100.19 An explicit formula for averaging sequences

1. *Introduction*

Starting with two real numbers a_1 and a_2, let a_3 be the arithmetic mean of a_1 and a_2, a_4 the arithmetic mean of a_2 and a_3, and so on, or more precisely, $a_{n+2} = \frac{1}{2}(a_n + a_{n+1})$ for any positive integer n. It is relatively straightforward to show that

$$a_n = \tfrac{1}{3}(a_1 + 2a_2) + \tfrac{2}{3}(a_1 - a_2)\left(-\tfrac{1}{2}\right)^{n-1}$$

for every positive integer n, and so $a_n \to \frac{1}{3}(a_1 + 2a_2)$ as $n \to \infty$ (see, for instance, in [1]). Investigating the behaviour of such a non-trivial sequence using computers as a tool is one of the most engaging activities for high school and college students.

Let k be a positive integer such that $k \geqslant 2$ and let $a_1, a_2, \ldots, a_k$ be real numbers. Define a sequence recursively by

$$a_{n+k} = \tfrac{1}{k}(a_n + a_{n+1} + \ldots + a_{n+k-1}) \tag{1}$$

for any positive integer n. The behaviour of the sequence $\{a_n\}_{n=1}^{\infty}$ is more sophisticated when $k > 2$ due to the chaos of terms. To illustrate, if $a_1 = 2$, $a_2 = 4$ and $a_3 = 5$ are specified, then the successive ten terms of the sequence $a_{n+3} = \frac{1}{3}(a_n + a_{n+1} + a_{n+2})$, rounding to four decimal places, are shown in the table below.

n	a_n	n	a_n
4	3.6667	9	4.1838
5	4.2222	10	4.1463
6	4.2963	11	4.1745
7	4.0617	12	4.1682
8	4.1934	13	4.1630

The convergence of $\{a_n\}_{n=1}^{\infty}$ has been well studied (see [2, 3, 4]), and its limit is given by

$$\frac{2}{k(k+1)}(a_1 + 2a_2 + 3a_3 + \ldots + ka_k).$$

Readers who are interested in generalisations of the sequence may consult [1] and [5].

To the best of the authors' knowledge, a closed form of $\{a_n\}_{n=1}^{\infty}$ has not been established. In this Article, an explicit formula for $\{a_n\}_{n=1}^{\infty}$ is presented. Knowing the explicit formula of the sequence leads to a deeper understanding of its behaviour, facilitates calculations, and opens up opportunities for further analysis and exploration. For example, consider the sequence starting with the three integers 2, 4, and 5 mentioned above again. It turns out that the nth term of the sequence involves trigonometric functions as follows (see Example 2)

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$$a_n = \frac{25}{6} + \frac{1}{6}\left(\frac{1}{\sqrt{3}}\right)^{n-1}\left[8\sqrt{2}\sin(n-1)\theta - 13\cos(n-1)\theta\right]$$

where $\theta = \pi + \arctan\sqrt{2}$.

2. *An explicit formula*

For $k \geqslant 2$, let us consider the polynomial

$$P_k(x) = x^k - k^{-1}x^{k-1} - k^{-1}x^{k-2} - \ldots - k^{-1}x - k^{-1}.$$

According to [4], Sanders proved the following properties.

1. If r is a root of $P_k(x)$, then $0 < |r| \leqslant 1$.
2. $r_1 = 1$ is a root of $P_k(x)$.
3. If r is a root of $P_k(x)$ and $|r| = 1$, then $r = 1$.
4. $P_k(x)$ has k distinct complex roots.

Let us make another observation that will be used to prove our main result.

5. If $r_2, r_3, \ldots, r_k$ are roots of $P_k(x)$ and $r_j \neq 1$ for all $j = 2, 3, \ldots, k$, then

$$(1 - r_2)(1 - r_3)\ldots(1 - r_k) = \frac{k+1}{2}. \tag{2}$$

To prove our observation, let us factorise $P_k(x)$ as

$$P_k(x) = \frac{1}{k}(x-1)\left[kx^{k-1} + (k-1)x^{k-2} + \ldots + 2x + 1\right].$$

As $r_1 = 1, r_2, \ldots, r_k$ are roots of $P_k(x)$, we then get

$$(x - r_2)(x - r_3)\ldots(x - r_k) = \frac{1}{k}\left(kx^{k-1} + (k-1)x^{k-2} + \ldots + 2x + 1\right) \tag{3}$$

Consequently, $(1 - r_2)(1 - r_3)\ldots(1 - r_k) = \frac{1}{2}(k+1)$.

Throughout this paper, let $r_1 = 1, r_2, \ldots, r_k$ be distinct roots of the polynomial $P_k(x)$. The main result of the Article is the following theorem.

Theorem 1. Let $k \geqslant 2$ and $a_1, a_2, \ldots, a_k$ be real numbers. For any positive integer n, define

$$a_{n+k} = \tfrac{1}{k}\left(a_n + a_{n+1} + \ldots + a_{n+k-1}\right).$$

Then,

$$a_n = \sum_{j=1}^{k} \frac{r_j^{n-1}}{\prod\limits_{\substack{1 \leqslant m \leqslant k \\ m \neq j}} (r_j - r_m)} \left(\sum_{l=1}^{k} (-1)^{k-l}\sigma_{k-l,j}\, a_l\right),$$

for every positive integer n, where $\sigma_{p,q}$ denotes the pth elementary symmetric polynomials in variables $r_1, r_2, \ldots, r_k$ with variable r_q set to zero, more precisely,

$$\sigma_{p,q} = \begin{cases} \displaystyle\sum_{\substack{1 \leqslant n_1 < \ldots < n_p \leqslant k \\ n_1, \ldots, n_p \neq q}} r_{n_1}\ldots r_{n_p}, & \text{if } p, q = 1, 2, \ldots, k, \\ 1. & \text{if } p = 0, \end{cases}$$

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and so

$$\lim_{n\to\infty} a_n = \frac{2}{k(k+1)}\left(a_1 + 2a_2 + 3a_3 + \ldots + ka_k\right).$$

The formulation of Theorem 1 is slightly complicated; however, it is easy to make use of. Let us illustrate by the following examples.

Example 1: Let a_1, a_2 be real numbers. For any positive integer n, define

$$a_{n+2} = \tfrac{1}{2}\left(a_n + a_{n+1}\right).$$

One can see that $r_1 = 1$ and $r_2 = -\frac{1}{2}$ are the roots of $P_2(x) = x^2 - \frac{1}{2}x - \frac{1}{2}$. Then, $\sigma_{0,1} = \sigma_{0,2} = 1$, $\sigma_{1,1} = r_2$, $\sigma_{1,2} = r_1$, $\sigma_{2,1} = r_2$, $\sigma_{2,2} = r_1$. Theorem 1 implies that

$$a_n = \left(\frac{-r_2a_1 + a_2}{r_1 - r_2}\right)r_1^{n-1} + \left(\frac{-r_1a_1 + a_2}{r_2 - r_1}\right)r_2^{n-1}$$

$$= \frac{1}{3}\left(a_1 + 2a_2\right) + \frac{2}{3}\left(a_1 - a_2\right)\left(-\frac{1}{2}\right)^{n-1}.$$

and hence $a_n \to \frac{1}{3}\left(a_1 + 2a_2\right)$ as $n \to \infty$.

Example 2: Let a_1, a_2, a_3 be real numbers. For any positive integer n, define

$$a_{n+3} = \tfrac{1}{3}\left(a_n + a_{n+1} + a_{n+2}\right).$$

We note that $r_1 = 1$, $r_2 = -\frac{1}{3} - \frac{\sqrt{2}i}{3}$ and $r_3 = -\frac{1}{3} + \frac{\sqrt{2}i}{3}$ are the roots of $P_3(x) = x^3 - \frac{1}{3}x^2 - \frac{1}{3}x - \frac{1}{3}$. Then,

$$\begin{cases} \sigma_{0,1} = \sigma_{0,2} = \sigma_{0,3} = 1, \\ \sigma_{1,1} = r_2 + r_3,\ \sigma_{1,2} = r_1 + r_3,\ \sigma_{1,3} = r_1 + r_2, \\ \sigma_{2,1} = r_2r_3,\ \sigma_{2,2} = r_1r_3,\ \sigma_{2,3} = r_1r_2. \end{cases}$$

Applying Theorem 1, we obtain that

$$a_n = C_1r_1^{n-1} + C_2r_2^{n-1} + C_3r_3^{n-1}, \tag{4}$$

where C_1, C_2 and C_3 are constants given by

$$C_1 = \frac{r_2r_3a_1 - (r_2 + r_3)a_2 + a_3}{(r_1 - r_2)(r_1 - r_3)},$$

$$C_2 = \frac{r_1r_3a_1 - (r_1 + r_3)a_2 + a_3}{(r_2 - r_1)(r_2 - r_3)},$$

$$C_3 = \frac{r_1r_2a_1 - (r_1 + r_2)a_2 + a_3}{(r_3 - r_1)(r_3 - r_2)}.$$

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Elementary computations show that $C_1 = \frac{1}{6}(a_1 + 2a_2 + 3a_3)$ and

$$C_2 = \tfrac{1}{12}\left[5a_1 - 2a_2 - 3a_3 + i\sqrt{2}\,(a_1 + 5a_2 - 6a_3)\right].$$

At first glance, the equation (4) seems not to give real numbers, but this is not the case. To simplify the equation (4), observe that $\overline{r_2} = r_3$, and so $\overline{C_2} = C_3$. It follows that

$$\begin{aligned} r_2^{n-1}C_2 + r_3^{n-1}C_3 &= r_2^{n-1}C_2 + \overline{r_2^{n-1}C_2} \\ &= 2\Re\left(r_2^{n-1}C_2\right) \\ &= 2\left[\Re\left(r_2^{n-1}\right)\Re\left(C_2\right) - \Im\left(r_2^{n-1}\right)\Im\left(C_2\right)\right] \\ &= \left(\frac{1}{\sqrt{3}}\right)^{n-1}\left[A_1\cos(n-1)\theta - A_2\sin(n-1)\theta\right], \end{aligned}$$

where $A_1 = \frac{1}{6}(5a_1 - 2a_2 - 3a_3)$, $A_2 = \frac{\sqrt{2}}{6}(a_1 + 5a_2 - 6a_3)$ and $\theta = \pi + \arctan\sqrt{2}$. Here, $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of the complex number z, respectively. Therefore,

$$a_n = \frac{1}{6}(a_1 + 2a_2 + 3a_3) + \left(\frac{1}{\sqrt{3}}\right)^{n-1}\left[A_1\cos(n-1)\theta - A_2\sin(n-1)\theta\right].$$

Since $\left(\frac{1}{\sqrt{3}}\right)^{n-1}\left[A_1\cos(n-1)\theta - A_2\sin(n-1)\theta\right] \to 0$ as $n \to \infty$, we obtain $a_n \to \frac{1}{6}(a_1 + 2a_2 + 3a_3)$ as $n \to \infty$. With $a_1 = 2$, $a_2 = 4$, $a_3 = 5$ this gives the formula

$$a_n = \frac{25}{6} + \frac{1}{6}\left(\frac{1}{\sqrt{3}}\right)^{n-1}\left[8\sqrt{2}\sin(n-1)\theta - 13\cos(n-1)\theta\right]$$

mentioned in the introduction.

3. *A proof of Theorem* 1

As the polynomial $P_k(x) = x^k - k^{-1}x^{k-1} - k^{-1}x^{k-2} - \ldots - k^{-1}x - k^{-1}$ has k distinct complex roots $r_1 = 1, r_2, \ldots, r_k$, using the standard method for solving linear recurrence relations with constant coefficients, the solution of the recurrence equation

$$a_{n+k} - k^{-1}a_n - k^{-1}a_{n+1} - \ldots - k^{-1}a_{n+k-1} = 0$$

can be expressed in the form

$$a_n = \sum_{j=1}^{k} c_j r_j^n \tag{5}$$

where the coefficients $c_1, \ldots, c_k$ are fixed by the initial conditions $a_l = \sum_{j=1}^{k} c_j r_j^l$ for all $l = 1, 2, \ldots, k$. To get the desired result, it suffices to

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find $c_1, c_2, \ldots, c_k$. Observe that the above initial conditions can be represented as

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & \ldots & r_k \\ r_1^2 & r_2^2 & \ldots & r_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ r_1^k & r_2^k & \ldots & r_k^k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}. \tag{6}$$

We note that $V(r_1, r_2, \ldots, r_k) = \begin{bmatrix} r_1 & r_2 & \ldots & r_k \\ r_1^2 & r_2^2 & \ldots & r_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ r_1^k & r_2^k & \ldots & r_k^k \end{bmatrix}$ is the Vandermonde matrix; its properties can be found, for example, in [6], pp. 37-38. Since

$$\det V(r_1, r_2, \ldots, r_k) = \prod_{1 \leqslant j \leqslant k} r_j \prod_{1 \leqslant i < j \leqslant k} (r_j - r_i) \neq 0.$$

the matrix $V(r_1, r_2, \ldots, r_k)$ is invertible. If $V(r_1, r_2, \ldots, r_k)^{-1} = \left[b_{jl}\right]$, then

$$b_{jl} = \frac{(-1)^{k-l}\sigma_{k-l,j}}{r_j \prod\limits_{\substack{1 \leqslant m \leqslant k \\ m \neq j}} (r_j - r_m)} \text{ for } j, l = 1, 2, \ldots, k.$$

The explicit formula given for the inverse of the Vandermonde matrix occurs in [6], but it is also more easily accessible via [7]. The equation (6) implies that $c_j = \sum\limits_{l=1}^{k} b_{jl} a_l$ for all $j = 1, 2, \ldots, k$. This fact, together with the equation (5), leads us to

$$a_n = \sum_{j=1}^{k} \frac{r_j^{n-1}}{\prod\limits_{\substack{1 \leqslant m \geqslant k \\ m \neq j}} (r_j - r_m)} \left(\sum_{l=1}^{k} (-1)^{k-l} \sigma_{k-l,j} a_l \right).$$

As $0 < |r_j| < 1$ for all $j = 2, 3, \ldots, k$, we deduce that 3.3

$$\lim_{n \to \infty} a_n = c_1 = \left(\prod_{m=2}^{k} (1 - r_m) \right)^{-1} \left[a_k + \sum_{l=1}^{k-1} (-1)^{k-l} \sigma_{k-l,1} a_l \right]. \tag{7}$$

To simplify the right-hand side of (7), let us consider the polynomial arising from the factorisation (3)

$$x(x - r_2)(x - r_3) \ldots (x - r_k) = x^k + \left(\frac{k-1}{k}\right) x^{k-1} + \ldots + \left(\frac{2}{k}\right) x^2 + \left(\frac{1}{k}\right) x.$$

For any $1 \leqslant j \leqslant k - 1$, Vieta's formulas give

$$\sigma_{j,1} = (-1)^j \left(\frac{k-j}{k}\right). \tag{8}$$

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It follows from the equations (2), (7) and (8) that

$$\lim_{n\to\infty} a_n = \left(\prod_{m=2}^{k}(1 - r_m)\right)^{-1}\left[a_k + \sum_{l=1}^{k-1}(-1)^{k-l}(-1)^{k-l}\left(\frac{l}{k}\right)a_l\right]$$

$$= \frac{2}{k(k+1)}\left(\sum_{l=1}^{k} l a_l\right).$$

The desired result follows.

4. *Remarks*

Theorem 1 provides a way to achieve explicit formulas for other interesting averaging sequences. To illustrate, let us consider the following examples. Suppose that positive real numbers $a_1, a_2, \ldots, a_k$ are given. If the arithmetic mean (1) is replaced by the geometric mean

$$a_{n+k} = \sqrt[k]{a_n a_{n+1} \ldots a_{n+k-1}},$$

applying Theorem 1 to the sequence $b_j = \ln a_j$ yields

$$a_n = \prod_{j=1}^{k}\left(\prod_{l=1}^{k} a_l^{(-1)^{k-l}\sigma_{k-l,j}}\right)^{r_j^{n-1}\Big/\left(\prod_{\substack{1\le m\le k\\ m\ne j}}(r_j - r_m)\right)},$$

and so

$$\lim_{n\to\infty} a_n = \left(\prod_{j=1}^{k} a_j^{j}\right)^{2/k(k+1)}.$$

If the arithmetic mean (1) is replaced by the harmonic mean

$$a_{n+k} = \frac{k}{1/a_n + 1/a_{n+1} + \ldots + 1/a_{n+k-1}},$$

using the similar technique with the sequence $b_j = 1/a_j$, we obtain that

$$a_n = \left(\sum_{j=1}^{k} \frac{r_j^{n-1}}{\prod_{\substack{1\le m\le k\\ m\ne j}}(r_j - r_m)}\left(\sum_{l=1}^{k}(-1)^{k-l}\sigma_{k-l,j}a_l^{-1}\right)\right)^{-1},$$

and

$$\lim_{n\to\infty} a_n = \frac{k(k+1)}{2}\left(\sum_{j=1}^{k}\frac{j}{a_j}\right)^{-1}.$$

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References

1. N. Lord, Sequences of averages revisited, *Math. Gaz*. **95** (July 2011), pp. 314-317.
2. J. Gowers, G. Davis and E. Miles, Sequences of averages, *Math. Gaz*. **70** (October 1986), pp. 200-203.
3. Problem 72.B, Problem Corner, *Math. Gaz*. **72** (October 1988), pp. 232-233.
4. P. Sanders, Averaging sequences, *Math. Gaz*. **78** (November 1994), pp. 326-328.
5. H. Flanders, Averaging sequences again, *Math. Gaz*. **80** (March 1996), pp. 219-222.
6. D. E. Knuth, *The Art of Computer Programming*, Vol. 1 (3rd edn.), Addison Wesley, Boston (1997).
7. ProofWiki: Inverse of Vandermonde Matrix. (24 March 2022). Retrieved August 30, 2023 from https://proofwiki.org/wiki/Inverse of Vandermonde Matrix

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109.20 Less than equable triangles on the Eisenstein lattice

1. *Introduction*

In this paper, we are interested in triangles that are *perimeter-dominant*, i.e., their perimeter exceeds their area. Recall that a *Heron triangle* is a triangle with integer sides and integer area. By Dolan's Theorem [1], the classification of perimeter-dominant Heronian triangles contains three exceptional examples and four infinite families determined by certain Pell, or Pell-like, equations.

We will obtain an analogous result for triangles with side lengths in $\sqrt{3}\mathbb{N}$ and area in $\frac{\sqrt{3}}{4}\mathbb{N}$. As we will consider in Section 3 of this Note, such triangles can be considered to be the equivalent of Heron triangles on the Eisenstein lattice.

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