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Triangles in Regular Graphs with Density Below One Half

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Let $k_3^{\mathrm{reg}}(n,d)$ be the minimum number of triangles in d-regular graphs with n vertices. We find the exact value of $k_3^{\mathrm{reg}}(n,d)$ for d between $2n/5+12\sqrt{n}/5$ and $n/2$. In addition, we identify the structure of the extremal graphs.

1. Introduction

What is the maximum degree d of a triangle-free regular graph? When n, the number of vertices, is even, it is easy to see that $d = n/2$ by taking G to be a complete bipartite graph. Andrásfai, Erdős and Sós [2] showed that a graph of minimum degree greater than $2n/5$ either contains a triangle or is bipartite, and hence, if n is odd, then $d \leqslant 2n/5$. This bound is tight, when n is divisible by 5 and G is a blow-up of a 5-cycle.

Let $k_3^{\mathrm{reg}}(n,d)$ be the minimum number of triangles in d-regular graphs with n vertices. Hence, $k_3^{\mathrm{reg}}(n,d) = 0$ implies that either n is even and $d \leqslant n/2$, or n is odd and $d \leqslant 2n/5$. The author [5] has evaluated k_3^{reg} for $n/2 \leqslant d \leqslant 3n/4$. In this paper we evaluate $k_3^{\mathrm{reg}}(n,d)$ for nearly all $d \leqslant n/2$.

Theorem 1.1. *For every odd integer $n \geqslant 10^7$ and even integer d with $2n/5+12\sqrt{n}/5 \leqslant d \leqslant n/2$,*

$$k_3^{\mathrm{reg}}(n,d) = \frac{d}{4}(3d-n-1).$$

In the next section, we give the structure of the extremal graphs and a sightly stronger statement, Theorem 2.1. In Section 3, we investigate the sum of the squares of the degrees of a graph, which turns out to be an important tool for our proof. Finally, we prove Theorem 2.1, and hence Theorem 1.1, in Section 4.

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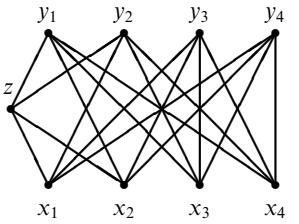


Figure 1. $\mathcal{G}(2,0)$.

2. Structure of the extremal graphs

Let k and l be integers with $k > l \geqslant 0$. Consider a complete bipartite graph $K_{2k+l,2k+l}$ with vertex sets $\{x_1,\ldots,x_{2k+l}\}$ and $\{y_1,\ldots,y_{2k+l}\}$. First, we remove a $(l+1)$-factor from the graph induced by the vertex set $x_1,\ldots,x_k,y_1,\ldots,y_k$, and an l-factor from the graph induced by $x_{k+1},\ldots,x_{2k+l},y_{k+1},\ldots,y_{2k+l}$. Join $x_1,\ldots,x_k,y_1,\ldots,y_k$ to a new vertex z. We denote by $\mathcal{G}(k,l)$ the family of graphs which can be obtained by the above construction. For $G \in \mathcal{G}(k,l)$, G is $2k$-regular with $4k+2l+1$ vertices. Each triangle in G contains the vertex z, so there are $k(k-l-1)$ triangles in G. Notice that $|\mathcal{G}(k,l)| = 1$ for $l = 0$. See Figure 1.

We prove that $\mathcal{G}(k,l)$ is in fact the extremal family for $k_3^{\mathrm{reg}}(4k+2l+1,2k)$.

Theorem 2.1. *Let $k \geqslant 2^{20}$ and $l \geqslant 0$ be integers with $k \geqslant 2l + 6\sqrt{10l} + 1$. Then,*

$$k_3^{\mathrm{reg}}(4k+2l+1,2k) = k(k-l-1).$$

Furthermore, the only extremal graphs are those in $\mathcal{G}(k,l)$.

Now we show that Theorem 1.1 is an easy consequence of Theorem 2.1. We write $n = 4k+2l+1$ and $d = 2k$. Since $d \geqslant 2n/5$, we have $k \geqslant 2^{20}$. Note that $l = (n-2d-1)/2 \leqslant n/10$. By the hypothesis of Theorem 1.1, we have $2k \geqslant 2(4k+2l+1)/5 + 12\sqrt{10l}/5$, that is, $k \geqslant 2l + 6\sqrt{10l} + 1$. Hence, by Theorem 2.1, $k_3^{\mathrm{reg}}(n,d) = k(k-l-1) = d(3d-n-1)/4$.

3. Sum of squared degrees

For a general graph H, we write $\phi(H)$ to be the sum of the squares of the degrees of the vertices of H.

Let a and b be the unique non-negative integers such that $e = \binom{a}{2} + b$ with $0 \leqslant b < a$. The *quasi-complete* graph C_n^e with e edges is the graph with vertex set $v_1,\ldots,v_n$ and $E(C_n^e) = \{v_iv_j : i < j \leqslant a\} \cup \{v_{a+1}v_i : i \leqslant b\}$. The *quasi-star* S_n^e is the complement graph of $C_n^{e'}$ where $e' = \binom{n}{2} - e$. Let $S(n,e) = \phi(S_n^e)$ and $C(n,e) = \phi(C_n^e)$. Ahlswede and Katona [1] proved that the maximum of $\phi(H)$ over all graphs with n vertices and e edges is either $S(n,e)$ or $C(n,e)$. This was rediscovered by Olpp [7]. We rephrase the theorem below.

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Theorem 3.1 (Ahlswede and Katona [1], Olpp [7]). *Let H be a graph with n vertices and e edges. Then*

$$\phi(H) \leqslant \begin{cases} S(n,e) & \text{if } 0 \leqslant e < \frac{1}{2}\binom{n}{2} - \frac{n}{2}, \\ \max\{S(n,e), C(n,e)\} & \text{if } \frac{1}{2}\binom{n}{2} - \frac{n}{2} \leqslant e \leqslant \frac{1}{2}\binom{n}{2} + \frac{n}{2}, \\ C(n,e) & \text{if } \frac{1}{2}\binom{n}{2} + \frac{n}{2} < e \leqslant \binom{n}{2}. \end{cases}$$

However, both $S(n,e)$ and $C(n,e)$ are difficult to express in terms of e and n. Clark, Székely and Entriger [8], de Caen [4] and Nikiforov [6] gave simpler but weaker bounds. Das [3] studied the problem with an extra constraint that the degrees of H are bounded. However, for our purposes, we need to determine the exact maximum of $\phi(H)$ when the degrees are bounded from above and e is small. Notice that in this problem we do not need to specify the number of vertices of H. To be precise, we are interested in the case when $e(H) = \beta r$ for $1 \leqslant \beta < 6/5$ and $\Delta(H) \leqslant r$.

Lemma 3.2. *Let r be a positive integer greater than 4500, and let H be a graph with $\Delta(H) \leqslant r$ and $e(H) = \beta r$ for $1 \leqslant \beta < 6/5$. Then*

$$\phi(H) \leqslant S(r+1, \beta r) = (\beta^2 - 2\beta + 2)r^2 + (5\beta - 4)r.$$

Furthermore, equality holds if and only if H is $S_{r+1}^{\beta r}$ with isolated vertices.

Proof. Suppose H is a graph achieving the maximum value of $\phi(H)$, so $\phi(H) \geqslant S(r+1, \beta r)$. First, we would like to determine the maximum degree Δ of H.

Suppose that $\Delta \leqslant (e(H)+1)/2$. Let W be the set of vertices $v \in H$ with $d(v) > e(H)/16$. Clearly, $|W| \leqslant 32$ and $\sum_{w \in W} d(w) \leqslant e(H) + \binom{|W|}{2} \leqslant e(H) + 496 \leqslant 10(e(H)-1)/9$. This means that we have

$$\begin{aligned} \phi(H) &= \sum_{w \in W} d(w)^2 + \sum_{v \notin W} d(v)^2 \\ &\leqslant \frac{1}{2}(e(H)+1) \sum_{w \in W} d(w) + \frac{1}{16} e(H) \sum_{v \notin W} d(v) \\ &\leqslant \frac{5}{9} e(H)^2 + \frac{1}{8} e(H)^2 = \frac{49}{72} e(H)^2 < r^2, \end{aligned}$$

contradicting the maximality of $\phi(H)$. Hence $\Delta \geqslant e(H)/2 + 1$ and there exists a unique vertex u with $d(u) = \Delta$.

Next, we would like to identify the isolated vertices of H. Suppose vv' is an edge of H with $v \notin \{u\} \cup N(u)$. If $\Delta < r$, the graph $H' = H - vv' + uv$ has

$$\phi(H') = \phi(H) + 2(d(u) + 1 - d(v')) > \phi(H).$$

This contradicts the maximality of $\phi(H)$, so we may assume $\Delta = r$. Since $d(v') < r$, there exists $u' \in N(u)$ not in the neighbourhood of v'. If $d(v) \geqslant d(u')$, $\phi(H - uu' + uv) > \phi(H)$, otherwise, $\phi(H - vv' + u'v') > \phi(H)$. This means that the set of isolated vertices of H is precisely the vertex set $V(H) \backslash (u \cup N(u))$. From now on, we may assume that H contains no isolated vertex.

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We now claim that Δ is in fact equal to r. Suppose the contrary. Since $\Delta < e(H)$, there exists an edge lying entirely in the neighbourhood of u. Let w_1w_2 be such an edge with $d(w_1)+d(w_2)$ minimal. Note that $d(w_1)+d(w_2) \leqslant e(H)-\Delta+3 \leqslant \Delta+1$ as $\Delta \geqslant e(H)/2+1$. Then the graph $\widetilde{H}$ which is obtained by removing the edge w_1w_2 and joining u to a new vertex w_3, has

$$\phi(\widetilde{H}) = \phi(H) + 2(\Delta - d_H(w_1) - d_H(w_2) + 2) > \phi(H),$$

a contradiction, and proves the claim.

Hence, $V(H) = r+1$ and $e(H) = \beta r$. This is a special case of Theorem 3.1, so $\phi(H) = S(r+1, \beta r)$. Furthermore, this is achieved if and only if $H = S^{\beta r}_{r+1}$. □

4. Proof of Theorem 2.1

From Section 2, $k_3^{\mathrm{reg}}(4k+2l+1, 2k) \leqslant k(k-l-1)$. Let G be a graph with $k_3^{\mathrm{reg}}(4k+2l+1, 2k)$ triangles. Partition $V(G)$ into X and Y with $e(X,Y)$ maximal. Without loss of generality $|X| = 2k+l+p+1$ and $|Y| = 2k+l-p$ for some non-negative integer p. We further define $e(X) = \beta k$. Since $2k|X| = \sum_{x\in X} d(x) = e(X,Y) + 2e(X) \leqslant 2k|Y| + 2e(X)$, we have $\beta \geqslant 2p+1$.

For any vertex v, $d_X(v)$ denotes the number of neighbours of v in X, and $d_Y(v)$ is defined similarly. By the maximality of $e(X,Y)$, $d_X(x), d_Y(y) \leqslant k$ for all $x \in X$ and for all $y \in Y$. For $xx' \in E(X)$, the number of triangles containing xx' is at least

$$d_Y(x) + d_Y(x') - |Y| = 2k + p - l - (d_X(x) + d_X(x')). \tag{4.1}$$

Let T be the set of triangles with two vertices in X and one vertex in Y. Summing (4.1) over all edges in X, we get

$$|T| \geqslant \sum_{xx'\in E(X)} 2k+p-l-(d_X(x)+d_X(x')) = e(X)(2k+p-l) - \phi(G[X]).$$

But $|T| \leqslant k(k-l-1)$ and so $k(k-l-1) \geqslant \beta k(2k+p-l) - \phi(G[X])$. Since $p \geqslant 0$, this implies

$$k(k-l-1) \geqslant \beta k(2k-l) - \phi(G[X]). \tag{4.2}$$

Moreover, since $\beta \geqslant 1$ and $l \leqslant k/2$, this gives

$$2\phi(G[X]) \geqslant (3\beta-1)k^2. \tag{4.3}$$

In order to apply Lemma 3.2 to $\phi(G[X])$, we need to bound β.

Given an edge e, let $w(e)$ be the number of triangles containing e. We call the edge e *heavy* if $w(e) > 2(k-l-1)/3$. Let T be a triangle and let e_1, e_2, e_3 be its edges. For $i = 0, 1, 2, 3$, let n_i denote the number of vertices with exactly i neighbours in T. Clearly, $\sum n_i = n$, $\sum in_i = 6k$ and $n_2 + 3n_3 = \sum w(e_i)$. Thus $\sum w(e_i) \geqslant n_2 + 2n_3 \geqslant \sum in_i - \sum n_i = 2k - 2l - 1$, so one of e_1, e_2, e_3 is heavy. This means that every triangle contains at least one heavy edge. Hence, G is triangle-free if we remove all the heavy edges. Let h be the number of heavy edges in G. Notice that $h \leqslant 9k/2$, because $2h(k-l-1)/3 \leqslant \sum w(e) = 3|k_3(G)| \leqslant 3k(k-l-1)$. Let W be the set of vertices incident with at least $3\sqrt{k}$ heavy edges. Then $|W| \leqslant 2h/(3\sqrt{k}) \leqslant 3\sqrt{k}$.

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Let G' be the subgraph formed by removing all the heavy edges and the vertex set W. Note that $|G'| = n - |W|$ and $\delta(G') \geqslant 2k - |W| - 3\sqrt{k}$. We claim that $2k - |W| - 3\sqrt{k} > 2(n - |W|)/5$. This is true provided $k \geqslant 2l + 12\sqrt{k} + 1$. If $l > 2k/5$, then $6\sqrt{10l} > 12\sqrt{k}$, so the inequality holds by the hypothesis of the theorem. If $l \leqslant 2k/5$, $k = 4k/5 + k/5 \geqslant 2l + k/5 > 2l + 12\sqrt{k} + 1$, as $k \geqslant 2^{20}$. Thus, the claim holds and so $\delta(G') > 2|G'|/5$. Moreover, G' is triangle-free and so is bipartite by the theorem of Andrásfai, Erdős and Sós [2]. Now, G' has at least

$$e(G) - 2k|W| - h \geqslant e(G) - 6k\sqrt{k} - 9k/2 \geqslant e(G) - 32k\sqrt{k}/5$$

edges. Since $e(X) + e(Y)$ is minimum, it follows that $e(X) \leqslant 32k\sqrt{k}/5$, in other words $\beta \leqslant 32\sqrt{k}/5$.

Now let U be the set of vertices $x \in X$ with $d_X(x) \geqslant \alpha\sqrt{\beta k}$, where $\alpha \leqslant \sqrt{k/\beta}$ is some number to be decided later. Then, $|U| \leqslant 2\beta k/\alpha\sqrt{\beta k} = 2\sqrt{\beta k}/\alpha$. Also $\sum_{u\in U} d_X(u) \leqslant e(X) + \binom{|U|}{2} \leqslant (1 + 2/\alpha^2)\beta k$. Hence,

$$\begin{aligned}
\phi(G[X]) &= \sum_{u\in U} d_X(u)^2 + \sum_{v\in X\setminus U} d_X(v)^2 \\
&\leqslant k\sum_{u\in U} d_X(u) + \alpha\sqrt{\beta k}\sum_{v\in X\setminus U} d_X(v) \\
&\leqslant (1 + 2/\alpha^2)\beta k^2 + \alpha(1 - 2/\alpha^2)(\beta k)^{3/2} \\
&\leqslant (1 + 2/\alpha^2)\beta k^2 + \alpha(\beta k)^{3/2}.
\end{aligned}$$

The second-to-last inequality is true as $\alpha < \sqrt{k/\beta}$. Since $k \geqslant 2^{20}$ and $\beta \leqslant 32\sqrt{k}/5$, we have $(16k/\beta)^{1/6} < \sqrt{k/\beta}$. Set $\alpha = (16k/\beta)^{1/6}$, so $\phi(G[X]) \leqslant \beta k^2 + 3(\beta^4 k^5/2)^{1/3}$. By (4.3), we know that

$$\beta - 1 - 3(4\beta^4/k)^{1/3}$$

is negative. Now we claim that $\beta < 6/5$. This is enough to show that the above equation is strictly positive when $\beta \in [6/5, 32\sqrt{k}/5]$. In fact, we only need to check the cases when $\beta = 6/5$ and $\beta = 32\sqrt{k}/5$ by convexity. If $\beta = 6/5$, we are done as $k \geqslant 2^{20}$. If $\beta = 32\sqrt{k}/5$, $\beta - 1 - 3(4\beta^4/k)^{1/3}$ is an increasing function in terms of k as $k \geqslant 2^{20}$. It is strictly positive when $k = 2^{20}$. Hence, this proves the claim, so $\beta < 6/5$.

Recall that $\beta \geqslant 2p + 1$, so we have $p = 0$. By Lemma 3.2, $\phi(G[X]) \leqslant (\beta^2 - 2\beta + 2)k^2 + (5\beta - 4)k$. Notice that

$$\begin{aligned}
\beta k(2k - l) - \phi(G[X]) &\geqslant \beta k(2k - l) - \left((\beta^2 - 2\beta + 2)k^2 + (5\beta - 4)k\right) \\
&= k(k - l - 1) + (\beta - 1)(3 - \beta)k^2 - (\beta - 1)(l + 5)k \\
&\geqslant k(k - l - 1) + (\beta - 1)(2 - \beta)k^2 \\
&\geqslant k(k - l - 1).
\end{aligned}$$

Thus equality holds in (4.2). For equality to hold, we need $\beta = 1$, so $e(X) = k$, and by Lemma 3.2, $G[X]$ is a star $K_{1,k}$. Let z be the centre of $K_{1,k}$, $X_z = X \cap N(z)$ and $Y_z = Y \cap N(z)$. By (4.1), each $x \in X_z$ has at least $k - l - 1$ neighbours in Y_z. In fact, each has exactly $k - l - 1$ neighbours in Y_z as $k(k - l - 1) = |k_3| \geqslant \sum_{x\in X_z} |N(x) \cap Y_z| \geqslant |X_z|(k - l - 1) =$

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$k(k-l-1)$. Thus, $G[X_z, Y\setminus Y_z]$ is complete bipartite. Similarly, $G[Y_z, X\setminus X_z]$ is complete bipartite by moving z to Y. By the regularity of the graph, $G[X\setminus(X_z\cup\{z\}), Y\setminus Y_z]$ is $(k-l)$-regular. Hence, G is a member of $\mathcal{G}(k,l)$. This completes the proof of Theorem 2.1.

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