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Convex structures revisited

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Abstract. We provide a complete characterization of extreme points of the space of sofic representations. We also show that the restriction map $\mathrm{Sof}(G, P^{\omega})$ to $\mathrm{Sof}(H, P^{\omega})$, where $H \subset G$, is not always surjective. The first part of the paper is a continuation of Păunescu [A convex structure on sofic embeddings. *Ergod. Th. & Dynam. Sys.* **34**(4) (2014), 1343–1352] and follows more closely the plan of Brown [Topological dynamical systems associated to II_1-factors. *Adv. Math.* **227**(4) (2011), 1665–1699].

1. *Introduction*

The first goal of this article is to review the results in [**Pă2**] and to obtain a complete characterization of extreme points, similar to the one in [**Br**]. There were two obstacles preventing us from obtaining a complete characterization.

Firstly, amplifications were a major issue. We managed to prove a link between extreme points and ergodicity of the commutant largely due to [**Pă2**, Proposition 2.8]. It proved that ergodicity is preserved under taking amplifications. In order to get an 'if and only if' statement, one would need a converse to this statement. Sadly, this kind of question proved to be rather hard to settle. Also, in [**Br**], there was no need to ask these questions, due to the diffuse nature of the hyperfinite factor, as opposed to our finite dimensional object P_n (permutation matrices). This problem will be solved by considering 'type II_1' permutations, i.e. the full group of an amenable type II_1 equivalence relation.

Secondly, it was hard to construct elements in the commutant of a sofic representation. Much to my surprise, there are sofic representations that act ergodically on the Loeb space (the sofic representation itself is ergodic, not its commutant). Such sofic representations are necessary extreme points and it seems that there are no tools to construct enough elements in the commutant to get ergodicity. This problem can only be solved by restricting the Loeb space to the commutant of the sofic representation.

Note, however, that the result in [**Pă2**, Theorem 2.10] is still useful in that form. Ergodicity of the commutant of a sofic representation is a question that appears in the study of sofic entropy, see [**Ke-Li**].

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Throughout the article ω denotes, as usual, a free ultrafilter on $\mathbb{N}$. We work with $M_n(\mathbb{C})$, the algebra of matrices in dimension n and its special subsets P_n, the subgroup of permutation matrices, and D_n, the subalgebra of diagonal matrices. This time we also need R, the unique hyperfinite type II_1 factor. We assume familiarity with ultraproducts of finite von Neumann algebras, there are many places in the literature where the construction can be checked.

1.1. *The convex structure on* $\mathrm{Hom}(N, R^\omega)$. Let us recall the construction from [**Br**]. Let N be a separable type II_1 factor. The following theorem is a fruitful result in the theory of type II_1 factors:

THEOREM 1.1. *The factor N is the hyperfinite factor if and only if any two unital $*$-homomorphisms $\pi, \rho : N \to R^\omega$ are unitary conjugate, i.e. there exists $u \in \mathcal{U}(R^\omega)$ such that $\pi(a) = u\rho(a)u^*$ for all $a \in N$.*

The direct implication is classic, while the converse is a much more recent result due to Jung, [**Ju**]. The question is now what happens outside of the hyperfinite world? One can always consider the set

$$\mathrm{Hom}(N, R^\omega) = \{\pi : N \to R^\omega : \text{unital } *\text{-homomorphism}\}/\sim,$$

where $\sim$ is unitary conjugacy defined as in the statement of Theorem 1.1.

This space has a natural topology given by pointwise convergence in the weak topology of R^ω. Due to the separability of N this turns out to be a metrizable topology. Ozawa showed in the appendix of [**Br**] that, for non-hyperfinite N, the space $\mathrm{Hom}(N, R^\omega)$ is non-separable (given that N satisfies Connes' Embedding Conjecture, i.e. $\mathrm{Hom}(N, R^\omega)$ is non-empty). This is quite an unfortunate fact, being just another example where the hyperfinite case is completely separated from the rest of the world.

Still, if you want to study the set $\mathrm{Hom}(N, R^\omega)$, the first observation is that the direct sum operation is constructing new elements out of old ones. Let $\pi, \rho : N \to R^\omega$ and consider the direct sum

$$\pi \oplus \rho : N \to R^\omega \oplus R^\omega = (R \oplus R)^\omega.$$

Choose a unital embedding $\theta : R \oplus R \to R$ so that $\theta^\omega : (R \oplus R)^\omega \to R^\omega$. Then

$$\theta^\omega \circ (\pi \oplus \rho) : N \to R^\omega$$

is a unital embedding whose class in $\mathrm{Hom}(N, R^\omega)$ can be different than the classes of π and ρ.

The map $\theta : R \oplus R \to R$ is unital so $\theta(1 \oplus 1) = 1$. Notice that $\theta(1 \oplus 0)$ is a projection in R. Let $\lambda = \mathrm{Tr}(\theta(1 \oplus 0))$. Then $\mathrm{Tr}(\theta(0 \oplus 1))$ must be equal to $1 - \lambda$. Denote by $[\xi]$ the class in $\mathrm{Hom}(N, R^\omega)$ of a unital morphism $\xi : N \to R^\omega$. We set by definition

$$[\theta^\omega \circ (\pi \oplus \rho)] = \lambda[\pi] + (1 - \lambda)[\rho].$$

The term $\lambda[\pi] + (1 - \lambda)[\rho]$ is just a formal writing of the element $[\theta^\omega \circ (\pi \oplus \rho)]$ that we constructed. Of course there are some well-definedness problems to be solved, but nothing more than routine work. The last observation to be made is that one can construct a map θ for any prescribed $\lambda \in [0, 1]$.

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It can be checked that if $[\pi] \neq [\rho]$ and $\lambda \neq 1$ then $\lambda[\pi] + (1-\lambda)[\rho] \neq [\pi]$ as expected. Furthermore, there are some axioms also involving a metric for the topology on $\mathrm{Hom}(N, R^\omega)$ that have to be settled. After these axioms are checked one can deduce due to a result by Capraro and Fritz, [**Ca-Fr**], that $\mathrm{Hom}(N, R^\omega)$ together with its metric and convex structures can be regarded as an honest closed convex subset of a Banach space.

Now we can state the main result of Brown's theory.

THEOREM 1.2. *The class* $[\pi] \in \mathrm{Hom}(N, R^\omega)$ *is an extreme point if and only if the relative commutant* $N' \cap R^\omega$ *is a factor.*

This is a nice result describing the extreme points of this convex structure, but their existence is still an open problem.

1.2. *The space* $\mathrm{Sof}(G, P^\omega)$. In [**Păz2**], we replaced the separable factor N and Connes' Embedding Conjecture by a countable group G and by the sofic property, respectively. We review here the construction of $\mathrm{Sof}(G, P^\omega)$ from that reference but, though this paper is pretty much self-contained, I assume some familiarity with notations and results from [**Păz2**].

We want to study embeddings of the group G into the universal sofic group $\Pi_{k\to\omega} P_{n_k}$ that are 'trace preserving', i.e. the trace of each non-trivial element is zero. We call such morphisms *sofic representations* of G. We first note that we have a similar result to Theorem 1.1 due to Elek and Szabo, [**El-Sz2**].

THEOREM 1.3. *The group* G *is amenable if and only if for any two group morphisms* $\Theta_1, \Theta_2 : G \to \Pi_{k\to\omega} P_{n_k}$ *such that* $\mathrm{Tr}(\Theta_i(g)) = 0$ *for any* $g \neq e$ *and any* $i = 1, 2$, *there exists* $p \in \Pi_{k\to\omega} P_{n_k}$ *such that* $\Theta_2(g) = p\Theta_1(g)p^*$ *for any* $g \in G$.

In order to be able to construct a convex structure on the set of sofic representations, we have to be flexible, considering universal sofic groups over any sequence of dimensions $\{n_k\}_k$ such that $n_k \to \infty$. This brings in certain complications, as we want to compare sofic representations over different universal sofic groups $\Pi_{k\to\omega} P_{n_k}$ and $\Pi_{k\to\omega} P_{m_k}$.

We notice that for a sequence $\{r_k\}_k \subset \mathbb{N}^*$ the universal sofic group $\Pi_{k\to\omega} P_{n_k}$ canonically embeds in $\Pi_{k\to\omega} P_{n_k r_k}$ by tensoring by identity:

$$\Pi_{k\to\omega} P_{n_k} \ni \Pi_{k\to\omega} p_k \to \Pi_{k\to\omega}(p_k \otimes 1_{r_k}) \in \Pi_{k\to\omega} P_{n_k r_k}.$$

Let $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$ be a sofic representation, $\Theta = \Pi_{k\to\omega}\theta_k$. We call an *amplification* of Θ the composition of Θ with the above canonic map:

$$\Theta \otimes 1_{r_k} : G \to \Pi_{k\to\omega} P_{n_k r_k}, \quad \Theta \otimes 1_{r_k}(g) = \Pi_{k\to\omega}\theta_k(g) \otimes 1_{r_k}.$$

The space of sofic representations is now

$$\mathrm{Sof}(G, P^\omega) = \{\Theta : G \to \Pi_{k\to\omega} P_{n_k} : \text{sofic representation } (n_k)_k \subset \mathbb{N}, n_k \to \infty\}/\sim,$$

where $\sim$ is amplifications and conjugacy as in Theorem 1.3. So two sofic representations are equivalent if they have amplifications that are conjugate. For $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$ we denote by $[\Theta]_P$ its class in $\mathrm{Sof}(G, P^\omega)$, to distinguish it from $[\Theta]_{\mathcal{E}}$ that we shall construct.

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2. *Type* II_1 *permutations*

2.1. *The* I_n *case.* One way of discussing sofic objects is by starting with a probability measure preserving equivalence relation. Usually one takes the full equivalence relation on a space with n elements, endowed with the normalized cardinal measure. The Feldman–Moore construction of this equivalence relation is the Cartain pair $D_n \subset M_n(\mathbb{C})$, where D_n is the subalgebra of diagonal matrices. The full group of this type I_n equivalence relation is $\mathrm{Sym}(n)$, the symmetric group. It embeds in M_n as P_n, the subgroup of permutation matrices. If $x \in \mathrm{Sym}(n)$ then the corresponding element in P_n is $\tilde{x}(i, j) = \delta_i^{p(j)}$ or $\tilde{x}$ is the characteristic function of the graph of p^{-1}.

The group P_n is in the normalizer of D_n, i.e. if $p \in P_n$ and $a \in D_n$ then $pap^* \in D_n$. What we have here is just a symmetric group, P_n, acting on a set with n elements in an obvious way. Passing to metric ultraproducts things become more interesting:

$$\Pi_{k\to\omega} P_{n_k} \curvearrowright \Pi_{k\to\omega} D_{n_k}.$$

The group $\Pi_{k\to\omega} P_{n_k}$ was introduced by Elek and Sazbo [**El-Sz1**] and it is called *the universal sofic group*. A countable group is *sofic* if and only if it is a subgroup of this group.

The algebra $\Pi_{k\to\omega} D_{n_k}$ is an abelian von Neumann algebra, isomorphic to $L^\infty(X_\omega, \mu_\omega)$, where (X_ω, μ_ω) is a Loeb measure space, i.e. an ultra product of probability spaces. For its construction see [**Lo**] or [**El-Sze**].

I call the action itself $\Pi_{k\to\omega} P_{n_k} \curvearrowright \Pi_{k\to\omega} D_{n_k}$ *the universal sofic action*. By definition, a standard action is *sofic* if it can be embedded into a universal sofic action.

2.2. *The hyperfinite case.* Instead of the space $\{1, \ldots, n\}$ with the normalized cardinal measure, we consider the unit interval $[0, 1]$ endowed with the Borel measure μ. On this space consider the equivalence relation

$$E = \{(x, y) : x - y \in \mathbb{Q}\}.$$

It is a standard fact that E is a measurable, countable, μ-preserving, ergodic, amenable equivalence relation. A consequence of the famous Connes–Feldman–Weiss theorem, [**CFW**], is the unicity of such an object: there is a unique ergodic, amenable type II_1 equivalence relation corresponding to the unique hyperfinite type II_1 factor.

We denote by $[E]$ *the full group* of E:

$$[E] = \{u : [0, 1] \to [0, 1] : u \text{ Borel bijection}, (x, u(x)) \in E \text{ for } \mu\text{-almost every } x\}.$$

We still have a Hamming distance on $[E]$ defined by $d_H(u, v) = \mu(\{x : u(x) \neq v(x)\})$. The Feldman–Moore construction of E, by definition, consists of some functions from E to $\mathbb{C}$ (we do not go into details here, they are not important):

$$M(E) = \{f : E \to \mathbb{C} : f \text{ is a multiplier}\}.$$

Operations on this algebra are defined as follows:

$$f \cdot g(x, z) = \sum_{yEx} f(x, y) g(y, z);$$

$$f^*(x, y) = \overline{f(y, x)},$$

$$\mathrm{Tr}(f) = \int_X f(x, x)\, d\mu(x).$$

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The Cartain subalgebra of $M(E)$ is composed of those functions with the support on the diagonal:

$$A = \{f : E \to \mathbb{C} : f \in M(E) \text{ and } f(x, y) = 0 \text{ if } x \neq y\}.$$

The algebra A is isomorphic to $L^\infty(X, \mu)$ and $M(E)$ is isomorphic to R, the hyperfinite type II_1 factor. The full group $[E]$ can be embedded in $M(E)$ and we denote its image by $\mathcal{E}$ (in the type I_n the full group was $\mathrm{Sym}(n)$ and its image in the Feldman–Moore construction was P_n). The group $\mathcal{E}$ is composed of those functions $f : E \to \mathbb{C}$ that have exactly one entry of one on each row and column:

$$\mathcal{E} = \{f \in M(E) : f(E) = \{0, 1\} \text{ and } \forall x\ \exists! y\ f(x, y) = 1 \text{ and } \forall y\ \exists! x\ f(x, y) = 1\}.$$

If $u \in [E]$ denote $\tilde{u} = \chi_{\mathrm{graph}(u^{-1})} \in \mathcal{E}$. The formula $d_H(u, v) = 1 - \mathrm{Tr}(\tilde{u}\tilde{v}^*)$ can be easily checked. Now we have $\mathcal{E}$ acting on A inside $M(E)$, to replace the old picture of P_n action on D_n inside $M_n(\mathbb{C})$. Passing to metric ultraproducts we get the diffuse universal sofic picture:

$$\mathcal{E}^\omega \curvearrowright A^\omega.$$

This is still a universal sofic action and $\mathcal{E}^\omega$ is still a universal sofic group. The benefit of these objects is that we no longer need amplifications in order to compare two sofic representations.

2.3. *The limit of symmetric groups.* As we have said, the group $\mathcal{E}$ is the type II_1 analogue of P_n. The next theorem will strengthen this idea. This result is stated in [**Pe-Kw**, p. 18, Example 5.5].

Let n, r be two natural numbers. Then $f_{n,r} : M_n \to M_{nr}$ defined by $f_{n,r}(x) = x \otimes 1_r$ is a trace preserving embedding. Moreover $f_{n,r}(P_n) \subset P_{nr}$ and $f_{n,r}(D_n) \subset D_{nr}$.

As $d_H(p, q) = 1 - \mathrm{Tr}(pq^*)$, the map $f_{n,r}$ restricted to P_n preserves the Hamming distance. We construct the direct limit of metric groups $\varinjlim(P_n, d_H)$ by taking the completion in the Hamming distance of the algebraic limit of the directed system $(P_n, f_{n,r})$. The same construction is available for (D_n, Tr).

THEOREM 2.1. *The direct limit* $\varinjlim(P_n, d_H)$ *is isomorphic to* $([E], d_H)$.

Proof. For $n \in \mathbb{N}$ embed P_n into $[E]$ by dividing the interval $[0, 1]$ into n equal parts and permuting these small intervals. The formula will look something like

$$\Phi_n : P_n \to [E], \quad \Phi_n(p) = x \to \frac{p(\lfloor nx \rfloor) + \{nx\}}{n},$$

where $\lfloor y \rfloor, \{y\}$ are the integer and fractional part of y and permutations $p \in P_n$ act on the set $\{0, 1, \ldots, n-1\}$. It is easy to check that the maps Φ_n preserve the metrics and they are compatible with the directed system. As $[E]$ is complete under d_H, it follows that we have an embedding $\Phi : \varinjlim P_n \to [E]$. To show that this map is surjective we need to check that $\bigcup_n \Phi_n(P_n)$ is dense in $[E]$.

For this, let $\phi \in [E]$ and $\varepsilon > 0$. For $q \in \mathbb{Q} \cap [0, 1)$ let

$$A_q = \{x \in [0, 1] : \phi(x) = x + q \pmod 1\}.$$

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For the rest of this proof addition with numbers in [0, 1) is always considered mod 1. The collection $\{A_q : q \in \mathbb{Q} \cap [0, 1)\}$ is a partition of [0, 1] and we can choose $\{q_1, \ldots, q_k\}$, a finite set, such that $\sum_{i=1}^{k} \mu(A_{q_i}) > 1 - \varepsilon$. To simplify notation, denote A_{q_i} by A_i. Note that both $\{A_i\}_i$ and $\{A_i + q_i\}_i$ are collections of pairwise disjoint sets.

Let $\mathcal{B}_n$ be the σ-*algebra* generated by the sets $[j/n, (j+1)/n)$, where $j = 0, \ldots, (n-1)$, such that elements in $\Phi_n(P_n)$ are measurable as functions from $([0, 1], \mathcal{B}_n)$ to $([0, 1], \mathcal{B}_n)$. We want to construct sets $B_i \in \mathcal{B}_n$, $i = 1, \ldots, k$, for some n, such that $\mu(A_i \Delta B_i)$ is small and $\{B_i\}_i$ and $\{B_i + q_i\}_i$ are also collections of disjoint sets.

Using the regularity properties of the Borel measure, we can find a sufficiently large n and sets $B_i' \in \mathcal{B}_n$ such that $\mu(A_i \Delta B_i') < \varepsilon/k^2$. As sets $\{A_i\}$ are disjoint it follows that $\mu(B_i' \cap B_j') < 2\varepsilon/k^2$ for $i \neq j$. Analogously, $\mu((B_i' + q_i) \cap (B_j' + q_j)) < 2\varepsilon/k^2$. Take out overlapping intervals to get disjoint sets $B_i \in \mathcal{B}_n$ such that $\{B_i + q_i\}_i$ are also disjoint and $\mu(A_i \Delta B_i) < 4\varepsilon/k$.

On increasing n, we can assume that each q_i is an integer multiple of $1/n$. Then there exists an element $\psi \in \Phi_n(P_n)$ such that $\psi(x) = x + q_i$ if $x \in B_i$. Then $\psi = \phi$ on $C = \bigcup_{i=1}^{k}(A_i \cap B_i)$. But $\mu(C) > \sum_{i=1}^{k}(\mu(A_i) - 2\varepsilon/k) > 1 - 3\varepsilon$.

It follows that the distance between ϕ and ψ is smaller than 3ε and we have finished. □

Actually, much more is true. By dividing the interval [0, 1] into n equal parts we can construct a trace preserving embedding $\Psi_n : M_n(\mathbb{C}) \to M(E)$. Then $\Psi_n(D_n) \subset A$ and $\Psi_n(P_n) \subset \mathcal{E}$, the latter being actually just Φ_n from the proof above composed with the canonical isomorphism between $[E]$ and $\mathcal{E}$. The maps $(\Psi_n)_n$ are also constructing isomorphisms $\varinjlim D_n \simeq A$ and $\varinjlim M_n \simeq M(E)$.

In this article we construct sofic representations in $\mathcal{E}^\omega$, but, due to the last theorem, most of the time we still deal just with $\Pi_{k\to\omega} P_{n_k}$. This is good for some definitions and for developing the intuition. So the next notation and theorem are quite important for the technical part of the article.

Notation 2.2. For a fixed sequence $\{n_k\}$ construct $\Psi : \Pi_{k\to\omega} M_{n_k} \to M(E)^\omega$ defined by $\Psi(\Pi_{k\to\omega} x_k) = \Pi_{k\to\omega} \Psi_{n_k}(x_k)$.

By construction, $\Psi(\Pi_{k\to\omega} D_{n_k}) \subset A^\omega$ and $\Psi(\Pi_{k\to\omega} P_{n_k}) \subset \mathcal{E}^\omega$. Also note that Ψ is trace preserving, in particular it is injective for a fixed sequence $\{n_k\}$.

THEOREM 2.3. *Let $\{u_i\}_{i\in\mathbb{N}}$ be a countable set of elements in $\mathcal{E}^\omega$ and $\{a_i\}_{i\in\mathbb{N}}$ be a countable set of elements in A^ω. Then there exists a sequence $\{n_k\}_k$ and elements $v_i \in \Pi_{k\to\omega} P_{n_k}$ and $b_i \in \Pi_{k\to\omega} D_{n_k}$ such that $\Psi(v_i) = u_i$ and $\Psi(b_i) = a_i$ for any $i \in \mathbb{N}$.*

Proof. The proof is just a consequence of Theorem 2.1 and of the analogous result $\varinjlim D_n \simeq A$, by a diagonal argument. For simplicity in writing (not in the argument) we only consider the family $\{u_i\}_i$. Let $u_i = \Pi_{k\to\omega} u_i^k$, where $u_i^k \in \mathcal{E}$. Choose $\{\varepsilon_k\}_k$ strictly positive numbers such that $\varepsilon_k \to_k 0$.

By Theorem 2.1, $\bigcup_n \Psi_n(P_n)$ is dense in $\mathcal{E}$. Then, for each k, there exists $n_k \in \mathbb{N}$ and $v_1^k, \ldots, v_k^k \in P_{n_k}$ such that $d_H(u_i^k, \Psi_{n_k}(v_i^k)) < \varepsilon_k$, for any $i \leqslant k$. Define $v_i = \Pi_{k\to\omega} v_i^k$. It is clear by construction that $\Psi(v_i) = u_i$. □

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3. *The space of diffuse sofic representations*

Definition 3.1. A *diffuse sofic representation* of a group G is a group morphism $\Theta : G \to \mathcal{E}^\omega$ such that $\mathrm{Tr}(\Theta(g)) = 0$ for any $g \neq e$.

Definition 3.2. For a countable group G denote by $\mathrm{Sof}(G, \mathcal{E}^\omega)$ the space of diffuse sofic representations of G factored by conjugacy: $\Theta_1 \sim \Theta_2$ if and only if there exists $u \in \mathcal{E}^\omega$ such that $\Theta_2 = A\, du \circ \Theta_1$.

Notation 3.3. For a diffuse sofic representation Θ we denote by $[\Theta]_{\mathcal{E}}$ its class in $\mathrm{Sof}(G, \mathcal{E}^\omega)$.

We prove now that there is a bijection between $\mathrm{Sof}(G, P^\omega)$ and $\mathrm{Sof}(G, \mathcal{E}^\omega)$.

PROPOSITION 3.4. *Let Θ_1, Θ_2 be two sofic representations such that $[\Theta_1]_P = [\Theta_2]_P$. Then $[\Psi \circ \Theta_1]_{\mathcal{E}} = [\Psi \circ \Theta_2]_{\mathcal{E}}$.*

Proof. Let $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$ be a sofic representation and $\{r_k\}_k \subset \mathbb{N}^*$. Inspecting the definition of Φ_n from the proof of Theorem 2.1 we see that $\Phi_n(s) = \Phi_{nr}(s \otimes 1_r)$, for any $s \in P_n$ and $r \in \mathbb{N}^*$. It follows that $\Psi(\Theta) = \Psi(\Theta \otimes 1_{r_k})$.

Assume now that Θ_1 and Θ_2 are conjugate. So there exists $u \in \Pi_{k\to\omega} P_{n_k}$ such that $u\Theta_1 u^* = \Theta_2$. Then $\Psi(u)\Psi(\Theta_1)\Psi(u)^* = \Psi(\Theta_2)$, implying that $[\Psi(\Theta_1)]_{\mathcal{E}} = [\Psi(\Theta_2)]_{\mathcal{E}}$. □

THEOREM 3.5. *The map $A : \mathrm{Sof}(G, P^\omega) \to \mathrm{Sof}(G, \mathcal{E}^\omega)$ defined by $A([\Theta]_P) = [\Psi \circ \Theta]_{\mathcal{E}}$ is a bijection.*

Proof. If Θ is a diffuse sofic representation then use Theorem 2.3 to construct a sofic representation Γ with $\Psi(\Gamma) = \Theta$. This shows that A is surjective.

Let now Θ_1 and Θ_2 be two sofic representations such that $A([\Theta_1]_P) = A([\Theta_2]_P)$. Then there is $u \in \mathcal{E}^\omega$ so that $u\Psi(\Theta_1)u^* = \Psi(\Theta_2)$. Again by Theorem 2.3 there is v in some $\Pi_{k\to\omega} P_{n_k}$ with $\Psi(v) = u$. Now, amplifying Θ_1, Θ_2 and v to a common sequence of dimensions and using the injectivity of Ψ we get $(v \otimes 1)(\Theta_1 \otimes 1)(v \otimes 1)^* = \Theta_2 \otimes 1$. It follows that $[\Theta_1]_P = [\Theta_2]_P$. □

A convex-like structure is defined on a metric space. We transport, via the bijection A, the usual metric defined on these spaces, see [**Pă2**, §1.4].

Definition 3.6. Let $G = \{g_0, g_1, \ldots\}$ where $g_0 = e$. For $[\Theta_1]_{\mathcal{E}}, [\Theta_2]_{\mathcal{E}} \in \mathrm{Sof}(G, \mathcal{E}^\omega)$ define

$$d([\Theta_1]_{\mathcal{E}}, [\Theta_2]_{\mathcal{E}}) = \inf\left\{\left(\sum_{i=1}^{\infty} \frac{1}{4^i} \|\Theta_1(g_i) - u\Theta_2(g_i)u^*\|_2^2\right)^{1/2} : u \in \mathcal{E}^\omega\right\}.$$

It follows that $\mathrm{Sof}(G, P^\omega)$ and $\mathrm{Sof}(G, \mathcal{E}^\omega)$ are isomorphic as metric spaces, via the map A.

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3.1. *The direct sum of the universal sofic group.* Let $\lambda \in [0, 1]$. We construct a morphism $\Phi_\lambda : \mathcal{E}^\omega \oplus \mathcal{E}^\omega \to \mathcal{E}^\omega$ to be used in the definition of the convex structure.

Let $u, v \in \mathcal{E}^\omega$. Use Theorem 2.3 to get $u_1, v_1 \in \Pi_{k\to\omega} P_{n_k}$ so that $\Psi(u_1) = u$ and $\Psi(v_1) = v$. Choose two sequences of natural numbers $\{r_k\}_k$ and $\{s_k\}_k$ such that $\lim_{k\to\omega} r_k/(r_k + s_k) = \lambda$. Construct $(u_1 \otimes 1_{r_k}) \oplus (v_1 \otimes 1_{s_k}) \in \Pi_{k\to\omega} P_{(r_k+s_k)n_k}$. Define

$$\Phi_\lambda(u \oplus v) = \Psi[(u_1 \otimes 1_{r_k}) \oplus (v_1 \otimes 1_{s_k})].$$

Note that $\Phi_\lambda(u \oplus v)$ does not depend on the particular choice of the sequences $\{r_k\}_k$ and $\{s_k\}_k$ as long as $\lim_{k\to\omega} r_k/(r_k + s_k) = \lambda$. The equality $\Psi(x) = \Psi(x \otimes 1)$ for any $x \in M_n(\mathbb{C})$ is important here. Also, the ultraproduct construction is factoring out small dependencies.

If $a_\lambda = \chi_{[0,\lambda]} \in A$ (characteristic function) then $\Phi_\lambda(u \oplus v)$ commutes with $(a_\lambda)^\omega \in A^\omega$ for any $u, v \in \mathcal{E}^\omega$. This is usual geometry in type II_1 factors. It is also a central observation for these convex structures that has to be made a theorem.

THEOREM 3.7. *The image of Φ_λ is composed of those elements that commute with $(a_\lambda)^\omega$, where a_λ is the characteristic function of $[0, \lambda]$:*

$$\Phi_\lambda(\mathcal{E}^\omega \oplus \mathcal{E}^\omega) = \{u \in \mathcal{E}^\omega : u \cdot (a_\lambda)^\omega = (a_\lambda)^\omega \cdot u\}.$$

Observation 3.8. The definition of Φ_λ can be extended to $M(E)^\omega \oplus M(E)^\omega$. A nice application is the formula $(a_\lambda)^\omega = \Phi_\lambda(1 \oplus 0)$.

3.2. *Cutting representations.* Cutting a diffuse sofic representation by a projection in A^ω represents the inverse operation of the direct sum. By Theorem 3.7, one needs a projection commuting with $\Theta(G)$, as $(a_\lambda)^\omega$ is in the role of the projection cutting a corner of the sofic representation.

Notation 3.9. Denote by $T_1 : \mathcal{E}^\omega \oplus \mathcal{E}^\omega \to \mathcal{E}^\omega$ the projection on the first summand, i.e. $T_1(u \oplus v) = u$. Similarly $T_2(u \oplus v) = v$.

Definition 3.10. Let p be a non-trivial projection in A^ω commuting with $\Theta(G)$. Let $\lambda = \mathrm{Tr}(p)$. Choose an element $u \in \mathcal{E}^\omega$ such that $upu^* = a_\lambda$. Define $\Theta_p^u = T_1(\Phi_\lambda^{-1}(u\Theta u^*))$.

PROPOSITION 3.11. *The class of Θ_p^u does not depend on the choice of u.*

Proof. Let $u, v \in \mathcal{E}^\omega$ be so that $upu^* = a_\lambda = vpv^*$. Then uv^* commutes with a_λ. We have

$$\begin{aligned}\Theta_p^u &= T_1(\Phi_\lambda^{-1}(u\Theta u^*)) = T_1(\Phi_\lambda^{-1}(uv^*v\Theta v^*vu^*))\\ &= T_1[\Phi_\lambda^{-1}(uv^*)\Phi_\lambda^{-1}(v\Theta v^*)\Phi_\lambda^{-1}(vu^*)]\\ &= T_1(\Phi_\lambda^{-1}(uv^*)) \cdot \Theta_p^v \cdot T_1(\Phi_\lambda^{-1}(uv^*))^*.\end{aligned}$$

As $T_1(\Phi_\lambda^{-1}(uv^*))$ is an element of $\mathcal{E}^\omega$, it follows that $[\Theta_p^u]_\mathcal{E} = [\Theta_p^v]_\mathcal{E}$. □

Definition 3.12. For a non-trivial projection $p \in A^\omega$ commuting with $\Theta(G)$, define $[\Theta_p]_\mathcal{E}$ to be the class of Θ_p^u for an element $u \in \mathcal{E}^\omega$ so that $upu^* = a_{\mathrm{Tr}(p)}$.

The following results are useful, both for the proof of the main result and also as an exercise to provide intuition into direct sums and amplifications of diffuse sofic representations.

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LEMMA 3.13. *Let p be a projection in $\Theta(G)' \cap A^\omega$ with $\mathrm{Tr}(p) = \lambda$. Assume that $0 < \lambda < 1$. Choose an element $u \in \mathcal{E}^\omega$ such that $upu^* = a_\lambda$. Then $[T_2(\Phi_\lambda^{-1}(u\Theta u^*))]_\mathcal{E} = [\Theta_{1-p}]_\mathcal{E}$.*

Proof. The main observation is that there exists $v \in \mathcal{E}^\omega$ such that $va_{1-\lambda}v^* = 1 - a_\lambda$ and $v\Phi_{1-\lambda}(x \oplus y)v^* = \Phi_\lambda(y \oplus x)$ for any $x, y \in \mathcal{E}^\omega$.

For $i = 1, 2$ define $\Theta_i : G \to \mathcal{E}^\omega$ by $\Theta_i = T_i(\Phi_\lambda^{-1}(u\Theta u^*))$. Then $u\Theta u^* = \Phi_\lambda(\Theta_1 \oplus \Theta_2) = v\Phi_{1-\lambda}(\Theta_2 \oplus \Theta_1)v^*$. It follows that $\Theta_2 = T_1(\Phi_{1-\lambda}^{-1}(v^*u\Theta u^*v))$. As $v^*u(1-p)u^*v = v^*(1-a_\lambda)v = a_{1-\lambda}$, by definition we have $\Theta_2 = \Theta_{1-p}^{v^*u}$. As $\Theta_2 = T_2(\Phi_\lambda^{-1}(u\Theta u^*))$ it follows that $[T_2(\Phi_\lambda^{-1}(u\Theta u^*))]_\mathcal{E} = [\Theta_{1-p}]_\mathcal{E}$. □

In a way the following Proposition is an anti-amplification. This feature is unique to diffuse sofic representations.

PROPOSITION 3.14. *For any diffuse sofic representation Θ and any $\lambda \in (0, 1)$ there exists a projection $p \in A^\omega$, commuting with $\Theta(G)$, such that $\mathrm{Tr}(p) = \lambda$ and $[\Theta]_\mathcal{E} = [\Theta_p]_\mathcal{E}$.*

Proof. Let $\Gamma : G \to \Pi_{k\to\omega} P_{n_k}$ be a sofic representation such that $\Theta = \Psi \circ \Gamma$. Let $\{r_k\}$ be a strictly increasing sequence of natural numbers. Construct $q \in \Pi_{k\to\omega} D_{r_k}$ a projection with $\mathrm{Tr}(q) = \lambda$. Then $1_{n_k} \otimes q$ commutes with $\Gamma \otimes 1_{r_k}$. Moreover $\Gamma_{1_{n_k}\otimes q}$ is still an amplification of Γ, so $[\Gamma]_P = [\Gamma_{1_{n_k}\otimes q}]_P$.

Let $p = \Psi(1_{n_k} \otimes q)$. Then p commutes with $\Theta = \Psi(\Gamma)$ and $\mathrm{Tr}(p) = \lambda$. Also the equality $[\Gamma]_P = [\Gamma_{1_{n_k}\otimes q}]_P$, transported via Ψ, becomes $[\Theta]_\mathcal{E} = [\Theta_p]_\mathcal{E}$. □

One problem in proving that the old action $\alpha(\Theta)$ is ergodic, is constructing elements in the commutant $\Theta(G)'$. The last proposition solved this problem, by an easy amplification. In the following lemma we note that there are plenty of projections inside $\Theta(G)' \cap A^\omega$.

LEMMA 3.15. *The algebra $\Theta(G)' \cap A^\omega$ is diffuse, i.e. it has no minimal projection.*

Proof. Let $p \in \Theta(G)' \cap A^\omega$ be a projection. Choose a sequence $\{n_k\}_k$ so that there exist $\Gamma : G \to \Pi_{k\to\omega} P_{n_k}$ a sofic representation and $q \in \Pi_{k\to\omega} D_{n_k}$ a projection such that $\Theta = \Psi \circ \Gamma$ and $p = \Psi(q)$. Because Ψ is injective on $\Pi_{k\to\omega} M_{n_k}$, q commutes with Γ.

Let $a \in D_2$ be a projection of trace $\frac{1}{2}$. Construct $q \otimes a \in \Pi_{k\to\omega} D_{2n_k}$. This is a projection with $\mathrm{Tr}(q \otimes a) = \frac{1}{2}\mathrm{Tr}(q)$ that commutes with $\Gamma \otimes 1_2$. Then $\Psi(q \otimes a)$ commutes with $\Psi(\Gamma \otimes 1_2) = \Theta$ and $\Psi(q \otimes a)$ is a sub-projection of p. □

4. *The convex structure*

Definition 4.1. For Θ_1, Θ_2 diffuse sofic representations and $\lambda \in [0, 1]$ define

$$\lambda[\Theta_1]_\mathcal{E} + (1-\lambda)[\Theta_2]_\mathcal{E} = [\Phi_\lambda(\Theta_1 \oplus \Theta_2)]_\mathcal{E}.$$

At this stage, we can consider $\lambda[\Theta_1]_\mathcal{E} + (1-\lambda)[\Theta_2]_\mathcal{E}$ to be just a formal notation for the element in $\mathrm{Sof}(G, \mathcal{E}^\omega)$ that we constructed. After the axioms of convex-like structures are checked, we can use the Capraro-Fritz theorem to deduce that $\mathrm{Sof}(G, \mathcal{E}^\omega)$ is endowed with the metric and this convex structure is a bounded closed convex subset of a Banach space. Then $\lambda[\Theta_1]_\mathcal{E} + (1-\lambda)[\Theta_2]_\mathcal{E}$ is a convex combination in this Banach space.

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As an observation, this definition is just the old convex structure on $\mathrm{Sof}(G, P^\omega)$ transported on $\mathrm{Sof}(G, \mathcal{E}^\omega)$ via the map A. This is enough to deduce that the axioms of convex-like structures, [**Br**, §2], are satisfied by the pair $(\mathrm{Sof}(G, \mathcal{E}^\omega), d)$. Also, they can be easily checked directly from the definitions presented in this paper.

PROPOSITION 4.2. *If* $[\Theta]_\mathcal{E} = \lambda[\Theta_1]_\mathcal{E} + (1-\lambda)[\Theta_2]_\mathcal{E}$, *with* $\lambda > 0$, *then there exists* $p \in A^\omega$ *a projection commuting with* $\Theta(G)$, $\mathrm{Tr}(p) = \lambda$ *such that* $[\Theta_p]_\mathcal{E} = [\Theta_1]_\mathcal{E}$.

Proof. We can assume that $\Theta = \Phi_\lambda(\Theta_1 \oplus \Theta_2)$. Then $p = a_\lambda$. By definition $\Theta_p^{\mathrm{Id}} = T_1(\Phi_\lambda^{-1}(\Theta)) = \Theta_1$. It follows that $[\Theta_p]_\mathcal{E} = [\Theta_1]_\mathcal{E}$. □

PROPOSITION 4.3. (Analogue of [**Br**, Proposition 3.3.4]) *Let* $p, q \in \Theta(G)' \cap A^\omega$ *be such that* $\mathrm{Tr}(p) = \mathrm{Tr}(q)$ *and* $\mathrm{Tr}(p) \notin \{0, 1\}$. *Then* $[\Theta_p]_\mathcal{E} = [\Theta_q]_\mathcal{E}$ *if and only if there exists an element* $u \in \Theta(G)' \cap \mathcal{E}^\omega$ *such that* $upu^* = q$.

Proof. Let $\lambda = \mathrm{Tr}(p) = \mathrm{Tr}(q)$ and let v_p, $v_q \in \mathcal{E}^\omega$ be such that $v_p p v_p^* = a_\lambda = v_q q v_q^*$. Let $u \in \Theta(G)' \cap \mathcal{E}^\omega$ be such that $upu^* = q$. Then

$$(v_q u v_p^*) a_\lambda (v_q u v_p^*)^* = (v_q u) p (v_q u)^* = v_q q v_q^* = a_\lambda,$$

so $(v_q u v_p^*)$ commutes with a_λ. Let $u_1 = T_1(\Phi_\lambda^{-1}(v_q u v_p^*))$ so that

$$\begin{aligned} u_1 \Theta_p^{v_p} u_1^* &= T_1(\Phi_\lambda^{-1}(v_q u v_p^*)) T_1(\Phi_\lambda^{-1}(v_p \Theta v_p^*)) T_1(\Phi_\lambda^{-1}(v_q u v_p^*))^* \\ &= T_1[\Phi_\lambda^{-1}(v_q u v_p^* v_p \Theta v_p^* (v_q u v_p^*)^*)] = T_1[\Phi_\lambda^{-1}(v_q \Theta v_q^*)] = \Theta_q^{v_q}. \end{aligned}$$

It follows that $[\Theta_p]_\mathcal{E} = [\Theta_q]_\mathcal{E}$.

Assume now that $[\Theta_p]_\mathcal{E} = [\Theta_q]_\mathcal{E}$. By the axioms of the convex-like structures (metric compatibility, see also Proof of [**Ca-Fr**, Corollary 6]) it follows that $[\Theta_{1-p}]_\mathcal{E} = [\Theta_{1-q}]_\mathcal{E}$. Recall that $[\Theta_p]_\mathcal{E} = [T_1(\Phi_\lambda^{-1}(v_p \Theta v_p^*))]_\mathcal{E}$ and $[\Theta_{1-p}]_\mathcal{E} = [T_2(\Phi_\lambda^{-1}(v_p \Theta v_p^*))]_\mathcal{E}$. So there exist u_1, $u_2 \in \mathcal{E}^\omega$ such that

$$\begin{aligned} u_1 T_1(\Phi_\lambda^{-1}(v_q \Theta v_q^*)) u_1^* &= T_1(\Phi_\lambda^{-1}(v_p \Theta v_p^*)) \quad \text{and} \\ u_2 T_2(\Phi_\lambda^{-1}(v_q \Theta v_q^*)) u_2^* &= T_2(\Phi_\lambda^{-1}(v_p \Theta v_p^*)). \end{aligned}$$

Let $u = \Phi_\lambda(u_1 \oplus u_2)$. Then $u_1 = T_1(\Phi_\lambda^{-1}(u))$ and $u_2 = T_2(\Phi_\lambda^{-1}(u))$. We have

$$\begin{aligned} v_p \Theta v_p^* &= \Phi_\lambda(\Phi_\lambda^{-1}(v_p \Theta v_p^*)) = \Phi_\lambda[T_1(\Phi_\lambda^{-1}(v_p \Theta v_p^*)) \oplus T_2(\Phi_\lambda^{-1}(v_p \Theta v_p^*))] \\ &= \Phi_\lambda[u_1 T_1(\Phi_\lambda^{-1}(v_q \Theta v_q^*)) u_1^* \oplus u_2 T_2(\Phi_\lambda^{-1}(v_q \Theta v_q^*)) u_2^*] \\ &= \Phi_\lambda[T_1(\Phi_\lambda^{-1}(u v_q \Theta v_q^* u^*)) \oplus T_2(\Phi_\lambda^{-1}(u v_q \Theta v_q^* u^*))] \\ &= \Phi_\lambda[\Phi_\lambda^{-1}(u v_q \Theta v_q^* u^*)] = u v_q \Theta v_q^* u^*. \end{aligned}$$

We have proved that $v_q^* u^* v_p$ commutes with $\Theta(G)$. As u is in the image of Φ_λ it commutes with a_λ. It follows that

$$(v_q^* u^* v_p) p (v_q^* u^* v_p)^* = (v_q^* u^*) a_\lambda (v_q^* u^*)^* = v_q^* a_\lambda v_q = q.$$ □

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4.1. *Actions on the Loeb space.* In $\mathrm{Sof}(G, \mathcal{E}^\omega)$ there is no need for amplifications. Another difference is that we consider only those elements of the Loeb space that commute with the diffuse sofic representation.

Notation 4.4. For a diffuse sofic representation $\Theta : G \to \mathcal{E}^\omega$ denote by $\gamma(\Theta)$ the action of $\Theta(G)' \cap \mathcal{E}^\omega$ on $\Theta(G)' \cap A^\omega$, defined by $\gamma(u)(a) = uau^*$.

The following lemma is easy, but it is one of the few tools that allows us to construct permutations. This is why it is so important. It was used in [**Pă1**, **Pă2**] (Lemma 1.6 in both articles, by a strange coincidence that I have just noticed). Here we need the diffuse version of this lemma. The proof is still the same, using Theorem 2.3.

LEMMA 4.5. *Let $\{p_i : i \in \mathbb{N}\}$ be projections in A^ω such that $\sum_i p_i = 1$. Let $\{u_i : i \in \mathbb{N}\}$ be unitary elements in $\mathcal{E}^\omega$ such that $\sum_i u_i p_i u_i^* = 1$. Then $v = \sum_i u_i p_i$ is an element in $\mathcal{E}^\omega$.*

PROPOSITION 4.6. *Let Θ be a diffuse sofic representation such that $\gamma(\Theta)$ is ergodic. Then if p, q are projections in $\Theta(G)' \cap A^\omega$ such that $\mathrm{Tr}(p) = \mathrm{Tr}(q)$ then there exists $u \in \Theta(G)' \cap \mathcal{E}^\omega$ such that $q = upu^*$.*

Proof. Assume first that $pq = 0$ (the underlying sets, on which p and q are projecting, are disjoint). We want to construct a partial isometry v such that $vpv^* = q$. As $\gamma(\Theta)$ is ergodic there exists $u \in \Theta(G)' \cap \mathcal{E}^\omega$ such that $upu^* \cdot q \neq 0$. By a maximality argument we can construct projections $\{p_i\}_i$ and $\{q_i\}_i$ in $\Theta(G)' \cap A^\omega$ and unitaries $\{u_i\}_i$ in $\Theta(G)' \cap \mathcal{E}^\omega$ such that $\sum_i p_i = p$, $\sum_i q_i = q$ and $u_i p_i u_i^* = q_i$ for any i.

Define $v = \sum_i u_i p_i$. It is easy to check that $vpv^* = q$, $vv^* = p$ and $v^*v = q$. Then $u = (1 - p - q) + v + v^*$ is a unitary commuting with $\Theta(G)$ such that $upu^* = q$. The proof is algebraic, but there is a lot of geometry behind the scenes. The unitary u is sending the underlying set of p onto the underlying set of q and vice-versa, while being identity on the rest of the space.

In order to prove that $u \in \mathcal{E}^\omega$, use the previous lemma with $\{p_i\}_i \cup \{q_i\}_i \cup \{1 - p - q\}$ as the set of projections and $\{u_i\}_i \cup \{u_i^*\}_i \cup \{\mathrm{Id}\}$ as the set of unitaries.

If $pq \neq 0$, replace p and q by $p_1 = p - pq$ and $q_1 = q - pq$. □

4.2. *The main result.*

PROPOSITION 4.7. *Let $\Theta : G \to \mathcal{E}^\omega$ be a sofic representation. Then $[\Theta]_{\mathcal{E}}$ is an extreme point in $\mathrm{Sof}(G, \mathcal{E}^\omega)$ if and only if $[\Theta]_{\mathcal{E}} = [\Theta_p]_{\mathcal{E}}$ for any projection $p \in \Theta(G)' \cap A^\omega$.*

Proof. This is just a consequence of Proposition 4.2. □

THEOREM 4.8. (Analogue of [**Br**, Proposition 5.2]) *Let $\Theta : G \to \mathcal{E}^\omega$ be a sofic representation. Then $[\Theta]$ is an extreme point in $\mathrm{Sof}(G, \mathcal{E}^\omega)$ if and only if the action $\gamma(\Theta)$ is ergodic.*

Proof. Assume that $[\Theta]_{\mathcal{E}} = \lambda[\Theta_1]_{\mathcal{E}} + (1 - \lambda)[\Theta_2]_{\mathcal{E}}$. Then by Proposition 4.2 there exists $p \in \Theta(G)' \cap A^\omega$ a projection with $\mathrm{Tr}(p) = \lambda$ such that $[\Theta_p]_{\mathcal{E}} = [\Theta_1]_{\mathcal{E}}$. Also by Proposition 3.14 there exists $q \in \Theta(G)' \cap A^\omega$ a projection with $\mathrm{Tr}(q) = \lambda$ such that $[\Theta_q]_{\mathcal{E}} = [\Theta]_{\mathcal{E}}$. If $\gamma(\Theta)$ is ergodic then by Proposition 4.6 there exists $u \in \Theta(G)' \cap \mathcal{E}^\omega$

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such that $upu^* = q$. Use now Proposition 4.3 to deduce that $[\Theta_p]_{\mathcal{E}} = [\Theta_q]_{\mathcal{E}}$. It follows that $[\Theta]_{\mathcal{E}} = [\Theta_1]_{\mathcal{E}}$ proving that $[\Theta]_{\mathcal{E}}$ is an extreme point.

For the converse, let $p, q \in \Theta(G)' \cap A^\omega$ be two projections such that $\mathrm{Tr}(p) = \mathrm{Tr}(q)$. By the previous proposition $[\Theta_p]_{\mathcal{E}} = [\Theta_q]_{\mathcal{E}}$. Then, by Proposition 4.3 there exists $u \in \Theta(G)' \cap \mathcal{E}^\omega$ such that $q = upu^*$. This is enough to deduce the ergodicity of $\gamma(\Theta)$ as $\Theta(G)' \cap A^\omega$ is diffuse (Lemma 3.15). □

Observation 4.9. The convex-like structures $\mathrm{Sof}(G, P^\omega)$ and $\mathrm{Sof}(G, \mathcal{E}^\omega)$ are isomorphic. This means that the extreme points constructed in [**Pă2**, §2.6] are still valid for $\mathrm{Sof}(G, \mathcal{E}^\omega)$. The existence of extreme points for any sofic group remains however an open question.

5. *Sofic representations that cannot be extended*

Let $G = \mathbb{Z} * \mathbb{Z}_2 = \langle a, b : b^2 = e \rangle$ and $c = bab$. Then $\mathbb{F}_2 = \langle a, c \rangle$ is a copy of the free group inside G. Let $R : \mathrm{Sof}(G, P^\omega) \to \mathrm{Sof}(\mathbb{F}_2, P^\omega)$ be the restriction map $R([\Theta]_P) = [\Theta|_{\mathbb{F}_2}]_P$. In this section we show that R is not surjective.

It is quite easy to show that most of the sofic representations $\Theta : \mathbb{F}_2 \to \Pi_{k\to\omega} P_{n_k}$ cannot be extended to a sofic representation $\tilde{\Theta} : G \to \Pi_{k\to\omega} P_{n_k}$ (same sequence of dimensions). Roughly speaking, a sofic representation of $\mathbb{F}_2$ is obtained by choosing two sequences of n_k-cycles, while a sofic representation of G is obtained by choosing two sequences of n_k-cycles (corresponding to the elements a and c) that are (almost) conjugated by an element of order two (corresponding to the element b). As a relatively low number of pairs of cycles are conjugated by an element of order two, it follows that most $\Theta : \mathbb{F}_2 \to \Pi_{k\to\omega} P_{n_k}$ cannot be extended to G.

However, when studying the function R, we must take into consideration amplifications. Indeed, there are sofic representations Θ that are not extendable, but they have amplifications that are. For example, assume that there exists an element $y \in \Pi_{k\to\omega} P_{n_k}$ such that $y\Theta(a)y^{-1} = \Theta(c)$ and $y^2\Theta(a) = \Theta(a)y^2$. Then, one can check that, if $\tilde{y} = \left[\begin{smallmatrix} 0 & y \\ y^{-1} & 0 \end{smallmatrix}\right]$, then $\tilde{y}^2 = \mathrm{Id}$ and $\tilde{y}(\Theta(a) \otimes 1_2)\tilde{y}^{-1} = \Theta(c) \otimes 1_2$.

I am quite sure that the existence of such an element $y \in \Pi_{k\to\omega} P_{n_k}$ is equivalent to the fact that $\Theta \otimes 1_2$ is extendable to G. We do not need this result. We shall prove that when Θ is an expander, which is known to happen most of the time, this is the only phenomenon that may make an amplification of Θ expandable.

5.1. *Hamming distance on matrices.*

For the technical part of the proof we need to extend the Hamming distance to the set of matrices.

Definition 5.1. For $x, y \in M_n(\mathbb{C})$ define the *Hamming distance on matrices*:

$$d_H(x, y) = \frac{1}{n}|\{i : \exists j \; x(i, j) \neq y(i, j)\}|.$$

The formula counts the number of rows that are different in x, y. Note that if $x, y \in P_n$ then this distance is the usual Hamming distance on a symmetric group.

Definition 5.2. We call a matrix $q \in M_n(\mathbb{C})$ a *piece of permutation* if q has only 0 and 1 entries and no more than one entry of 1 on each row and each column. Alternatively $q = pa$, where $p \in P_n$ and a is a projection in D_n.

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PROPOSITION 5.3. *Let* $x, y \in M_n(\mathbb{C})$ *and* $p \in P_n$. *Then* $d_H(x, y) = d_H(px, py) = d_H(xp, yp)$. *Instead, if* p *is a piece of permutation then*

$$d_H(px, py) \leqslant d_H(x, y) \quad \text{and} \quad d_H(xp, yp) \leqslant d_H(x, y).$$

5.2. *The main theorem.* The following lemma is the key to the proof. From the existence of an element $y \in P_{nr}$ with some properties, we infer the existence of an element $w \in P_n$ with similar properties. We need these types of results in order to get rid of an amplification.

LEMMA 5.4. *Let* $x, z \in P_n$ *and* $y \in P_{nr}$ *be such that* $y^2 = \mathrm{Id}_{nr}$ *and* $d_H(y(x \otimes 1_r), (z \otimes 1_r)y) < \varepsilon$. *Assume that for any projection* $p \in D_n$, $0 < \mathrm{Tr}(p) \leqslant \frac{1}{2}$ *implies* $\lambda\,\mathrm{Tr}(p) < d_H(p, xpx^*) + d_H(p, zpz^*)$, *where* λ *is a constant,* $0 < \lambda \leqslant 1$. *Then there exists* $w \in P_n$ *such that* $d_H(wx, zw) < 72\varepsilon/\lambda$ *and* $d_H(xw, wz) < 72\varepsilon/\lambda$.

Proof. As $M_{nr}(\mathbb{C}) \simeq M_r(\mathbb{C}) \otimes M_n(\mathbb{C})$, elements in $M_{nr}(\mathbb{C})$ can be viewed as functions from $\{1, \ldots r\}^2$ to $M_n(\mathbb{C})$. Then $(x \otimes 1_r)(i, j) = \delta_i^j x$ and $[y(x \otimes 1_r)](i, j) = \sum_k y(i, k)(x \otimes 1_r)(k, j) = y(i, j)x$. Similarly $[(z \otimes 1_r)y](i, j) = z \cdot y(i, j)$.

Let $A, B \in P_{nr}$. We want to compare $d_H(A, B)$ to $\sum_{i,j=1}^r d_H(A(i, j), B(i, j))$. If A and B are different on a row it may happen that we count this error twice in $\sum_{i,j=1}^r d_H(A(i, j), B(i, j))$. It follows that

$$2d_H(A, B) \geqslant \frac{1}{r} \sum_{i,j=1}^{r} d_H(A(i, j), B(i, j)).$$

By hypothesis $d_H(y(x \otimes 1_r), (z \otimes 1_r)y) < \varepsilon$ and we can also deduce $d_H((x \otimes 1_r)y, y(z \otimes 1_r)) < \varepsilon$. Let $d_H(y(i, j)x, zy(i, j)) = \varepsilon_{i,j}^1$ and $d_H(xy(i, j), y(i, j)z) = \varepsilon_{i,j}^2$. Then

$$\frac{1}{r} \sum_{i,j=1}^{r} \varepsilon_{i,j}^1 \leqslant 2d_H(y(x \otimes 1_r), (z \otimes 1_r)y) < 2\varepsilon;$$

$$\frac{1}{r} \sum_{i,j=1}^{r} \varepsilon_{i,j}^2 \leqslant 2d_H((x \otimes 1_r)y, y(z \otimes 1_r)) < 2\varepsilon.$$

From these inequalities we can deduce the existence of an $i \in \{1, \ldots, r\}$ for which

$$\sum_{j=1}^{r} \varepsilon_{i,j}^1 < 8\varepsilon, \quad \sum_{j=1}^{r} \varepsilon_{i,j}^2 < 8\varepsilon, \quad \sum_{j=1}^{r} \varepsilon_{j,i}^1 < 8\varepsilon \quad \text{and} \quad \sum_{j=1}^{r} \varepsilon_{j,i}^2 < 8\varepsilon.$$

From now on i is fixed with this property. Noting that $y(i, j)$ is a piece of permutation, we get

$$\begin{aligned} &d_H(y(i, j)y(j, i)x, xy(i, j)y(j, i)) \\ &\quad \leqslant d_H(y(i, j)y(j, i)x, y(i, j)zy(j, i)) + d_H(y(i, j)zy(j, i), xy(i, j)y(j, i)) \\ &\quad \leqslant d_H(y(j, i)x, zy(j, i)) + d_H(y(i, j)z, xy(i, j)) = \varepsilon_{j,i}^1 + \varepsilon_{i,j}^2. \end{aligned}$$

Denote $p_j = y(i, j)y(j, i)$ and note that $y^2 = \mathrm{Id}_{nr}$ implies $\sum_j p_j = \mathrm{Id}_n$. The above inequality is $d_H(p_j, xp_jx^*) = d_H(p_jx, xp_j) \leqslant \varepsilon_{j,i}^1 + \varepsilon_{i,j}^2$. Analogously, $d_H(p_j, zp_jz^*) \leqslant \varepsilon_{i,j}^1 + \varepsilon_{j,i}^2$.

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For $S \subset \{1, \ldots, r\}$ define $p_S = \sum_{j \in S} p_j$. Both p_j and xp_jx^* are elements in D_n and this implies that $d_H(p_S, xp_Sx^*) \leqslant \sum_{j \in S} d_H(p_j, xp_jx^*)$. Using the above inequalities we get that, for any subset S,

$$d_H(p_S, xp_Sx^*) \leqslant \sum_{j \in S} (\varepsilon^1_{j,i} + \varepsilon^2_{i,j}) < 16\varepsilon.$$

The same statement is true for $d_H(p_S, zp_Sz^*)$. Assume that $0 < \mathrm{Tr}(p_S) \leqslant \frac{1}{2}$. Then, by hypothesis, $\lambda \,\mathrm{Tr}(p_S) < d_H(p_S, xp_Sx^*) + d_H(p_S, zp_Sz^*)$. Hence $\mathrm{Tr}(p_S) < 32\varepsilon/\lambda$ in this case. At this point, as $\sum_{j=1}^{r} p_j = \mathrm{Id}_n$, it is easy to see that there exists j such that $\mathrm{Tr}(p_j) \geqslant \frac{1}{2}$. Let $S = \{1, \ldots, r\} \backslash \{j\}$. Then $\mathrm{Tr}(p_S) < 32\varepsilon/\lambda$, so $\mathrm{Tr}(p_j) > 1 - 32\varepsilon/\lambda$.

Let $w \in P_n$ be such that $d_H(w, y(i, j)) < 32\varepsilon/\lambda$. It is easy to see that $d_H(wx, zw) \leqslant 32\varepsilon/\lambda + 8\varepsilon + 32\varepsilon/\lambda < 72\varepsilon/\lambda$. The same is true for $d_H(xw, wz)$. □

PROPOSITION 5.5. *Let $\Theta : \mathbb{F}_2 \to \Pi_{k \to \omega} P_{n_k}$ be a sofic representation of $\mathbb{F}_2$. Choose $a_k, c_k \in P_{n_k}$ such that $\Theta(a) = \Pi_{k \to \omega} a_k$ and $\Theta(c) = \Pi_{k \to \omega} c_k$. Assume that*

(1) *$\{a_k, c_k\}_k$ is an expander, i.e. there exists $\lambda > 0$ such that for any k and any projection $p \in D_{n_k}$ with $0 < \mathrm{Tr}(p) \leqslant \frac{1}{2}$ we have $\lambda \,\mathrm{Tr}(p) < d_H(p, a_k pa_k^*) + d_H(p, c_k pc_k^*)$;*

(2) *there is no $w \in \Pi_{k \to \omega} P_{n_k}$ such that $w\Theta(a)w^{-1} = \Theta(c)$ and $w^2\Theta(a) = \Theta(a)w^2$.*

Then there is no Ψ sofic representation of G such that $R([\Psi]_P) = [\Theta]_P$.

Proof. Assume that there exists a sofic representation $\Psi : G \to \Pi_{k \to \omega} P_{n_k r_k}$ such that $\Psi|_{\mathbb{F}_2} = \Theta \otimes 1_{r_k}$. Let $y = \Psi(b)$. Then $y^2 = \mathrm{Id}$ and $y \cdot [\Theta \otimes 1_{r_k}](a) = [\Theta \otimes 1_{r_k}](c) \cdot y$.

Find $y_k \in P_{n_k r_k}$ such that $y_k^2 = \mathrm{Id}_{n_k r_k}$ and $y = \Pi_{k \to \omega} y_k$. Then $d_H(y_k(a_k \otimes 1_{r_k}), (c_k \otimes 1_{r_k})y_k) \to 0$ when $k \to \omega$. Use Lemma 5.4 to construct $w_k \in P_{n_k}$ such that $d_H(w_k a_k, c_k w_k) \to_{k \to \omega} 0$ and $d_H(a_k w_k, w_k c_k) \to_{k \to \omega} 0$. Let $w = \Pi_{k \to \omega} w_k$. Then $w\Theta(a) = \Theta(c)w$ and $\Theta(a)w = w\Theta(c)$. This is in contradiction with condition (2). □

In the following sections we show that there exist sofic representations of $\mathbb{F}_2$ satisfying conditions (1) and (2) from Proposition 5.5. Fix a sequence $\{n_k\}_k$ increasing to infinity. For each k, arbitrarily choose two n_k-cycles (a_k, c_k) from the $[(n_k - 1)!]^2$ pairs available. It is well-known that the sequence $(a_k, c_k)_k$ is generating a sofic representation of the free group with probability 1 (see §5.5 for a reference). The theory of expander graphs tells us that the first condition required in Proposition 5.5 is also attained with probability 1 (for small enough λ). Some estimations will show that condition (2) is also satisfied with probability 1.

5.3. *First condition.* We review here some basic facts about expanders. For a comprehensive introduction to the subject, see [**Lu**].

Notation 5.6. For any two cycles $a, c \in P_n$ denote by $G_{a,c}$ the 4-regular graph (V, E), where $V = \{1, \ldots, n\}$ and $E = \{(i, a(i)); (i, a^{-1}(i)); (i, c(i)); (i, c^{-1}(i)) : i \in V\}$. These graphs may have multiple edges.

Definition 5.7. For a graph $G = (V, E)$ the *Cheeger constant* $h(G)$ is defined as

$$h(G) = \min_{0 < |S| \leqslant |V|/2} \frac{|\partial S|}{|S|},$$

where $S \subset V$ and ∂S is the set of edges in E with exactly one vertex in S.

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The link between the Cheeger constant and condition (1) from Proposition 5.5 is clear. Choose $a, c \in P_n$ and a projection $p \in D_n$. Construct $G_{a,c} = (V, E)$. Let S be the subset of V corresponding to p, so that $\mathrm{Tr}(p) = (1/n)|S|$. We can see that $d_H(p, apa^*) + d_H(p, cpc^*) = (1/n)|\partial S|$. It follows that condition (1) is satisfied if and only if $\{h(G_{a_k,c_k})\}_k$ is bonded away from zero.

The Cheeger constant is strongly connected to the spectral gap. For a graph G we shall denote by $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$ the eigenvalues of the adjacency matrix. If G is a 4-regular graph, as is always the case in this paper, then $\lambda_1(G) = 4$. The second eigenvalue is of interest to us.

PROPOSITION 5.8. (Cheeger inequality) *For any d-regular graph the following holds:*

$$\tfrac{1}{2}(d - \lambda_2(G)) \leqslant h(G).$$

A reference for this inequality is [**Al-Mi**, Lemma 2.1]. The following theorem is the missing piece of the puzzle.

THEOREM 5.9. [**Fr**, Theorem 1.2] *For any $\varepsilon > 0$ there exists a constant μ_ε such that for at least $(1 - \mu_\varepsilon/n)[(n-1)!]^2$ pairs of n-cycles $\{a, c\}$ we have, for all $i > 1$,*

$$|\lambda_i(G_{a,c})| \leqslant 2\sqrt{3} + \varepsilon.$$

From now on we fix $a \in P_n$ to be the cycle $(1, 2, \ldots, n)$. As any two n-cycles are conjugate, we can deduce the following from the above theorem.

THEOREM 5.10. *There exists a constant μ_1 such that for at least $(1 - \mu_1/n)[(n-1)!]$ n-cycles c we have, for all $i > 1$,*

$$|\lambda_i(G_{a,c})| \leqslant 3.6.$$

Altogether, setting $\lambda = 0.2$, we have the following result.

PROPOSITION 5.11. *There exists a constant μ_1 such that for at least $(1 - \mu_1/n)[(n-1)!]$ n-cycles c the following holds: for any projection $p \in D_n$ with $0 < \mathrm{Tr}(p) \leqslant \frac{1}{2}$ we have $\lambda\,\mathrm{Tr}(p) < d_H(p, apa^*) + d_H(p, cpc^*)$.*

5.4. *Second condition.* Now we try to estimate the number of elements $w \in P_n$ so that $d_H(w^2 a, aw^2) < \varepsilon$. We stick to our choice $a = (1, \ldots, n)$.

5.4.1. *Number of permutations close to a fixed element.*

PROPOSITION 5.12. *Let $\varepsilon > 0$ and $x \in P_n$. Then the number of permutations $y \in P_n$ such that $d_H(x, y) < \varepsilon$ is less than $n^{\lfloor \varepsilon n \rfloor}$.*

Proof. For $x \in P_n$ we construct a permutation y such that $d_H(x, y) < \varepsilon$ as follows:

(1) choose a set $A \subset \{1, \ldots, n\}$ such that $|A| = \lfloor \varepsilon n \rfloor$; this set will contain the points where x and y differ;
(2) choose a bijection $\phi : A \to x(A)$;
(3) define $y(t) = \phi(t)$ if $t \in A$ and $y(t) = x(t)$ otherwise.

All the permutations $y \in P_n$ such that $d_H(x, y) < \varepsilon$ are obtained this way. In total we get at most $C_n^{\lfloor \varepsilon n \rfloor} \cdot (\lfloor \varepsilon n \rfloor !)$. It is easy to see that this is smaller than $n^{\lfloor \varepsilon n \rfloor}$. □

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5.4.2. *Number of permutations almost commuting with a fixed element.*

PROPOSITION 5.13. *Let $\varepsilon > 0$. Then the number of permutations $y \in P_n$ such that $d_H(ay, ya) < \varepsilon$ is less than $n^{\lfloor \varepsilon n \rfloor + 1}$.*

Proof. We construct elements almost commuting with a as follows: divide $\{1, \ldots, n\}$ into k subsets composed of consecutive numbers, then permute these subsets.

Formally, choose $1 = i_1 < i_2 < \cdots < i_k < i_{k+1} = n + 1$. Define $l_j = i_{j+1} - i_j$, $j = 1, \ldots, k$ (the length of the jth segment). Define $s : \{1, \ldots, n\} \to \{1, \ldots, k\}$, $s(v)$ is the unique number such that $i_{s(v)} \leqslant v < i_{s(v)+1}$.

Choose $r \in \mathrm{Sym}(k)$ and let $t_{r(w)} = 1 + \sum_{r(j)<r(w)} l_j$ (these are the new starting points of the segments to replace the numbers i_j). Define

$$y(v) = t_{r(s(v))} + (v - i_{s(v)}).$$

If $v \in \{1, \ldots, n\} \backslash \{i_1, \ldots, i_k\}$ then $s(v-1) = s(v)$. Then $y(v-1) = t_{r(s(v))} + ((v-1) - i_{s(v)}) = y(v) - 1$. This can be rewritten as $ya(v-1) = ay(v-1)$, so $d_H(ay, ya) \leqslant k/n$.

All permutations y for which $d_H(ay, ya) \leqslant (k-1)/n$ can be constructed this way. We need to consider $k - 1 = \lfloor \varepsilon n \rfloor$. The number of such permutations is at most $C_n^{k-1} \cdot k! = (n! \cdot k)/(n-k+1)! < n^k$. □

For an exact formula and a more detailed proof the reader can check [**Mo-Ri**, Theorem 4.13].

5.4.3. *Number of solutions to the equation $x^2 = y$.*

Notation 5.14. For $y \in \mathrm{Sym}(n)$ denote by $S_2(y)$ the number of solutions in $\mathrm{Sym}(n)$ of the equation $x^2 = y$.

We now compute $S_2(y)$ for some $y \in \mathrm{Sym}(n)$.

Notation 5.15. For $x \in \mathrm{Sym}(n)$ denote by $c_x(i)$ the number of x-cycles of length i.

From this definition we see that $c_x(1)$ is the number of fixed points of x and $\sum_i i c_x(i) = n$.

Let $x(1) = t$. Then $x(t) = x^2(1) = y(1)$ and $x(y(1)) = x^2(t) = y(t)$. Inductively, we get

$$x(y^k(t)) = y^{k+1}(1) \quad \text{and} \quad x(y^k(1)) = y^k(t) \quad \text{for any } k \geqslant 0.$$

There are two cases. Assume that 1 and t are in the same y-cycle, i.e. there exists $a \in \mathbb{N}$ so that $t = y^a(1)$. Take a to be the smallest such value. It follows that $x(y^{k+a}(1)) = y^{k+1}(1)$ and $x(y^k(1)) = y^{k+a}(1)$. Combining the two equations we get $y^{2a-1}(1) = 1$. It is easy to get a contradiction if $y^k(1) = 1$ for some $0 < k < 2a - 1$, so 1 must be in a y-cycle of length $2a - 1$. All the values of x on the elements composing this y-cycle are determined ($x(1) = y^a(1)$ and the rest will follow).

Assume now that 1 and t are in two distinct y-cycles. Then the two cycles must be of equal length. The values of x on the elements composing the two cycles are determined once we chose the value of $x(1)$.

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Let us determine the number of solutions of the equation $x^2 = y$ when y has only cycles of length i. If i is even then $c_y(i)$ must be even, otherwise we have no solution. We have to group these cycles in pairs of two and there are $(c_y(i))!/[(c_y(i)/2)!2^{c_y(i)/2}]$ possibilities to do so. For each coupling we have $i^{c_y(i)/2}$ associated solutions. All in all, the number of solutions in this case is

$$S_2(y) = \frac{(c_y(i))!(i/2)^{c_y(i)/2}}{(c_y(i)/2)!}.$$

If i is odd then we can group $2k$ cycles in k pairs for $k = 0, \ldots, \lfloor c_y(i)/2 \rfloor$. The cycles left unpaired are not adding to the number of solutions, as the permutation x is perfectly determined on the elements of those cycles. We reach the formula

$$S_2(y) = \sum_{k=0}^{\lfloor c_y(i)/2 \rfloor} \frac{(c_y(i))!i^k}{(c_y(i)-2k)!k!2^k}.$$

If y is an arbitrary element of $\mathrm{Sym}(n)$ then

$$S_2(y) = \left[\Pi_i \frac{(c_y(2i))!(i)^{c_y(2i)/2}}{(c_y(2i)/2)!}\right]\left[\Pi_i \left(\sum_{k=0}^{\lfloor c_y(2i+1)/2 \rfloor} \frac{(c_y(2i+1))!(2i+1)^k}{(c_y(2i+1)-2k)!k!2^k}\right)\right]$$

if and only if $c_y(2i)$ is even for each i, otherwise $S_2(y) = 0$.

PROPOSITION 5.16. *The maximal number of solutions of the equation $x^2 = y$ is attained when y is identity. In other words,*

$$S_2(\mathrm{Id}) = \max\{S_2(y) : y \in \mathrm{Sym}(n)\}.$$

Proof. Assume first that y is composed only of cycles of length i. So $n = ic_y(i)$. Then

$$S_2(y) \leqslant \sum_{k=0}^{\lfloor c_y(i)/2 \rfloor} \frac{(c_y(i))!i^k}{(c_y(i)-2k)!k!2^k}.$$

As $S_2(\mathrm{Id}) = \sum_{k=0}^{\lfloor n/2 \rfloor} (n!)/(n-2k)!k!2^k$ it is enough to prove that

$$\frac{(c_y(i))!i^k}{(c_y(i)-2k)!k!2^k} \leqslant \frac{n!}{(n-2k)!k!2^k} \quad \text{for } k = 0, \ldots, [c_y(i)/2].$$

This inequality is equivalent to

$$c_y(i)(c_y(i)-1)\cdots(c_y(i)-2k+1)i^k \leqslant n(n-1)\cdots(n-2k+1).$$

As $n = ic_y(i)$ we see that $n(n-i)(n-2i)\cdots(n-i(2k-1))$ is an intermediate value in the inequality above.

Let now y be an arbitrary element in $\mathrm{Sym}(n)$. By the first part of the proof $S_2(y) \leqslant \Pi_i S_2(\mathrm{Id}_{ic_y(i)})$. The inequality $\Pi_i S_2(\mathrm{Id}_{ic_y(i)}) \leqslant S_2(\mathrm{Id}_n)$ is clear as $\Pi_i S_2(\mathrm{Id}_{ic_y(i)})$ counts only some of the solutions of the equation $x^2 = \mathrm{Id}_n$. □

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5.4.4. *Finishing the counting argument.*

Notation 5.17. Denote $\mathrm{Acyc}(n,\varepsilon)=\{w\in \mathrm{Sym}(n) : d_H(w^2a, aw^2)<\varepsilon\}$ and $\mathrm{Bcyc}(n,\varepsilon)=\{c\in \mathrm{Sym}(n) : \exists w\in \mathrm{Acyc}(n,\varepsilon), d_H(waw^{-1}, c)<\varepsilon\}$.

PROPOSITION 5.18. *For small enough ε and large enough n we have* $|\mathrm{Bcyc}(n,\varepsilon)|<1/n\cdot(n-1)!$.

Proof. Combining Propositions 5.13 and 5.16, we get that $|\mathrm{Acyc}(n,\varepsilon)|<n^{\lfloor \varepsilon n\rfloor+1}S_2(\mathrm{Id}_n)$. Using Proposition 5.12, we get $|\mathrm{Bcyc}(n,\varepsilon)|<n^{\lfloor \varepsilon n\rfloor}\cdot|\mathrm{Acyc}(n,\varepsilon)|<n^{2\lfloor \varepsilon n\rfloor+1}S_2(\mathrm{Id}_n)$.

Clearly $(n-2k)!\cdot k!>\lfloor n/3\rfloor!$ for any $k=0,\ldots,[n/2]$. It follows that

$$S_2(\mathrm{Id}_n)=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{n!}{(n-2k)!k!2^k}<\left\lfloor\frac{n}{2}\right\rfloor\cdot(n!)\cdot\left(\left\lfloor\frac{n}{3}\right\rfloor!\right)^{-1}.$$

It is easy to see that there exists a constant $t>0$ so that $\lfloor n/3\rfloor!>n^{tn}$ for large enough n (one can use Stirlings's formula to deduce that t can be chosen arbitrarily close to $\frac{1}{3}$, but we do not need this). Altogether,

$$|\mathrm{Bcyc}(n,\varepsilon)|<n^{2\lfloor \varepsilon n\rfloor+1}\cdot n\cdot n^{-tn}\cdot(n!)=n^{2\lfloor \varepsilon n\rfloor+4-tn}\cdot\frac{1}{n}(n-1)!.$$

The conclusion can now be deduced. □

5.5. *Random sofic representations of the free group.* Let ξ be a non-trivial, reduced word in $\mathbb{F}_2$. We show that if we randomly choose $a, c\in P_n$ two n-cycles, then most of the time $\mathrm{Tr}(\xi(a,c))$ is small. Here $\xi(a,c)$ is the element in P_n obtained by replacing the two generators of $\mathbb{F}_2$ by a and c, respectively.

Again, by symmetry, we can assume that $a\in P_n$ is the cycle $(1,2,\ldots,n)$. Following notation in [**AGV**], for a finite graph $G=(V,E)$, we denote by $\gamma(G,r)$ the number of non-trivial r-cycles in G divided by $|V|$. Here *trivial* refers to backtracking the same edges. Using different edges between the same two vertices does not count as *trivial*. Having this is mind, it is easy to see that $\mathrm{Tr}(\xi(a,c))\leqslant\gamma(G_{a,c},r)$, where r is the length of ξ.

The proof relies on the following theorem.

THEOREM 5.19. [**AGV**, Theorem 1] *Let $G=(V,E)$ be a finite d-regular graph with $|V|\geqslant 8d$. Then for any $r\geqslant 1$ we have*

$$\frac{\lambda_2(G)}{2\sqrt{d-1}}\geqslant 1+\frac{\gamma(G,r)}{rv_r}-\frac{3/2\log\log_d|V|+6}{\log_d|V|},$$

where $v_r=2\cdot 10^{11}2^{4r}(d-1)^{3r}r$.

We can now prove the main result of this section. In our case $d=4$.

THEOREM 5.20. *For any $\varepsilon>0$ there exists a natural number n_0 such that for all $n>n_0$ for at least $(1-\varepsilon)[(n-1)!]$ n-cycles c we have* $\mathrm{Tr}(\xi(a,c))<\varepsilon$.

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Proof. Let $\delta = \varepsilon/(rv_r)$. Theorem 5.9, applied for this δ, yields the existence of a constant μ_δ such that for at least $(1 - \mu_\delta/n)[(n-1)!]$ n-cycles c we have $\lambda_2(G_{a,c}) \leqslant 2\sqrt{3} + \delta$. Then, by the previous theorem,

$$\frac{\delta}{2\sqrt{3}} + \frac{3/2 \log\log_d n + 6}{\log_d n} \geqslant \frac{\gamma(G_{a,c}, r)}{rv_r}.$$

For large enough n, we get that $(\varepsilon/2\sqrt{3}) + rv_r(3/2 \log\log_d n + 6)/(\log_d n) < \varepsilon$ and $\mu_\delta/n < \varepsilon$. As $\mathrm{Tr}(\xi(a, c)) \leqslant \gamma(G_{a,c}, r)$, the conclusion follows. □

It is an easy consequence of this Theorem that arbitrarily choosing two n_k-cycles (a_k, c_k) yields a sofic representation of the free group with probability 1. Now we can put all the pieces together.

PROPOSITION 5.21. *Let $\{n_k\}_k$ be a sequence, $n_k \to \infty$ and $a_k = (1, \ldots, n_k)$. Choose $c_k \in P_{n_k}$ a random n_k-cycle from the $(n_k - 1)!$ possibilities. Then $\Theta : \mathbb{F}_2 \to \Pi_{k\to\omega} P_{n_k}$ defined by $\Theta(a) = \Pi_{k\to\omega} a_k$ and $\Theta(c) = \Pi_{k\to\omega} c_k$ is a sofic representation of $\mathbb{F}_2$, satisfying the conditions in Proposition 5.5 with probability* 1.

Proof. Combine Propositions 5.11, 5.18 and Theorem 5.20. □

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