Generalisation of a quadrilateral duality theorem

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We will prove a generalisation to our Theorem 10 about duality between orthodiagonal quadrilaterals and equidiagonal quadrilaterals in [1, p. 134]. These are quadrilaterals with perpendicular diagonals and diagonals of equal lengths respectively. Unfortunately we made a careless mistake in the proof of Theorem 10 (ii) in [1], which was found by Zoltán Szilasi at the University of Debrecen in Hungary. When reviewing that theorem, we realised that it's just a special case of a more general theorem. The equilateral triangles in Theorem 10 can be replaced by almost any regular polygons.

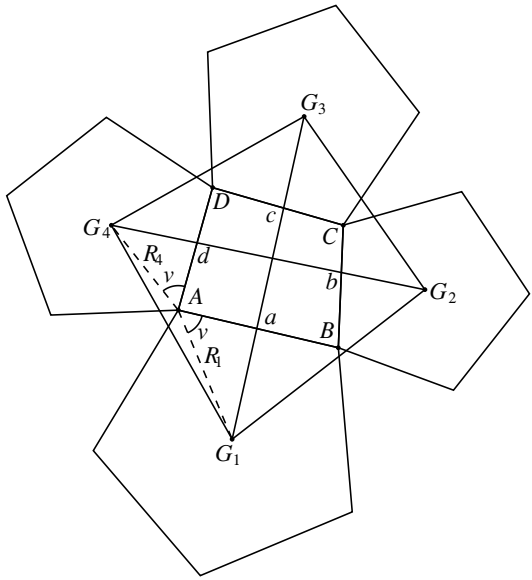


FIGURE 1: Illustration of the theorem in the pentagon case

Theorem: Suppose regular polygons with the same number of sides are erected outwards on the sides of a convex quadrilateral *ABCD*. Then the following characteristic properties hold:

(i) *ABCD* is an equidiagonal quadrilateral if, and only if, the polygon centroids are the vertices of an orthodiagonal quadrilateral.

(ii) *ABCD* is an orthodiagonal quadrilateral if, and only if, the polygon centroids are the vertices of an equidiagonal quadrilateral.

There is one exception: the regular polygons cannot be squares. If they are, then the polygon centroids are the vertices of a midsquare quadrilateral

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for all convex quadrilaterals $ABCD$.* This last result is known as van Aubel's theorem.†

Proof:

(i) Let the centroids of the regular polygons be G_1, G_2, G_3, G_4 and their corresponding circumradii be R_1, R_2, R_3, R_4 (see Figure 1). We define

$$v = \angle G_1AB = \angle G_2BC = \angle G_3CD = \angle G_4DA. \tag{1}$$

Since $\cos(360° - \psi) = \cos\psi$ for all angles ψ, we have by the cosine rule

$$\begin{aligned}
(G_1G_2)^2 &= R_1^2 + R_2^2 - 2R_1R_2\cos(B + 2v),\\
(G_2G_3)^2 &= R_2^2 + R_3^2 - 2R_2R_3\cos(C + 2v),\\
(G_3G_4)^2 &= R_3^2 + R_4^2 - 2R_3R_4\cos(D + 2v),\\
(G_4G_1)^2 &= R_4^2 + R_1^2 - 2R_4R_1\cos(A + 2v).
\end{aligned}$$

Thus

$$\begin{aligned}
&\tfrac{1}{2}\left((G_1G_2)^2 + (G_3G_4)^2 - (G_2G_3)^2 - (G_4G_1)^2\right)\\
&= -R_1R_2\cos(B + 2v) - R_3R_4\cos(D + 2v)\\
&\quad + 2R_2R_3\cos(C + 2v) + R_4R_1\cos(A + 2v).
\end{aligned} \tag{2}$$

The circumradii are given by

$$R_1 = \frac{a}{2\cos v}, \qquad R_2 = \frac{b}{2\cos v}, \qquad R_3 = \frac{c}{2\cos v}, \qquad R_4 = \frac{d}{2\cos v}$$

where a, b, c, d are the sides in the quadrilateral $ABCD$. Inserting these expressions in (2), applying the addition formula for cosine and the double angle formulas for sine and cosine, we find that the right-hand side of (2) becomes

$$\begin{aligned}
&-\frac{ab}{4}\cos B\left(1 - \tan^2 v\right) + \frac{ab}{4}\sin B \cdot 2\tan v\\
&-\frac{cd}{4}\cos D\left(1 - \tan^2 v\right) + \frac{cd}{4}\sin D \cdot 2\tan v\\
&+\frac{bc}{4}\cos C\left(1 - \tan^2 v\right) - \frac{bc}{4}\sin C \cdot 2\tan v\\
&+\frac{da}{4}\cos A\left(1 - \tan^2 v\right) - \frac{da}{4}\sin A \cdot 2\tan v
\end{aligned}$$

* A *midsquare quadrilateral* is a quadrilateral that is both equidiagonal and orthodiagonal (see [1, p.137]). They have previously been called skewsquares and pseudosquares according to [2] and were recently called equiorthodiagonal quadrilaterals in [3, p.102].

† Two beautiful proofs of van Aubel's theorem are given in [4] and elegant special cases are considered in [5].

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or equivalently

$$\tfrac{1}{4}\left(1 - \tan^2 v\right)\left(bc\cos C + da\cos A - ab\cos B - cd\cos D\right)$$

$$+\tfrac{1}{2}\tan v\left(ab\cos B + cd\cos D - bc\cos C - da\cos A\right).$$

The last parenthesis is equal to zero in all convex quadrilaterals according to Lemma 7 in [1]. Hence we have proved that

$$\tfrac{1}{2}\left((G_1G_2)^2 + (G_3G_4)^2 - (G_2G_3)^2 - (G_4G_1)^2\right)$$

$$= \tfrac{1}{4}\left(1 - \tan^2 v\right)\left(bc\cos C + da\cos A - ab\cos B - cd\cos D\right).$$

Now there are two possibilities. Since $v \in [0,\ 90°]$, the first parenthesis on the right-hand side is zero only when $v = 45°$. In this case the regular polygons are squares, and then the quadrilateral $G_1G_2G_3G_4$ is always orthodiagonal. If the regular polygons are not squares, then

$$(G_1G_2)^2 + (G_3G_4)^2 = (G_2G_3)^2 + (G_4G_1)^2 \tag{3}$$

is equivalent to

$$bc\cos C + da\cos A = ab\cos B + cd\cos D \tag{4}$$

where (3) is a well-known necessary and sufficient condition for $G_1G_2G_3G_4$ to be orthodiagonal and (4) is a necessary and sufficient condition for $ABCD$ to be equidiagonal according to Proposition 6 in [1]. This proves the first part of the theorem.

(ii) Let the vertices A, B, C and D of a convex quadrilateral be represented by the complex numbers z_1, z_2, z_3 and z_4 respectively. Also, let the centroids G_1, G_2, G_3 and G_4 of the regular polygons be represented by the complex numbers g_1, g_2, g_3 and g_4 respectively. Then

$$g_1 = \frac{z_1 - z_2}{2\cos v}\left(\cos v + i\sin v\right) + z_2 = \frac{1}{2}\left(z_1 + z_2\right) + \frac{i}{2}\left(z_1 - z_2\right)\tan v,$$

where v is still given by (1). (The mistake in our original proof was that we forgot to include the second z_2 in this expression.) In the same way we have

$$g_2 = \frac{1}{2}\left(z_2 + z_3\right) + \frac{i}{2}\left(z_2 - z_3\right)\tan v,$$

$$g_3 = \frac{1}{2}\left(z_3 + z_4\right) + \frac{i}{2}\left(z_3 - z_4\right)\tan v,$$

$$g_4 = \frac{1}{2}\left(z_4 + z_1\right) + \frac{i}{2}\left(z_4 - z_1\right)\tan v.$$

Thus

$$g_3 - g_1 = \frac{1}{2}\left(z_3 + z_4 - z_1 - z_2\right) + \frac{i}{2}\left(z_3 - z_4 - z_1 + z_2\right)\tan v,$$

$$g_4 - g_2 = \frac{1}{2}\left(z_4 + z_1 - z_2 - z_3\right) + \frac{i}{2}\left(z_4 - z_1 - z_2 + z_3\right)\tan v.$$

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We now define the two new complex numbers $w_1 = z_3 - z_1$ and $w_2 = z_4 - z_2$. Then

$$2(g_3 - g_1) = w_1 + w_2 + i(w_1 - w_2)\tan v, \tag{5}$$

$$2(g_4 - g_3) = -w_1 + w_2 + i(w_1 + w_2)\tan v. \tag{6}$$

Since w_1 and w_2 are parallel to the diagonals of $ABCD$, we have to prove that the angle between w_1 and w_2 is a right angle if, and only if, $|g_3 - g_1| = |g_4 - g_2|$, where the latter is a necessary and sufficient condition for $G_1G_2G_3G_4$ to be equidiagonal. To do so, we rewrite w_1 and w_2 in polar form. Thus $w_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$ and $w_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$. The angle between them is $\varphi_1 - \varphi_2$. Inserting the polar expressions into (5) and (6) yields

$$\begin{aligned} 2(g_3 - g_1) = {} & r_1(\cos\varphi_1 - \sin\varphi_1\tan v) + r_2(\cos\varphi_2 + \sin\varphi_2\tan v) \\ & + i(r_1(\sin\varphi_1 + \cos\varphi_1\tan v) + r_2(\sin\varphi_2 - \cos\varphi_2\tan v)) \end{aligned}$$

and

$$\begin{aligned} 2(g_4 - g_2) = {} & -r_1(\cos\varphi_1 + \sin\varphi_1\tan v) + r_2(\cos\varphi_2 - \sin\varphi_2\tan v) \\ & + i(r_1(-\sin\varphi_1 + \cos\varphi_1\tan v) + r_2(\sin\varphi_2 + \cos\varphi_2\tan v)). \end{aligned}$$

Thus

$$\begin{aligned} 4|g_3 - g_1|^2 = {} & (r_1(\cos\varphi_1 - \sin\varphi_1\tan v) + r_2(\cos\varphi_2 + \sin\varphi_2\tan v))^2 \\ & + (r_1(\sin\varphi_1 + \cos\varphi_1\tan v) + r_2(\sin\varphi_2 - \cos\varphi_2\tan v))^2 \end{aligned}$$

and

$$\begin{aligned} 4|g_4 - g_2|^2 = {} & (-r_1(\cos\varphi_1 + \sin\varphi_1\tan v) + r_2(\cos\varphi_2 - \sin\varphi_2\tan v))^2 \\ & + (r_1(-\sin\varphi_1 + \cos\varphi_1\tan v) + r_2(\sin\varphi_2 + \cos\varphi_2\tan v))^2. \end{aligned}$$

Now expanding these expressions using $(x + y)^2 = x^2 + y^2 + 2xy$ several times, simplifying, applying the Pythagorean trigonometric identity and the trigonometric subtraction formulas yields

$$\begin{aligned} 4|g_3 - g_1|^2 = {} & r_1^2 + r_2^2 + r_1^2\tan^2 v + r_2^2\tan^2 v \\ & + 2r_1r_2(\cos(\varphi_1 - \varphi_2) - \tan^2 v\cos(\varphi_1 - \varphi_2) - 2\tan v\sin(\varphi_1 - \varphi_2)) \end{aligned}$$

and

$$\begin{aligned} 4|g_4 - g_2|^2 = {} & r_1^2 + r_2^2 + r_1^2\tan^2 v + r_2^2\tan^2 v \\ & - 2r_1r_2(\cos(\varphi_1 - \varphi_2) - \tan^2 v\cos(\varphi_1 - \varphi_2) - 2\tan v\sin(\varphi_1 + \varphi_2)). \end{aligned}$$

From the last two expressions we get the difference

$$4|g_3 - g_1|^2 - 4|g_4 - g_2|^2 = 4r_1r_2(\cos(\varphi_1 - \varphi_2) - \tan^2 v\cos(\varphi_1 - \varphi_2))$$

or equivalently

$$|g_3 - g_1|^2 - |g_4 - g_2|^2 = r_1r_2(1 - \tan^2 v)\cos(\varphi_1 - \varphi_2).$$

Again there are two possibilities. As $1 - \tan^2 v = 0$ only when $v = 45°$, and then the regular polygons are squares. In this case the quadrilateral $G_1G_2G_3G_4$ is always equidiagonal (since $|g_3 - g_1| - |g_4 - g_2| = 0$ by

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factorising the left-hand side). If the regular polygons are not squares, then

$$|g_3 - g_1| = |g_4 - g_2| \qquad \Leftrightarrow \qquad \varphi_1 - \varphi_2 = \pm 90°.$$

This proves that $G_1G_2G_3G_4$ is equidiagonal if and only if $ABCD$ is orthodiagonal.

Combining the results of the two parts of the proof we have that when the regular polygons are squares, then their centroids are always vertices in a quadrilateral that is both orthodiagonal and equidiagonal (van Aubel's theorem), i.e. it is a midsquare quadrilateral (a name we invented in [1]).

We note that when the regular polygons only have two sides, they degenerate into line segments, which are the sides of the quadrilateral $ABCD$ and their centroids are the midpoints of these sides. Thus the theorem we just proved contains, as a special case, the following characteristic properties: a convex quadrilateral is equidiagonal if, and only if, its Varignon parallelogram is orthodiagonal, and the quadrilateral is orthodiagonal if, and only if, its Varignon parallelogram is equidiagonal. We have given a short proof of this duality before, see Theorem 7 in [6], where it was formulated differently.

Previous results

The referee kindly provided several references to previous similar theorems that are either special cases or generalisations of our theorem. Note however, regarding generalisations, that only direct theorems and no converses were considered. As far as we know, the present note is the first time this duality with regular polygons between orthodiagonal and equidiagonal quadrilaterals has been stated and proved. Also, none of the previous proofs used complex numbers.

The special case that in an equidiagonal quadrilateral with equilateral triangles constructed on the sides, then the line segments connecting opposite triangle centroids are orthogonal was Problem 5 on the Shortlist of problems proposed for the International Mathematical Olympiad in 1992 [7, p. 269]. A proof using vectors was given on page 560 in that book. A pure geometric solution to this problem by Waldemar Pompe from Poland was published in [8, pp. 47-48].

The following generalisation of the direct theorem involving the equidiagonal quadrilateral was proved by G. Kandall in [8, p. 49]: Given a quadrilateral $ABCD$ with diagonals AC and BD of equal lengths, if similar isosceles triangles EAB and GCD are erected outwardly (or inwardly) on AB and CD respectively, and similar isosceles triangles HAD and FBC are erected outwardly (or inwardly) on DA and BC respectively, then EG is orthogonal to FH. Here it's actually only required that the opposite pairs of isosceles triangles have the same orientation, so one pair can be outward and the other pair inward. A decade later, the same problem was proposed in another journal, where it was solved by Michael Fox from the UK using congruence and similarity as well as by means of transformations, see [9, pp. 43-45].

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The dual problem about orthodiagonal quadrilaterals can be formulated as follows: Given a quadrilateral $ABCD$ with orthogonal diagonals AC and BD, if similar triangles EAB, FBC, GCD and HAD are erected outwardly (or inwardly) on sides AB, BC, CD and DA respectively, then $EG = FH$. This generalisation was discovered by Michael de Villiers from South Africa using dynamic geometry and posed as a Reader Problem in [10, pp. 38-39]. It was solved in two different elegant ways by Michael Fox from the UK in [11, pp. 64-66].

In all the proofs given or cited the quadrilateral is assumed to be convex. We leave it as a task for the interested reader to investigate if any of these results also hold in concave and crossed quadrilaterals.

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