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ALGEBRAIC ASPECTS OF SPECTRAL THEORY

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Abstract. We describe some aspects of spectral theory that involve algebraic considerations but need no analysis. Some of the important applications of the results are to the algebra of $n \times n$ matrices with entries that are polynomials or more general analytic functions.

§1. *Introduction.* This paper describes a range of results in spectral theory that can be formulated and proved in an algebraic context without assuming that the algebra has a norm or even that the underlying scalar field is the field of complex numbers. In some of the applications the "scalars" involved are polynomials or rational functions. The author undertook this research in order to obtain some insights into a range of contexts in which such ideas are used, particularly control theory, polynomial pencils and so-called nonlinear spectral theory. Such problems are often not easily formulated in traditional Banach algebra terms. As well as its potential applications, indicated below, this study may provide a framework for those aspects of spectral theory that do not depend on analytic tools—norms, continuity, complex analysis, etc.

A very limited account of spectral theory at our degree of generality may be found in [**4**, pp. 1–12, 87]. A fuller treatment is given in [**7**, Ch. 1], but it still has little overlap with the contents of this paper. More relevant is [**29**, Appendix 4], which gives an extensive account of various rings and fields of polynomials and analytic and meromorphic functions from the algebraic perspective. Some elements of this theory are described in the final section of the present paper. Some examples of algebras to which the theorems in this paper might be applied are the following.

(1) Algebras of bounded or unbounded operators on a Hilbert space and infinite tensor product algebras are simple examples.

(2) The theorems in this paper may also be applied to the algebra $M(n, \mathcal{B})$ of all $n \times n$ matrices with entries in a commutative ring $\mathcal{B}$. The case in which $\mathcal{B}$ is the ring of polynomials (or rational functions) in a single complex variable is well-established in control theory, and provided the inspiration for this paper; see [**2, 6, 11, 23**]. This example also features in the theory of loop groups [**21**, §3.5] and in the study of evolution equations that involve matrices with coefficients that depend polynomially or rationally on time. Another important case, in which the matrix coefficients are formal power series, is discussed in the final section.

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(3) The space $\mathcal{H}_\infty^n$ of all bounded, analytic $n \times n$ matrix-valued functions on the upper half plane or the unit disc plays a key role in optimal control theory; see [17, 30]. References to the early literature on bounded operator-valued functions on the unit disc may be found in [26, Ch. 5]. In both cases one may regard the coefficient ring as the integral domain consisting of all bounded, complex-valued analytic functions on a domain in $\mathbf{C}$; the quotient field is a certain class of meromorphic functions on the same domain.

(4) Voiculescu has developed an algebraic approach to non-commutative probability theory that has substantial applications to random matrix theory; see [27, 28], where the term "free probability theory" is used. In this application one considers $n \times n$ matrices whose entries lie in a certain algebra of unbounded random variables, by using highly algebraic methods.

We emphasize that much of the material in this paper is classical if $\mathcal{A}$ is the algebra of all $n \times n$ complex matrices; see [16]. We are mainly concerned with the extent to which the results hold for more general choices of the algebra and base field. The final section describes some examples of relevant fields. We give a number of basic algebraic definitions in the paper for the benefit of those who are not familiar with the subject.

§2. *Some elementary results.* Let $\mathcal{A}$ be an algebra over a field $\mathbf{F}$; we always assume that $\mathcal{A}$ contains an identity element, which we denote by e. We say that $\lambda \in \mathbf{F}$ does not lie in the spectrum of $a \in \mathcal{A}$ if there exists $b \in \mathcal{A}$ such that $(a - \lambda e)b = b(a - \lambda e) = e$. The spectrum of a might be empty or equal to $\mathbf{F}$.

The formula $L_a(x) = ax$ defines a one-to-one algebra homomorphism from $\mathcal{A}$ to the algebra $\mathcal{L}(\mathcal{A})$ of linear operators on $\mathcal{A}$. The similar formula $R_a(x) = xa$ defines an anti-homomorphism, in the sense that $R_a R_b = R_{ba}$ for all $a, b \in \mathcal{A}$. The following lemma has some similarity to a corresponding result for C*-algebras, but it is applicable much more generally, for example to the algebra of $n \times n$ upper triangular matrices with entries in any field.

LEMMA 1. *We have $L_a R_b = R_b L_a$ for every $a, b \in \mathcal{A}$. If $M \in \mathcal{L}(\mathcal{A})$ and $MR_b = R_b M$ for all $b \in \mathcal{A}$ then there exists $a \in \mathcal{A}$ such that $M = L_a$.*

Proof. The first statement is elementary. To prove the second we put $a = M(e)$ and observe that

$$M(b) = M(R_b e) = R_b M(e) = M(e)b = ab$$

for all $b \in \mathcal{A}$, and so $M = L_a$. □

LEMMA 2. *If L_a is invertible as an element of $\mathcal{L}(\mathcal{A})$ then a is an invertible element of $\mathcal{A}$ and $(L_a)^{-1} = L_{a^{-1}}$. Hence*

$$\operatorname{spec}(L_a) = \operatorname{spec}(a)$$

for all $a \in \mathcal{A}$.

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Proof. Let $M \in \mathcal{L}(\mathcal{A})$ and

$$ML_a(x) = x = L_a M(x) \tag{1}$$

for all $x \in \mathcal{A}$. Putting $M(e) = b$ and $x = e$ in the second equality of (1) yields $e = ab$. Putting $x = bs$ in the first equality of (1) yields $M(s) = M(abs) = ML_a(bs) = bs$ for all $s \in \mathcal{A}$. Therefore $M = L_b$. Finally putting $x = e$ in the first equality of (1) yields $ba = L_b L_a e = e$, and so $b = a^{-1}$. □

We say that a non-zero element a in an algebra $\mathcal{A}$ over $\mathbf{F}$ is *algebraic* if there exists a non-zero polynomial p with coefficients in $\mathbf{F}$ such that $p(a) = 0$. The set of all such polynomials is a non-zero, proper ideal $\mathcal{J}$ in the algebra $\mathcal{P}$ of all polynomials, and contains a unique monic polynomial m of lowest degree, called the minimum polynomial of a. Moreover, $\mathcal{J} = \{mq : q \in \mathcal{P}\}$. If $\mathcal{A}$ is finite-dimensional then all $a \in \mathcal{A}$ are algebraic, but there are important infinite-dimensional algebras in which the same holds.

Example 3. Let $\mathcal{A}_r = M(n_r, \mathbf{F})$ for all positive integers r and put $\mathcal{A} = \bigotimes_{r=1}^{\infty} \mathcal{A}_r$. The algebra $\mathcal{A}$ is generated by expressions $x = \bigotimes_{r=1}^{\infty} x_r$ where all but a finite number of the x_r equal the identity element of $\mathcal{A}_r$. Equivalently $\mathcal{A} = \bigcup_{m=1}^{\infty} \mathcal{B}_m$ where

$$\mathcal{B}_m = \bigotimes_{r=1}^{m} \mathcal{A}_r \simeq M(n_1 n_2 \cdots n_m, \mathbf{F}).$$

Clearly every element of $\mathcal{A}$ is algebraic, but there is no upper bound on the degrees of the minimum polynomials.

THEOREM 4. *Let a be an algebraic element of the subalgebra $\mathcal{A}$ of the algebra $\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ have the same identity element e. If a has a left or right inverse b in $\mathcal{B}$ then it has a two-sided inverse in $\mathcal{A}$. The spectrum of a in either $\mathcal{A}$ or $\mathcal{B}$ equals the set of zeros of its minimum polynomial, and must be finite.*

Proof. Let m be the minimum polynomial of a and assume that a has a right inverse $b \in \mathcal{B}$. If $m(\lambda) = \lambda^n + \sum_{r=0}^{n-1} \alpha_r \lambda^r$ and $\alpha_0 = 0$ then one would have

$$\begin{aligned} &a^{n-1} + \alpha_{n-1} a^{n-2} + \cdots + \alpha_2 a + \alpha_1 e \\ &\quad = (a^{n-1} + \alpha_{n-1} a^{n-2} + \cdots + \alpha_2 a + \alpha_1 e)ab \\ &\quad = m(a)b \\ &\quad = 0. \end{aligned}$$

This contradicts the assumption that m is the minimum polynomial, so we deduce that $\alpha_0 \neq 0$. The same conclusion holds if we assume that a has a left inverse in $\mathcal{B}$. It now follows immediately that

$$a^{-1} = -\alpha_0^{-1}(a^{n-1} + \alpha_{n-1} a^{n-2} + \cdots + \alpha_1 e) \in \mathcal{A}.$$

The above calculations show that a is invertible (in either $\mathcal{A}$ or $\mathcal{B}$) if and only if $m(0) \neq 0$. Since the minimum polynomial of $a - \lambda e$ is $m_\lambda(s) = m(s + \lambda)$, we conclude that $\lambda \in \operatorname{spec}(a)$ if and only if $m(\lambda) = 0$. □

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It is sometimes useful to distinguish between the left and right spectra of an element a of the algebra $\mathcal{A}$. One says that $\lambda \in \mathrm{spec}_L(a)$ if there does not exist $b \in \mathcal{A}$ such that $b(\lambda e - a) = e$, and that $\lambda \in \mathrm{spec}_R(a)$ if there does not exist $c \in \mathcal{A}$ such that $(\lambda e - a)c = e$. One sees immediately that

$$\mathrm{spec}(a) = \mathrm{spec}_L(a) \cup \mathrm{spec}_R(a).$$

The elements b, c above need not be unique. Indeed, if there is a unique solution $b \in \mathcal{A}$ of $ab = e$ then the identity

$$a(b + ba - e) = ab + (ab)a - a = e$$

implies that b is the two-sided inverse of a.

LEMMA 5. *If a is an algebraic element of $\mathcal{A}$ then*

$$\mathrm{spec}_L(a) = \mathrm{spec}_R(a) = \mathrm{spec}(a)$$

for all $a \in \mathcal{A}$.

Proof. Let m be the minimum polynomial of $\lambda e - a$. If $\lambda \notin \mathrm{spec}_L(a)$ then there exists $b \in \mathcal{A}$ such that $b(\lambda e - a) = e$. Since $\lambda e - a$ is algebraic, Theorem 4 implies that $\lambda \notin \mathrm{spec}(a)$. Hence $\mathrm{spec}(a) \subseteq \mathrm{spec}_L(a)$. The inclusion $\mathrm{spec}(a) \subseteq \mathrm{spec}_R(a)$ has a similar proof, and the reverse inclusions are elementary. □

Example 6. The condition in Lemma 5 that the element a is algebraic cannot be omitted. Let $\mathcal{A}$ be the algebra of all bounded linear operators on the Hilbert space $\ell^2(\mathbf{Z}^+)$ and let $a \in \mathcal{A}$ be defined by $(a\phi)(n) = \phi(n + 1)$ for all $\phi \in \ell^2(\mathbf{Z}^+)$ and all $n \geq 1$. Then

$$\mathrm{spec}_L(a) = \{z \in \mathbf{C} : |z| \leq 1\},$$
$$\mathrm{spec}_R(a) = \{z \in \mathbf{C} : |z| = 1\}.$$

The proof of the first identity uses $(\lambda e - a)\phi_\lambda = 0$ whenever $|\lambda| < 1$, where $\phi_\lambda \in \ell^2(\mathbf{Z}^+)$ is defined by $\phi_\lambda(n) = \lambda^n$ for all $n \geq 1$. The proof of the second identity uses $aa^* = e$ and $(\lambda e - a)c_\lambda = e$ whenever $|\lambda| < 1$, where $c_\lambda = -a^*(e - \lambda a^*)^{-1}$.

We next prove a spectral mapping theorem for polynomials.

THEOREM 7. *If the field $\mathbf{F}$ is algebraically closed, $a \in \mathcal{A}$ and p is a polynomial then*

$$\mathrm{spec}(p(a)) = \{p(z) : z \in \mathrm{spec}(a)\}.$$

Proof. Suppose that p is monic and of degree n. The statement of the theorem can be rewritten in several ways, each of which is equivalent to the next one.

- Given $y \in \mathbf{F}$ then $y \in \mathrm{spec}(p(a))$ if and only if there exists $x \in \mathrm{spec}(a)$ such that $y = p(x)$.
- Let

$$q(s) = p(s) - y = \prod_{r=1}^{n}(s - s_r),$$

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where the s_r depend on y. Then $0 \in \operatorname{spec}(q(a))$ if and only if there exists $x \in \operatorname{spec}(a)$ such that $q(x) = 0$.

- $0 \in \operatorname{spec}(q(a))$ if and only if there exists $x \in \operatorname{spec}(a)$ and $r \in \{1, \ldots, n\}$ such that $x - s_r = 0$.
- $0 \in \operatorname{spec}(q(a))$ if and only if there exists $r \in \{1, \ldots, n\}$ such that $s_r \in \operatorname{spec}(a)$.
- $q(a)$ is not invertible if and only if there exists $r \in \{1, \ldots, n\}$ such that $a - s_r e$ is not invertible.
- $q(a)$ is invertible if and only if $a - s_r e$ is invertible for all $r \in \{1, \ldots, n\}$.

The truth of the final statement is an elementary consequence of the fact that $q(a)$ is the product of the $a - s_r e$, all of which commute in $\mathcal{A}$. □

Without the hypothesis that $\mathbf{F}$ is algebraically closed, the theorem is false. For example, if $\mathbf{F} = \mathbf{R}$ and $a = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ then $\operatorname{spec}(a) = \emptyset$ but $\operatorname{spec}(a^4) = \{-4\}$.

§3. *Commutative algebras.* In this section we present an algebraic version of Gelfand's representation of commutative Banach algebras as function algebras.

Let $\mathcal{A}$ be a commutative algebra with identity e over a field $\mathbf{F}$ and let J_m denote its maximal proper ideals, which are parametrized by $m \in M$. For each $m \in M$, $\mathbf{F}_m = \mathcal{A}/J_m$ is a commutative algebra with no proper ideals, and hence is a field containing $\mathbf{F}$. The natural homomorphism $\phi_m : \mathcal{A} \to \mathbf{F}_m$ is called a character of $\mathcal{A}$.

The set M of all maximal ideals in $\mathcal{A}$ is called the max-spectrum of $\mathcal{A}$ in algebraic geometry to distinguish it from the spectrum, namely the set N of all prime ideals. Elements of M correspond to algebra homomorphisms mapping $\mathcal{A}$ onto a field, which we call characters, while elements of N correspond to algebra homomorphisms mapping $\mathcal{A}$ into a field, or equivalently onto an integral domain. Every maximal ideal is obviously prime.

Determining the maximal ideals, prime ideals or characters in an algebra is a highly non-trivial matter. Hilbert's Nullstellensatz implies that if $\mathcal{A}$ is the algebra of polynomials on $\mathbf{C}^n$ then every character is obtained by evaluation at some point of $\mathbf{C}^n$; see [1, p. 85]. For some other algebras of unbounded continuous functions this is not true and the question is much harder.

The Gelfand formula

$$\widehat{a}(m) = \phi_m(a)$$

takes an element $a \in \mathcal{A}$ and maps it by an algebra homomorphism to $\widehat{a} \in \mathcal{F}(M)$, where $\mathcal{F}(M)$ is the algebra of all functions f on M such that $f(m) \in \mathbf{F}_m$. (More correctly $\mathcal{F}(M)$ is the algebra of sections of a certain bundle over M.)

THEOREM 8 [12, Proposition IX.2.5 and Theorem IX.2.4]. *The element $a \in \mathcal{A}$ is invertible if and only if $\widehat{a}(m) \neq 0$ for all $m \in M$. If $a \in \mathcal{A}$ then*

$$\operatorname{spec}(a) = \mathbf{F} \cap \{\widehat{a}(m) : m \in M\}.$$

Proof. If a is invertible and $ab = e$ then

$$\widehat{a}(m)\widehat{b}(m) = \phi_m(a)\phi_m(b) = \phi_m(ab) = e,$$

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so $\widehat{a}(m) \neq 0$ for every $m \in M$. Conversely if a is not invertible then $a\mathcal{A}$ is a proper ideal in $\mathcal{A}$ and is contained in some maximal proper ideal J_m. Since $a \in J_m$ one has $\widehat{a}(m) = 0$.

The final statement of the theorem is proved by applying the above to $a - \lambda e$ for every choice of $\lambda \in \mathbf{F}$. □

Example 9. Let $I = [0, 1]$ and let $\mathcal{A}_1$ be the algebra over $\mathbf{Q}$ consisting of all continuous functions $f : I \to \mathbf{R}$ such that $f(\mathbf{Q} \cap I) \subseteq \mathbf{Q}$, where $\mathbf{Q}$ is the field of rational numbers. Given a positive integer n and any sequence $a_0, a_1, \ldots, a_n \in \mathbf{Q}$ one may define a function $f \in \mathcal{A}_1$ by linear interpolation starting from the values $f(r/n) = a_r$ where $0 \leq r \leq n$. This construction may be used to prove that $\mathcal{A}_1$ is uniformly dense in the algebra of all continuous real-valued functions on I.

For each $a \in I$ one may define the character ϕ_a on $\mathcal{A}_1$ by

$$\phi_a(f) = f(a). \tag{2}$$

One notes that $\mathbf{F}_a = \mathbf{Q}$ if $a \in \mathbf{Q} \cap I$ but $\mathbf{F}_a = \mathbf{R}$ otherwise.

THEOREM 10. *Every character ϕ of $\mathcal{A}_1$ is of the form ϕ_a for some $a \in I$.*

Proof. Suppose that ϕ is not of the stated form. For every $a \in I$ the kernels J and J_a of ϕ and ϕ_a respectively are maximal proper ideals, so there exists $f_a \in J$ such that $f_a(a) \neq 0$. By a compactness argument there exist $a_1, \ldots, a_m \in I$ such that $h := \sum_{r=1}^{m} |f_{a_r}|^2 > 0$ on I. This yields a contradiction, because $h \in J$ and a proper ideal cannot contain an invertible element. □

Example 11. The algebra $\mathcal{A}_2 = C_b(\mathbf{R})$ of all bounded, continuous, complex-valued functions f on $\mathbf{R}$ is a C*-algebra for the norm

$$\|f\| = \sup\{|f(x)| : x \in \mathbf{R}\},$$

and for no other norm; see [7, Theorem 3.2.6]. The characters of $\mathcal{A}_2$ are all norm continuous and are in one-to-one correspondence with the points in the Stone–Čech compactification of $\mathbf{R}$, and all characters are complex-valued; see [7, Definition 4.2.6]. The only characters that can be identified constructively are those given by the formula (2) where $a \in \mathbf{R}$.

Example 12. The algebra $\mathcal{A}_3 = C(\mathbf{R})$ of all continuous, complex-valued functions on $\mathbf{R}$ has "finite" characters given by (2). We say that a character ϕ on $\mathcal{A}_3$ is "at infinity", and write $\phi \in M_\infty$, if it is not of this form. Lemma 13 and Theorem 14 give some insight into the form of the characters at infinity.

LEMMA 13. *The set M_∞ is non-empty. Every character ϕ at infinity has a kernel J_ϕ that contains the ideal*

$$J = \{f \in \mathcal{A}_3 : \operatorname{supp}(f) \text{ is compact}\}. \tag{3}$$

Proof. Suppose that $f \in \mathcal{A}_3$ has support contained in $(-n, n)$ for some n. Let $a \in [-n, n]$. Since J_ϕ and $J_a = \{f \in \mathcal{A}_3 : f(a) = 0\}$ are both maximal proper ideals, there exists $f_a \in J_\phi$ such that $f_a(a) \neq 0$. By a compactness argument

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there exist $a_1, \ldots, a_m \in [-n, n]$ such that $h := \sum_{r=1}^{m} |f_{a_r}|^2 > 0$ on $[-n, n]$. Since $f = hk$ for some $k \in \mathcal{A}_3$, it follows that $f \in J_\phi$. This proves the second statement of the lemma.

The existence of characters at infinity follows from the fact that every ideal K in $\mathcal{A}_3$ is contained in a maximal such ideal by Zorn's lemma. One may construct an initial ideal from any diverging sequence $a = \{a_n\}_{n=1}^{\infty}$ of real numbers by the formula

$$K_a = \{f \in \mathcal{A}_3 : f(a_n) = 0 \text{ for all large enough } n\}.$$

Since each such K_a contains J, the character ϕ corresponding to any maximal ideal containing K_a must lie in M_∞. □

THEOREM 14. *If ϕ is a character of $\mathcal{A}_3$ at infinity then the field $\phi(\mathcal{A})$ contains the field G of all complex-valued rational functions.*

Proof. Every $g \in G$ has a finite number of poles, so there exists a smallest integer $n \geq 0$ such that $g(x)$ is continuous on $\{x \in \mathbf{R} : |x| \geq n\}$. We define $\rho(g) \in \mathcal{A}_3$ to be equal to g on $\{x \in \mathbf{R} : |x| \geq n\}$ and by linear interpolation in $[-n, n]$.

If J is defined by (3), then

$$\rho(g_1) + \rho(g_2) - \rho(g_1 + g_2) \in J, \qquad \rho(g_1)\rho(g_2) - \rho(g_1 g_2) \in J$$

for all $g_1, g_2 \in G$. Therefore $g \to \rho(g) + J$ defines a one-to-one structure-preserving map from G into $\mathcal{A}/J$. By an abuse of notation one may say that $G \subseteq \mathcal{A}/J$.

There is a one-to-one correspondence between ideals in $\mathcal{A}/J$ and ideals in $\mathcal{A}$ that contain J. Similarly, the characters on $\mathcal{A}$ whose kernels contain J may be identified with the characters on the algebra $\mathcal{A}/J$. The restriction of such characters to G, regarded as a subfield of $\mathcal{A}/J$, must be one-to-one, because a field contains no proper ideals, so the field $\phi(\mathcal{A}/J)$ must contain a copy of G. □

§4. *Resolvent families.* We now turn to the study of (pseudo) resolvent families; the ideas here are taken from [9]. From this point onwards $\mathcal{A}$ will denote a (typically non-commutative) algebra over a field $\mathbf{F}$. A resolvent family (r, S) in $\mathcal{A}$ consists of a set $S \subseteq \mathbf{F}$ together with a map $r : S \to \mathcal{A}$ such that

$$r_\lambda - r_\mu = (\mu - \lambda) r_\lambda r_\mu \tag{4}$$

for all $\lambda, \mu \in S$. By interchanging the roles of λ and μ one readily sees that this implies that r_λ and r_μ commute for all $\lambda, \mu \in S$. We say that $(\tilde{r}, \tilde{S})$ extends (r, S) if $S \subseteq \tilde{S}$ and $\tilde{r}(\lambda) = r(\lambda)$ for all $\lambda \in S$. We say that (r, S) is maximal if it has no proper extension.

If $a \in \mathcal{A}$ and $S = \mathbf{F} \backslash \mathrm{spec}(a)$ then one obtains a (true) resolvent family by putting $r_\lambda = (\lambda e - a)^{-1}$ for all $\lambda \in S$. (Pseudo) resolvent families are not all of this type; if a is an unbounded closed linear operator on a Banach space $\mathcal{B}$ and $\mathcal{A}$ is the algebra of all bounded operators on $\mathcal{B}$, then one may define the resolvent

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family associated with a just as in the case in which a is bounded. The theory of such resolvent families is very highly developed because of the role it plays in the theory of differential operators and one-parameter semigroups, [8].

THEOREM 15. *A resolvent family (r, S) is the family associated with an element $a \in \mathcal{A}$ if and only if it is maximal and r_λ is invertible for some (equivalently all) $\lambda \in S$. If this holds then* $\operatorname{spec}(a) = \mathbf{F} \backslash S$.

Proof. Suppose that $br_\mu = r_\mu b = e$ for some $b \in \mathcal{A}$ and $\mu \in S$. Multiplying (4) on the left (right) by b yields $e = (b - \mu e + \lambda e) r_\lambda = r_\lambda (b - \mu e + \lambda e)$, so r_λ is invertible for all $\lambda \in S$. Putting $a = \mu e - b$ then yields $r_\lambda = (\lambda e - a)^{-1}$ for all $\lambda \in S$.

Given $a \in \mathcal{A}$ let (r_0, S_0) be the resolvent family associated with a, where $S_0 = \mathbf{F} \backslash \operatorname{spec}(a)$. Suppose that it is not maximal, that (r, S) is a proper extension and that $\lambda \in S \cap \operatorname{spec}(a)$. Then r_λ is invertible with an inverse that we denote b, and $r_\mu = (\mu e - a)^{-1}$ for some chosen $\mu \in S$. If we multiply (4) on the left by b and on the right by $\mu e - a$ we obtain

$$\mu e - a - b = (\mu - \lambda) e.$$

Therefore $b = \lambda e - a$; since b is invertible we conclude that $\lambda \notin \operatorname{spec}(a)$. The contradiction implies that (r_0, S_0) is maximal. □

THEOREM 16. *Every resolvent family (r, S) is uniquely determined on S by r_μ for a single chosen $\mu \in S$. Moreover, (r, S) has a unique maximal extension $(\widetilde{r}, \widetilde{S})$.*

Proof. Given $\alpha, \lambda \in S$ the resolvent equation implies that

$$(e + (\lambda - \alpha) r_\alpha)(e + (\alpha - \lambda) r_\lambda) = e.$$

Therefore $e + (\lambda - \alpha) r_\alpha$ is invertible and

$$r_\lambda = (\alpha - \lambda)^{-1} \{(e + (\lambda - \alpha) r_\alpha)^{-1} - e\} = (e + (\lambda - \alpha) r_\alpha)^{-1} r_\alpha.$$

If we define

$$\widetilde{r}_\lambda = (e + (\lambda - \alpha) r_\alpha)^{-1} r_\alpha$$

on the set $\widetilde{S}$ of all $\lambda \in \mathbf{F}$ for which $e + (\lambda - \alpha) r_\alpha$ is invertible, then $\widetilde{r}$ must be the maximal extension of r provided $\widetilde{r}$ satisfies the resolvent equation. If $\lambda, \mu \in \widetilde{S}$ then

$$\begin{aligned}
(\mu - \lambda) \widetilde{r}_\lambda \widetilde{r}_\mu &= (\mu - \lambda) r_\alpha (e + (\lambda - \alpha) r_\alpha)^{-1} (e + (\lambda - \alpha) r_\alpha)^{-1} r_\alpha \\
&= \{(e + (\mu - \alpha) r_\alpha) - (e + (\lambda - \alpha) r_\alpha)\} \\
&\quad . (e + (\lambda - \alpha) r_\alpha)^{-1} (e + (\mu - \alpha) r_\alpha)^{-1} r_\alpha \\
&= \{(e + (\lambda - \alpha) r_\alpha)^{-1} - (e + (\mu - \alpha) r_\alpha)^{-1}\} r_\alpha \\
&= \widetilde{r}_\lambda - \widetilde{r}_\mu.
\end{aligned}$$

□

Motivated by the last two theorems, we define the spectrum $\operatorname{spec}(r, S)$ of a resolvent family (r, S) to be $\mathbf{F} \backslash \widetilde{S}$ where $(\widetilde{r}, \widetilde{S})$ is its unique maximal extension. Our next task is to explain how one can use algebraic ideas to classify the spectrum into parts.

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If $\mathcal{J}$ is a two-sided ideal in $\mathcal{A}$, there is a natural algebra homomorphism $\pi_{\mathcal{J}}$ from $\mathcal{A}$ onto the quotient algebra $\mathcal{A}/\mathcal{J}$. Given $a \in \mathcal{A}$ we put

$$\begin{aligned} \operatorname{spec}_{\mathcal{J}}(a) &= \operatorname{spec}(\pi_{\mathcal{J}}(a)), \\ \operatorname{spec}_{L,\mathcal{J}}(a) &= \operatorname{spec}_L(\pi_{\mathcal{J}}(a)), \\ \operatorname{spec}_{R,\mathcal{J}}(a) &= \operatorname{spec}_R(\pi_{\mathcal{J}}(a)). \end{aligned}$$

See [12, Ch. 1.3] for a full discussion of the various definitions of the essential spectrum of an operator on a Hilbert space $\mathcal{H}$; one widely used definition arises by taking $\mathcal{A}$ to be the algebra of all bounded operators on $\mathcal{H}$, and $\mathcal{J}$ to be the ideal of all compact operators. The quotient algebra $\mathcal{A}/\mathcal{J}$ is called the Calkin algebra and arises in the study of Fredholm operators and index theory.

LEMMA 17. *Let (r, S) be a maximal resolvent family in $\mathcal{A}$. If one defines*

$$\operatorname{spec}_{\mathcal{J}}(r, S) = \operatorname{spec}(\pi_{\mathcal{J}} r, S)$$

then

$$\operatorname{spec}_{\mathcal{J}}(r, S) \subseteq \operatorname{spec}(r, S).$$

Proof. One sees immediately that $(\pi_{\mathcal{J}} r, S)$ is a resolvent family in $\mathcal{A}/\mathcal{J}$. If $\widetilde{S}$ is the domain of its maximal extension then the lemma is simply a restatement of the fact that $S \subseteq \widetilde{S}$. □

Note that if (r, S) is associated with a closed unbounded operator a on $\mathcal{H}$, then the resolvent family $(\pi_{\mathcal{J}} r, S)$ has a much less direct relationship with a because $\pi_{\mathcal{J}} a$ is not easy to define.

We turn to some applications to perturbation theory. The algebra $\mathcal{L}(\mathcal{H})$ of all bounded operators on a Hilbert space $\mathcal{H}$ contains many two-sided ideals, but the set $\mathcal{C}_\infty$ of all compact operators and the set $\mathcal{C}_1$ of all trace class operators are particularly important. The following results are abstractions of theorems known in the above context; they have proved important in quantum theory, because of the existence of physically relevant C^*-algebras with many two-sided ideals. See [9, 14, 15] for many references to these recent developments. These ideals allow the division of the classical essential spectrum into parts with different natures.

Let $\mathcal{J}$ be a two-sided ideal in an algebra $\mathcal{A}$ over the field $\mathbf{F}$. If $a, b \in \mathcal{A}$ and $\lambda \notin \operatorname{spec}(a) \cup \operatorname{spec}(b)$ then the identity

$$(\lambda e - a)^{-1} - (\lambda e - b)^{-1} = (\lambda e - a)^{-1}(a - b)(\lambda e - b)^{-1}$$

implies that $a - b \in \mathcal{J}$ if and only if the difference of the two resolvents lies in $\mathcal{J}$. Moreover, the definition of the $\mathcal{J}$-spectrum implies immediately that $\operatorname{spec}_{\mathcal{J}}(a) = \operatorname{spec}_{\mathcal{J}}(b)$ provided $a - b \in \mathcal{J}$.

If A, B are two unbounded closed operators on a Hilbert space $\mathcal{H}$ the analogous result is proved by treating the resolvent operators directly as follows; see [12, Theorem IX.2.4], where $\mathcal{J}$ is the ideal of compact operators in $\mathcal{L}(\mathcal{H})$.

THEOREM 18. *Let (r_1, S_1) and (r_2, S_2) be two resolvent families in $\mathcal{A}$ and let $S = S_1 \cap S_2$. If $r_{1,\lambda} - r_{2,\lambda}$ lies in the two-sided ideal $\mathcal{J}$ for some $\lambda \in S$ then this holds for all $\lambda \in S$, and $\operatorname{spec}_{\mathcal{J}}(r_1) = \operatorname{spec}_{\mathcal{J}}(r_2)$.*

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Proof. If we put $\widetilde{r}_i = \pi_{\mathcal{J}}(r_i)$ then the hypothesis implies that $\widetilde{r}_{1,\lambda} = \widetilde{r}_{2,\lambda}$ for some $\lambda \in S$. Theorem 16 now implies that $\widetilde{r}_1 = \widetilde{r}_2$ throughout S. Therefore they have the same maximal extension, and the original resolvents have the same $\mathcal{J}$-spectrum. □

The following theorem is adapted from [**20**, Lemma 10.1.8], where applications to the essential spectra of Hankel operators are given.

THEOREM 19. *If $a_1, \ldots, a_n \in \mathcal{A}$ and $a_r a_s \in \mathcal{J}$ for all $r \neq s$ then*

$$\operatorname{spec}_{\mathcal{J}}(a_1 + \cdots + a_n) \cup \{0\} = \bigcup_{r=1}^{n} \operatorname{spec}_{\mathcal{J}}(a_r).$$

Proof. Put $a = \sum_{r=1}^{n} a_r$ and $\widetilde{x} = \pi_{\mathcal{J}} x \in \mathcal{A}/\mathcal{J}$ for every $x \in \mathcal{A}$. The hypotheses imply that $\widetilde{a}_r \widetilde{a}_s = 0$ for all $r \neq s$, and hence that

$$(\lambda\widetilde{e} - \widetilde{a}_1) \cdots (\lambda\widetilde{e} - \widetilde{a}_n) = \lambda^{n-1}(\lambda\widetilde{e} - \widetilde{a})$$

for all $\lambda \in \mathbf{F}$. Since all the terms in this equation commute we conclude that if $\lambda \neq 0$ then $\lambda\widetilde{e} - \widetilde{a}$ is invertible if and only if $\lambda\widetilde{e} - \widetilde{a}_r$ are invertible for all r. This completes the proof as far as non-zero λ are concerned.

If $0 \notin \operatorname{spec}(\widetilde{a}_1)$ then $\widetilde{a}_1$ is invertible and the identity $\widetilde{a}_1 \widetilde{a}_r = 0$ implies that $\widetilde{a}_r = 0$ for all $r \neq 1$. Therefore $0 \in \bigcup_{r=1}^{n} \operatorname{spec}(\widetilde{a}_r)$ in all cases. □

Example 20. The following application of Theorem 19 is well-known in the spectral analysis of Schrödinger operators. Let $p_1, \ldots, p_n \in \mathcal{A}$ satisfy $p_r p_s = \delta_{r,s} p_r$ for all r, s and $\sum_{r=1}^{n} p_r = e$. Let $a \in \mathcal{A}$ and assume that $a_{r,s} := p_r a p_s \in \mathcal{J}$ for all $r \neq s$. Then by applying Theorem 19 to the double sum $a = \sum_{r,s} a_{r,s}$ one obtains

$$\operatorname{spec}_{\mathcal{J}}(a) \cup \{0\} = \bigcup_{r=1}^{n} \operatorname{spec}_{\mathcal{J}}(a_{r,r}).$$

Example 21. Let $\mathcal{A}$ be the algebra $M(n, \mathcal{P})$ of all $n \times n$ matrices whose entries are polynomials in a variable $x \in \mathbf{F}$. The spectrum of an element of $\mathcal{A}$ is normally equal to $\mathbf{F}$ because an element of $\mathcal{A}$ is invertible if and only if its determinant is a non-zero constant. However, if $n = 2$ and

$$a = \begin{pmatrix} 1 & 1 \\ x & 1+x \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 1+x^3 \\ 0 & 0 \end{pmatrix}$$

with $x \in \mathbf{F}$ being the polynomial variable, then $\operatorname{spec}(a) = \mathbf{F} \backslash \{0\}$ while $\operatorname{spec}(b) = \{0\}$.

By passing to suitable quotient algebras one obtains more interesting spectra. Let $\mathcal{A} = M(n, \mathcal{P})$ and let S be a finite subset of $\mathbf{F}$. We define $\mathcal{B}_S$ to be the algebra of all functions $f : S \to M(n, \mathbf{F})$ with the obvious pointwise operations, and define $\pi_S : \mathcal{A} \to \mathcal{B}_S$ by $\pi_S(a) = f$ where $f(s) = a(s)$ for all $s \in S$. The kernel of π_S is a two-sided ideal $\mathcal{J}_S$ in $\mathcal{A}$ and π_S is an algebra homomorphism from $\mathcal{A}$ onto $\mathcal{B}_S$. Moreover,

$$\operatorname{spec}_{\mathcal{J}_S}(a) = \operatorname{spec}(\pi_S a) = \bigcup_{s \in S} \operatorname{spec}(a(s))$$

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for all $a \in \mathcal{A}$. Hence $\#(\mathrm{spec}_{\mathcal{J}_S}(a)) \leq n(\#S)$. As S increases, the kernels of the ideals $\mathcal{J}_S$ decrease and $\mathrm{spec}_{\mathcal{J}_S}(a)$ increases to $\bigcup_{x \in \mathbf{F}} \mathrm{spec}(a(x))$.

If one puts $\mathbf{F} = \mathbf{R}$, $n = 2$ and defines $a \in \mathcal{A}$ by

$$a = \begin{pmatrix} x^2 & 1 \\ 0 & x^2 \end{pmatrix},$$

then $\mathrm{spec}(a) = \mathbf{R}$ but $\bigcup_{x \in \mathbf{R}} \mathrm{spec}(a(x)) = [0, \infty)$.

§5. *Connections between ab and ba.* If a, b lie in an algebra $\mathcal{A}$ over $\mathbf{F}$, an elementary calculation shows that

$$bp(ab) = p(ba)b$$

for every polynomial p with coefficients in $\mathbf{F}$. The calculations in this section are based on the idea that this relationship might be valid for other classes of function. The following lemma may be found in [7, Proposition 1.5.29].

LEMMA 22. *If $a, b \in \mathcal{A}$ then*

$$\mathrm{spec}(ab) \cup \{0\} = \mathrm{spec}(ba) \cup \{0\}.$$

If $\dim(\mathcal{A}) < \infty$ *then*

$$\mathrm{spec}(ab) = \mathrm{spec}(ba).$$

Moreover, ab is invertible if and only if both a and b are invertible.

Proof. If $\lambda \neq 0$ and $\lambda e - ab$ is invertible then a direct calculation shows that

$$c(\lambda e - ba) = (\lambda e - ba)c = e$$

where

$$c = \lambda^{-1}(e + b(\lambda e - ab)^{-1}a).$$

The first part of the lemma follows by interchanging the roles of a and b.

In order to prove the second part we have to show that ab is invertible if and only if ba is invertible, whenever $\dim(\mathcal{A}) < \infty$. If ab is invertible then there exists $c \in \mathcal{A}$ such that $(ab)c = e$. Therefore $L_a L_b L_c = I$, and so

$$\det(L_a) \det(L_b) \det(L_c) = 1.$$

It follows that L_a, L_b are invertible. Lemma 2 now implies that a, b are invertible and hence that ba is invertible. The converse is similar. □

Example 23. The following arises in the theory of quantum groups, and more specifically the quantum torus. If $a, b \in \mathcal{A}$ and $ab = zba$ for some non-zero $z \in \mathbf{F}$ then Lemma 22 implies that $\mathrm{spec}(ab)$ is invariant under multiplication by $z^{\pm 1}$. If $\mathcal{A}$ is a Banach algebra then the boundedness of the spectrum of ab further implies that $|z| = 1$.

We say that $J \subseteq \mathcal{A}$ is a right ideal if it is a linear subspace of $\mathcal{A}$ and $xy \in J$ whenever $x \in J$ and $y \in \mathcal{A}$. If $a \in \mathcal{A}$ one defines the principal right ideal J_a by $J_a = \{ax : x \in \mathcal{A}\}$. It follows immediately that $J_a = J_b$ if and only if there exist $x, y \in \mathcal{A}$ such that $ax = b$ and $by = a$.

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If $\mathcal{A}$ is an algebra of linear maps on V, and W is a linear subspace of V, then one may define a right ideal of $\mathcal{A}$ by

$$J_W = \{x : x(V) \subseteq W\}.$$

If $\lambda \in \mathbf{F}$ lies in the spectrum of $a \in \mathcal{A}$ then one may define a class of "Jordan" right ideals $J(a, \lambda, n)$ by

$$J(a, \lambda, n) = \{x \in \mathcal{A} : (a - \lambda e)^n x = 0\}.$$

Our next lemma lists some elementary consequences of this definition.

LEMMA 24. *Under the above assumptions*

$$J(a, \lambda, n) \subseteq J(a, \lambda, n + 1)$$

for all natural numbers n. Either all $J(a, \lambda, n)$ *are distinct, or there exists N such that* $J(a, \lambda, n) \neq J(a, \lambda, n + 1)$ *if* $n < N$ *but* $J(a, \lambda, n) = J(a, \lambda, n + 1)$ *if* $n \geq N$. *If* $\dim(\mathcal{A}) < \infty$ *then the latter case occurs.*

LEMMA 25. *If* $a, b \in \mathcal{A}$ *and* $\lambda \neq 0$ *then there is a natural isomorphism of* $J(ab, \lambda, n)$ *with* $J(ba, \lambda, n)$ *for all n.*

Proof. We start by putting $J_1 = J(ba, \lambda, n)$ and $J_2 = J(ab, \lambda, n)$. If $x \in J_1$ then

$$(ab - \lambda e)^n ax = a(ba - \lambda e)^n x = 0$$

and so $ax \in J_2$. Similarly $x \in J_2$ implies that $bx \in J_1$. Now put

$$c = \sum_{r=1}^{n} \frac{n!}{r!(n-r)!} a(ba)^{r-1}(-\lambda)^{n-r}.$$

By applying the two results just proved one sees that $x \in J_1$ implies $cx \in J_2$. Since $(ba - \lambda e)^n = bc + (-\lambda)^n e$, it follows that $b(cx) = -(-\lambda)^n x$ for all $x \in J_1$. Therefore $L_b : J_2 \to J_1$ is onto. A similar argument based on $(ab - \lambda e)^n = cb + (-\lambda)^n e$ yields $c(by) = -(-\lambda)^n y$ for all $y \in J_2$. Therefore $L_b : J_2 \to J_1$ is one-to-one.

We have now proved that L_b provides an isomorphism of J_1 with J_2. In the particular case $n = 1$ it proves that $\lambda \neq 0$ is an eigenvalue of ab if and only if it is an eigenvalue of ba. □

If $\dim(\mathcal{A}) < \infty$ and we put $d(a, \lambda, n) = \dim(J(a, \lambda, n))$, then the above lemma establishes that

$$d(ab, \lambda, n) = d(ba, \lambda, n) \quad \text{for all } \lambda \neq 0, n \geq 1.$$

If $\lambda = 0$ these identities do not hold, even if $\mathcal{A}$ is the algebra of 2×2 matrices and

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

§6. *Transformations of polynomial pencils.* Let $\mathrm{SL}(2, \mathbf{F})$ denote the multiplicative group of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with entries in the field $\mathbf{F}$ and determinant one. There is an extensive literature on such groups [5], particularly when $\mathbf{F}$

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is finite, which we do not assume. It is easy to show that SL(2, **F**) is generated by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and the set of all $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ where $x \in \mathbf{F}$; see [5, Lemma 6.1.1]. From this it easily follows that the centre of SL(2, **F**) consists of $\pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Therefore the centre contains two elements unless **F** has characteristic two, in which case it contains only one.

We define $\mathcal{P}_n$ to be the space of polynomials with degree at most n in the variable $x \in \mathbf{F}$, where the coefficients lie in the algebra $\mathcal{A}$. In the theory of nonlinear eigenvalue problems the spectrum of $p \in \mathcal{P}_n$ is defined to be the set

$$\operatorname{spec}(p) = \{x \in \mathbf{F} : p(x) \text{ is not invertible in } \mathcal{A}\}.$$

We also include a symbolic element ∞ in $\operatorname{spec}(p)$ if $p(x) = \sum_{r=0}^{n} a_r x^r$ and a_n is not invertible in $\mathcal{A}$. Clearly $\operatorname{spec}(p) = \operatorname{spec}(bpc)$ whenever $b, c \in \mathcal{A}$ are both invertible.

LEMMA 26. *Given $g \in G = \mathrm{SL}(2, \mathbf{F})$ and $x \in \tilde{\mathbf{F}} = \mathbf{F} \cup \{\infty\}$, the formula*

$$g \cdot x = \frac{ax + b}{cx + d}$$

defines a map on $\tilde{\mathbf{F}}$ such that $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and all $x \in \tilde{\mathbf{F}}$. Moreover, the formula

$$(T_g p)(x) = (a - cx)^n p(g^{-1}x)$$

defines a linear map on $\mathcal{P}_n$ such that $T_g T_h p = T_{gh} p$ for all $g, h \in G$ and $p \in \mathcal{P}_n$.

Proof. If $p(x) = \sum_{r=0}^{n} a_r x^r$ then

$$(T_g p)(x) = \sum_{r=0}^{n} a_r (dx - b)^r (a - cx)^{n-r}.$$

The statements of the lemma now reduce to direct computations. □

LEMMA 27. *If $g \in G = \mathrm{SL}(2, \mathbf{F})$ and $p \in \mathcal{P}_n$ then*

$$\operatorname{spec}(T_g p) = g \cdot \operatorname{spec}(p).$$

Proof. There are four cases.

(1) $x \neq \infty$ and $cx \neq a$, which imply $g^{-1}x \neq \infty$.
(2) $x \neq \infty$ and $cx = a$, which imply $dx \neq b$ and then $g^{-1}x = \infty$.
(3) $x = \infty$ and $c \neq 0$, which imply $g^{-1}x \neq \infty$.
(4) $x = \infty$ and $c = 0$, which imply $d \neq 0$ and then $g^{-1}x = \infty$.

The first and generic case depends on the equivalence of the statements: $x \notin \operatorname{spec}(T_g p)$; $(a - cx)^n p(g^{-1}x)$ is invertible; $p(g^{-1}x)$ is invertible; $g^{-1}x \notin \operatorname{spec}(p)$; $x \notin g \cdot \operatorname{spec}(p)$. The other cases are similar, and are omitted. □

LEMMA 28. *If **F** is infinite, $\dim(\mathcal{A}) < \infty$ and $p \in \mathcal{P}_n$ is regular in the sense that $p(x)$ is invertible for some $x \in \mathbf{F}$, then there exists $g \in G$ such that $T_g(p) = q$ where $q(x) = \sum_{r=0}^{n} a_r x^r$ and a_0, a_n are both invertible.*

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Proof. We first observe that $\det(p(x))$ is a non-zero $\mathbf{F}$-valued polynomial with degree at most mn where m is the dimension of $\mathcal{A}$ and n is the degree of p. Therefore the number of $x \in \mathbf{F}$ for which $p(x)$ is not invertible is no larger than mn, and there must exist two distinct $x, y \in \mathbf{F}$ such that $p(x)$, $p(y)$ are both invertible. If $g \in G$ is chosen so that $g \cdot x = 0$ and $g \cdot y = \infty$ then $q = T_g(p)$ has the required properties. □

In many respects it is more natural to consider homogeneous polynomial pencils, i.e. functions

$$p(x, y) = \sum_{r=0}^{n} x^r y^{n-r} a_r$$

where $x, y \in \mathbf{F}$ and $a_r \in \mathcal{A}$ for all r. The group $\mathrm{SL}(2, \mathbf{F})$ acts on such polynomials in a similar fashion, but one no longer needs to consider points at infinity separately. If $\mathbf{F} = \mathbf{R}$ then such a homogeneous polynomial is determined by its values on the unit circle in $\mathbf{R}^2$. However, it is possible to analyse the eigenvalue problem in the projective space $P(\mathbf{F}^2)$ directly, without choosing a particular parametrization of $P(\mathbf{F}^2)$ by points in $\tilde{F}$; see [**10**].

§7. *Factorization of polynomial pencils.* Given a monic polynomial pencil with values in $\mathcal{A}$, i.e. an expression of the form

$$p(x) = \sum_{r=0}^{m} a_r x^r$$

where $a_r \in \mathcal{A}$ for all r, $a_m = e$ and $x \in \mathbf{F}$, there may or may not exist a factorization

$$p(x) = \prod_{r=1}^{m} (xe - b_r)$$

such that $b_r \in \mathcal{A}$ for all r. Putting $m = 2$, suppose that $p(x) = x^2 e - xa + b$ for some $a, b \in \mathcal{A}$ and all $x \in \mathbf{F}$. If p can be factorized as

$$p(x) = (xe - c)(xe - d) \tag{5}$$

for some $c, d \in \mathcal{A}$ and all $x \in \mathbf{F}$, a routine calculation shows that

$$c^2 - ca + b = 0. \tag{6}$$

Conversely if this equation is soluble then p can be factorized. See [**16**, p. 75]. Note also that (6) is equivalent to

$$d^2 - ad + b = 0.$$

The following example, essentially due to [**16**, Example 3.3], shows that it may not be possible to factorize a monic quadratic polynomial even if $\mathbf{F} = \mathbf{C}$ and $\dim(\mathcal{A}) < \infty$.

Example 29. Let $\mathcal{A}$ be the algebra of $n \times n$ complex matrices and let $a \in \mathcal{A}$ be the elementary Jordan matrix such that $a_{r,s} = 1$ if $s = r + 1$ and $a_{r,s} = 0$ otherwise. Then $a^{n-1} \neq 0$ but $a^n = 0$. Suppose that there exist $c, d \in \mathcal{A}$

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such that

$$x^2e - a = (xe - c)(xe - d)$$

for all $x \in \mathbf{C}$. Then $d = -c$ and $c^2 = a$. Therefore $c^{2n-2} \neq 0$ but $c^{2n} = 0$. By writing down the Jordan form of c one readily sees that this is impossible for an $n \times n$ matrix.

If $\dim(\mathcal{A}) < \infty$ then equation (5) and Lemma 22 together imply that

$$\operatorname{spec}(p) = \operatorname{spec}(c) \cup \operatorname{spec}(d).$$

The following example shows that this need not be true if $\mathcal{A}$ is infinite-dimensional.

Example 30. This example requires some familiarity with the spectral theorem for bounded self-adjoint operators. Let $c : \ell^2(\mathbf{Z}^+) \to \ell^2(\mathbf{Z}^+)$ be the bounded operator given by $(cf)(n) = f(n+1)$ for all $n \geq 0$, and let $d = c^*$. Then $cd = e$ but $dc \neq e$. One has

$$p(x) := x^2e - ax + e = (xe - c)(xe - d)$$

for all $x \in \mathbf{C}$, where $a = c + d$ is self-adjoint with $\operatorname{spec}(a) = [-2, 2]$. The spectrum of p is $\{z : |z| = 1\}$ but c, d both have spectra equal to $\{z : |z| \leq 1\}$, so

$$\operatorname{spec}(p) \neq \operatorname{spec}(c) \cup \operatorname{spec}(d).$$

However, there exists a different factorization

$$p(x) = (xe - u)(xe - u^*)$$

where u is unitary and u, u^*, a all commute. For this factorization

$$\operatorname{spec}(p) = \operatorname{spec}(u) = \operatorname{spec}(u^*) = \{z : |z| = 1\}.$$

Given an algebra $\mathcal{A}$, let $\mathcal{A}_m$ denote the algebra of all $m \times m$ matrices with entries in $\mathcal{A}$, and let e_m denote the identity element in $\mathcal{A}_m$.

LEMMA 31. *Let* $p(x) = \sum_{r=0}^{m} x^r a_r$ *where* $x \in \mathbf{F}$, $a_r \in \mathcal{A}$ *for all* r *and* $a_m = e$. *Then* x *lies in* $\operatorname{spec}(p)$ *if and only if it lies in the spectrum of the matrix* $X \in \mathcal{A}_m$ *defined by*

$$X_{r,s} = \begin{cases} 1 & \text{if } s = r + 1, \\ -a_{s-1} & \text{if } r = m, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This follows directly from the standard identity

$$G(x)(xe_m - X) = \begin{pmatrix} p(x) & 0 \\ 0 & e_{m-1} \end{pmatrix} H(x) \tag{7}$$

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in which $G(x), H(x) \in \mathcal{A}_m$ are invertible for all $x \in \mathbf{F}$. If $m = 4$ they are given by

$$G(x) = \begin{pmatrix} x^3e + x^2a_3 + xa_2 + a_1 & x^2e + xa_3 + a_2 & xe + a_3 & e \\ -e & 0 & 0 & 0 \\ 0 & -e & 0 & 0 \\ 0 & 0 & -e & 0 \end{pmatrix}$$

and

$$H(x) = \begin{pmatrix} e & 0 & 0 & 0 \\ -xe & e & 0 & 0 \\ 0 & -xe & e & 0 \\ 0 & 0 & -xe & e \end{pmatrix},$$

from which the general formula can be inferred. □

Our next two results will be used in the proof of Theorem 34.

LEMMA 32 (Euclid). *Let*

$$p(x) = \sum_{r=0}^{m} a_r x^r$$

for all $x \in \mathbf{F}$, where $a_r \in \mathcal{A}$ for all r and $a_m = e$. Given $d \in \mathcal{A}$ put

$$q(x) = \sum_{r=0}^{m-1} b_r x^r,$$

where b_r are defined inductively by $b_{m-1} = e$ and

$$\begin{aligned} a_{m-1} &= b_{m-2} - b_{m-1}d, \\ a_{m-2} &= b_{m-3} - b_{m-2}d, \\ &\vdots \\ a_1 &= b_0 - b_1 d. \end{aligned}$$

Then

$$p(x) = q(x)(xe - d) + \text{rem}$$

for all $x \in \mathbf{F}$, where

$$\text{rem} = a_0 + b_0 d = \sum_{r=0}^{m} a_r d^r.$$

The proof involves simple substitutions, as does that of the corollary below. We continue with the notation of Lemma 32.

COROLLARY 33. *Let $\mathcal{A}$ be an algebra of linear operators on some vector space V. Let $\mu \in \mathbf{F}$ and $v \in V$. If $dv = \mu v$ and $p(\mu)v = 0$ then*

$$p(x)v = q(x)(xe - d)v$$

for all $x \in \mathbf{F}$.

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From this point on we assume that $\mathcal{A}$ is the algebra of all $n \times n$ matrices over a field $\mathbf{F}$ and that $P(x) = \sum_{r=0}^{m} x^r A_r$ where A_m is the identity matrix I. The following theorem may be found in [18] and [16, Theorem 3.21].

THEOREM 34. *If the polynomial* $\det(P(x))$ *has* mn *distinct roots in* $\mathbf{F}$ *then there exist* $C_1, \ldots, C_m \in \mathcal{A}$ *such that*

$$P(x) = (xI - C_1)(xI - C_2) \cdots (xI - C_m)$$

for all $x \in \mathbf{F}$.

Proof. By calculating the determinants of every matrix in (7), one sees that the hypothesis is equivalent to the assumption that the $mn \times mn$ matrix X has mn distinct eigenvalues $\lambda_1, \ldots, \lambda_{mn}$. The corresponding (right, column) eigenvectors $w_r \in \mathbf{F}^{mn}$ of X form a basis in $\mathbf{F}^{mn}$.

If we write the eigenvector w_r in the form $(u_{1,r}, \ldots, u_{m,r})'$, where each $u_r \in \mathbf{F}^n$, then (7) implies that

$$\begin{pmatrix} P(\lambda_r) & 0 \\ 0 & e_{m-1} \end{pmatrix} H(\lambda_r) w_r = 0,$$

and so

$$P(\lambda_r) u_{1,r} = 0 \quad \text{for all } r \in \{1, \ldots, mn\}.$$

If $u \in \mathbf{F}^n$ then using the expansion

$$\begin{pmatrix} u \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \sum_{r=1}^{mn} \alpha_r \begin{pmatrix} u_{1,r} \\ u_{2,r} \\ \vdots \\ u_{m,r} \end{pmatrix}$$

we deduce that $u = \sum_{r=1}^{mn} \alpha_r u_{1,r}$. Therefore $\{u_{1,r}\}_{r=1}^{mn}$ spans $\mathbf{F}^n$. This sequence must contain a basis of $\mathbf{F}^n$ and, after relabelling, we may donate it by $\{u_{1,r}\}_{r=1}^{n}$.

Let C_m denote the matrix such that $C_m u_{1,r} = \lambda_r u_{1,r}$ for all r such that $1 \le r \le n$. Corollary 33 implies that

$$P(x) = Q(x)(xI - C_m) u_{1,r}$$

for all $1 \le r \le n$. Therefore

$$P(x) = Q(x)(xI - C_m).$$

The proof is completed by an induction, noting that $Q(x)$ satisfies the same hypotheses as $P(x)$, but with m replaced by $m - 1$. □

Note that a monic quadratic polynomial may have many different factorizations; see [16, Corollary 3.22] for an explicit lower bound on the number.

Example 35. Define the diagonal $n \times n$ matrix B by $B_{r,s} = \pm \delta_{r,s} b_r$ where the sign choices are independent and $b_r \in \mathbf{F}$ are arbitrary. The quadratic pencil $p(x) = x^2 I - B^2$ does not depend on the sign choices, and it has the factorization $p(x) = (xI - B)(xI + B)$. We conclude that generically $p(x)$ has at least 2^n different factorizations, unless the field $\mathbf{F}$ has characteristic 2.

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§8. *Quadratic equations in rings.* In this section we study quadratic polynomials in which the variable s or x lies in a (generically non-commutative) ring $\mathcal{R}$ rather than in some base field. If $a, b, c, d \in \mathcal{R}$ the algebraic Riccati equation

$$sds + as + sb + c = 0 \tag{8}$$

is of great importance in a variety of contexts, but particularly in optimal control theory; see [**30**, Ch. 13], where it is assumed that c, d are symmetric real matrices and that $a = b^{\mathrm{T}}$. If d is invertible (which is not generally assumed for the Riccati equation) and we put $x = ds$, then equation (8) may be rewritten in the form $x^2 + ux + xv + w = 0$ where $u = dad^{-1}$, $v = b$ and $w = dc$. We shall study this equation within a general ring $\mathcal{R}$.

THEOREM 36. *Given $u, v, w, u', v', w' \in \mathcal{R}$, the identity*

$$x^2 + ux + xv + w = x^2 + u'x + xv' + w' \tag{9}$$

holds for all $x \in \mathcal{R}$ if and only if

$$w = w' \quad \text{and} \quad u - u' = v' - v \in \mathcal{C}, \tag{10}$$

where $\mathcal{C}$ denotes the centre of $\mathcal{R}$.

Proof. The proof of (9) given (10) is a matter of direct substitution. Conversely, given (9), by putting $x = 0$ and then $x = e$ one obtains $w = w'$ and $u + v = u' + v'$. It only remains to prove that $c := u - u' (= v' - v)$ lies in $\mathcal{C}$. Given what we have just proved, (9) can be reduced to $ux + xv = u'x + xv'$ and then rewritten in the form $(u - u')x = x(v' - v)$. This implies that $cx = xc$ for all $x \in \mathcal{A}$, so $c \in \mathcal{C}$. □

Let $p : \mathcal{R} \to \mathcal{R}$ be a monic quadratic polynomial of the form

$$p(x) = x^2 + ux + xv + w.$$

We define a factorization of p to be an identity of the form

$$p(x) = (x - a)(x - b), \tag{11}$$

valid for all $x \in \mathcal{R}$, where a, b is an ordered pair in $\mathcal{R}$. In general a factorization need not exist, and, if one does, it need not be unique.

Example 37. Let

$$p(x) = x^2 - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x - x \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

for all $x \in \mathcal{A} = M(2, \mathbf{C})$. A direct calculation shows that $p(a) = 0$ for $a = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$. If $p(x) = (x - a)(x - b)$ for some $b \in \mathcal{A}$ and all $x \in \mathcal{A}$ then by putting $x = 0$ one sees that $b = -\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. By evaluating the other coefficients we deduce that such a factorization does not exist.

THEOREM 38. *If the centre $\mathcal{C}$ of $\mathcal{R}$ is a field and p has the factorization (11), then that factorization is unique unless $c := a - b \in \mathcal{C}$, in which case one also has $p(x) = (x - b)(x - a)$ for all $x \in \mathcal{R}$. There is no other factorization.*

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Proof. If $c \in \mathcal{C}$ then it follows immediately that a and b commute and that the second factorization in valid.

Conversely suppose that in addition to (11) one has $p(x) = (x - a')(x - b')$ for all $x \in \mathcal{R}$. On expanding both sides of the two factorizations a direct application of Theorem 36 yields

$$a - a' = b' - b \in \mathcal{C} \quad \text{and} \quad ab = a'b'.$$

If one puts $c = a - a' (\in \mathcal{C})$ then these identities imply that $a' = a - c$, $b' = b + c$ and $c(a - b) - c^2 = 0$. Assuming that the two factorizations are indeed distinct, $c \neq 0$ and multiplying by c^{-1} yields $a - b = c$. Hence $a' = b$ and $b' = a$. □

§9. *The equation* $ax - xb = c$. This section treats a generalization of the Sylvester equation for matrices, conventionally written in the form $AX + XB = C$. The continuous Lyapunov equation for matrices $AX + XA^{\mathrm{T}} + Q = 0$, where the superscript T denotes the matrix transpose, often arises in applied mathematics. We consider the problem at an algebraic level, but the main novelties are that, motivated by the examples described in §10, we do not assume that the algebra is finite-dimensional or that the base field is algebraically closed.

If a, b, c lie in the algebra $M(n, \mathbf{F})$ of $n \times n$ matrices with entries in a field $\mathbf{F}$, the equation $ax - xb = c$ is not always soluble for x within $M(n, \mathbf{F})$. Lemma 48 describes some constraints on the c for which the equation $ax - xa = c$ is soluble. Detailed analyses of the conditions under which a matrix equation in $M(n, \mathbf{C})$ of the form $ax - xb = c$ is soluble are given in [**16**, Ch. S2] and [**3, 13, 24, 30**]. Our own discussion considers the case in which a, b, c, x all lie in an algebra $\mathcal{A}$ over a general field $\mathbf{F}$.

LEMMA 39. *Given* $a, b \in \mathcal{A}$ *the following are equivalent.*

(1) a, b *have relatively prime minimum polynomials* p, q.
(2) *There exists* $f \in \mathcal{P}$ *such that* $f(a) = e$ *and* $f(b) = 0$.

Proof. We first show that (1) ⇒ (2). There exist $h, k \in \mathcal{P}$ such that $ph + qk = 1$. If we put $f = qk$ then $f(b) = q(b)k(b) = 0$ and $f(a) = e - p(a)h(a) = e$.

To prove that (2) ⇒ (1), observe that the identity $f(a) = e$ implies $f \neq 0$ as an element of $\mathcal{P}$, and $f(b) = 0$ then implies that b is algebraic with a minimum polynomial q; moreover, there exists $k \in \mathcal{P}$ such that $f = qk$. Putting $g = 1 - f$, a similar argument implies that a is algebraic with a minimum polynomial p and that $g = ph$ for some $h \in \mathcal{P}$. Since $ph + qk = 1$, p and q are relatively prime. □

We say that $a, b \in \mathcal{A}$ are *spectrally disjoint* if they are both algebraic and the conditions of Lemma 39 hold.

THEOREM 40. *If* $a, b \in \mathcal{A}$ *are spectrally disjoint then* $ax - xb = c$ *has a unique solution* $x \in \mathcal{A}$ *for all* $c \in \mathcal{A}$.

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Proof. Let $\mathcal{B}$ be the algebra of all linear maps from $\mathcal{A}$ to $\mathcal{A}$. Define $A, B \in \mathcal{B}$ by $A(x) = ax$ and $B(x) = xb$. If $f \in \mathcal{P}$ is defined as in Lemma 39 then $f(A)(x) = f(a)x = x$ and $f(B)(x) = xf(b) = 0$ for all $x \in \mathcal{A}$. Therefore $f(A) = I$ and $f(B) = 0$ as elements of $\mathcal{B}$. Since A and B commute we have

$$I = f(A) - f(B) = (A - B)g(A, B) = g(A, B)(A - B) \tag{12}$$

where g is a polynomial in two commuting variables; this may be proved by using the particular case

$$A^n - B^n = (A - B)(A^{n-1} + A^{n-2}B \cdots + AB^{n-2} + B^{n-1}).$$

Equation (12) implies that $A - B$ is invertible as an element of $\mathcal{B}$, or equivalently one-to-one and onto as a linear transformation on $\mathcal{A}$. □

An Banach space version of the above theorem and its corollary below may be based on the following theorem; see also [**22**].

THEOREM 41. *Let A and B be two commuting operators on the complex Banach space $\mathcal{B}$ and let f be an analytic function on a region U containing $S = \mathrm{spec}(A) \cup \mathrm{spec}(B)$. If γ is a closed curve in U that winds once around S then*

$$f(A) - f(B) = (A - B)g(A, B)$$

where the bounded operator $g(A, B)$ is defined by

$$g(A, B) = \frac{1}{2\pi i} \int_\gamma f(z)R(z, A)R(z, B)\, dz.$$

Proof. This follows by combining the holomorphic functional calculus [**8**, Theorem 1.5.1] with the identity

$$R(z, A) - R(z, B) = (A - B)R(z, A)R(z, B). \qquad \square$$

COROLLARY 42. *Let $\mathcal{J}$ be an ideal in $\mathcal{A}$. Under the assumptions of Theorem 40 one has $x \in \mathcal{J}$ if and only if $c \in \mathcal{J}$.*

Proof. The simplest proof involves applying Theorem 40 to the relevant elements of $\mathcal{A}/\mathcal{J}$. The following is a more constructive approach. The statement $(x \in \mathcal{J}) \Rightarrow (c \in \mathcal{J})$ follows directly from the defining properties of an ideal. Conversely (12) implies

$$x = (A - B)^{-1}c = g(A, B)c.$$

Since A and B map $\mathcal{J}$ into $\mathcal{J}$, one immediately concludes that $c \in \mathcal{J}$ implies $x \in \mathcal{J}$. □

The following is a more general version of Theorem 40.

THEOREM 43. *If $a, b \in \mathcal{A}$ commute and are spectrally disjoint then $a - b$ is invertible.*

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Proof. Let $f \in \mathcal{P}$ satisfy $f(a) = e$ and $f(b) = 0$. There exists a polynomial g in two commuting variables x, y such that

$$f(x) - f(y) = (x - y)g(x, y) \quad \text{for all } x, y \in \mathbf{F}.$$

This implies that

$$e = f(a) - f(b) = (b - a)g(a, b)$$

and so $a - b$ is invertible with inverse $g(a, b)$. □

Example 44. Let V denote the vector space of all $m \times n$ matrices with entries in the field $\mathbf{F}$. Given $c \in V$, $a \in M(m, \mathbf{F})$ and $b \in M(n, \mathbf{F})$, one might wish to solve the equation $ax - xb = c$ for $x \in V$. This problem may be analysed by using Theorem 43, where $\mathcal{A}$ is taken to be the algebra of all linear maps from V to V. If $L, R \in \mathcal{A}$ are defined by $L(v) = av$ and $R(v) = vb$ then one sees immediately that L and R commute. Moreover, L has the same spectrum and minimum polynomial as a, while R has the same spectrum and minimum polynomial as b. Therefore the equation $ax - xb = c$ is uniquely soluble if a and b are spectrally disjoint.

In order to apply Theorem 40 or 43 in a purely algebraic context we need two lemmas. We define $\mathcal{P}^*$ to be the ring of polynomials with entries in the algebraic closure $\mathbf{F}^*$ of $\mathbf{F}$. We put $\mathcal{A}^* = \mathcal{A} \otimes \mathbf{F}^*$ and identify $a \in \mathcal{A}$ with $a \otimes 1 \in \mathcal{A}^*$ where 1 is the multiplicative identity in $\mathbf{F}^*$. We define $\mathcal{B}$ to be the algebra of all $\mathbf{F}$-linear operators from $\mathcal{A}$ to $\mathcal{A}$ and $\mathcal{B}^*$ to be the algebra of all $\mathbf{F}^*$-linear operators from $\mathcal{A}^*$ to $\mathcal{A}^*$.

LEMMA 45. *Let $\mathcal{A}$ be an algebra over $\mathbf{F}$, and let $a \neq 0$ be an algebraic element of $\mathcal{A}$. The following four polynomials coincide:*

(1) *the minimum polynomial p_1 of $a \in \mathcal{A}$;*
(2) *the minimum polynomial p_2 of $L_a \in \mathcal{B}$;*
(3) *the minimum polynomial p_3 of $a \in \mathcal{A}^*$;*
(4) *the minimum polynomial p_4 of $L_a \in \mathcal{B}^*$.*

Proof. The equivalence of items (1) and (2) uses the fact that the homomorphism $a \to L_a$ is one-to-one. Similarly for items (3) and (4). We prove the equivalence of items (1) and (3) by using the fact that $\mathcal{P}$ and $\mathcal{P}^*$ are principal ideal domains.

Since $p_1(a) = 0$ in $\mathcal{A}$, this also holds in $\mathcal{A}^*$, so p_3 is a factor of p_1. Since both are monic, in order to establish that they are equal we need only prove that $\deg(p_3) \geq \deg(p_1)$. Let P be a projection on $\mathbf{F}^*$ with range $\mathbf{F}$, where $\mathbf{F}^*$ is regarded as a vector space over $\mathbf{F}$. The polynomial $p_5 = P(p_3) \in \mathcal{P}$ is monic with $p_5(a) = 0$. Therefore p_1 is a factor of p_5 and $\deg(p_3) = \deg(p_5) \geq \deg(p_1)$. □

LEMMA 46. *Let $\mathcal{A}$ be an algebra over a field $\mathbf{F}$, and let a and b be two algebraic elements of $\mathcal{A}$. If p and q are their minimum polynomials then the following are equivalent.*

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(1) *p and q are relatively prime in $\mathcal{P}$.*
(2) *p and q are relatively prime in $\mathcal{P}^*$.*
(3) *L_a and L_b have disjoint spectra regarded as operators on $\mathcal{A}^*$.*

Proof. To see that (1) $\Rightarrow$ (2), observe that the existence of an identity $hp + kq = 1$ in $\mathcal{P}$ implies that the same identity holds in $\mathcal{P}^*$.

We now show that (2) $\Rightarrow$ (1). Since p and q are relatively prime in $\mathcal{P}^*$, there exist $h^*, k^* \in \mathcal{P}^*$ such that $h^*p + k^*q = 1$. Defining the projection P as in the proof of Lemma 45 and putting $h = P(h^*)$, $k = P(k^*)$ we obtain $hp + kq = 1$; hence p and q are relatively prime in $\mathcal{P}$.

Finally we prove (2) $\Leftrightarrow$ (3). It follows from Theorem 4 that both statements are equivalent to p and q having no common zeros. □

Example 47. Let $\mathcal{Q}$ denote the algebra of quaternions over a base field $\mathbf{F}$. If

$$a = a_0 + a_1 i + a_2 j + a_3 k$$

then the minimum polynomial of a is

$$p(z) = z^2 - 2a_0 z + (a_0^2 + a_1^2 + a_2^2 + a_3^2).$$

The field $\mathbf{F}$ is said to be formally real if, whenever $a_r \in \mathbf{F}$ for $1 \leq r \leq n$ and $\sum_{r=1}^n a_r^2 = 0$, one has $a_r = 0$ for all $\leq r \leq n$. Examples are $\mathbf{R}$ and the field of all rational functions in one variable whose coefficients are all real.

If $\mathbf{F}$ is formally real then $\mathcal{Q}$ is a division algebra (skew field). In this case $\operatorname{spec}(a) = \emptyset$ unless $a_1 = a_2 = a_3 = 0$, in which case $\operatorname{spec}(a) = a_0$. Given $a, b \in \mathcal{Q}$ the equation $ax - xb = c$ has a unique solution $x \in \mathcal{Q}$ for all $c \in \mathcal{A}$ if and only if either $a_0 \neq b_0$ or $a_1^2 + a_2^2 + a_3^2 \neq b_1^2 + b_2^2 + b_3^2$.

If $a = b$ then Theorem 40 cannot be applied. The next two results provide some insight into this case. A trace on an algebra $\mathcal{A}$ is by definition a non-zero linear functional $\operatorname{tr} : \mathcal{A} \to \mathbf{F}$ such that $\operatorname{tr}(ab) = \operatorname{tr}(ba)$ for all $a, b \in \mathcal{A}$. Important infinite-dimensional examples arise in the theory of finite von Neumann algebras and free probability theory [**27, 28**], as well as Example 3.

LEMMA 48. *Let $\mathcal{A}$ be an algebra with a trace* tr. *If $a, x, c \in \mathcal{A}$ and $ax - xa = c$ then c satisfies the constraints $\operatorname{tr}(a^m c) = 0$ for all $m \geq 0$.*

Proof. One takes the trace of both sides of the identity

$$[a^{m+1}, x] = a^m[a, x] + a^{m-1}[a, x]a + \cdots + a[a, x]a^{m-1} + [a, x]a^m. \quad \square$$

The constraints on c in Lemma 48 are not necessarily independent for different m.

THEOREM 49. *Let $\mathcal{A} = M(n, \mathbf{F})$ and let d be the dimension of the set of C for which $AX - XA = C$ is soluble. Then $0 \leq d \leq n^2 - n$. The case $d = 0$ occurs if $A = I$, while the case $d = n^2 - n$ occurs if A is the $n \times n$ elementary*

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Jordan matrix $J_{n,\lambda}$ defined by

$$J_{n,\lambda,r,s} = \begin{cases} \lambda & \text{if } s = r, \\ 1 & \text{if } s = r+1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We define $L : \mathcal{A} \to \mathcal{A}$ by $L(X) = AX - XA$. Since the rank of a linear transformation does not alter if one increases the base field, we may assume that $\mathbf{F}$ is algebraically closed. We have to prove that

$$\dim(\ker(L)) \geq n,$$

with equality in the stated case. We choose a basis of $\mathbf{F}^n$ for which A is a block diagonal matrix whose blocks are elementary Jordan matrices. More precisely, we assume that the diagonal blocks are J_{n_s,λ_s} where $1 \leq s \leq S$, so that $\sum_{s=1}^{S} n_s = n$. Since $\ker(L)$ contains the space of all diagonal block matrices X such that $XA = AX$, we need only prove that the space V_s of $n_s \times n_s$ matrices X_s satisfying

$$X_s J_{n_s,\lambda_s} = J_{n_s,\lambda_s} X_s$$

has $\dim(V_s) = n_s$. A direct calculation shows that V_s is the space of $n_s \times n_s$ upper triangular Toeplitz matrices, which has the required dimension. □

§10. *Fields of analytic functions.* The ring $\mathbf{P}$ of all polynomials with complex coefficients is an integral domain, and its quotient field is the space $\mathbf{Q}$ of all rational functions on $\mathbf{C}$. $\mathbf{Q}$ is not algebraically closed. If $a, b, c \in \mathbf{Q}$ and $a \neq 0$ then the quadratic equation $ax^2 + bx + c = 0$ has a solution in $\mathbf{Q}$ if and only if the discriminant $\Delta = b^2 - 4ac$ has a square root in $\mathbf{Q}$. This holds if and only if every zero and pole of f has even order.

The elements of $\mathbf{Q}$ have a finite number of poles and so, strictly speaking, are not functions. The proper description involves the theory of germs of analytic functions. To define these we start with expressions of the form

$$f(z) = \sum_{r=0}^{\infty} a_r z^{m+r} \tag{13}$$

where $m \in \mathbf{Z}$ and $a_0 \neq 0$ (unless f is identically zero); we assume that the series converges in some deleted neighbourhood $U = \{z : 0 < |z| < r\}$ of 0. We identify two "germs" (f, U) and (g, V) if $g = f$ on $U \cap V$, and define the sum and product of two germs by pointwise operations, excluding the origin. The set of such germs is a field $\mathbf{F}_1$ that contains $\mathbf{Q}$. The series in (13) has a radius of convergence $R \in (0, \infty]$, so the corresponding germ has a canonical representative in which one puts $U = \{z : 0 < |z| < R\}$. The ring $\mathbf{E}$ of entire functions on $\mathbf{C}$ is contained in $\mathbf{F}_1$, and the factorization theorem of Weierstrass, [**19**, §13.3], implies that its quotient field, also contained in $\mathbf{F}_1$, is the field of (germs of) meromorphic functions.

The roots of a polynomial with coefficients in $\mathbf{Q}$ may be described algebraically or geometrically, and the two perspectives are significantly different. Elements of the algebraic closure of $\mathbf{F}$ may be associated with a single-valued

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branch of a multi-valued analytic function, but geometrically speaking the natural entity is the Riemann surface associated with the multi-valued function. The following description presents the elements of the algebraic closure $\mathbf{F}_2$ of $\mathbf{Q}$ as analytic functions on a particular domain in $\mathbf{C}$; this domain is far from unique and assigns the origin in $\mathbf{C}$ a status that it does not deserve, but it has the merit of being simple and explicit.

We define $\mathbf{G} \supset \mathbf{F}_2$ to consist of all pairs (f, U) where f is an analytic function on a region U of the form

$$U = \mathbf{C} \Big\backslash \bigcup_r S(w_r), \tag{14}$$

where each $S(w_r)$ is a semi-infinite ray starting at w_r and the number of rays is finite or countable; in the second case it is assumed that $|w_r| \to \infty$ as $r \to \infty$. If $w = 0$ we define $S(w) = [0, \infty)$, and for all other w we define $S(w) = w[1, \infty)$. Each such "star-shaped" region U is dense in $\mathbf{C}$ and simply connected, and the intersection of any finite number of such U is of the same form.

The sum of (f, U) and (g, V) is defined to be the pointwise sum of f and g with domain $U \cap V$, with a similar definition for the product. We again identify (f, U) and (g, V) as elements of $\mathbf{G}$ if $f = g$ on $U \cap V$. This turns $\mathbf{G}$ into an integral domain, which has the following important property.

THEOREM 50. *Let*

$$p(z, w) = \sum_{r=0}^{n} a_r(w) z^r$$

where a_r are entire or meromorphic functions of $w \in \mathbf{C}$ and $a_n = 1$. Then

$$p(z, w) = \prod_{r=1}^{n} (z - b_r(w))$$

where each $b_r \in \mathbf{G}$.

Proof. According to [25, Theorem IV.14.2] the roots of the polynomial $z \to p(z, w)$ are branches of analytic functions of w whose singularities are all algebraic, and the singularities form a discrete set in $\mathbf{C}$. If we denote these by w_r then each root has one or more single-valued branches on the set obtained from $\mathbf{C}$ by removing the rays through all w_r. □

Example 51. The solutions of the equation

$$z^3 - (1 - w^2) = 0$$

are the three functions

$$f_r(w) = \alpha^r (1 - w^2)^{1/3}$$

where $r = 0, 1, 2$ and $\alpha = e^{2\pi i/3}$. These are different branches of the multi-valued cube root. One may define a single-valued branch of $(1 - w^2)^{1/3}$ on

$$U = \mathbf{C} \backslash ([1, +\infty) \cup (-\infty, -1])$$

by analytically continuing the obvious power series from the unit ball.

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Similar ideas may be expressed in a more algebraic form. The set of formal power series $f(z) = \sum_{r=0}^{\infty} a_r z^r$, with no convergence assumption and the obvious "pointwise" definitions of addition and multiplication, is an integral domain $\mathbf{D}$. Although this is not closed under fractional powers, the quantity f^α has an obvious expansion as a formal power series in $\mathbf{D}$ with the expected properties for all $\alpha \in \mathbf{C}$ provided $a_0 \neq 0$, as do $\exp(f)$ and many other standard functions of f. In particular, $f \in \mathbf{D}$ is invertible within $\mathbf{D}$ if and only if $a_0 \neq 0$. The quotient field of $\mathbf{D}$ may be identified with the set of formal series $\sum_{r=0}^{\infty} a_r z^{m+r}$ where $m \in \mathbf{Z}$.

One may wish to solve a matrix problem in which the matrix entries are polynomials whose degree is at most N, where N is specified. This amounts to assuming that the variable z in the polynomial is small enough that higher-order times can be neglected. Such problems can often be solved rigorously by first considering the problem in the algebra of $n \times n$ matrices with entries in $\mathbf{D}$, and then passing to the quotient algebra with respect to the two-sided ideal consisting of all $n \times n$ matrices whose entries lie in $\mathbf{K} = \sum_{n=0}^{\infty} a_n z^{n+N+1}$. This ideal is not maximal, and the space of polynomials with degree at most N is not an integral domain or a field.

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