Ergod. Th. & Dynam. Sys. (2003), **23**, 885–893
DOI: 10.1017/S0143385702001414
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Printed in the United Kingdom

An invariant for non-singular isomorphism

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(*Received* 5 *February* 2002 *and accepted in revised form* 3 *December* 2002)

Abstract. A dimension invariant for isomorphism of non-singular ergodic integer actions is presented. A result analogous to Shannon's Theorem is used to calculate the critical dimension of type III odometer actions. It is shown that the dimension is non-trivial and that it is not an invariant of orbit equivalence.

1. *Introduction*

In the study of measure-preserving actions, Kolmogorov–Sinai (KS) entropy has been highly useful as an invariant for isomorphism. Some efforts have been made to extend the use of entropy and similar invariants to the non-singular domain (see, for example, [**5**]); however, these attempts have not generally been very successful. The invariant we present is not the result of broadening the definition of KS entropy, although it does seem related to the concept of average coordinate (AC) entropy for odometer actions first presented in [**3**].

Our construction arose from attempts to extend the results of Maharam in [**2**]. These results relate the behaviour of the Radon–Nikodym derivative sequences

$$\frac{\sum_{i=0}^{n-1}(dT^{-i}\mu/d\mu)(x)}{n}$$

to the existence of measures preserved by the action of a non-singular measurable transformation T.

The rate of growth of Radon–Nikodym sums is a fundamental invariant of an ergodic non-singular dynamical system. In §2 we formulate a dimension which quantifies this growth and show that it is invariant for non-singular isomorphism.

Calculations in §3 show that this critical dimension is not an invariant of orbit equivalence. Using a result similar to Shannon's Theorem we provide a formula for the critical dimension for a wide class of type III product odometers on the space

$$X = \prod_{i=1}^{\infty} \mathbb{Z}_2.$$

This formula coincides with that given in [**3**] and [**4**] for the AC entropy of a product measure on X. As the range of AC entropy values within orbit equivalence classes has

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already been investigated, it is easy to produce examples of orbit-equivalent systems with different critical dimensions. Finally in §4 we provide upper and lower bounds on the critical dimension of a Markov odometer by generalizing the calculation techniques used for the product measure case.

2. *Definitions and notation*

We use the notation $\omega_i(x)$ to denote the ith the Radon–Nikodym derivatives of (T, X, μ) i.e. $\omega_i(x)$ is that L^1_μ function for which $\mu(T^i(E)) = \int_E \omega_i(x)\, d\mu$ for all μ-measurable sets $E \subseteq X$.

Definition 2.1. Let (T, X, μ) be a non-singular conservative ergodic dynamical system with $\mu(X) = 1$. For $\eta > 0$, let

$$X^\eta_\alpha = \left\{ x \in X : \liminf_{n\to\infty} \left(\sum_{i=0}^{n-1} \omega_i(x) \Big/ n^\alpha \right) > \eta \right\}$$

and X_α be the invariant set $\bigcup_{\eta>0} X^\eta_\alpha$. If $\mu(X_\alpha) = 1$, we say that α is a positive dimension for (T, X, μ).

LEMMA 2.2. *For two isomorphic systems (T_1, X_1, μ_1) and (T_2, X_2, μ_2), if α is a positive dimension for (T_1, X_1, μ_1), then it is a positive dimension for (T_2, X_2, μ_2).*

Proof. Let ϕ be the conjugacy map between the two systems. So $\phi : X_1 \to X_2$ is invertible with $\phi^{-1} \equiv \psi$, so that $\mu_1 \circ \phi \sim \mu_2$, $\mu_1 \sim \mu_2 \circ \psi$ and of course

$$\phi(T_1^i(x)) = T_2^i(\phi(x)) \quad \mu_1\text{-a.e.}$$

and

$$T_1^i(\psi(y)) = \psi(T_2^i(y)) \quad \mu_2\text{-a.e.}$$

First note that

$$\frac{d\mu_2 \circ \psi}{d\mu_1}(T_1^i(x))\omega_i^1(x) = \omega_i^2(\phi(x))\frac{d\mu_2 \circ \psi}{d\mu_1}(x).$$

Then take the set $A \subset X_1$ $(\mu_1(A) > 0)$, on which $d\mu_2 \circ \psi / d\mu_1$ is bounded above and below. So for $x \in A$ we have

$$\sum_{i=0}^{n-1} \chi_A(T_1^i(x))\omega_i^1(x) \le K \sum_{i=0}^{n-1} \chi_{\phi(A)}(T_2^i(\phi(x)))\omega_i^2(\phi(x))$$

for some $K > 0$. Now by the Hurewicz Theorem, for $\epsilon < \mu_1(A)$, there is $N(x)$ so that, for $n > N$,

$$\begin{aligned}(\mu_1(A) - \epsilon) \sum_{i=0}^{n-1} \omega_i^1(x) &< \sum_{i=0}^{n-1} \chi_A(T_1^i(x))\omega_i^1(x) \\ &\le K \sum_{i=0}^{n-1} \omega_i^2(\phi(x)).\end{aligned}$$

The result easily follows. □

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Definition 2.3. If $\alpha = \{\sup \alpha' : \mu(X_{\alpha'}) = 1\}$ then we call α the critical dimension for (T, X, μ).

The reader will have noted that the proof of Lemma 2.2 is easily extended to give the following statement.

LEMMA 2.4. *If α and α_A are the critical dimensions of T and the induced transformation T_A $(A \subset X, \mu(A) > 0)$, respectively, then $\alpha \leq \alpha_A$.*

LEMMA 2.5. (i) *The critical dimension of a II_1 system is equal to one.*
(ii) *The critical dimension of a II_∞ or type III system is less than or equal to one.*

Proof. Both these statements follow from Theorem 1 of [**2**], by which we know that the following statements are equivalent:

(a) there exists a finite T-invariant measure equivalent to μ;

(b)
$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \omega_i(x)$$
exists as a finite strictly positive real number μ-almost everywhere;

(c)
$$0 < \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \omega_i(x)$$
for μ-almost all $x \in X$. □

3. *Odometer actions*

In this section we show that the critical dimension is non-trivial; that is, it is not simply equal to one or zero. We do this by calculating the critical dimension for type III product odometer actions.

Definition 3.1. Let
$$X = \prod_{i=1}^{\infty} \mathbb{Z}_2.$$
The odometer action on X is that invertible transformation $T : X \to X$ which is defined by the following. If j is the least positive integer such that $x_j = 0$, then
$$(Tx)_j = 1$$
$$(Tx)_i = 0 \quad i < j$$
$$(Tx)_i = x_i \quad i > j.$$
T is not defined for $x = (1, 1, 1, \ldots)$; T^{-1} is not defined for $x = (0, 0, 0, \ldots)$.

We include the following theorem taken from [**1**, p. 127]. The result will be used for the proof of Theorem 3.3.

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THEOREM 3.2. [**1**] *Let* $y_1, y_2, \ldots$ *be mutually independent random variables with* $\mathbf{E}(y_i) = 0$ *and* $\mathbf{E}(y_i^2) = \sigma_i^2 < \infty$. *Then if*

$$\sum_1^\infty \frac{\sigma_i^2}{i^2} < \infty,$$

$$\lim_{n\to\infty} \frac{y_1 + \cdots + y_n}{n} = 0$$

for μ*-almost all* $x \in X$.

In this section we apply this theorem to estimate the number of zeroes in a coordinate string of a certain length. If we take our random variables to be $y_i(x) = x_i - \mu_i(1)$, then $E(y_i) = 0$ and we see that

$$\lim_{n\to\infty} \frac{\text{number of ones in first } n \text{ coordinates} - \sum_{i=1}^n \mu_i(1)}{n} = 0$$

for μ-almost all $x \in X$.

Now we begin our calculations to determine the critical dimension of a product odometer. Let

$$X = \prod_{i=1}^\infty \mathbb{Z}_2$$

and let T be the odometer action on X which will be equipped with a product measure

$$\mu = \bigotimes_{i=1}^\infty \mu_i.$$

We will assume that $\mu_i(1)$ is bounded away from zero for all $i \in \mathbb{N}$. Each $x \in X$ shall be identified with a unique sequence $S_x = \{n_1, n_2, \ldots, n_p \ldots\} \subseteq \mathbb{N}$, where $n_i \in S_x$ if and only if $x_{n_i} = 0$.

Suppose for simplicity that $n_1 = 1$. Let

$$[0]_q = \{x \in X : x_1 = x_2 = \cdots = x_q = 0\}.$$

Then for all $x \in [0]_q$ we have that

$$\sum_{i=0}^{2^q-1} \omega_i(x)\mu([0]_q) = 1.$$

More generally, let $I_p = 2^{n_1} + 2^{n_2-1} + \cdots + 2^{n_p-1} - 1$ be the first integer k for which $(T^k(x))_i = 1$ for all $i = 1, 2, \ldots, n_p$. Then

$$\sum_{I_{p-1}+1}^{I_p} \omega_i(x) = \frac{\omega_{I_{p-1}+1}(x)}{\prod_{i=1}^{n_p-1} \mu_i(0)} = \frac{\mu_{n_p}(1)}{\mu_{n_p}(0)} \frac{1}{\prod_{i=1}^{n_p-1} \mu_i(0)} \prod_{i=1}^{n_p-1} \frac{\mu_i(0)}{\mu_i(x_i)} = \frac{\mu_{n_p}(1)}{\prod_{i=1}^{n_p} \mu_i(x_i)}.$$

This simple fact will allow us to obtain the following result which is clearly an analogue of Shannon's Theorem.

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THEOREM 3.3. *Let T be a product odometer on*

$$\left(X = \prod_{i=1}^{\infty} \mathbb{Z}_2, \mu\right)$$

where $\mu_i(1) \geq \rho > 0$ for each $i \in \mathbb{N}$ and let α be the critical dimension. Then for μ-almost all $x \in X$, we have the following statements:

(i)

$$\liminf_{p\to\infty} -\frac{1}{n_{p+1}} \sum_{i=1}^{n_p} \log(\mu_i(x_i)) \leq \alpha \leq \liminf_{p\to\infty} -\frac{1}{n_p} \sum_{i=1}^{n_p} \log(\mu_i(x_i));$$

(ii)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log(\mu_i(x_i)) - \frac{1}{n} \sum_{i=1}^{n} \mu_i(0)\log(\mu_i(0)) + \mu_i(1)\log(\mu_i(1)) = 0.$$

Note that the right-hand inequality in part (i) of the theorem does not require the assumption that $\mu_i(1)$ is bounded away from zero.

Proof. We estimate the critical dimension by first identifying a point with the sequence $[n_1, n_2, \ldots, n_p \ldots]$ which specifies where each of the zeroes occurs on the coordinate sequence and then examining the growth of the Radon–Nikodym derivative sums $\sum_{i=0}^{I_p} \omega_i(x)$, where $I_p = 2^{n_1} + 2^{n_2-1} + \cdots + 2^{n_p-1} - 1$. Now

$$\sum_{i=0}^{I_p} \omega_i(x) = \sum_{j=1}^{p} \mu_{n_j}(1) \prod_{i=1}^{n_j} \frac{1}{\mu_i(x_i)}.$$

We have the following inequality for any $\alpha \geq 0$:

$$\frac{\sum_{i=0}^{I_p} \omega_i(x)}{(I_p+1)^\alpha} \leq \frac{p}{2^{(n_p-1)\alpha} \prod_{i=1}^{n_p} \mu_i(x_i)} = 2^{n_p((\log p/n_p) - (1/n_p)\sum_{i=1}^{n_p} \log \mu_i(x_i) - \alpha + (\alpha/n_p))}.$$

Now if

$$\alpha > \liminf_{p\to\infty} -\frac{1}{n_p} \sum_{i=1}^{n_p} \log \mu_i(x_i)$$

then

$$\liminf_{p\to\infty} \frac{\sum_{i=0}^{I_p} \omega_i(x)}{(I_p+1)^\alpha} = 0$$

and α is strictly greater than the critical dimension. To obtain the lower bound on the critical dimension, note that for any n with $I_p \leq n \leq I_{p+1}$ we have for any $\alpha \geq 0$,

$$\frac{\sum_{i=0}^{I_p} \omega_i(x)}{2^{\alpha n_{p+1}}} \leq \frac{\sum_{i=0}^{I_p} \omega_i(x)}{(I_{p+1}+1)^\alpha} \leq \frac{\sum_{i=0}^{n} \omega_i(x)}{(n+1)^\alpha}.$$

We use the inequality

$$\sum_{i=0}^{I_p} \omega_i(x) \geq \rho \prod_{i=1}^{n_p} \frac{1}{\mu_i(x_i)}$$

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to show that

$$2^{n_{p+1}(-(1/n_{p+1})\sum_{i=1}^{n_p}\log\mu_i(x_i)-\alpha+(\log\rho/n_{p+1}))} \leq \frac{1}{(n+1)^\alpha}\sum_{i=0}^{n}\omega_i(x).$$

If

$$\alpha < \liminf_{p\to\infty} -\frac{1}{n_{p+1}}\sum_{i=1}^{n_p}\log\mu_i(x_i)$$

for all $p \in \mathbb{N}$, then α is clearly less than the critical dimension. Hence we have proven (i).

To obtain (ii) note that using $y_i(x) = \log\mu_i(x_i) - \mu_i(0)\log\mu_i(0) - \mu_i(1)\log\mu_i(1)$ the result can be established by applying Theorem 3.2. □

An immediate consequence of the theorem is that for any product odometer we can easily calculate the critical dimension provided that

$$\lim_{p\to\infty}\frac{n_{p+1}}{n_p} = 1$$

for μ-almost all $x \in X$. Using Theorem 3.2, it can easily be shown that this condition is satisfied for a wide class of product measures.

LEMMA 3.4. *If there exists $m, n \in \mathbb{N}$ $(m < n)$ and $c > 0$ such that, for all $k \in \mathbb{N}$,*

$$\#\{i \in \{k+1, \ldots, k+n\} : \mu_i(0) \geq c\} \geq m$$

then

$$\lim_{p\to\infty}\frac{n_{p+1}}{n_p} = 1.$$

It would appear then that the critical dimension of a product odometer is intimately related to its AC entropy. In [**3**], the AC entropy of a system of finite coordinate changes (X, Γ, μ) (or alternatively of the odometer action on (X, μ)) is defined as

$$h_{\mathrm{AC}}(\mu) = \lim_{n\to\infty}\frac{1}{n}H(\beta_n)$$

where this exists. Here β_n is the partition of X into 2^n cylinder sets of the form

$$(x_1, x_2, \ldots, x_n) \times \prod_{i=n+1}^{\infty}\mathbb{Z}_2.$$

For μ a product measure, the AC entropy formula reduces to

$$h_{\mathrm{AC}}(\mu) = -\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mu_i(1)\log\mu_i(1) + \mu_i(0)\log\mu_i(0)$$

where the limit exists.

It should be noted that AC entropy was not formulated as an isomorphism invariant. It is not known at this stage the extent to which AC entropy can be said to be invariant, even within measure equivalence classes. Further, AC entropy is only defined on infinite product spaces $\prod_{i=1}^{\infty}\mathbb{Z}_{l(p)}$. Hence, the critical dimension should not be confused with AC entropy, even though in the case of product odometers the two concepts roughly coincide.

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LEMMA 3.5. *For μ, a product measure on X with $\mu_i(1) \geq \rho > 0$, for all $i \in \mathbb{N}$, if $h_{\mathrm{AC}}(\mu)$ exists, then it is equal to the critical dimension of the odometer action on (X, μ).*

Proof. To prove the result we will show that if the AC entropy exists, then either

$$\lim_{p\to\infty} \frac{n_{p+1}}{n_p} = 1$$

for μ-almost all x or $h_{\mathrm{AC}}(\mu) = 0$. Suppose that there is some $\epsilon > 0$ such that, for some infinite sequence $p_k, k = 1, 2, \ldots$, we have

$$\frac{n_{p_k+1}}{n_{p_k}} = 1 + \epsilon_k$$

where $\epsilon_k > \epsilon$. Then by Theorem 3.2

$$\begin{aligned}
&\lim_{k\to\infty} \frac{1}{n_{p_k+1}} \sum_{i=1}^{n_{p_k+1}} x_i - \mu_i(1) \\
&\quad = \lim_{k\to\infty} \frac{1}{n_{p_k+1}} \left(\sum_{i=1}^{n_{p_k}} (x_i - \mu_i(1)) + \sum_{i=n_{p_k}+1}^{i=n_{p_k+1}-1} (1 - \mu_i(1)) + (0 - \mu_{n_{p_k+1}}(1)) \right) \\
&\quad = \lim_{k\to\infty} \frac{1}{n_{p_k+1}} \sum_{i=n_{p_k}+1}^{n_{p_k+1}-1} \mu_i(0) = 0 \quad \mu\text{-a.e.}
\end{aligned}$$

However, as $0 \leq -\log(1-t) \leq t/\rho$ for $t \leq 1 - \rho$, this means that

$$\lim_{k\to\infty} \frac{1}{n_{p_k+1}} \sum_{i=n_{p_k}+1}^{n_{p_k+1}-1} \log \mu_i(1) = 0.$$

By Theorem 3.3(ii) this would imply that

$$\lim_{k\to\infty} \frac{1}{n_{p_k+1}} \sum_{i=1}^{n_{p_k}} \log \mu_i(x_i) = \lim_{k\to\infty} \frac{1}{(1+\epsilon_k) n_{p_k}} \sum_{i=1}^{n_{p_k}} \log(\mu_i(x_i)) = h_{\mathrm{AC}}(\mu).$$

However, as $\epsilon_k > \epsilon > 0$, the only way for this to be true and for the AC entropy limit to exist, would be if $h_{\mathrm{AC}}(\mu) = 0$; but by Theorem 3.3(i) this would immediately imply that the critical dimension was zero. □

It is easy to provide examples of type III product odometers of III_λ product measures on X to show that the critical dimension is not an invariant of orbit equivalence.

As a quick demonstration of this, suppose for $\lambda \in (0, 1)$ we have

$$\mu_i(0) = \frac{\lambda}{1+\lambda}, \quad \mu_i(1) = \frac{1}{1+\lambda}$$

for all $i \in \mathbb{N}$. We can then vary this measure by replacing μ_i with $\mu_i(0) = \mu_i(1) = 1/2$ for, say, i odd. Clearly we will have two III_λ odometers with different critical dimensions.

In fact, the product measure examples used in [**3**] and [**4**] to establish the range of possible values of AC entropy within orbit equivalence classes allow us to state the following result.

THEOREM 3.6. *For all $\lambda \in [0, 1]$ and all $c \in [0, 1]$ there is a III_λ product odometer action with critical dimension equal to c.*

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4. *Markov odometers*

Suppose μ is a Markov measure on

$$X = \prod_{i=1}^{\infty} \mathbb{Z}_2,$$

that is, for each cylinder set,

$$C_n^x = (x_1, x_2, \ldots, x_n) \times \prod_{i=n+1}^{\infty} \mathbb{Z}_2$$

we have $\mu(C_n^x) = K_{x_1} P^1_{x_1,x_2} P^2_{x_2,x_3} \ldots P^{n-1}_{x_{n-1},x_n}$ where $K_0 + K_1 = 1$ and $P^i_{1,0} + P^i_{1,1} = P^i_{0,0} + P^i_{0,1} = 1$. Then using techniques similar to those demonstrated in the previous section, we can provide estimates for the critical dimension of a Markov odometer. First we introduce the notation X_i^j to represent the subset of all $x \in X$ with $x_j = i$. Now for a Markov odometer

$$\sum_{i=0}^{2^k-1} \omega_i(x) = \frac{\mu(X_{x_{k+1}}^{k+1})}{\mu([0]_k \cap X_{x_{k+1}}^{k+1})}$$

for all $x \in [0]_k$. Further, it is easily shown that

$$\begin{aligned} \sum_{I_{p-1}+1}^{I_p} \omega_i(x) &= \frac{\mu(X_1^{n_p}) P^{n_p}_{1,x_{n_p+1}}}{\mu(C^x_{n_p+1})} \\ &= 2^{\log \mu(X_1^{n_p}) + \log P^{n_p}_{1,x_{n_p+1}} - \log K_{x_1} - \sum_{i=1}^{n_p} \log P^i_{x_i,x_{i+1}}}. \end{aligned}$$

Then, following the same arguments used in the proof of Theorem 3.3, we establish the following result.

THEOREM 4.1. *For a Markov odometer as described above and for μ-almost all $x \in X$ we have:*

(i)

$$\alpha \le \liminf_{p\to\infty} -\frac{1}{n_p} \log \mu(C^x_{n_p+1}) = \liminf_{p\to\infty} -\frac{1}{n_p} \sum_{i=1}^{n_p} \log P^i_{x_i,x_{i+1}}$$

(ii)

$$\begin{aligned} &\liminf_{p\to\infty} -\frac{1}{n_{p+1}} (\log \mu(X_1^{n_p}) + \log P^{n_p}_{1,x_{n_p+1}} - \log \mu(C^x_{n_p+1})) \\ &= \liminf_{p\to\infty} -\frac{1}{n_{p+1}} \left(\log \mu(X_1^{n_p}) + \log P^{n_p}_{1,x_{n_p+1}} - \sum_{i=1}^{n_p} \log P^i_{x_i,x_{i+1}} \right) \le \alpha. \end{aligned}$$

Note that we have made no assumptions regarding the weights of the measure μ to obtain (ii). If required, lower bounds could be placed on $\mu(X_1^{n_p})$ and $P_{1,x_{n+1}}$ to give a result more like Theorem 3.3(i).

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Acknowledgements. The author would like to acknowledge the comments and suggestions of Dan Rudolph, Jane Hawkins and Tony Dooley in the preparation of this paper, and the Australian Research Council for their financial support.

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