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Second Order Convergence of the Interpolation based on Q_1^c-Element

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Abstract. In this paper, the second order convergence of the interpolation based on Q_1^c-element is derived in the case of d=1, 2 and 3. Using the integral average on each element, the new basis functions of tensor product type is builded up and we can easily extend it to the higher dimensional case. Finally, some numerical tests are made to show the analytical results of the interpolation errors.

AMS subject classifications: 35Q30, 65N30

Key words: Q_1^c-finite element, Tensor-product polynomial space, Integral average, Non-uniform mesh, Interpolation error.

1. Introduction

The interpolation theory of functions is very important in the numerical analysis of finite element methods [1]. In this paper, we establish the local interpolation error estimates of Q_1^c-element for the scalar-valued functions living in the Sobolev spaces.

Given a set of d intervals $\{[0, L_i]\}_{1\le i\le d}$, the set $K = \prod_{i=1}^{d}[0, L_i]$ is called a cuboid. Let Q_1 be the polynomial space in the variables $x_1, \ldots, x_d$, with real coefficients and of degree at most 1 in each variable. For $x \in K$, there exists a unique vector $(t_1, \ldots, t_d) \in [0, 1]^d$ such that, for all $1 \le i \le d$, $x_i = t_i L_i$. In the case of dimension 1, $Q_1 = P_1$; in the case of dimension $d \ge 2$,

$$Q_1 = \left\{ p(x) : p(x) = \sum_{0\le i_1,\ldots,i_d\le 1} \alpha_{i_1,\ldots,i_d} x_1^{i_1} \cdots x_d^{i_d} \right\}.$$

One readily verifies that Q_1 is a vector space with 2^d dimensions. Obviously, Q_1 is the tensor product Lagrange finite elements. Let $\{T^h\}$, $0 < h \le 1$, be a non-degenerate

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rectangular family of subdivisions of K. Setting $\widetilde{I}_h$ be the interpolation operator constructed by Q_1. Its sharper interpolation error estimate is also well-known [1–3]. If the function u is smooth enough, i.e., $u \in H^2(K)$, then

$$\|u - \widetilde{I}_h u\|_{0,K} + h|u - \widetilde{I}_h u|_{1,K} \leq ch^2|u|_{2,K}.$$

This estimate is optimal. Here and below $c > 0$ is a constant independent of h. $\|\cdot\|_{i,K}$ and $|\cdot|_{i,K}$ denote the norm and semi-norm respectively on the standard Sobolev space $H^i(K)$ with $i = 0,\ 1,\ 2$.

On the other hand, we know that the approximation of order k can be achieved as long as polynomials of degree k are used [1, 4, 5]. These results apply to both the standard tensor-product polynomial spaces and the serendipity elements as well. However, one may wonder what the fewer extra terms in the tensor-product polynomial spaces can be added to obtain the approximate accuracy of $k + 1$ order. In this paper, we consider a new polynomial space Q_1^c by using Q_1 conforming element and the fewer extra basis functions to enhance the approximation order in case of R^d. Here d quadratic polynomials of each direction variable are added in R^d space. In other words, for the case of dimension 1, the function space is

$$Q_1 \oplus \text{span}\left\{ -\frac{h_i^2}{2}\phi_{i-1}\phi_i \right\}.$$

For the case of dimension 2, the basis function space is

$$Q_1 \oplus \text{span}\left\{ -\frac{h_i^2}{2}\phi_{i-1}\phi_i,\ -\frac{\tau_j^2}{2}\psi_{j-1}\psi_j \right\}.$$

For the case of dimension 3, the basis function space is

$$Q_1 \oplus \text{span}\left\{ -\frac{h_i^2}{2}\phi_{i-1}\phi_i,\ -\frac{\tau_j^2}{2}\psi_{j-1}\psi_j,\ -\frac{\kappa_k^2}{2}\theta_{k-1}\theta_k \right\}.$$

Here $h_i,\ \tau_j$ and κ_k are step sizes of each element in directions x, y and z respectively. The basis functions $\phi_i(x)$, $\psi_j(y)$ and $\theta_k(z)$ are linear. Setting I_h be the interpolation operator constructed by Q_1^c. In the cases of dimension 1 and 2, if $u \in H^3(K)$, then the following interpolation error estimate holds

$$\|u - I_h u\|_{0,K} + h|u - I_h u|_{1,K} + h^2|u - I_h u|_{2,K} \leq ch^3|u|_{3,K},$$

(see Theorem 2.1 and Theorem 3.1 for details). In dimension 3, if $u \in H^4(K)$, then the following interpolation error estimate also holds

$$\begin{aligned} &\|u - I_h u\|_{0,K} + h|u - I_h u|_{1,K} + h^2|u - I_h u|_{2,K} \\ \leq& c\Big(h^3|u|_{3,K} + h^4|u|_{4,K}\Big), \end{aligned}$$

(see Theorem 4.1 for details).

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Remark 1.1. Here Q_1^c element is differ from Q_2 element, where Q_1^c element only needs $2^d + d$ degrees of freedom and Q_2 element needs 3^d degrees of freedom in the d-dimensional case. Hence, Q_1^c element is simpler than Q_2 element. Furthermore, we have proved that Q_1^c element is of the convergence rate of same order as Q_2 element in this paper.

The paper is structured as follows. The convergence of the Q_1^c interpolation function for one dimension is provided in Section 2. The convergence of the Q_1^c interpolation function for two dimensions is studied in Section 3. The convergence of the Q_1^c interpolation function for three dimensions is proved in Section 4. In Section 5, numerical experiments are presented to confirm the theoretical analysis for Q_1^c interpolation function. Finally, conclusions are drawn in the last section.

2. Convergence of the Q_1^c interpolation function: 1D case

In this paper, we always assume that $\Omega \subset R^d$ with $d = 1,\ 2,\ 3$ is a bounded domain. Also, we use $\|\cdot\|_{i,\omega}$ and $|\cdot|_{i,\omega}$ denote the norm and semi-norm respectively on the standard Sobolev space $H^i(\omega)$ with $i = 0,\ 1,\ 2,\ 3$ for $\omega \subset \Omega$.

In this section, we consider $J_x = \{I_i; i = 1, \cdots, n_1\}$ be a mesh of $\Omega = (0, L_1)$ with $0 = x_0 < x_1 < \cdots < x_{n_1} = L_1$ and $I_i = [x_{i-1}, x_i]$ and $h_i = x_i - x_{i-1}$. Also, we define $x_{i-\frac{1}{2}} = \frac{1}{2}(x_i + x_{i-1})$ with $i = 1, \cdots, n_1$ and set $h = \max\limits_{1\le i\le n_1} h_i$.

We set that $\{\phi_0(x),\ \phi_1(x), \cdots, \phi_{n_1}(x)\}$ is a set of piecewise linear functions in $(0, L_1)$ such that

$$\phi_i(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}},\ \ x \in I_i; \quad \phi_i(x) = \frac{x_{i+1} - x}{x_{i+1} - x_i},\ \ x \in I_{i+1},$$

for $i = 1, \cdots, n_1 - 1$ and

$$\phi_0(x) = \frac{x_1 - x}{x_1 - x_0},\ \ x \in I_1; \quad \phi_{n_1}(x) = \frac{x - x_{n_1-1}}{x_{n_1} - x_{n_1-1}},\ \ x \in I_{n_1}.$$

Then we establish the following Q_1^c finite element space:

$$X_h = \text{span}\Big\{v_h \in C(\bar{\Omega}) \cap H^1(\Omega); v_h|_{I_i} \in Q_1^c(I_i),\ \ i = 1, \cdots, n_1\Big\},$$

where

$$Q_1^c(I_i) = \text{span}\Big\{\phi_{i-1},\ \phi_i,\ \ -\frac{h_i^2}{2}\phi_{i-1}\phi_i\Big\}.$$

Here each element only has three degrees of freedom in the 1D case. Now, we define the interpolation operator $I_h : H^2(\Omega) \to X_h$ as follows:

$$I_h u|_{I_i} = u(x_{i-1})\phi_{i-1}(x) + u(x_i)\phi_i(x) - \overline{\partial_{xx} u_i}\frac{h_i^2}{2}\phi_{i-1}(x)\phi_i(x),$$

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where

$$\bar{u}_i = \frac{1}{h_i}\int_{x_{i-1}}^{x_i} u(x)dx,$$

i.e. $\bar{u}_i$ denotes the integral average on the element I_i. From the definition of $I_h u$, we have

$$I_h u(x_i) = u(x_i), \qquad I_h u(x_{i-1}) = u(x_{i-1}), \tag{2.1a}$$
$$\partial_x (I_h u)(x_{i-\frac{1}{2}}) = \overline{\partial_x u_i}, \qquad \partial_{xx}(I_h u)(x) = \overline{\partial_{xx} u_i}. \tag{2.1b}$$

From some simple calculations, we obtain the following integration relations.

Lemma 2.1. *There hold*

$$\int_a^b |\phi(x) - \frac{1}{b-a}\int_a^b \phi(t)dt|^2 dx \le \frac{1}{3}(b-a)^2 \int_a^b |\phi_t(t)|^2 dt, \tag{2.2}$$

for each $\phi \in H^1(a,b)$, *and*

$$\frac{1}{2}\Big[\phi(a)+\phi(b)\Big] - \frac{1}{b-a}\int_a^b \phi(t)dt = \frac{1}{2(b-a)}\int_a^b (t-a)(b-t)\phi_{tt}(t)dt, \tag{2.3}$$

$$\phi\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \phi(t)dt = -\frac{1}{2(b-a)}\int_a^{\frac{a+b}{2}} (t-a)^2\phi_{tt}(t)dt - \frac{1}{2(b-a)}\int_{\frac{a+b}{2}}^b (t-b)^2\phi_{tt}(t)dt, \tag{2.4}$$

for each $\phi \in H^2(a,b)$.

Next, we will give some L^2, H^1 *and* H^2 *estimates of the error* $e = u - I_h u$.

Lemma 2.2. *For each* $u \in H^1(I_i)$, *there holds*

$$\|e\|_{0,I_i}^2 \le \frac{h_i^2}{8}\|\partial_x e\|_{0,I_i}^2. \tag{2.5}$$

Proof. From (2.1b), we have

$$\begin{aligned}\|e\|_{0,I_i}^2 &= \int_{x_{i-1}}^{x_{i-\frac{1}{2}}} |e(x) - e(x_{i-1})|^2 dx + \int_{x_{i-\frac{1}{2}}}^{x_i} |e(x) - e(x_i)|^2 dx \\ &=: K_1 + K_2,\end{aligned}$$

where

$$K_1 \le \int_{x_{i-1}}^{x_{i-\frac{1}{2}}} (x - x_{i-1}) \int_{x_{i-1}}^{x_{i-\frac{1}{2}}} |\partial_x e(s)|^2 ds dx \le \frac{h_i^2}{8}\int_{x_{i-1}}^{x_{i-\frac{1}{2}}} |\partial_x e(r)|^2 dr,$$
$$K_2 \le \int_{x_{i-\frac{1}{2}}}^{x_i} (x - x_i) \int_{x_{i-\frac{1}{2}}}^{x_i} |\partial_x e(s)|^2 ds dx \le \frac{h_i^2}{8}\int_{x_{i-\frac{1}{2}}}^{x_i} |\partial_x e(r)|^2 dr,$$

which yields (2.5). □

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Lemma 2.3. *For each $u \in H^2(I_i)$, there holds*

$$\|\partial_x e\|_{0,I_i}^2 \le 2h_i|\partial_x e(x_{i-\frac{1}{2}})|^2 + \frac{h_i^2}{2}\|\partial_{xx} e\|_{0,I_i}^2. \tag{2.6}$$

Proof. From (2.1b), we obtain

$$\begin{aligned}\|\partial_x e\|_{0,I_i}^2 &= \int_{x_{i-1}}^{x_i} |\partial_x e(x) - \partial_x e(x_{i-\frac{1}{2}}) + \partial_x e(x_{i-\frac{1}{2}})|^2 dx \\ &\le 2\int_{x_{i-1}}^{x_i} |x - x_{i-\frac{1}{2}}| \int_{x_{i-1}}^{x_i} |\partial_{xx} e(r)|^2 dr dx + 2h_i|\partial_x e(x_{i-\frac{1}{2}})|^2 \\ &\le \frac{h_i^2}{2}\|\partial_{xx} e\|_{0,I_i}^2 + 2h_i|\partial_x e(x_{i-\frac{1}{2}})|^2,\end{aligned}$$

which yields (2.6). □

Lemma 2.4. *For each $u \in H^3(I_i)$, there holds*

$$2h_i\|\partial_x e(x_{i-\frac{1}{2}})\|_{0,I_i}^2 \le \frac{h_i^4}{160}\|\partial_{xxx} u\|_{0,I_i}^2. \tag{2.7}$$

Proof. From Lemma 2.1 and (2.1b), we have

$$\begin{aligned}\partial_x e(x_{i-\frac{1}{2}}) &= \partial_x u(x_{i-\frac{1}{2}}) - \frac{1}{h_i}\int_{x_{i-1}}^{x_i} \partial_x u(r) dr \\ &= -\frac{1}{2h_i}\int_{x_{i-\frac{1}{2}}}^{x_i} (r - x_i)^2 \partial_{xxx} u(r) dr - \frac{1}{2h_i}\int_{x_{i-1}}^{x_{i-\frac{1}{2}}} (r - x_{i-1})^2 \partial_{xxx} u(r) dr.\end{aligned}$$

Moreover,

$$|\partial_x e(x_{i-\frac{1}{2}})|^2 \le \frac{h_i^3}{320}\|\partial_{xxx} u\|_{0,I_i}^2.$$

Hence we deduce (2.7). □

Lemma 2.5. *For each $u \in H^3(I_i)$, there holds*

$$\|\partial_{xx} e\|_{0,I_i}^2 \le \frac{h_i^2}{3}\|\partial_{xxx} u\|_{0,I_i}^2. \tag{2.8}$$

Proof. Using Lemma 2.1 and (2.1b), we have

$$\int_{x_{i-1}}^{x_i} |\partial_{xx} e(x)|^2 dx = \int_{x_{i-1}}^{x_i} |\partial_{xx} u(x) - \overline{\partial_{xx} u}|^2 dx \le \frac{h_i^2}{3}\|\partial_{xxx} u\|_{0,I_i}^2,$$

which gives (2.8). □

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From Lemmas 2.2–2.5, we conclude the following convergence results.

Theorem 2.1. For each $u \in H^3(I_i)$, there holds

$$\|e\|_{0,I_i} \leq \frac{h_i^3}{4}\|\partial_{xxx}u\|_{0,I_i}, \tag{2.9}$$

$$\|\partial_x e\|_{0,I_i} \leq \frac{h_i^2}{2}\|\partial_{xxx}u\|_{0,I_i}, \tag{2.10}$$

$$\|\partial_{xx} e\|_{0,I_i} \leq \frac{h_i}{\sqrt{3}}\|\partial_{xxx}u\|_{0,I_i}. \tag{2.11}$$

3. Convergence of the Q_1^c interpolation function: 2D case

In this section, we consider $\Omega = (0, L_1) \times (0, L_2)$. Let $J_y = \{J_j; j = 1, \cdots, n_2\}$ be a mesh of $(0, L_2)$ with $0 = y_0 < y_1 < \cdots < y_{n_2} = L_2$ and $J_j = [y_{j-1}, y_j]$ and $\tau_j = y_j - y_{j-1}$. Also, we define $y_{j-\frac{1}{2}} = \frac{1}{2}(y_j + y_{j-1})$ with $j = 1, \cdots, n_2$. So that we establish a mesh of Ω:

$$J_h = \{K_{ij}; i = 1, \cdots, n_1,\ j = 1, \cdots, n_2\},$$

here, $K_{ij} = I_i \times J_j$ and the center of K_{ij} is $(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}})$, see Fig. 2.

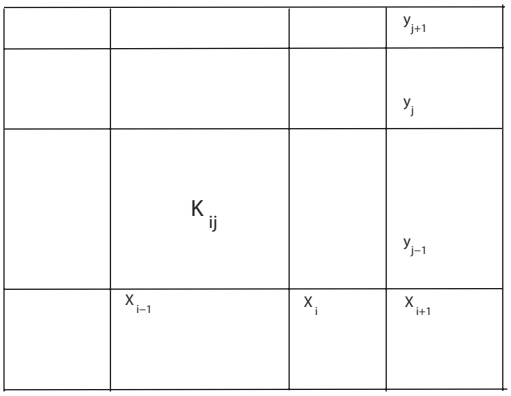


Figure 1: The non-uniform mesh J_h in 2D domain.

We set that $\psi_0(y),\ \psi_1(y), \cdots, \psi_{n_2}(y)$ is a set of piecewise linear functions in $(0, L_2)$ such that

$$\psi_j(y) = \frac{y - y_{j-1}}{y_j - y_{j-1}},\ \ y \in J_j; \quad \psi_j(y) = \frac{y_{j+1} - y}{y_{j+1} - y_j},\ \ y \in J_{j+1},$$

for $j = 1, \cdots, n_2 - 1$ and

$$\psi_0(y) = \frac{y_1 - y}{y_1 - y_0},\ \ y \in J_1; \quad \psi_{n_2}(y) = \frac{y - y_{n-1}}{y_{n_2} - y_{n_2-1}},\ \ y \in J_{n_2}.$$

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Then we establish the following finite element space based on Q_1^c element:

$$X_h = \text{span}\Big\{v_h \in C(\bar{\Omega}) \cap H^1(\Omega); v_h|_{K_{ij}} \in Q_1^c(K_{ij}),\ \ i = 1, \cdots, n_1,\ j = 1, \cdots, n_2\Big\},$$

where

$$Q_1^c(K_{ij}) = \text{span}\Big\{\phi_{i-1}\psi_{j-1},\ \phi_{i-1}\psi_j,\ \phi_i\psi_{j-1},\ \phi_i\psi_j,\ \ -\frac{h_i^2}{2}\phi_{i-1}\phi_i,\ \ -\frac{\tau_j^2}{2}\psi_{j-1}\psi_j\Big\}.$$

Here each element only has six degrees of freedom in the 2D case. Now, for each $u \in H^2(\Omega)$ we define the interpolation function $I_h u(x, y)$ as follows:

$$\begin{aligned} I_h u|_{K_{ij}} = {} & u(x_{i-1}, y_{j-1})\phi_{i-1}(x)\psi_{j-1}(y) + u(x_{i-1}, y_j)\phi_{i-1}(x)\psi_j(y) \\ & + u(x_i, y_{j-1})\phi_i(x)\psi_{j-1}(y) + u(x_i, y_j)\phi_i(x)\psi_j(y) \\ & - \overline{\partial_{xx}u}_{ij}\frac{h_i^2}{2}\phi_{i-1}\phi_i - \overline{\partial_{yy}u}_{ij}\frac{\tau_j^2}{2}\psi_{j-1}\psi_j, \end{aligned}$$

where

$$\bar{u}_{ij} = \frac{1}{h_i\tau_j}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i} u(r, s)drds,$$

i.e. $\bar{u}_{ij}$ denotes the integral average on the element K_{ij}. From the definition of $I_h u_h$, we have

$$I_h u(x_k, y_l) = u(x_k, y_l), \quad k = i-1, i,\ \ l = j-1,\ j, \tag{3.1a}$$

$$\partial_x(I_h u)(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}}) = \frac{1}{2h_i}\int_{x_{i-1}}^{x_i}[\partial_x u(r, y_{j-1}) + \partial_x u(r, y_j)]dr, \tag{3.1b}$$

$$\partial_y(I_h u)(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}}) = \frac{1}{2\tau_j}\int_{y_{j-1}}^{y_j}[\partial_x u(x_{i-1}, s) + \partial_x u(x_i, s)]ds, \tag{3.1c}$$

$$\begin{aligned} &\partial_{xy}(I_h u)(x, y) = \overline{\partial_{xy}u}_{ij}, \quad \partial_{xx}(I_h u)(x, y) = \overline{\partial_{xx}u}_{ij}, \\ &\partial_{yy}(I_h u)(x, y) = \overline{\partial_{yy}u}_{ij}. \end{aligned} \tag{3.1d}$$

From some simple calculations, we obtain the following integration relations.

Lemma 3.1. *For each $\phi \in H^1((a, b) \times (c, d))$, there hold*

$$\begin{aligned} &(b-a)\int_c^d |\phi(\bar{x}, y)|^2 dy \\ &\quad \le (b-a)^2\int_c^d\int_a^b |\partial_x\phi(x, y)|^2 dxdy + 2\int_c^d\int_a^b |\phi(x, y)|^2 dxdy, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &(d-c)\int_a^b |\phi(x, \bar{y})|^2 dx \\ &\quad \le (d-c)^2\int_c^d\int_a^b |\partial_y\phi(x, y)|^2 dxdy + 2\int_c^d\int_a^b |\phi(x, y)|^2 dxdy, \end{aligned} \tag{3.3}$$

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$$\int_c^d \int_a^b |u(x,y)-\bar{u}|^2 dxdy \leq \frac{2}{3}\int_c^d \int_a^b \Big[(b-a)^2|\partial_x u(x,y)|^2 + (d-c)^2|\partial_y u(x,y)|^2\Big] dxdy. \tag{3.4}$$

with $\bar{x} \in [a,b]$ *and* $\bar{y} \in [c,d]$.

Next, we will give some L^2, H^1 *and* H^2 *estimates of the eror* $e = u - I_h u$.

Lemma 3.2. *For each* $u \in H^2(K_{ij})$, *there holds*

$$\|e\|^2_{0,K_{ij}} \leq \frac{h_i^2}{4}\|\partial_x e\|^2_{0,K_{ij}} + \frac{\tau_j^2}{2}\|\partial_y e\|^2_{0,K_{ij}} + \frac{h_i^2\tau_j^2}{4}\|\partial_{xy} e\|^2_{0,K_{ij}}. \tag{3.5}$$

Proof. From the integration by parts, Lemma 3.1 and (3.1d), we have

$$\begin{aligned}
&\|e\|^2_{0,K_{ij}} \\
&= \left(\int_{y_{j-1}}^{y_{j-\frac{1}{2}}}\int_{x_{i-1}}^{x_{i-\frac{1}{2}}} + \int_{y_{j-1}}^{y_{j-\frac{1}{2}}}\int_{x_{i-\frac{1}{2}}}^{x_i} + \int_{y_{j-\frac{1}{2}}}^{y_j}\int_{x_{i-1}}^{x_{i-\frac{1}{2}}} + \int_{y_{j-\frac{1}{2}}}^{y_j}\int_{x_{i-\frac{1}{2}}}^{x_i}\right)|e(x,y)|^2 dxdy \\
&=: K_1 + K_2 + K_3 + K_4,
\end{aligned}$$

where

$$\begin{aligned}
K_1 &= \int_{y_{j-1}}^{y_{j-\frac{1}{2}}}\int_{x_{i-1}}^{x_{i-\frac{1}{2}}} |e(x,y) - e(x_{i-1},y) + e(x_{i-1},y) - e(x_{i-1},y_{j-1})|^2 dxdy \\
&\leq 2\int_{y_{j-1}}^{y_{j-\frac{1}{2}}}\int_{x_{i-1}}^{x_{i-\frac{1}{2}}} \Big[(x-x_{i-1})\int_{x_{i-1}}^{x_{i-\frac{1}{2}}} |\partial_x e(r,y)|^2 ds \\
&\qquad + (y-y_{j-1})\int_{y_{j-1}}^{y_{j-\frac{1}{2}}} |\partial_x e(x_{i-1},s)|^2 ds\Big] dxdy \\
&\leq \frac{h_i^2}{4}\int_{y_{j-1}}^{y_{j-\frac{1}{2}}}\int_{x_{i-1}}^{x_{i-\frac{1}{2}}} |\partial_x e(r,y)|^2 drdy + \frac{h_i\tau_j^2}{8}\int_{y_{j-1}}^{y_{j-\frac{1}{2}}} |\partial_y e(x_{i-1},s)|^2 ds.
\end{aligned}$$

Similar results can be obtained for K_2, K_3 and K_4. Hence

$$\begin{aligned}
\|e\|^2_{0,K_{ij}} &= K_1 + K_2 + K_3 + K_4 \\
&\leq \frac{h_i^2}{4}\|\partial_x e\|^2_{0,K_{ij}} + \frac{h_i\tau_j^2}{8}\int_{y_{j-1}}^{y_j} \Big[|\partial_y e(x_{i-1},t)|^2 + |\partial_y e(x_i,t)|^2\Big] dt \\
&\leq \frac{h_i^2}{4}\|\partial_x e\|^2_{0,K_{ij}} + \frac{\tau_j^2}{2}\|\partial_y e\|^2_{0,K_{ij}} + \frac{h_i^2\tau_j^2}{4}\|\partial_{xy} e\|^2_{0,K_{ij}}.
\end{aligned}$$

which yields (3.5). □

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Lemma 3.3. *For each $u \in H^3(K_{ij})$, there holds*

$$\|\partial_x e\|^2_{0,K_{ij}} \le 3h_i\tau_j|\partial_x e(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}})|^2 + \frac{3}{2}h_i^2\|\partial_{xx}e\|^2_{0,K_{ij}} + \frac{3}{4}\tau_j^2\|\partial_{xy}e\|^2_{0,K_{ij}} + \frac{3}{4}h_i^2\tau_j^2\|\partial_{xxy}u\|^2_{0,K_{ij}}, \quad (3.6)$$

$$\|\partial_y e\|^2_{0,K_{ij}} \le 3h_i\tau_j|\partial_y e(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}})|^2 + \frac{3}{2}\tau_j^2\|\partial_{yy}e\|^2_{0,K_{ij}} + \frac{3}{4}h_i^2\|\partial_{xy}e\|^2_{0,K_{ij}} + \frac{3}{4}h_i^2\tau_j^2\|\partial_{xyy}u\|^2_{0,K_{ij}}. \quad (3.7)$$

Proof. From the integration by parts, Lemma 3.1, Lemma 3.2 and (3.1d), we have

$$\begin{aligned}
&\|\partial_x e\|^2_{0,K_{ij}} \\
&= \int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i} |\partial_x e(x,y) - \partial_x e(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}}) + \partial_x e(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}})|^2 dxdy \\
&\le 3h_i\tau_j|\partial_x e(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}})|^2 + 3\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}|y - y_{j-\frac{1}{2}}|\int_{y_{j-1}}^{y_j}|\partial_{xy}e(x,s)|^2 dsdxdy \\
&\quad +3\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}|x - x_{i-\frac{1}{2}}|\int_{x_{i-1}}^{x_i}|\partial_{xx}e(r, y_{j-\frac{1}{2}})|^2 drdxdy \\
&\le 3h_i\tau_j|\partial_x e(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}})|^2 + \frac{3}{4}\tau_j^2\|\partial_{xy}e\|^2_{0,K_{ij}} + \frac{3}{4}h_i^2\int_{x_{i-1}}^{x_i}|\partial_{xx}e(r, y_{j-\frac{1}{2}})|^2 dr \\
&\le 3h_i\tau_j|\partial_x e(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}})|^2 + \frac{3}{4}\tau_j^2\|\partial_{xy}e\|^2_{0,K_{ij}} + \frac{3}{2}h_i\|\partial_{xx}e\|^2_{0,K_{ij}} + \frac{3}{4}h_i^2\tau_j^2\|\partial_{xxy}u\|^2_{0,K_{ij}}.
\end{aligned}$$

which yields (3.6). Similarly, (3.7) can be proven. □

Lemma 3.4. *For each $u \in H^3(K_{ij})$, there holds*

$$h_i\tau_j|\partial_x e(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}})|^2 \le 3\tau_j^4\|\partial_{xyy}u\|^2_{0,K_{ij}} + 3h_i^4\|\partial_{xxx}u\|^2_{0,K_{ij}} + 3h_i^2\tau_j^2\|\partial_{xxy}u\|^2_{0,K_{ij}}, \quad (3.8)$$

$$h_i\tau_j|\partial_y e(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}})|^2 \le 3h_i^4\|\partial_{xxy}u\|_{0,K_{ij}} + 3h_i^2\tau_j^2\|\partial_{xyy}u\|^2_{0,K_{ij}} + 3\tau_j^4\|\partial_{yyy}u\|^2_{0,K_{ij}}. \quad (3.9)$$

Proof. From the integration by parts, Lemma 2.1, Lemma 3.1, Lemma 3.2 and (3.1d), we have

$$\begin{aligned}
&\partial_x e(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}}) \\
&= \frac{1}{\tau_j}\int_{y_{j-1}}^{y_j}[\partial_x u(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}}) - \partial_x u(x_{i-\frac{1}{2}}, y)]dy \\
&\quad + \frac{1}{h_i\tau_j}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}[\partial_x u(x_{i-\frac{1}{2}}, y) - \partial_x u(x,y)]dxdy
\end{aligned}$$

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$$
\begin{aligned}
&+\frac{1}{h_i\tau_j}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}[\partial_x u(x,y)-\frac{1}{2}(\partial_x u(x,y_j)+\partial_x u(x,y_{j-1}))]dxdy\\
=&\frac{1}{\tau_j}\int_{y_{j-\frac{1}{2}}}^{y_j}(y-y_j)\partial_{xy}u(x_{i-\frac{1}{2}},y)dy+\frac{1}{\tau_j}\int_{y_{j-1}}^{y_{j-\frac{1}{2}}}(y-y_{j-1})\partial_{xy}u(x_{i-\frac{1}{2}},y)dy\\
&-\frac{1}{2h_i\tau_j}\int_{y_{j-1}}^{y_j}\int_{x_{i-\frac{1}{2}}}^{x_i}(x-x_i)^2\partial_{xxx}u(x,y)dxdy\\
&-\frac{1}{2h_i\tau_j}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_{i-\frac{1}{2}}}(x-x_{i-1})^2\partial_{xxx}u(x,y)dxdy\\
&-\frac{1}{2h_i\tau_j}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}(y-y_{j-1})(y-y_j)\partial_{xyy}u(x,y)dxdy\\
=&\frac{1}{2h_i\tau_j}\int_{y_{j-\frac{1}{2}}}^{y_j}\int_{x_{i-1}}^{x_i}(y-y_j)^2\partial_{xyy}u(x,y)dxdy\\
&+\frac{1}{2h_i\tau_j}\int_{y_{j-1}}^{y_{j-\frac{1}{2}}}\int_{x_{i-1}}^{x_i}(y-y_{j-1})^2\partial_{xyy}u(x,y)dxdy\\
&+\frac{1}{2h_i\tau_j}\int_{y_{j-\frac{1}{2}}}^{y_j}\int_{x_{i-\frac{1}{2}}}^{x_i}(y-y_j)(x-x_i)\partial_{xxy}u(x,y)dxdy\\
&+\frac{1}{2h_i\tau_j}\int_{y_{j-\frac{1}{2}}}^{y_j}\int_{x_{i-1}}^{x_{i-\frac{1}{2}}}(y-y_j)(x-x_{i-1})\partial_{xxy}u(x,y)dxdy\\
&+\frac{1}{2h_i\tau_j}\int_{y_{j-1}}^{y_{j-\frac{1}{2}}}\int_{x_{i-\frac{1}{2}}}^{x_i}(y-y_j)(x-x_i)\partial_{xxy}u(x,y)dxdy\\
&+\frac{1}{2h_i\tau_j}\int_{y_{j-1}}^{y_{j-\frac{1}{2}}}\int_{x_{i-1}}^{x_{i-\frac{1}{2}}}(y-y_j)(x-x_{i-1})\partial_{xxy}u(x,y)dxdy\\
&-\frac{1}{2h_i\tau_j}\int_{y_{j-1}}^{y_j}\int_{x_{i-\frac{1}{2}}}^{x_i}(x-x_i)^2\partial_{xxx}u(x,y)dxdy\\
&-\frac{1}{2h_i\tau_j}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_{i-\frac{1}{2}}}(x-x_{i-1})^2\partial_{xxx}u(x,y)dxdy\\
&-\frac{1}{2h_i\tau_j}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}(y-y_{j-1})(y-y_j)\partial_{xyy}u(x,y)dxdy\\
\le&\frac{1}{\sqrt{h_i\tau_j}}\left(\tau_j^2\|\partial_{xyy}u\|_{0,K_{ij}}+h_i^2\|\partial_{xxx}u\|_{0,K_{ij}}+h_i\tau_j\|\partial_{xxy}u\|_{0,K_{ij}}\right),
\end{aligned}
$$

which yields (3.8). Similarly, (3.9) can be proven. □

Lemma 3.5. *For each $u\in H^3(K_{ij})$, there holds*

$$\|\partial_{xy}e\|_{0,K_{ij}}^2\le\frac{2}{3}\left(h_i^2\|\partial_{xxy}u\|_{0,K_{ij}}^2+\frac{2}{3}\tau_j^2\|\partial_{xyy}u\|_{0,K_{ij}}^2\right),\tag{3.10}$$

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$$\|\partial_{xx}e\|^2_{0,K_{ij}} \le \frac{2}{3}\left(h_i^2\|\partial_{xxx}u\|^2_{0,K_{ij}} + \frac{2}{3}\tau_j^2\|\partial_{xxy}u\|^2_{0,K_{ij}}\right), \tag{3.11}$$

$$\|\partial_{yy}e\|^2_{0,K_{ij}} \le \frac{2}{3}\left(h_i^2\|\partial_{xyy}u\|^2_{0,K_{ij}} + \frac{2}{3}\tau_j^2\|\partial_{yyy}u\|^2_{0,K_{ij}}\right). \tag{3.12}$$

Proof. From the integration by parts, Lemma 2.1, Lemma 3.1, Lemma 3.2 and (3.1d), we have

$$\begin{aligned}\|\partial_{xy}e\|^2_{0,K_{ij}} &= \int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i} |\partial_{xy}u(x,y) - \overline{\partial_{xy}u}|^2 dxdy \\ &\le \frac{2}{3}\left(h_i^2\|\partial_{xxy}u\|^2_{0,K_{ij}} + \frac{2}{3}\tau_j^2\|\partial_{xyy}u\|^2_{0,K_{ij}}\right),\end{aligned}$$

which yields (3.10). Similarly, (3.11) and (3.12) can be proven. □

We deduce from Lemma 3.2-Lemma 3.5 that the following convergence results.

Theorem 3.1. *For each $u \in H^3(K_{ij})$, there hold*

$$\|\nabla\nabla e\|_{0,K_{ij}} \le h_i\|\partial_x\nabla\nabla u\|_{0,K_{ij}} + \tau_j\|\partial_y\nabla\nabla u\|_{0,K_{ij}}, \tag{3.13}$$

$$\|\nabla e\|_{0,K_{ij}} \le 4\Big(h_i^2\|\partial_{xx}\nabla u\|_{0,K_{ij}} + 2h_i\tau_j\|\partial_{xy}\nabla u\|_{0,K_{ij}} + \tau_j^2\|\partial_{yy}\nabla u\|_{0,K_{ij}}\Big), \tag{3.14}$$

$$\begin{aligned}\|e\|_{0,K_{ij}} \le{}& 2h_i^3\|\partial_{xxx}u\|_{0,K_{ij}} + 4h_i^2\tau_j\|\partial_{xxy}u\|_{0,K_{ij}} \\ &+4h_i\tau_j^2\|\partial_{xyy}u\|_{o,K_{ij}} + 3\tau_j^3\|\partial_{yyy}u\|_{0,K_{ij}}.\end{aligned} \tag{3.15}$$

4. Convergence of the Q_1^c interpolation function: 3D case

In this section, we assume that $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$. First, we consider $J_z = \{K_k; k = 1, \cdots, n_3\}$ be a mesh of $(0, L_3)$ with $0 = z_0 < z_1 < \cdots < z_{n_3} = L_3$ and $K_k = [z_{k-1}, z_k]$ and $\kappa_k = z_k - z_{k-1}$. Also, we define $z_{k-\frac{1}{2}} = \frac{1}{2}(z_k + z_{k-1})$ with $k = 1, \cdots, n_3$. So that we establish a mesh

$$J_h = \big\{K_{ijk}; i = 1, \cdots, n_1,\ j = 1, \cdots, n_2,\ k = 1, \cdots, n_3\big\},$$

here, $K_{ijk} = I_i \times J_j \times K_k$ and the center of K_{ijk} is $(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}}, z_{k-\frac{1}{2}})$, see Figure 1.

We set that $\theta_0(z),\ \theta_1(z), \cdots, \theta_{n_3}(z)$ is a set of piecewise linear functions in $(0, L_3)$ such that

$$\theta_k(z) = \frac{z - z_{k-1}}{z_k - z_{k-1}},\ \ z \in K_k;\ \ \theta_k(z) = \frac{z_{k+1} - z}{z_{k+1} - z_k},\ \ z \in K_{k+1},$$

for $k = 1, \cdots, n_3 - 1$ and

$$\theta_0(z) = \frac{z_1 - z}{z_1 - z_0},\ \ z \in K_1; \quad \theta_{n_3}(z) = \frac{z - z_{n_3-1}}{z_{n_3} - z_{n_3-1}},\ \ z \in K_{n_3}.$$

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Then we establish the following finite element space based on Q_1^c element:

$$S_h = \text{span}\{v_h \in C(\bar{\Omega}) \cap H^1(\Omega); v_h|_{K_{ijk}} \in Q_1^c(K_{ijk}), \\ i = 1, \cdots, n_1,\ j = 1, \cdots, n_2,\ k = 1, \cdots, n_3\},$$

where

$$Q_1^c(K_{ijk}) = \text{span}\Big\{\phi_{i-1}\psi_{j-1}\theta_{k-1},\ \phi_{i-1}\psi_j\theta_{k-1},\ \phi_i\psi_{j-1}\theta_{k-1},\ \phi_i\psi_j\theta_{k-1},\ \phi_{i-1}\psi_{j-1}\theta_k,$$

$$\phi_{i-1}\psi_j\theta_k,\ \phi_i\psi_{j-1}\theta_k,\ \phi_i\psi_j\theta_k,\ -\frac{h_i^2}{2}\phi_{i-1}\phi_i,\ -\frac{\tau_j^2}{2}\psi_{j-1}\psi_j,\ -\frac{\kappa_k^2}{2}\theta_{k-1}\theta_k\Big\}.$$

Here each element only has eleven degrees of freedom in the 3D case. Now, for each $u \in H^2(\Omega)$ we define the interpolation function $I_h u$ as follows:

$$\begin{aligned} &I_h u(x,y,z)|_{K_{ijk}} \\ &= u(x_{i-1},y_{j-1},z_{k-1})\phi_{i-1}(x)\psi_{j-1}(y)\theta_{k-1}(z) + u(x_{i-1},y_j,z_{k-1})\phi_{i-1}(x)\psi_j(y)\theta_{k-1}(z) \\ &\quad + u(x_i,y_{j-1},z_{k-1})\phi_i(x)\psi_{j-1}(y)\theta_{k-1}(z) + u(x_i,y_j,z_{k-1})\phi_i(x)\psi_j(y)\theta_{k-1}(z) \\ &\quad + u(x_{i-1},y_{j-1},z_k)\phi_{i-1}(x)\psi_{j-1}(y)\theta_k(z) + u(x_{i-1},y_j,z_k)\phi_{i-1}(x)\psi_j(y)\theta_k(z) \\ &\quad + u(x_i,y_{j-1},z_k)\phi_i(x)\psi_{j-1}(y)\theta_k(z) + u(x_i,y_j,z_k)\phi_i(x)\psi_j(y)\theta_k(z) \\ &\quad - \overline{\partial_{xx}u}_{ijk}\frac{h_i^2}{2}\phi_{i-1}(x)\phi_i(x) - \overline{\partial_{yy}u}_{ijk}\frac{\tau_j^2}{2}\psi_{j-1}(y)\psi_j(y) - \overline{\partial_{zz}u}_{ijk}\frac{\kappa_k^2}{2}\theta_{k-1}(z)\theta_k(z), \end{aligned}$$

where

$$\bar{u}_{ijk} = \frac{1}{h_i\tau_j\kappa_k}\int_{z_{k-1}}^{z_k}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i} u(r,s,t)drdsdt,$$

i.e. $\bar{u}_{ijk}$ denotes the integral average on the element K_{ijk}.

From the definition of $I_h u_h$, we have

$$\begin{aligned} &I_h u(x_l,y_m,z_n) = u(x_l,y_m,z_n),\ \ l = i-1,\ i,\ \ m = j-1,\ j,\ \ n = k-1,\ k \\ &\partial_x(I_h u)(x_{i-\frac{1}{2}},y_{j-\frac{1}{2}},z_{k-\frac{1}{2}}) = \frac{1}{4h_i}\int_{x_{i-1}}^{x_i}[\partial_x u(r,y_{j-1},z_{k-1}) + \partial_x u(r,y_{j-1},z_k) \\ &\qquad\qquad + \partial_x u(r,y_j,z_{k-1}) + \partial_x u(r,y_j,z_k)]dr, \\ &\partial_y(I_h u)(x_{i-\frac{1}{2}},y_{j-\frac{1}{2}},z_{k-\frac{1}{2}}) = \frac{1}{4\tau_j}\int_{y_{j-1}}^{y_j}[\partial_y u(x_{i-1},s,z_{k-1}) + \partial_y u(x_{i-1},s,z_k) \\ &\qquad\qquad + \partial_y u(x_i,s,z_{k-1}) + \partial_y u(x_i,s,z_k)]ds, \\ &\partial_z(I_h u)(x_{i-\frac{1}{2}},y_{j-\frac{1}{2}},z_{k-\frac{1}{2}}) = \frac{1}{4\kappa_k}\int_{z_{k-1}}^{z_k}[\partial_z u(x_{i-1},y_{j-1},t) + \partial_z u(x_{i-1},y_j,t) \\ &\qquad\qquad + \partial_z u(x_i,y_{j-1},t) + \partial_z u(x_i,y_j,t)]dt, \\ &\partial_{xy}(I_h u)(x,y,z_{k-\frac{1}{2}}) = \frac{1}{2h_i\tau_j}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}[\partial_{xy}u(r,s,z_{k-1}) + \partial_{xy}u(r,s,z_k)]drds, \end{aligned}$$

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$$\partial_{xz}(I_hu)(x,y_{j-\frac{1}{2}},z)=\frac{1}{2h_i\kappa_k}\int_{z_{k-1}}^{z_k}\int_{x_{i-1}}^{x_i}[\partial_{xz}u(r,y_{j-1},t)+\partial_{xz}u(r,y_j,t)]drdt,$$
$$\partial_{yz}(I_hu)(x_{i-\frac{1}{2}},y,z)=\frac{1}{2\tau_j\kappa_k}\int_{z_{k-1}}^{z_k}\int_{y_{j-1}}^{y_j}[\partial_{yz}u(x_{i-1},s,t)+\partial_{yz}u(x_i,s,t)]dsdt,$$
$$\partial_{xyz}(I_hu)(x,y,z)=\frac{1}{h_i\tau_j\kappa_k}\int_{z_{k-1}}^{z_k}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}\partial_{xyz}u(r,s,t)drdsdt,$$
$$\partial_{xx}(I_hu)(x,y,z)=\overline{\partial_{xx}u}_{ijk},\qquad \partial_{yy}(I_hu)(x,y,z)=\overline{\partial_{yy}u}_{ijk},$$
$$\partial_{zz}(I_hu)(x,y,z)=\overline{\partial_{zz}u}_{ijk}. \tag{4.1}$$

From some simple calculations, we obtain the following integration relations.

Lemma 4.1. *For each $\phi\in H^1((a,b)\times(c,d)\times(e,f))$, there hold*

$$(b-a)\int_e^f\int_c^d|\phi(\bar{x},y,z)|^2dydz$$
$$\le(b-a)^2\int_e^f\int_c^d\int_a^b|\partial_x\phi(x,y,z)|^2dxdydz+2\int_e^f\int_c^d\int_a^b|\phi(x,y,z)|^2dxdydz, \tag{4.2}$$
$$(d-c)\int_e^f\int_a^b|\phi(x,\bar{y},z)|^2dxdz$$
$$\le(d-c)^2\int_e^f\int_c^d\int_a^b|\partial_y\phi(x,y,z)|^2dxdydz+2\int_e^f\int_c^d\int_a^b|\phi(x,y,z)|^2dxdydz, \tag{4.3}$$
$$(f-e)\int_c^d\int_a^b|\phi(x,y,\bar{z})|^2dxdy$$
$$\le(f-e)^2\int_e^f\int_c^d\int_a^b|\partial_y\phi(x,y,z)|^2dxdydz+2\int_e^f\int_c^d\int_a^b|\phi(x,y,z)|^2dxdydz, \tag{4.4}$$
$$\int_e^f\int_c^d\int_a^b|u(x,y,z)-\bar{u}|^2dxdydz$$
$$\le\int_e^f\int_c^d\int_a^b\Big[(b-a)^2|\partial_xu(x,y,z)|^2+(d-c)^2|\partial_yu(x,y,z)|^2$$
$$+(f-e)^2|\partial_zu(x,y,z)|^2\Big]dxdydz. \tag{4.5}$$

with $\bar{x}\in[a,b]$, $\bar{y}\in[c,d]$ and $\bar{z}\in[e,f]$.

Next, we will give some L^2, H^1 and H^2 estimates of the error $e=u-I_hu$.

Lemma 4.2. *For each $u\in H^3(K)$, there holds*

$$\|e\|_{0,K}^2\le\frac{3h_i^2}{8}\|\partial_xe\|_{0,K}^2+\frac{3}{4}\tau_j^2\|\partial_ye\|_{0,K}^2+\frac{3}{2}\kappa_k^2\|\partial_ze\|_{0,K}^2+\frac{3}{2}h_i^2\tau_j^2\kappa_k^2\|\partial_{xyz}e\|_{0,K}^2$$
$$+\frac{3h_i^2\tau_j^2}{8}\|\partial_{xy}e\|_{0,K}^2+\frac{3}{4}h_i^2\kappa_k^2\|\partial_{xz}e\|_{0,K}^2+\frac{3}{4}\tau_j^2\kappa_k^2\|\partial_{yz}e\|_{0,K}^2. \tag{4.6}$$

where $K=K_{ijk}$.

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Proof. From the integration by parts, Lemma 3.1, Lemma 4.1 and (4.1), we have

$$\begin{aligned}
\|e\|_{0,K}^2 &= \sum_{i=1}^{8} K_i \\
&= \int_{z_{k-1}}^{z_{k-\frac{1}{2}}} \left(\int_{y_{j-1}}^{y_{j-\frac{1}{2}}} \int_{x_{i-1}}^{x_{i-\frac{1}{2}}} + \int_{y_{j-1}}^{y_{j-\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_i} + \int_{y_{j-1}}^{y_{j-\frac{1}{2}}} \int_{x_{i-1}}^{x_{i-\frac{1}{2}}} + \int_{y_{j-1}}^{y_{j-\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_i} \right) |e(x,y,z)|^2 \\
&\quad + \int_{z_{k-\frac{1}{2}}}^{z_k} \left(\int_{y_{j-1}}^{y_{j-\frac{1}{2}}} \int_{x_{i-1}}^{x_{i-\frac{1}{2}}} + \int_{y_{j-1}}^{y_{j-\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_i} + \int_{y_{j-1}}^{y_{j-\frac{1}{2}}} \int_{x_{i-1}}^{x_{i-\frac{1}{2}}} \right. \\
&\quad \left. + \int_{y_{j-1}}^{y_{j-\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_i} \right) |e(x,y,z)|^2,
\end{aligned}$$

where

$$\begin{aligned}
K_1 &= \int_{z_{k-1}}^{z_{k-\frac{1}{2}}} \int_{y_{j-1}}^{y_{j-\frac{1}{2}}} \int_{x_{i-1}}^{x_{i-\frac{1}{2}}} |e(x,y,z) - e(x_{i-\frac{1}{2}}, y_{j-1}, z_{k-1})|^2 dxdydz \\
&\le 3 \int_{z_{k-1}}^{z_{k-\frac{1}{2}}} \int_{y_{j-1}}^{y_{j-\frac{1}{2}}} \int_{x_{i-1}}^{x_{i-\frac{1}{2}}} \Big[(x - x_{i-1}) \int_{x_{i-1}}^{x_{i-\frac{1}{2}}} |\partial_x e(r,y,z)|^2 dr \\
&\qquad + (y - y_{j-1}) \int_{y_{j-1}}^{y_{j-\frac{1}{2}}} |\partial_y e(x_{i-1}, s, z)|^2 ds \\
&\qquad + (z - z_{k-1}) \int_{z_{k-1}}^{z_{k-\frac{1}{2}}} |\partial_z e(x_{i-1}, y_{j-1}, t)|^2 dt \Big] dxdydz \\
&\le \frac{3h_i^2}{8} \int_{z_{k-1}}^{z_{k-\frac{1}{2}}} \int_{y_{j-1}}^{y_{j-\frac{1}{2}}} \int_{x_{i-1}}^{x_{i-\frac{1}{2}}} |\partial_x e(r,y,z)|^2 drdydz \\
&\qquad + \frac{3h_i\tau_j^2}{16} \int_{z_{k-1}}^{z_{k-\frac{1}{2}}} \int_{y_{j-1}}^{y_{j-\frac{1}{2}}} |\partial_x e(x_{i-1}, s, z)|^2 dsdz \\
&\qquad + \frac{h_i\tau_j\kappa_k}{8} \int_{z_{k-1}}^{z_{k-\frac{1}{2}}} |\partial_x e(x_{i-1}, y_{j-1}, t)|^2 dt.
\end{aligned}$$

Similar results can be obtained for $K_i,\ i = 2, \cdots, 8$. Hence

$$\begin{aligned}
\|e\|_{0,K}^2 &= \sum_{i=1}^{8} K_i \\
&\le \frac{3h_i^2}{8} \|\partial_x e\|_{0,K}^2 + \frac{3h_i\tau_j^2}{16} \int_{z_{k-1}}^{z_k} \int_{y_{j-1}}^{y_j} (|\partial_y e(x_{i-1}, s, z)|^2 + |\partial_y e(x_i, s, z)|^2) dsdz \\
&\qquad + \frac{3h_i\tau_j\kappa_k^2}{32} \int_{z_{k-1}}^{z_k} \Big[|\partial_z e(x_{i-1}, y_{j-1}, t)|^2 + |\partial_z e(x_{i-1}, y_j, t)|^2 \\
&\qquad + |\partial_z e(x_i, y_{j-1}, t)|^2 + |\partial_z e(x_i, y_j, t)|^2 \Big] dt \\
&\le \frac{3h_i^2}{8} \|\partial_x e\|_{0,K}^2 + \frac{3}{4}\tau_j^2 \|\partial_y e\|_{0,K}^2 + \frac{3h_i^2\tau_j^2}{8} \|\partial_{xy} e\|_{0,K}^2
\end{aligned}$$

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$$
\begin{aligned}
&+\frac{3h_i\kappa_k^2}{16}\int_{z_{k-1}}^{z_k}\int_{y_{j-1}}^{y_j}\Big[2|\partial_z e(x_{i-1},s,t)|^2+2|\partial_z e(x_i,s,t)|^2\\
&+\tau_j^2|\partial_{yz}e(x_{i-1},s,t)|^2+\tau_j^2|\partial_{yz}e(x_i,s,t)|^2\Big]dsdt\\
\le\ &\frac{3h_i^2}{8}\|\partial_x e\|_{0,K}^2+\frac{3}{4}\tau_j^2\|\partial_y e\|_{0,K}^2+\frac{3h_i^2\tau_j^2}{8}\|\partial_{xy}e\|_{0,K}^2\\
&+\frac{3}{2}\kappa_k^2\|\partial_z e\|_{0,K}^2+\frac{3}{4}h_i^2\kappa_k^2\|\partial_{xz}e\|_{0,K}^2+\frac{3}{4}\tau_j^2\kappa_k^2\|\partial_{yz}e\|_{0,K}^2+\frac{3}{2}h_i^2\tau_j^2\kappa_k^2\|\partial_{xyz}e\|_{0,K}^2,
\end{aligned}
$$

which is (4.6). □

Lemma 4.3. *For each $u\in H^4(K)$, there holds*

$$
\begin{aligned}
\|\partial_x e\|_{0,K}^2 \le\ & 4h_i\tau_j\kappa_k|\partial_x e(x_{i-\frac12},y_{j-\frac12},z_{k-\frac12})|^2+h_i^2\|\partial_{xx}e\|_{0,K}^2+2\tau_j^2\|\partial_{xy}e\|_{0,K}^2\\
&+4\kappa_k^2\|\partial_{xz}e\|_{0,K}^2+h_i^2\tau_j^2\|\partial_{xxy}u\|_{0,K}^2+2h_i^2\kappa_k^2\|\partial_{xxz}u\|_{0,K}^2\\
&+4h_i^2\tau_j^2\kappa_k^2\|\partial_{xxyz}u\|_{0,K}^2+2\tau_j^4\kappa_k^2\|\partial_{xyyz}u\|_{0,K}^2+2\tau_j^2\kappa_k^4\|\partial_{xyzz}u\|^2, \quad (4.7)\\
\|\partial_y e\|_{0,K}^2 \le\ & 4h_i\tau_j\kappa_k|\partial_y e(x_{i-\frac12},y_{j-\frac12},z_{k-\frac12})|^2+\tau_j^2\|\partial_{yy}e\|_{0,K}^2+2h_i\|\partial_{xy}e\|_{0,K}^2\\
&+4\kappa_k^2\|\partial_{yz}e\|_{0,K}^2+h_i^2\tau_j^2\|\partial_{xyy}u\|_{0,K}^2+2\tau_j^2\kappa_k^2\|\partial_{yyz}u\|_{0,K}^2\\
&+4h_i^2\tau_j^2\kappa_k^2\|\partial_{xyyz}u\|_{0,K}^2+2h_i^4\kappa_k^2\|\partial_{xxyz}u\|_{0,K}^2+2h_i^2\kappa_k^4\|\partial_{xyzz}u\|^2, \quad (4.8)\\
\|\partial_z e\|_{0,K}^2 \le\ & 4h_i\tau_j\kappa_k|\partial_z e(x_{i-\frac12},y_{j-\frac12},z_{k-\frac12})|^2+\kappa_k^2\|\partial_{zz}e\|_{0,K}^2+2\tau_j^2\|\partial_{yz}e\|_{0,K}^2\\
&+4h_i^2\|\partial_{xz}e\|_{0,K}^2+\tau_j^2\kappa_k^2\|\partial_{yzz}u\|_{0,K}^2+2h_i^2\kappa_k^2\|\partial_{xzz}u\|_{0,K}^2\\
&+4h_i^2\tau_j^2\kappa_k^2\|\partial_{xyzz}u\|_{0,K}^2+2h_i^2\tau_j^4\|\partial_{xyyz}u\|_{0,K}^2+2h_i^4\tau_j^2\|\partial_{xxyz}u\|^2. \quad (4.9)
\end{aligned}
$$

Proof. From the integration by parts, Lemma 3.1, Lemma 3.2 and (3.1d), we have

$$
\begin{aligned}
&\|\partial_x e\|_{0,K}^2=\int_K|\partial_x e(x,y,z)-\partial_x e(x_{i-\frac12},y_{j-\frac12},z_{k-\frac12})+\partial_x e(x_{i-\frac12},y_{j-\frac12},z_{k-\frac12})|^2dxdydz\\
\le\ &4h_i\tau_j\kappa_k|\partial_x e(x_{i-\frac12},y_{j-\frac12},z_{k-\frac12})|^2\\
&+4\int_K\Big[|x-x_{i-\frac12}|\int_{x_{i-1}}^{x_i}|\partial_{xx}e(r,y,z)|^2dr\Big]dxdydz\\
&+4\int_K\Big[|y-y_{j-\frac12}|\int_{y_{j-1}}^{y_j}|\partial_{xy}e(x_{i-\frac12},s,z)|^2ds\Big]dxdydz\\
&+\int_K\Big[|z-z_{k-\frac12}|\int_{z_{k-1}}^{z_k}|\partial_{xz}e(x_{i-\frac12},y_{j-\frac12},t)|^2dt\Big]dxdydz\\
\le\ &4h_i\tau_j|\partial_x e(x_{i-\frac12},y_{j-\frac12},z_{k-\frac12})|^2+h_i^2\|\partial_{xx}e\|_{0,K}^2\\
&+h_i\tau_j^2\int_{z_{k-1}}^{z_k}\int_{y_{j-1}}^{y_j}|\partial_{xy}e(x_{i-\frac12},s,z)|^2dsdz+h_i\tau_j\kappa_k^2\int_{z_{k-1}}^{z_k}|\partial_{xz}e(x_{i-\frac12},y_{j-\frac12},t)|^2dt
\end{aligned}
$$

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$$\begin{aligned}
&\le 4h_i\tau_j\kappa_k|\partial_x e(x_{i-\frac{1}{2}},y_{j-\frac{1}{2}},z_{k-\frac{1}{2}})|^2+h_i^2\|\partial_{xx}e\|_{0,K}^2+2\tau_j^2\|\partial_{xy}e\|_{0,K}^2+h_i^2\tau_j^2\|\partial_{xxy}e\|_{0,K}^2\\
&\quad+2h_i\kappa_k^2\int_{z_{k-1}}^{z_k}\int_{y_{j-1}}^{y_j}|\partial_{xz}e(x_{i-\frac{1}{2}},s,t)|^2dsdt+h_i\tau_j^2\kappa_k^2\int_{z_{k-1}}^{z_k}|\partial_{xyz}e(x_{i-\frac{1}{2}},s,t)|^2dt\\
&\le 4h_i\tau_j\kappa_k|\partial_x e(x_{i-\frac{1}{2}},y_{j-\frac{1}{2}},z_{k-\frac{1}{2}})|^2+h_i^2\|\partial_{xx}e\|_{0,K}^2+2\tau_j^2\|\partial_{xy}e\|_{0,K}^2\\
&\quad+4\kappa_k^2\|\partial_{xz}e\|_{0,K}^2+h_i^2\tau_j^2\|\partial_{xxy}u\|_{0,K}^2+2h_i^2\kappa_k^2\|\partial_{xxz}u\|_{0,K}^2\\
&\quad+2\tau_j^2\kappa_k^2\Big(2h_i^2\|\partial_{xxyz}u\|_{0,K}^2+\tau_j^2\|\partial_{xyyz}u\|_{0,K}^2+\kappa_k^2\|\partial_{xyzz}u\|^2\Big),
\end{aligned}$$

which yields (4.7). Similarly, (4.8) and (4.9) can be proven. □

Lemma 4.4. *For each $u\in H^4(K)$, there holds*

$$\begin{aligned}
&|\partial_x e(x_{i-\frac{1}{2}},y_{j-\frac{1}{2}},z_{k-\frac{1}{2}})|^2\\
&\le\frac{3}{2h_i\tau_j\kappa_k}\Big(\tau_j^4\kappa_k^2\|\partial_{yyz}\partial_x u\|_{0,K}^2+\tau_j^2\kappa_k^4\|\partial_{yzz}\partial_x u\|_{0,K}^2+h_i^2\tau_j^2\kappa_k^2\|\partial_{xyz}\partial_x u\|_{0,K}^2\Big)\\
&\quad+\frac{5}{2h_i\tau_j\kappa_k}\Big(h_i^4\|\partial_{xx}\partial_x u\|_{0,K}^2+\tau_j^4\|\partial_{yy}\partial_x u\|_{0,K}^2+\kappa_k^4\|\partial_{zz}\partial_x u\|_{0,K}^2\Big)\\
&\quad+\frac{5}{2h_i\tau_j\kappa_k}\Big(h_i^2\tau_j^2\|\partial_{xy}\partial_x u\|_{0,K}^2+h_i^2\kappa_k^2\|\partial_{xz}\partial_x u\|_{0,K}^2\Big),
\end{aligned}\tag{4.10}$$

$$\begin{aligned}
&|\partial_y e(x_{i-\frac{1}{2}},y_{j-\frac{1}{2}},z_{k-\frac{1}{2}})|^2\\
&\le\frac{3}{2h_i\tau_j\kappa_k}\Big(h_i^4\kappa_k^2\|\partial_{xxz}\partial_y u\|_{0,K}^2+h_i^2\kappa_k^4\|\partial_{xzz}\partial_y u\|_{0,K}^2+h_i^2\tau_j^2\kappa_k^2\|\partial_{xyz}\partial_y u\|_{0,K}^2\Big)\\
&\quad+\frac{5}{2h_i\tau_j\kappa_k}\Big(h_i^4\|\partial_{xx}\partial_y u\|_{0,K}^2+\tau_j^4\|\partial_{yy}\partial_y u\|_{0,K}^2+\kappa_k^4\|\partial_{zz}\partial_y u\|_{0,K}^2\Big)\\
&\quad+\frac{5}{2h_i\tau_j\kappa_k}\Big(h_i^2\tau_j^2\|\partial_{xy}\partial_y u\|_{0,K}^2+\tau_j^2\kappa_k^2\|\partial_{yz}\partial_y u\|_{0,K}^2\Big),
\end{aligned}\tag{4.11}$$

$$\begin{aligned}
&|\partial_z e(x_{i-\frac{1}{2}},y_{j-\frac{1}{2}},z_{k-\frac{1}{2}})|^2\\
&\le\frac{3}{2h_i\tau_j\kappa_k}\Big(h_i^4\tau_j^2\|\partial_{xxy}\partial_z u\|_{0,K}^2+h_i^2\tau_j^4\|\partial_{xyy}\partial_z u\|_{0,K}^2+h_i^2\tau_j^2\kappa_k^2\|\partial_{xyz}\partial_z u\|_{0,K}^2\Big)\\
&\quad+\frac{5}{2h_i\tau_j\kappa_k}\Big(h_i^4\|\partial_{xx}\partial_z u\|_{0,K}^2+\tau_j^4\|\partial_{yy}\partial_z u\|_{0,K}^2+\kappa_k^4\|\partial_{zz}\partial_z u\|_{0,K}^2\Big)\\
&\quad+\frac{5}{2h_i\tau_j\kappa_k}\Big(h_i^2\kappa_k^2\|\partial_{xz}\partial_z u\|_{0,K}^2+\tau_j^2\kappa_k^2\|\partial_{yz}\partial_z u\|_{0,K}^2\Big).
\end{aligned}\tag{4.12}$$

Proof. From the integration by parts, Lemma 2.1, Lemma 3.1, Lemma 4.1, Lemma 3.2 and (4.1), we have

$$\begin{aligned}
&\partial_x e(x_{i-\frac{1}{2}},y_{j-\frac{1}{2}},z_{k-\frac{1}{2}})=K_1+K_2+K_3+K_4\\
&=\frac{1}{h_i\tau_j\kappa_k}\int_{z_{k-1}}^{z_k}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}\Big[\partial_x u(x_{i-\frac{1}{2}},y_{j-\frac{1}{2}},z_{k-\frac{1}{2}})-\partial_x u(x,y_{i-\frac{1}{2}},z_{k-\frac{1}{2}})\Big]
\end{aligned}\tag{4.13}$$

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$$
\begin{aligned}
&+\frac{1}{h_i\tau_j\kappa_k}\int_{z_{k-1}}^{z_k}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}\Big[\partial_x u(x,y_{j-\frac12},z_{k-\frac12})\\
&\quad-\frac{\partial_x u(x,y_{j-1},z_{k-\frac12})+\partial_x u(x,y_j,z_{k-\frac12})}{2}\Big]\\
&+\frac{1}{h_i\tau_j\kappa_k}\int_{z_{k-1}}^{z_k}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}\Big[\frac12\partial_x u(x,y_{j-1},z_{k-\frac12})\\
&\quad-\frac{\partial_x u(x,y_{j-1},z_{k-1})+\partial_x u(x,y_{j-1},z_k)}{4}\Big]\\
&+\frac{1}{h_i\tau_j\kappa_k}\int_{z_{k-1}}^{z_k}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}\Big[\frac12\partial_x u(x,y_j,z_{k-\frac12})\\
&\quad-\frac{\partial_x u(x,y_j,z_{k-1})+\partial_x u(x,y_j,z_k)}{4}\Big],
\end{aligned}
$$

where

$$
\begin{aligned}
K_1&=\frac{1}{h_i\tau_j\kappa_k}\int_{z_{k-1}}^{z_k}\int_{y_{j-1}}^{y_j}\Big[\int_{x_{i-1}}^{x_{i-\frac12}}(x-x_{i-1})\partial_{xx}u(x,y_{j-\frac12},z_{k-\frac12})dx\\
&\quad+\int_{x_{i-\frac12}}^{x_i}(x-x_i)\partial_{xx}u(x,y_{i-\frac12},z_{k-\frac12})dx\Big]dydz\\
&=-\frac{1}{2h_i\tau_j\kappa_k}\int_{z_{k-1}}^{z_k}\int_{y_{j-1}}^{y_j}\Big[\int_{x_{i-1}}^{x_{i-\frac12}}(x-x_{i-1})^2\partial_{xxx}u(x,y,z)dx\\
&\quad+\int_{x_{i-\frac12}}^{x_i}(x-x_i)^2\partial_{xxx}u(x,y,z)dx\Big]dydz\\
&\quad+\frac{1}{h_i\tau_j\kappa_k}\int_{z_{k-1}}^{z_k}\int_{y_{j-1}}^{y_j}\Big[\int_{x_{i-1}}^{x_{i-\frac12}}(x-x_{i-1})\int_y^{y_{j-\frac12}}\partial_{xxy}u(x,s,z)ds\\
&\quad+(x-x_{i-1})\int_z^{z_{k-\frac12}}\partial_{xxz}u(x,y,t)dt\\
&\quad+(x-x_{i-1})\int_{z_{k-1}}^{z_{k-\frac12}}\int_y^{y_{j-\frac12}}\partial_{xxyz}u(x,s,t)dsdt\\
&\quad+\frac{1}{h_i\tau_j\kappa_k}\int_{z_{k-1}}^{z_k}\int_{y_{j-1}}^{y_j}\Big[\int_{x_{i-\frac12}}^{x_i}(x-x_i)\int_y^{y_{j-\frac12}}\partial_{xxy}u(x,s,z)ds\\
&\quad+(x-x_{i-1})\int_z^{z_{k-\frac12}}\partial_{xxz}u(x,y,t)dt\\
&\quad+(x-x_i)\int_{z_{k-1}}^{z_{k-\frac12}}\int_y^{y_{j-\frac12}}\partial_{xxyz}u(x,s,t)dsdt\\
&\le\frac{1}{2\sqrt{h_i\tau_j\kappa_k}}\Big(h_i^2\|\partial_{xxx}u\|_{0,K}+h_i\tau_j\|\partial_{xxy}u\|_{0,K}\\
&\quad+h_i\kappa_k\|\partial_{xxz}u\|_{0,K}+h_i\tau_j\kappa_k\|\partial_{xxyz}u\|_{0,K}\Big),
\end{aligned}
\tag{4.14}
$$

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$$
\begin{aligned}
K_2 &= \frac{1}{h_i\tau_j\kappa_k}\int_{z_{k-1}}^{z_k}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}[\partial_x u(x,y_{j-\frac12},z_{k-\frac12})-\partial_x u(x,y,z_{k-\frac12})]dxdydz\\
&\quad+\frac{1}{h_i\tau_j\kappa_k}\int_{z_{k-1}}^{z_k}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}\Big[\partial_x u(x,y,z_{k-\frac12})\\
&\quad-\frac{\partial_x u(x,y_{j-1},z_{k-\frac12})+\partial_x u(x,y_j,z_{k-\frac12})}{2}\Big]dxdydz\\
&= -\frac{1}{2h_i\tau_j\kappa_k}\int_{z_{k-1}}^{z_k}\int_{x_{i-1}}^{x_i}\Big[\int_{y_{j-1}}^{y_{j-\frac12}}(y-y_{j-1})^2\partial_{xyy}u(x,y,z_{k-\frac12})dy\\
&\quad+\int_{y_{j-\frac12}}^{y_j}(y-y_j)^2\partial_{xyy}u(x,y,z_{k-\frac12})dy\Big]dxdz\\
&\quad-\frac{1}{2h_i\tau_j\kappa_k}\int_{z_{k-1}}^{z_k}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}(y_j-y)(y-y_{j-1})\partial_{xyy}u(x,y,z_{k-\frac12})dxdydz\\
&\le \frac{1}{8\sqrt{h_i\tau_j\kappa_k}}\Big(2\tau_j^2\|\partial_{xyy}u\|_{0,K}+\tau_j^2\kappa_k\|\partial_{xyyz}u\|_{0,K}\Big), \qquad (4.15)\\
K_3 &= -\frac{1}{4h_i\tau_j\kappa_k}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}\Big[\int_{z_{k-1}}^{z_{k-\frac12}}(z-z_{k-1})^2\partial_{xzz}u(x,y_{j-1},z)dz\\
&\quad+\int_{z_{k-\frac12}}^{z_k}(z-z_k)^2\partial_{xzz}u(x,y_{j-1},z)dz\Big]dxdz\\
&\quad-\frac{1}{4h_i\tau_j\kappa_k}\int_{z_{k-1}}^{z_k}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}(z_k-z)(z-z_{k-1})\partial_{xzz}u(x,y_{j-1},z)dxdydz\\
&\le \frac{1}{16\sqrt{h_i\tau_j\kappa_k}}\Big(2\kappa_k^2\|\partial_{xzz}u\|_{0,K}+\tau_j\kappa_k^2\|\partial_{xyzz}u\|_{0,K}\Big),\\
K_4 &\le \frac{1}{16\sqrt{h_i\tau_j\kappa_k}}\Big(2\kappa_k^2\|\partial_{xzz}u\|_{0,K}+\tau_j\kappa_k^2\|\partial_{xyzz}u\|_{0,K}\Big),
\end{aligned}
$$

which give (4.10). Similarly, (4.11) and (4.12) can be proven. □

Lemma 4.5. *For each $u\in H^4(K)$, there holds*

$$
\begin{aligned}
\|\partial_{xy}e\|_{0,K}^2 &\le \frac{16}{3}\Big(h_i^2\|\partial_x\partial_{xy}u\|_{0,K}^2+\tau_j^2\|\partial_y\partial_{xy}u\|_{0,K}^2+\kappa_k^2\|\partial_z\partial_{xy}u\|_{0,K}^2\Big)\\
&\quad+\frac{8}{3}\Big(h_i^2\kappa_k^2\|\partial_{xz}\partial_{xy}u\|_{0,K}^2+\tau_j^2\kappa_k^2\|\partial_{yz}\partial_{xy}u\|_{0,K}^2+\kappa_j^4\|\partial_{zz}\partial_{xy}u\|_{0,K}^2\Big), \qquad (4.16)\\
\|\partial_{xz}e\|_{0,K}^2 &\le \frac{16}{3}\Big(h_i^2\|\partial_x\partial_{xz}u\|_{0,K}^2+\tau_j^2\|\partial_y\partial_{xz}u\|_{0,K}^2+\kappa_k^2\|\partial_z\partial_{xz}u\|_{0,K}^2\Big)\\
&\quad+\frac{8}{3}\Big(h_i^2\tau_j^2\|\partial_{xy}\partial_{xz}u\|_{0,K}^2+\tau_j^4\|\partial_{yy}\partial_{xz}u\|_{0,K}^2+\tau_j^2\kappa_k^2\|\partial_{yz}\partial_{xz}u\|_{0,K}^2\Big), \qquad (4.17)\\
\|\partial_{yz}e\|_{0,K}^2 &\le \frac{16}{3}\Big(h_i^2\|\partial_x\partial_{yz}u\|_{0,K}^2+\tau_j^2\|\partial_y\partial_{yz}u\|_{0,K}^2+\kappa_k^2\|\partial_z\partial_{yz}u\|_{0,K}^2\Big)\\
&\quad+\frac{8}{3}\Big(h_i^4\|\partial_{xx}\partial_{yz}u\|_{0,K}^2+h_i^2\tau_j^2\|\partial_{xy}\partial_{yz}u\|_{0,K}^2+h_i^2\kappa_k^2\|\partial_{xz}\partial_{yz}u\|_{0,K}^2\Big), \qquad (4.18)
\end{aligned}
$$

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$$\|\partial_{xx}e\|_{0,K}^2 \le h_i^2\|\partial_x\partial_{xx}u\|_{0,K}^2 + \tau_j^2\|\partial_y\partial_{xx}u\|_{0,K}^2 + \kappa_k^2\|\partial_z\partial_{xx}u\|_{0,K}^2, \tag{4.19}$$

$$\|\partial_{yy}e\|_{0,K}^2 \le h_i^2\|\partial_x\partial_{yy}u\|_{0,K}^2 + \tau_j^2\|\partial_y\partial_{yy}u\|_{0,K}^2 + \kappa_k^2\|\partial_z\partial_{yy}u\|_{0,K}^2, \tag{4.20}$$

$$\|\partial_{zz}e\|_{0,K}^2 \le h_i^2\|\partial_x\partial_{zz}u\|_{0,K}^2 + \tau_j^2\|\partial_y\partial_{zz}u\|_{0,K}^2 + \kappa_k^2\|\partial_z\partial_{zz}u\|_{0,K}^2. \tag{4.21}$$

Proof. From Lemma 2.1, Lemma 3.1, Lemma 4.1 and (4.1), we have

$$\begin{aligned}
&\|\partial_{xy}e\|_{0,K}^2\\
&\le 4\int_K \Big|\partial_{xy}u(x,y,z) - \partial_{xy}u(x,y,z_{k-\frac12})|^2 dxdydz\\
&\quad +4\int_K \Big|\partial_{xy}u(x,y,z_{k-\frac12}) - \frac{1}{h_i\tau_j}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}\partial_{xy}u(r,s,z_{k-\frac12})drds\Big|^2 dxdydz\\
&\quad +\frac{4}{h_i^2\tau_j^2\kappa_k^2}\int_K\Big|\int_K[\partial_{xy}u(r,s,z_{k-\frac12}) - \partial_{xy}u(r,s,t)]drdsdt|^2 dxdydz\\
&\quad +\frac{4}{h_i^2\tau_j^2\kappa_k^2}\int_K\Big|\int_K\left[\partial_{xy}u(r,s,t) - \frac{\partial_{xy}u(r,s,z_{k-1}) + \partial_{xy}u(r,s,z_k)}{2}\right]drdsdt|^2 dxdydz\\
&\le \frac83\int_{z_{k-1}}^{z_k}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}\Big(h_i^2|\partial_{xxy}u(x,y,z_{k-\frac12})|^2 + \tau_j^2|\partial_{xyy}u(x,y,z_{k-\frac12})|^2\Big)dxdydz\\
&\quad +2\kappa_k^2\|\partial_{xyz}u\|_{0,K}^2 + \frac{\kappa_k^4}{2}\|\partial_{xyzz}u\|_{0,K}^2\\
&\le 2\kappa_k^2\|\partial_{xyz}u\|_{0,K}^2 + \frac{16}{3}h_i^2\|\partial_{xxy}u\|_{0,K}^2 + \frac{16}{3}\tau_j^2\|\partial_{xyy}u\|_{0,K}^2\\
&\quad +\frac83 h_i^2\kappa_k^2\|\partial_{xxyz}u\|_{0,K}^2 + \frac83\tau_j^2\kappa_k^2\|\partial_{xyyz}u\|_{0,K}^2 + \frac{\kappa_k^4}{2}\|\partial_{xyzz}u\|_{0,K}^2,
\end{aligned}$$

which yields (4.16). Similarly, (4.17) and (4.18) can be proven. Finally, (4.19)-(4.21) can be proven by Lemma 4.1 and (4.1). □

We deduce from Lemma 3.2-Lemma 3.5 that the following convergence results.

Theorem 4.1. *For each $u \in H^4(K)$, there hold*

$$\begin{aligned}
\|\nabla\nabla e\|_{0,K} &\le \frac{4}{\sqrt3}[h_i\|\partial_x\nabla\nabla u\|_{0,K} + \tau_j\|\partial_y\nabla\nabla u\|_{0,K} + \kappa_k\|\partial_z\nabla\nabla u\|_{0,K}]\\
&\quad +\frac{4}{\sqrt3}[h_i^2\|\partial_{xx}\nabla\nabla u\|_{0,K} + \tau_j^2\|\partial_{yy}\nabla\nabla u\|_{0,K} + \kappa_k^2\|\partial_{zz}\nabla\nabla u\|_{0,K}]\\
&\quad +\frac{4}{\sqrt3}[h_i\tau_j\|\partial_{xy}\nabla\nabla u\|_{0,K} + h_i\kappa_k\|\partial_{xz}\nabla\nabla u\|_{0,K} + \tau_j\kappa_k\|\partial_{yz}\nabla\nabla u\|_{0,K}],
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
\|\nabla e\|_{0,K} &\le c_0[h_i^2\|\partial_{xx}\nabla u\|_{0,K} + \tau_j^2\|\partial_{yy}\nabla u\|_{0,K} + \kappa_k^2\|\partial_{zz}\nabla u\|_{0,K}\\
&\quad +h_i\tau_j\|\partial_{xy}\nabla u\|_{0,K}^2 + h_i\kappa_k\|\partial_{xz}\nabla u\|_{0,K} + \tau_j\kappa_k\|\partial_{yz}\nabla u\|_{0,K}^2]\\
&\quad +c_0[h_i^2\tau_j\|\partial_{xxy}\nabla u\|_{0,K} + h_i^2\kappa_k\|\partial_{xxz}\nabla u\|_{0,K} + h_i\tau_j^2\|\partial_{xyy}\nabla u\|_{0,K} + \tau_j^2\kappa_k\|\partial_{yyz}\nabla u\|_{0,K}]
\end{aligned}$$

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Table 1: Numerical results of the 1D case using mesh T_h.

Mesh	$\|u - I_h u\|_0$	$\|\partial_x(u - I_h u)\|_0$	$\|\partial_{xx}(u - I_h u)\|_0$	Rate	Rate	Rate
h	5.0420e-3	6.2994e-2	1.1046e+0			
$h/2$	6.4225e-4	1.6381e-2	5.6839e-1	2.9728	1.9432	9.5852e-1
$h/4$	8.0672e-5	4.1368e-3	2.8631e-1	2.9930	1.9854	9.8930e-1
$h/8$	1.0096e-5	1.0368e-3	1.4342e-1	2.9982	1.9963	9.9730e-1
$h/16$	1.2624e-6	2.5937e-4	7.1746e-2	2.9996	1.9991	9.9932e-1

$$+c_0[h_i\kappa_k^2\|\partial_{xzz}\nabla u\|_{0,K} + \tau_j\kappa_k^2\|\partial_{yzz}\nabla u\|_{0,K}^2 + h_i\tau_j\kappa_k\|\partial_{xyz}\nabla u\|_{0,K}], \tag{4.23}$$

$$\|e\|_{0,K} \le c_0 \sum_{\alpha+\beta+\gamma=3} h_i^\alpha \tau_j^\beta \kappa_k^\gamma \|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma u\|_{0,K} + c_0 \sum_{\alpha+\beta+\gamma=4} h_i^\alpha \tau_j^\beta \kappa_k^\gamma \|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma u\|_{0,K}. \tag{4.24}$$

Here we set that α, β and γ are non-negative integers.

Remark 4.1. For d-dimensional case, using the integral average and tensor product type approach, only $(2^d + d)$ freedoms of degree are used on each element. Here 2^d denotes the number of dth-linear functions. And d denotes the number of quadratic polynomial of each direction variable.

5. Numerical experiments

In this section, we present four test problems in 1D, 2D and 3D to illustrate the theoretical results obtained in the previous section.

Problem 1: We consider the following function:

$$u(x) = \exp(x)x(x-1), \quad x \in [0,1]$$

and present the interpolation error. Here we use the following two coarse meshes produced by the random numbers in [0,1].

$$T_h = \{0,\ 0.097540,\ 0.63236,\ 0.91338,\ 1\},$$
$$T_{\tilde{h}} = \{0,\ 0.42176,\ 0.65574,\ 0.79221,\ 0.91574,\ 0.95949,\ 1\}$$

From Tables 1-2, we can see that the convergence rate are optimal in terms of three different norms.

Problem 2: We consider the following 2D function:

$$u(x,y) = \exp(2x+y)x(x-1)y(y-1), \quad x,y \in [0,1],$$

and also present the interpolation error on two nonuniform coarse meshes produced by the random numbers in $[0,1] \times [0,1]$. Namely

$$T_h = \{0, 0.13229, 0.26530, 0.60295, 0.87048, 1\} \times \{0, 0.058109, 0.45775, 0.86480, 1\}.$$

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Table 2: Numerical results of the 1D case using mesh $T_{\widetilde{h}}$.

Mesh	$\|u - I_h u\|_0$	$\|\partial_x(u - I_h u)\|_0$	$\|\partial_{xx}(u - I_h u)\|_0$	Rate	Rate	Rate
$\widetilde{h}$	1.5386e-3	2.5114e-2	6.2505e-1			
$\widetilde{h}/2$	1.9488e-4	6.4522e-3	3.1771e-1	2.9809	1.9606	9.7627e-1
$\widetilde{h}/4$	2.4442e-5	1.6243e-3	1.5952e-1	2.9951	1.9900	9.9393e-1
$\widetilde{h}/8$	3.0579e-6	4.0677e-4	7.9846e-2	2.9988	1.9975	9.9847e-1
$\widetilde{h}/16$	3.8232e-7	1.0174e-4	3.9934e-2	2.9997	1.9994	9.9962e-1

Table 3: Numerical results of the 2D case using mesh T_h.

Mesh	$\|u - I_h u\|_0$	$\|\nabla(u - I_h u)\|_0$	$\|\nabla\nabla(u - I_h u)\|_0$	Rate	Rate	Rate
h	7.5219e-3	1.4051e-1	1.7053 e+0			
$h/2$	9.9923e-4	3.8879e-2	8.9263e-1	2.9122	1.8536	9.3393e-1
$h/4$	1.2703e-4	9.9815e-3	4.5219e-1	2.9756	1.9617	9.8112e-1
$h/8$	1.5948e-5	2.5122e-3	2.2686e-1	2.9937	1.9903	9.9513e-1
$h/16$	1.9957e-6	6.2911e-4	1.1353e-1	2.9984	1.9976	9.9877e-1

Table 4: Numerical results of the 2D case using mesh $T_{\widetilde{h}}$.

Mesh	$\|u - I_h u\|_0$	$\|\nabla(u - I_h u)\|_0$	$\|\nabla\nabla(u - I_h u)\|_0$	Rate	Rate	Rate
$\widetilde{h}$	9.2130e-3	1.5728e-1	1.9463e+0			
$\widetilde{h}/2$	1.2205e-3	4.2980e-2	9.9743e-1	2.9162	1.8716	9.6447e-1
$\widetilde{h}/4$	1.5497e-4	1.1003e-2	5.0227e-1	2.9774	1.9658	9.8976e-1
$\widetilde{h}/8$	1.9449e-5	2.7673e-3	2.5160e-1	2.9942	1.9913	9.9734e-1
$\widetilde{h}/16$	2.4336e-6	6.9287e-4	1.2586e-1	2.9985	1.9978	9.9933e-1

$$T_{\widetilde{h}} = \{0, 0.33900, 0.40122, 0.72221, 1\} \times \{0, 0.069368, 0.52698, 0.77836, 0.89424, 1\}.$$

The numerical results are presented respectively in Table 3 and Table 4 in terms of three different norms. We can see that the convergence rates are optimal on different meshes, which confirm our theoretical analysis.

Problem 3: We consider the following 3D function:

$$u(x, y, z) = \exp(x + 2y + 3z)\sin(\pi x)\sin(\pi y)\sin(\pi z), \quad x, y, z \in [0, 1],$$

and also present the interpolation error on two nonuniform coarse meshes produced by the random numbers in $[0, 1] \times [0, 1] \times [0, 1]$. Namely

$$\begin{aligned} T_h = &\{0,\ 0.36582,\ 0.62790,\ 0.76350,\ 1\} \\ &\times \{0, 0.77198,\ 0.93285,\ 0.97274,\ 1\} \\ &\times \{0,\ 0.13887, 0.19203,\ 0.69627,\ 1\} \end{aligned}$$

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Table 5: Numerical results of the 3D case using mesh T_h.

Mesh	$\|u - I_h u\|_0$	$\|\nabla(u - I_h u)\|_0$	$\|\nabla\nabla(u - I_h u)\|_0$	Rate	Rate	Rate
h	2.6577e+0	2.7286e+1	3.4242e+2			
$h/2$	4.8101e-1	1.0430e+1	1.7150e+2	2.4660	1.3875	9.9759e-1
$h/4$	6.5984e-2	2.9168e+0	8.7361e+1	2.8659	1.8382	9.7311e-1
$h/8$	8.4389e-3	7.4913e-1	4.3988e+1	2.9670	1.9611	9.8988e-1
$h/16$	1.0609e-3	1.8854e-1	2.2036e+1	2.9918	1.9904	9.9726e-1

Table 6: Numerical results of the 3D case using mesh $T_{\widetilde{h}}$.

Mesh	$\|u - I_h u\|_0$	$\|\nabla(u - I_h u)\|_0$	$\|\nabla\nabla(u - I_h u)\|_0$	Rate	Rate	Rate
$\widetilde{h}$	1.4245e+0	2.3035e+1	2.3572e+2			
$\widetilde{h}/2$	2.0035e-1	6.4588e+0	1.3139e+2	2.8299	1.8345	8.4323e-1
$\widetilde{h}/4$	2.5777e-2	1.6610+0	6.7267e+1	2.9584	1.9593	9.6588e-1
$\widetilde{h}/8$	3.2453e-3	4.1818e-1	3.3828e+1	2.9896	1.9898	9.9169e-1
$\widetilde{h}/16$	4.0639e-4	1.0473e-1	1.6938e+1	2.9974	1.9975	9.9794e-1

and

$$\begin{aligned} T_{\widetilde{h}} = &\{0,\ 0.093820,\ 0.52540,\ 0.53034,\ 1\} \\ &\times\{0,\ 0.39346,\ 0.48485,\ 0.67143,\ 0.86114,\ 1\} \\ &\times\{0,\ 0.15000,\ 0.34771,\ 0.52005,\ 0.58609,\ 0.74126,\ 1\}. \end{aligned}$$

The numerical results are presented respectively in Table 5 and Table 6 in terms of three different norms. We can see that the convergence rates are also optimal on different meshes, which further confirm our theoretical analysis.

Problem 4: We consider the following 2D function on the unit square domain Ω

$$u = \sin^{\alpha}(\pi x)\sin^{\alpha}(\pi y),$$

where the real parameter $\alpha > 0$ is a constant. Note that u satisfies different kinds of smoothness conditions if the suitable choices of α are given. Here we consider five cases in numerical experiments by choosing different values of α.

(i) If $\alpha = \frac{5}{2}$, then $u \in H^2(\Omega) \cap H_0^1(\Omega)$, but $u \notin H^3(\Omega) \cap H_0^1(\Omega)$.

(ii) If $\alpha = \frac{7}{2}$, then $u \in H^3(\Omega) \cap H_0^1(\Omega)$, but $u \notin H^4(\Omega) \cap H_0^1(\Omega)$.

(iii) If $\alpha = \frac{9}{2}$, then $u \in H^4(\Omega) \cap H_0^1(\Omega)$, but $u \notin H^5(\Omega) \cap H_0^1(\Omega)$.

(iv) If $\alpha = \frac{11}{2}$, then $u \in H^5(\Omega) \cap H_0^1(\Omega)$, but $u \notin H^6(\Omega) \cap H_0^1(\Omega)$.

(v) If $\alpha = 1$, then $u \in H^n(\Omega) \cap H_0^1(\Omega), \quad \forall n \geq 2$.

We present the interpolation error on the nonuniform coarse mesh produced by the random numbers in $[0,1] \times [0,1]$. Namely

$$T_h = \{0, 0.011902, 0.16218, 0.33712, 1\} \times \{0, 0.16565, 0.31122, 0.52853, 0.79428, 1\}.$$

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Table 7: Numerical results of the 2D case $u \in H^2(\Omega) \cap H_0^1(\Omega)$, but $u \notin H^3(\Omega) \cap H_0^1(\Omega)$.

Mesh	$\|u - I_h u\|_0$	$\|\nabla(u - I_h u)\|_0$	$\|\nabla\nabla(u - I_h u)\|_0$	Rate	Rate	Rate
h	9.4798e-2	9.6524e-1	1.1853e+1			
$h/2$	1.5739e-2	3.7255e-1	5.3668e+0	2.5905	1.3734	1.1431e+0
$h/4$	2.0714e-3	9.7979e-2	3.6887e+0	2.9257	1.9269	5.4094e-1
$h/8$	2.6429e-4	2.5270e-2	1.9538e+0	2.9704	1.9550	9.1682e-1
$h/16$	3.3658e-5	6.4308e-3	1.0063e+0	2.9731	1.9744	9.5727e-1

Table 8: Numerical results of the 2D case $u \in H^3(\Omega) \cap H_0^1(\Omega)$, but $u \notin H^4(\Omega) \cap H_0^1(\Omega)$.

Mesh	$\|u - I_h u\|_0$	$\|\nabla(u - I_h u)\|_0$	$\|\nabla\nabla(u - I_h u)\|_0$	Rate	Rate	Rate
h	1.1018e-1	1.0765e+0	1.4643e+1			
$h/2$	2.1640e-2	4.9664e-1	6.3381e+0	2.3480	1.1160	1.2081e+0
$h/4$	2.8794e-3	1.3125e-1	5.1547e+0	2.9099	1.9199	2.9815e-1
$h/8$	3.6010e-4	3.3674e-2	2.6398e+0	2.9993	1.9626	9.6546e-1
$h/16$	4.4983e-5	8.4743e-3	1.3293e+0	3.0009	1.9905	9.8974e-1

Table 9: Numerical results of the 2D case $u \in H^4(\Omega) \cap H_0^1(\Omega)$, but $u \notin H^5(\Omega) \cap H_0^1(\Omega)$.

Mesh	$\|u - I_h u\|_0$	$\|\nabla(u - I_h u)\|_0$	$\|\nabla\nabla(u - I_h u)\|_0$	Rate	Rate	Rate
h	1.2083e-1	1.1809e+0	1.6894e+1			
$h/2$	2.7661e-2	6.1378e-1	7.9856e+0	2.1271	0.94403	1.0810e+0
$h/4$	3.7208e-3	1.6418e-1	6.6223e+0	2.8942	1.9025	2.7006e-1
$h/8$	4.6559e-4	4.2614e-2	3.4090e+0	2.9985	1.9459	9.5798e-1
$h/16$	5.8171e-5	1.0753e-2	1.7190e+0	3.0007	1.9866	9.8776e-1

Table 10: Numerical results of the 2D case $u \in H^5(\Omega) \cap H_0^1(\Omega)$, but $u \notin H^6(\Omega) \cap H_0^1(\Omega)$.

Mesh	$\|u - I_h u\|_0$	$\|\nabla(u - I_h u)\|_0$	$\|\nabla\nabla(u - I_h u)\|_0$	Rate	Rate	Rate
h	1.2836e-1	1.2919e+0	1.8553e+1			
$h/2$	3.4324e-2	7.2427e-1	9.8741e+0	1.9029	0.83488	9.0993e-1
$h/4$	4.6050e-3	1.9860e-1	8.3267e+0	2.8980	1.8667	2.4590e-1
$h/8$	5.7433e-4	5.1784e-2	4.2041e+0	3.0032	1.9393	9.8597e-1
$h/16$	7.1724e-5	1.3095e-2	2.1218e+0	3.0014	1.9835	9.8650e-1

The numerical results are presented respectively in Tables 7-11 in terms of different kinds of smoothness conditions. We can see that the convergence rates are almost optimal, which confirm our theoretical analysis further. Notice that the L^2-norm errors of three different functions are not good, if the smoothness condition of the function $\nabla\nabla(u - I_h u)$ is weak.

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Table 11: Numerical results of the 2D case $u \in H^n(\Omega) \cap H_0^1(\Omega),\ n \geq 2$.

Mesh	$\|u - I_h u\|_0$	$\|\nabla(u - I_h u)\|_0$	$\|\nabla\nabla(u - I_h u)\|_0$	Rate	Rate	Rate
h	3.8218e-2	6.0287e-1	3.9051e+0			
$h/2$	5.1745e-3	1.5451e-1	2.2912e+0	2.8848	1.9642	7.6925e-1
$h/4$	6.5962e-4	3.9074e-2	1.1777e+0	2.9717	1.9834	9.6015e-1
$h/8$	8.2856e-5	9.7986e-3	5.9270e-1	2.9929	1.9956	9.9058e-1
$h/16$	1.0370e-5	2.4516e-3	2.9683e-1	2.9982	1.9989	9.9768e-1

6. Conclusions

In this work, we have presented the second order convergence of the interpolation based on Q_1^c-element in the case of d=1, 2 and 3. Numerical experiments are made to demonstrate the theoretical analysis. Using the integral average and tensor product type approach, this idea can be extended to higher dimensions ($d \geq 4$). In further work, we aim to use this element to numerical methods for PDEs.

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