-measurable multifunction, then Φ admits a Σ-measurable single-valued selection.*

Proof. Due to Theorem 6.6, it suffices to show that Σ has the $\aleph_1$-weak reduction principle Σ-hereditary. So let $G \in \Sigma$, $\{U_i\}_{i\in I} \subseteq \Sigma$ with $|I| \leqslant \aleph_0$ such that $G = \bigcup_{i\in I} U_i$. We assume a strict well-order $<$ on I and for all $i \in I$ we define $G_i = U_i \setminus \bigcup_{j<i} U_j$, which obviously belongs to the σ-algebra Σ. Hence, $\{G_i\}_{i\in I} \subseteq \Sigma$ is a disjoint precise refinement of $\{U_i\}_{i\in I}$. □

Let Γ be a class of sets and $\Gamma(X)$ the collection of subsets of X which are in Γ for any set X. Recall that Γ satisfies the generalized reduction property (see [**16**, Definition 22.14]), if for any set X and $\{A_n\}_{n\in\omega} \subseteq \Gamma(X)$ there exists $\{B_n\}_{n\in\omega} \subseteq \Gamma(X)$ such that $B_n \cap B_m = \emptyset$ for $n \neq m$, $B_n \subseteq A_n$ for all $n \in \omega$ and $\bigcup_{n\in\omega} B_n = \bigcup_{n\in\omega} A_n$.

LEMMA 6.8. *Let Γ be a class of sets that satisfies the generalized reduction property and X a non-empty set such that $\emptyset \in \Gamma(X)$. Then $\Gamma(X)$ satisfies the $\aleph_1$-weak reduction principle $\Delta(X)$-hereditary.*

Proof. Let $G \in \Delta(X)$ and $\{U_i\}_{i\in I} \subseteq \Gamma(X)$ such that $|I| \leqslant \aleph_0$ and $G = \bigcup_{i\in I} U_i$. We may assume that $|I| = \aleph_0$, otherwise we can add the empty set so many times until we reach $\aleph_0$. There exists a pairwise disjoint family $\{G_i\}_{i\in I} \subseteq \Gamma(X)$ such that $G_i \subseteq U_i$ for all $i \in I$ and $\bigcup_{i\in I} G_i = \bigcup_{i\in I} U_i$. □

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THEOREM 6.9. *If X is a metrizable space, ξ an ordinal, $1 < \xi < \omega_1$, Y a Polish space and $\Phi : X \to \mathcal{F}(Y)$ a $\Sigma^0_\xi(X)$-measurable multifunction, then Φ admits a $\Sigma^0_\xi(X)$-measurable single-valued selection.*

Proof. It is known that $\Sigma^0_\xi(X)$ is closed under countable unions, finite intersections, satisfies the generalized reduction property and $\emptyset, X \in \Sigma^0_\xi(X)$ (see [16, Proposition 22.1, Theorem 22.16]). The result follows from the previous lemma and Theorem 6.6. □

The classes $\Sigma^1_{2n+2}(X)$ and $\Pi^1_{2n+1}(X)$, for X Polish and $n \in \omega$, of the projective hierarchy are closed under countable unions, finite intersections, satisfy the generalized reduction property and contain $\emptyset$ and X (see [16, p. 327]), and thus, we can prove exactly in the same way the following two results.

THEOREM 6.10. *If X, Y are Polish spaces, $n \in \omega$ and $\Phi : X \to \mathcal{F}(Y)$ a $\Sigma^1_{2n+2}(X)$-measurable multifunction, then Φ admits a $\Sigma^1_{2n+2}(X)$-measurable single-valued selection.*

THEOREM 6.11. *If X, Y are Polish spaces, $n \in \omega$ and $\Phi : X \to \mathcal{F}(Y)$ a $\Pi^1_{2n+1}(X)$-measurable multifunction, then Φ admits a $\Pi^1_{2n+1}(X)$-measurable single-valued selection.*

6.3. *Single-valued continuous selections of continuous multifunctions and the hyperspace selection problem.* Let us consider two topological spaces X, Y and a multifunction $\Phi : X \to \mathcal{F}(Y)$, where $\mathcal{F}(Y)$ is equipped with the Vietoris topology. Recall that the Vietoris topology is generated by the sets $\{A \in \mathcal{F}(Y) : A \cap U \neq \emptyset\}$ and $\{A \in \mathcal{F}(Y) : A \subseteq U\}$, where U is an arbitrary open subset of Y. Michael [20] proposed that the problem of finding a single-valued continuous selection for Φ can be reduced in the following two problems: the first is to determine whether Φ is continuous and the second is to find a continuous map $\phi : \mathcal{F}(Y) \to Y$ such that $\phi(A) \in A$ for all $A \in \mathcal{F}(Y)$. The second problem is known as the hyperspace selection problem; for more details we refer to Repovš and Semenov [26], Gutev [11], Gutev and Nogura [12].

Generally, let X be a topological space and $\mathcal{D}$ a subspace of $\mathcal{F}(X)$. A continuous map $\phi : \mathcal{D} \to X$ such that $\phi(A) \in A$ for all $A \in \mathcal{D}$, is called a continuous selection for $\mathcal{D}$. A famous result in this direction that was proved independently by Choban [4] and Engelking, Heath, and Michael [9] is the following.

> If X is a completely metrizable space with $\dim(X) = 0$, then there exists a continuous selection for $\mathcal{F}(X)$.

This result, as Michael [20] first observed, evidently implies the following selection theorem.

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If X is a topological space, Y a completely metrizable space with $\dim(Y) = 0$ and $\Phi : X \to \mathcal{F}(Y)$ a continuous multifunction, then Φ admits a single-valued continuous selection.

Here, using the core idea of Theorem 4.2, we provide an elementary proof of the above solution of the hyperspace selection problem, based on the concept of the leftmost branch of a tree (see [**16**, p. 9]).

THEOREM 6.12. *Let X be a completely metrizable space with $\dim(X) = 0$. Then there exists a continuous selection for $\mathcal{F}(X)$.*

Proof. Consider the identity multifunction $\Phi : \mathcal{F}(X) \to \mathcal{F}(X)$; that is, $\Phi(A) = A$ for all $A \in \mathcal{F}(X)$. It is obvious that Φ is continuous. From Proposition 3.7 there exists a complete clopen disjoint tree scheme $(W_t)_{t\in T}$ on X. From Lemma 4.1 $(\Phi^l(W_t))_{t\in T}$ is a clopen tree scheme on X. Assume that T is tree on a set I and let $\leqslant$ be a well-order on I. For each $A \in \mathcal{F}(X)$ we define the pruned subtree $T_A = \{t \in T : A \in \Phi^l(W_t)\}$. Now we can correspond to each $A \in \mathcal{F}(X)$ the leftmost branch $b_A \in [T_A]$; that is, $b_A(n) \leqslant t(n)$ for each $n \in \omega$ and $t \in T_A$ with $|t| = n$. We define $\phi : \mathcal{F}(X) \to X$ by $\{\phi(A)\} = \bigcap_{n\in\omega} W_{b_A|n}$. To complete the proof it suffices to show that ϕ is a continuous selection of Φ. Let $A \in \mathcal{F}(X)$ and W be an open neighborhood of $\phi(A)$. Since $(W_t)_{t\in T}$ is complete, there is $n_0 \in \omega$ such that $\phi(A) \in W_{b_A|n_0} \subseteq W$. Let

$$T_A^n = \{t \in T_A : |t| = n\} \quad \text{for any } n \in \omega$$

and

$$\mathcal{U} = \bigcap_{n\leqslant n_0} \left\{ B \in \mathcal{F}(X) : B \subseteq \bigcup_{t\in T_A^n} W_t \;\&\; B \cap W_{b_A|n} \neq \emptyset \right\}.$$

Since $\Phi(A) = A$, observe that $A \cap W_t \neq \emptyset$ if and only if $t \in T_A$. Furthermore, $\{W_t : |t| = n\}$ is a cover of X for any $n \in \omega$, and hence, $A \subseteq \bigcup_{t\in T_A^n} W_t$ and $A \cap W_{b_A|n} \neq \emptyset$ for any $n \leqslant n_0$. Thus, $\mathcal{U}$ is an open neighborhood of A (with respect to the Vietoris topology). Moreover, by the definition of the leftmost branch and the fact that $\{W_t : |t| = n\}$ is a pairwise disjoint cover of X for any $n \leqslant n_0$, we have $b_B|n_0 = b_A|n_0$ for all $B \in \mathcal{U}$. Therefore, $\phi(B) \in W_{b_A|n_0} \subseteq W$ for all $B \in \mathcal{U}$, and so ϕ is continuous. It remains to show that ϕ is a selection of Φ. Again let $A \in \mathcal{F}(X)$. By the definition of b_A we have $A \in \Phi^l(W_{b_A|n})$ for all $n \in \omega$, equivalently $\Phi(A) \cap W_{b_A|n} \neq \emptyset$ for all $n \in \omega$. But $\{W_{b_A|n} : n \in \omega\}$ is a neighborhood base at $\phi(A)$ and Φ is closed-valued, and thus, $\phi(A) \in \Phi(A)$, and the proof is complete. □

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