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AN EXPLICIT MATRIX UNIT REALISATION OF THE WEDDERBURN DECOMPOSITION

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Abstract

We construct explicit group-algebra elements corresponding to the elementary matrices in the Wedderburn decomposition of a finite group algebra. These elements yield an explicit algebra isomorphism that coincides with the classical Wedderburn isomorphism. As applications, we apply our construction to the symmetric group to compute the transition matrix between two natural bases of the centre and reinterpret the Frobenius characteristic map through the Wedderburn decomposition. Consequently, we show that the Schur basis is naturally compatible with the identity-matrix basis arising from the Wedderburn decomposition.

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1. Introduction

The Wedderburn decomposition theorem is one of the fundamental results in the representation theory of finite groups. It asserts that the group algebra of a finite group decomposes as a direct sum of full matrix algebras,

$$\mathbb{C}[G] \cong \bigoplus_{\lambda \in \widehat{G}} \mathrm{End}(V_\lambda),$$

where $\widehat{G}$ denotes the set of isomorphism classes of irreducible representations of G. This decomposition provides a complete description of the algebraic structure of $\mathbb{C}[G]$, and establishes a bridge between the group algebra and the representation theory of G [2, 4, 13]. From this viewpoint, understanding an element of the group algebra is relatively straightforward. Every element of $\mathbb{C}[G]$ can be regarded as a collection of matrices acting on irreducible representation spaces through the Wedderburn decomposition. In contrast, the inverse problem appears to be considerably more subtle: given elementary matrices (or matrix units) in one of the matrix algebras

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End(V_λ), can one explicitly determine the corresponding elements of the group algebra? Although the existence of such elements is guaranteed by the Wedderburn theorem, explicit realisations inside the group algebra are much less transparent. This problem is of independent interest in representation theory. Explicit realisation of matrix units plays an important role in the study of primitive idempotents, Fourier analysis on finite groups, the representation theory of symmetric groups and the centre of universal enveloping algebras [3, 5, 7–10]. Nevertheless, to the best of our knowledge, there has been no systematic treatment constructing matrix units explicitly inside the group algebra and showing that they realise the Wedderburn decomposition itself. The main purpose of the present paper is to develop such a theory.

For every irreducible representation (π^λ, V_λ) of a finite group G, we explicitly construct elements $e^\lambda_{ij} \in \mathbb{C}[G]$ that satisfy the *matrix-unit relations* $e^\lambda_{ij} e^\mu_{kl} = \delta_{\lambda\mu}\delta_{jk} e^\lambda_{il}$ (see [11] for the details of the matrix-unit relations). These elements generate matrix subalgebras naturally isomorphic to End(V_λ), yielding an explicit decomposition of the group algebra into simple matrix algebras (see Theorem 3.1 and Corollary 3.3). More importantly, we prove that the algebra isomorphism obtained from these matrix units coincides exactly with the classical Wedderburn decomposition (see Theorem 3.4). Consequently, our construction provides an explicit matrix unit realisation of the Wedderburn decomposition rather than merely an abstract existence theorem.

We then apply this general construction to the symmetric group algebra $\mathbb{C}[S_m]$. Using the Young basis of irreducible representations, we obtain explicit formulae for the matrix units in terms of Young basis vectors. These formulae provide a concrete realisation of matrix units entirely inside the symmetric group algebra and establish a direct connection between the Young basis and the algebraic structure of $\mathbb{C}[S_m]$. As another application, we investigate the centre of the symmetric group algebra. The centre possesses two distinguished bases. One is the conjugacy-class basis, while the other consists of the identity matrices associated with the simple matrix blocks arising from the Wedderburn decomposition. By exploiting our explicit matrix unit realisation, we compute the transition matrix between these two bases and prove that its entries are precisely the irreducible characters of the symmetric group. This gives a new interpretation of the character table as the transition matrix between two natural bases of the centre.

Finally, we reconsider the Frobenius characteristic map from the perspective of the Wedderburn decomposition. Classically, the Frobenius characteristic map is a linear isomorphism from the centre of the symmetric group algebra to the ring of symmetric functions, sending the conjugacy-class basis to the normalised power-sum basis [1, 6, 12]. However, the Wedderburn decomposition identifies the centre of the group algebra with the direct sum of scalar identity matrices in the simple matrix blocks. Motivated by this observation, we introduce an analogous characteristic map whose target is this direct sum of identity matrices. Comparing the two characteristic maps, we show that the Schur-function basis corresponds naturally to the identity-matrix basis under the Wedderburn decomposition. This provides a conceptual explanation for the distinguished role of Schur functions in the Frobenius characteristic theory,

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showing that the Schur basis is precisely the basis compatible with the identity matrices arising from the Wedderburn decomposition.

2. Preliminaries

2.1. Wedderburn decomposition [2, 4]. Let G be a finite group. Write $\widehat{G}$ for the index set of isomorphic classes of the irreducible representations of G over the complex numbers $\mathbb{C}$ and let $\{V_\lambda \mid \lambda \in \widehat{G}\}$ be a complete set of representatives of the isomorphism classes of irreducible representations of G. Set $d_\lambda = \dim V_\lambda$ for each $\lambda \in \widehat{G}$. Denote by $\mathrm{Mat}_{d_\lambda}(\mathbb{C})$ the set of all $d_\lambda \times d_\lambda$ matrices over $\mathbb{C}$. Note that the representations $\pi_\lambda : G \to \mathrm{Aut}(V_\lambda)$ extend uniquely to algebra homomorphisms $\widetilde{\pi}_\lambda : \mathbb{C}[G] \to \mathrm{Mat}_{d_\lambda}(\mathbb{C})$ for $\lambda \in \widehat{G}$, where $\mathbb{C}[G]$ denotes the group algebra associated with G over $\mathbb{C}$. These linear maps give a canonical algebra isomorphism called the *Wedderburn decomposition*:

$$\widetilde{\pi} : \mathbb{C}[G] \longrightarrow \bigoplus_{\lambda \in \widehat{G}} \mathrm{Mat}_{d_\lambda}(\mathbb{C}). \tag{2.1}$$

2.2. Schur orthogonality relation [13]. Let (π^λ, V_λ) and (π^μ, V_μ) be irreducible representations of a finite group G. Consider any linear transformation $T : V_\mu \to V_\lambda$ and set

$$A_T = \sum_{g \in G} \pi^\lambda(g) T \pi^\mu(g^{-1}) : V_\mu \longrightarrow V_\lambda. \tag{2.2}$$

Then, for $h \in G$ and $x \in V_\mu$,

$$\begin{aligned} \pi^\lambda(h) A_T(x) &= \sum_{g \in G} \pi^\lambda(hg) T \pi^\mu(g^{-1})(x) \\ &= \sum_{t \in G} \pi^\lambda(t) T \pi^\mu(t^{-1}h)(x) \quad (\text{set } t = hg) \\ &= \sum_{t \in G} \pi^\lambda(t) T \pi^\mu(t^{-1}) \pi^\mu(h)(x) = A_T(\pi^\mu(h)x). \end{aligned}$$

This implies that the linear map A_T is a G-module homomorphism from V_μ to V_λ. It follows from Schur's lemma that

$$A_T = \begin{cases} 0 & \text{if } V_\mu \not\simeq V_\lambda, \\ c_T \, \mathrm{id}_{V_\lambda} & \text{if } V_\mu \simeq V_\lambda \quad (c_T \in \mathbb{C}). \end{cases} \tag{2.3}$$

In the case of $V_\mu \simeq V_\lambda$, we see from (2.2) and (2.3) that

$$\mathrm{tr}\, A_T = \sum_{g \in G} \mathrm{tr}(\pi^\lambda(g) T \pi^\lambda(g^{-1})) = |G| \,\mathrm{tr}\, T = c_T \dim V_\lambda. \tag{2.4}$$

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It follows from (2.4) that $c_T = (|G|/d_\lambda)\,\mathrm{tr}\,T$ and this yields

$$\sum_{g\in G} \pi^\lambda(g) T \pi^\lambda(g^{-1}) = \frac{|G|}{d_\lambda}\,\mathrm{tr}\,T \cdot \mathrm{id}_{V_\lambda}. \tag{2.5}$$

Fix a basis $\mathcal{B}_\lambda = \{v_1, \dots, v_{d_\lambda}\}$ for V_λ and take the linear map T in (2.2) to be $T_{jk} : V_\lambda \to V_\lambda$, which sends v_k to v_j and every other basis vector to 0. Then, (2.5) gives

$$\sum_{g\in G} \pi^\lambda(g) T_{jk} \pi^\lambda(g^{-1}) = \frac{|G|}{d_\lambda}\delta_{jk}\,\mathrm{id}_{V_\lambda}, \tag{2.6}$$

where δ_{jk} denotes the Kronecker delta. Comparing the (i,l)-entries on both sides of (2.6) gives

$$\sum_{g\in G} \pi^\lambda_{ij}(g) \pi^\lambda_{kl}(g^{-1}) = \frac{|G|}{d_\lambda}\delta_{jk}\delta_{il}, \tag{2.7}$$

where $\pi^\lambda_{ab}(g)$ denotes the (a,b)-entry of the matrix of the linear map $\pi^\lambda(g)$ with respect to the basis $\mathcal{B}_\lambda$. Combining (2.3) and (2.7) yields the Schur orthogonality relations:

$$\sum_{g\in G} \pi^\lambda_{ij}(g) \pi^\mu_{kl}(g^{-1}) = \frac{|G|}{d_\lambda}\delta_{\lambda\mu}\delta_{il}\delta_{jk}. \tag{2.8}$$

2.3. Partitions [6, 12]. A *partition* λ of m is a weakly decreasing sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ of integers such that:

(1) $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$;
(2) $\lambda_1 + \cdots + \lambda_n = m$.

We write $\lambda \vdash m$ if λ is a partition of m, and identify a partition with its Young diagram, which is a left-justified array of rows of unit boxes so that the first row contains λ_1 boxes, the second row contains λ_2 boxes and so on. A *tableau* $\mathcal{T}$ of shape $\lambda \vdash m$ is obtained from the Young diagram of λ by filling in the boxes of the Young diagram with the numbers in a given set. In this case, we denote by $\mathrm{sh}(\mathcal{T}) = \lambda$ the tableau $\mathcal{T}$ of shape λ. A tableau of shape $\lambda \vdash m$ is called *standard* if the boxes are filled bijectively with the entries $1, \dots, m$ and the entries strictly increase along both the rows from left to right and the columns from top to bottom. A tableau is also called *semistandard* if the entries weakly increase along each row from left to right and strictly increase in each column from top to bottom.

2.4. Symmetric functions [1, 6, 12]. Let S_n be the symmetric group on the set $\{1, \dots, n\}$. The symmetric group S_n acts on the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ by permuting the variables. A polynomial invariant under this action is called *symmetric* and we denote by $\Lambda_n = \mathbb{C}[x_1, \dots, x_n]^{S_n}$ the ring of symmetric polynomials. For each $k \in \mathbb{Z}_{\geq 0}$, write Λ^k_n for the set of homogeneous symmetric polynomials of degree k. The kth power-sum symmetric function is defined by $p_k := \sum_{i=1}^n x_i^k$ and we set $p_\lambda := p_{\lambda_1} \cdots p_{\lambda_r}$ for a partition $\lambda = (\lambda_1, \dots, \lambda_r) \vdash m$. Then, the set

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$\{p_\mu \mid \mu \vdash m,\ \ell(\mu) \le n\}$ forms a basis of Λ_n^m, where $\ell(\mu)$ denotes the number of parts of the partition μ.

Let $\mathcal{T}$ be a semistandard tableau of shape $\mu \vdash m$ with entries in $\{1, \dots, n\}$. For each i, let m_i denote the number of entries of $\mathcal{T}$ equal to i. Then, we call $\mathrm{wt}(\mathcal{T}) = (m_1, \dots, m_n)$ the weight of $\mathcal{T}$. Write x^α for the monomial $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\ge 0}$. The *Schur function* associated with the partition $\mu \vdash m$ is

$$S_\mu(x_1, \dots, x_n) := \sum_{\mathrm{sh}(\mathcal{T})=\mu} x^{\mathrm{wt}(\mathcal{T})},$$

where the sum is taken over all semistandard tableaux $\mathcal{T}$ of shape μ with $\ell(\mu) \le n$. The Schur functions $\{S_\mu \mid \mu \vdash m\}$ are symmetric and form a basis of Λ_n^m.

2.5. The characteristic map [1, 6, 12]. Assume that a permutation $\sigma \in S_m$ is written as a product of disjoint cycles $\sigma = \sigma_1 \cdots \sigma_k$, where the cycle σ_i has length l_i for $i = 1, \dots, k$. If $l_1 \ge \cdots \ge l_k$ and $\mu = (l_1, \dots, l_k)$ is a partition of m, then σ is said to be of type μ, and we write $\mathrm{type}(\sigma) = \mu$. Observe that two permutations σ and τ have the same type if and only if they are conjugate, that is, $\sigma = \rho\tau\rho^{-1}$ for some $\rho \in S_m$. Hence, the conjugacy classes of S_m are naturally indexed by the partitions of m.

Now, let $z = \sum_{\sigma \in S_m} c_\sigma \sigma$ be an element of the centre $Z(\mathbb{C}[S_m])$ and define $C_\mu = \sum_{\mathrm{type}(\sigma)=\mu} \sigma$. Thus, C_μ is the sum of all permutations belonging to the conjugacy class corresponding to the partition $\mu \vdash m$. Since the coefficients c_σ are constant on each conjugacy class of S_m, we may express z as

$$z = \sum_{\mu \vdash m} a_\mu C_\mu. \tag{2.9}$$

Let $R(S_m)$ denote the $\mathbb{Z}$-module generated by the irreducible characters χ_μ of S_m, where χ_μ is the character of the irreducible representation V_μ. The *characteristic map* $ch : R(S_m) \to \Lambda_n^m$ is defined by

$$ch(\chi_\mu) = \frac{1}{m!} \sum_{\sigma \in S_m} \chi_\mu(\sigma) p_{\mathrm{type}(\sigma)} \tag{2.10}$$

and is extended linearly to all of $R(S_m)$. The image $ch(\chi_\mu)$ is called the *characteristic* of χ_μ. The $\mathbb{Z}$-module $R(S_m)$ may be identified with the centre $Z(\mathbb{C}[S_m])$, so the characteristic map can equally be regarded as a map on $Z(\mathbb{C}[S_m])$. More precisely, let χ_μ^ν denote the common value of the character χ_μ on the conjugacy class corresponding to the partition $\nu \vdash m$. Then, each character $\chi_\mu \in R(S_m)$ can be identified with the element $\sum_{\nu \vdash m} \chi_\mu^\nu C_\nu$ of the centre by (2.9). Moreover, (2.10) can be rewritten as

$$ch(\chi_\mu) = \frac{1}{m!} \sum_{\nu \vdash m} \chi_\mu^\nu n_\nu p_\nu, \tag{2.11}$$

where n_ν denotes the cardinality of the conjugacy class corresponding to $\nu \vdash m$. If z_ν is the order of the centraliser $Z_{S_m}(\sigma)$ for any permutation σ satisfying $\mathrm{type}(\sigma) = \nu$, then $n_\nu = m!/z_\nu$. Therefore, (2.11) becomes $ch(\chi_\mu) = \sum_{\nu \vdash m} \chi_\mu^\nu / z_\nu p_\nu$. Identifying χ_μ with

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$\sum_{\nu\vdash m}\chi^\nu_\mu C_\nu$ yields the equivalent description

$$ch : Z(\mathbb{C}[S_m]) \to \Lambda^m_n$$

given by $ch(C_\nu) = p_\nu/z_\nu$ for every conjugacy class C_ν.

3. Matrix units in the group algebra

Matrix units are matrices having a single nonzero entry equal to 1 [11]. We denote by E_{ij} the matrix unit whose (i,j)-entry is 1 and all other entries are zero. In this section, we calculate explicitly the elements in $\mathbb{C}[G]$ corresponding to the matrix units in $\mathrm{Mat}_{d_\lambda}(\mathbb{C})$ under the isomorphism (2.1).

Let (π^λ, V_λ) be an irreducible representation of a finite group G. Choose a basis B_λ of V_λ and consider the matrix $(\pi^\lambda_{ab}(g))_{1\le a,b\le d_\lambda}$ of $\pi^\lambda(g)$ with respect to the basis B_λ. For $\lambda \in \widehat{G}$ and $1 \le i,j \le d_\lambda$, set

$$e^\lambda_{ij} = \frac{d_\lambda}{|G|}\sum_{g\in G}\pi^\lambda_{ji}(g^{-1})g \in \mathbb{C}[G]. \tag{3.1}$$

THEOREM 3.1. *Let M_λ be the subspace of $\mathbb{C}[G]$ spanned by $\{e^\lambda_{ij} \mid 1 \le i,j \le d_\lambda\}$. Then:*

(1) $e^\lambda_{ij}e^\mu_{kl} = \delta_{\lambda\mu}\delta_{jk}e^\lambda_{il}$ *in* $\mathbb{C}[G]$;
(2) M_λ *is isomorphic to* $\mathrm{Mat}_{d_\lambda}(\mathbb{C})$ *as algebras.*

PROOF. To compute $e^\lambda_{ij}e^\mu_{kl}$, note that

$$e^\lambda_{ij}e^\mu_{kl} = \frac{d_\lambda d_\mu}{|G|^2}\sum_{g,h\in G}\pi^\lambda_{ji}(g^{-1})\pi^\mu_{lk}(h^{-1})gh = \frac{d_\lambda d_\mu}{|G|^2}\sum_{g,x\in G}\pi^\lambda_{ji}(g^{-1})\pi^\mu_{lk}(x^{-1}g)x. \tag{3.2}$$

Since $\pi^\mu(x^{-1}g) = \pi^\mu(x^{-1})\pi^\mu(g)$ as matrices,

$$\pi^\mu_{lk}(x^{-1}g) = \sum_{1\le r\le d_\mu}\pi^\mu_{lr}(x^{-1})\pi^\mu_{rk}(g) = \sum_{1\le r\le d_\mu}\pi^\mu_{rk}(g)\pi^\mu_{lr}(x^{-1}).$$

So, (3.2) implies that

$$\begin{aligned} e^\lambda_{ij}e^\mu_{kl} &= \frac{d_\lambda d_\mu}{|G|^2}\sum_{\substack{g,x\in G\\ 1\le r\le d_\mu}}\pi^\lambda_{ji}(g^{-1})\pi^\mu_{rk}(g)\pi^\mu_{lr}(x^{-1})x \\ &= \frac{d_\lambda d_\mu}{|G|^2}\sum_{\substack{x\in G\\ 1\le r\le d_\mu}}\Big(\sum_{g\in G}\pi^\lambda_{ji}(g^{-1})\pi^\mu_{rk}(g)\Big)\pi^\mu_{lr}(x^{-1})x \end{aligned}$$

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$$= \frac{d_\lambda d_\mu}{|G|^2} \sum_{\substack{x\in G \\ 1\le r\le d_\mu}} \frac{|G|}{d_\lambda}\delta_{\lambda\mu}\delta_{jk}\delta_{ir}\pi^\mu_{lr}(x^{-1})x \quad \text{(by (2.8))}$$

$$= \frac{d_\mu}{|G|}\sum_{x\in G}\delta_{\lambda\mu}\delta_{jk}\pi^\mu_{li}(x^{-1})x = \delta_{\lambda\mu}\delta_{jk}\frac{d_\mu}{|G|}\sum_{x\in G}\pi^\mu_{li}(x^{-1})x = \delta_{\lambda\mu}\delta_{jk}e^\mu_{il}.$$

The statement (1) now follows.

Now, we define

$$\varphi : M_\lambda \to \mathrm{Mat}_{d_\lambda}(\mathbb{C}), \quad e^\lambda_{ij} \mapsto E^{d_\lambda}_{ij},$$

where $E^{d_\lambda}_{ij}$ denotes the matrix unit in $\mathrm{Mat}_{d_\lambda}(\mathbb{C})$ whose (i,j)-entry is 1 and all other entries are zero. Then, φ is an algebra isomorphism by statement (1). The results now follow. □

COROLLARY 3.2. *Let M_λ be defined as in Theorem 3.1. Then, $\mathbb{C}[G] = \bigoplus_{\lambda\in\widehat{G}} M_\lambda$.*

PROOF. By the definition of M_λ, $\dim(\bigoplus_{\lambda\in\widehat{G}} M_\lambda) = \sum_{\lambda\in\widehat{G}} d_\lambda^2$. Also from (2.1), $\dim \mathbb{C}[G] = \sum_{\lambda\in\widehat{G}} d_\lambda^2$. Thus, $\bigoplus_{\lambda\in\widehat{G}} M_\lambda$ is a subalgebra of $\mathbb{C}[G]$, which has the same dimension as $\mathbb{C}[G]$. The result now follows. □

COROLLARY 3.3. *Let $\Psi : \mathbb{C}[G] \to \bigoplus_{\lambda\in\widehat{G}} \mathrm{Mat}_{d_\lambda}(\mathbb{C})$ be a linear map defined by $e^\lambda_{ij} \mapsto E^\lambda_{ij}$ for each $\lambda \in \widehat{G}$ and $1 \le i,j \le d_\lambda$. Then, Ψ is an algebra isomorphism.*

PROOF. The result is immediate from Theorem 3.1 and Corollary 3.2. □

In the following theorem, we show that the isomorphism Ψ appearing in Corollary 3.3 coincides with the isomorphism $\widetilde{\pi}$ defined in (2.1).

THEOREM 3.4. *Let e^λ_{ij} be defined as in (3.1). Then, $\pi(e^\lambda_{ij}) = \Psi(e^\lambda_{ij})$ for $\lambda \in \widehat{G}$ and $1 \le i,j \le d_\lambda$.*

PROOF. Let $\widetilde{\pi}^\mu : \mathbb{C}[G] \to \mathrm{Mat}_{d_\mu}(\mathbb{C})$ be the associated algebra homomorphism with the irreducible representation (π^μ, V^μ). Then,

$$\widetilde{\pi}^\mu(e^\lambda_{ij}) = \widetilde{\pi}^\mu\Big(\frac{d_\lambda}{|G|}\sum_{g\in G}\pi^\lambda_{ji}(g^{-1})g\Big) = \frac{d_\lambda}{|G|}\sum_{g\in G}\pi^\lambda_{ji}(g^{-1})\pi^\mu(g).$$

Comparing the (a,b)-entries on both sides,

$$\widetilde{\pi}^\mu_{ab}(e^\lambda_{ij}) = \frac{d_\lambda}{|G|}\sum_{g\in G}\pi^\lambda_{ji}(g^{-1})\pi^\mu_{ab}(g) = \frac{d_\lambda}{|G|}\frac{|G|}{d_\lambda}\delta_{\lambda\mu}\delta_{jb}\delta_{ia} = \delta_{\lambda\mu}\delta_{jb}\delta_{ia},$$

where the second equality follows from (2.8). This implies that the (i,j)-entry of $\widetilde{\pi}^\lambda(e^\lambda_{ij})$ is 1 and all other entries are zero under the condition $\mu = \lambda$. Thus,

$$\widetilde{\pi}^\mu(e^\lambda_{ij}) = \delta_{\lambda\mu}E^\lambda_{ij}. \tag{3.3}$$

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It follows from (2.1) and (3.3) that

$$\widetilde{\pi}(e_{ij}^{\lambda}) = \sum_{\rho \in \widehat{G}} \overline{\pi}^{\rho}(e_{ij}^{\lambda}) = E_{ij}^{\lambda} = \Psi(e_{ij}^{\lambda}).$$

The result now follows. □

In the following result, we present an inversion formula of (3.1).

COROLLARY 3.5. *For $g \in G$,*

$$g = \sum_{\substack{\lambda \in \widehat{G} \\ 1 \le i,j \le d_\lambda}} \pi_{ij}^{\lambda}(g) e_{ij}^{\lambda}.$$

PROOF. By Corollary 3.2, $\{e_{ij}^{\lambda} \mid \lambda \in \widehat{G} \text{ and } 1 \le i,j \le d_\lambda\}$ forms a basis of $\mathbb{C}[G]$. So, we can write $g \in G$ as

$$g = \sum_{\lambda,i,j} c_{ij}^{\lambda}(g) e_{ij}^{\lambda} \quad \text{for some } c_{ij}^{\lambda}(g) \in \mathbb{C}. \tag{3.4}$$

Applying $\widetilde{\pi}^{\mu} : \mathbb{C}[G] \to \mathrm{Mat}_{d_\mu}(\mathbb{C})$ to both sides of (3.4),

$$\pi^{\mu}(g) = \sum_{\lambda,i,j} c_{ij}^{\lambda}(g) \widetilde{\pi}^{\mu}(e_{ij}^{\lambda}). \tag{3.5}$$

By (3.3), (3.5) gives rise to

$$\pi^{\mu}(g) = \sum_{\lambda,i,j} c_{ij}^{\lambda}(g) \delta_{\lambda\mu} E_{ij}^{\lambda} = \sum_{i,j} c_{ij}^{\mu}(g) E_{ij}^{\mu}.$$

This implies that $\pi^{\mu}(g)$ is the $d_\mu \times d_\mu$ matrix whose (i,j)-entry is $c_{ij}^{\mu}(g)$. So, we have $c_{ij}^{\mu}(g) = \pi_{ij}^{\mu}(g)$ for each $\mu \in \widehat{G}$ and $1 \le i,j \le d_\mu$. The result now follows from (3.4). □

4. Application to symmetric groups

4.1. Matrix units in $\mathbb{C}[S_m]$. By the representation theory of S_m, the irreducible representations of S_m over $\mathbb{C}$ can be parametrised by the partitions of m [4, 12]. For a partition $\lambda \vdash m$, write V_λ for the corresponding irreducible representation of S_m. The representation space V_λ is equipped with an S_m-invariant inner product $(\,|\,)$, and there is an orthonormal basis $\{v_{\mathcal{U}}\}$ parametrised by the set of standard tableaux $\mathcal{U}$ of shape λ [1, 6]. We call this orthonormal basis the *Young basis*. Applying (2.1) to the case of S_m, we have an algebra isomorphism

$$\widetilde{\pi} : \mathbb{C}[S_m] \to \bigoplus_{\lambda \vdash m} \mathrm{Mat}_{f_\lambda}(\mathbb{C}), \tag{4.1}$$

where f_λ is the dimension of V_λ. For standard tableaux $\mathcal{U}$ and $\mathcal{V}$ of shape λ, let $E_{\mathcal{U}\mathcal{V}}^{\lambda}$ be the matrix unit in $\mathrm{Mat}_{f_\lambda}(\mathbb{C})$ whose $(\mathcal{U},\mathcal{V})$-entry is 1 and all other entries are 0.

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Then, under the isomorphism (4.1), the matrix unit $E^{\lambda}_{\mathcal{UV}} \in \mathrm{Mat}_{f_\lambda}(\mathbb{C})$ corresponds to

$$e^{\lambda}_{\mathcal{UV}} = \frac{f_\lambda}{m!} \sum_{s \in S_m} (s\, v_{\mathcal{U}} \mid v_{\mathcal{V}}) s^{-1} \in \mathbb{C}[S_m]$$

by Theorem 3.4.

4.2. Centre of $\mathbb{C}[S_m]$. Recall from Section 2.5 that $\{C_\mu \mid \mu \vdash m\}$ forms a basis of the centre $Z(\mathbb{C}[S_m])$. However, from Section 4.1, $\{\sum_{\mathrm{sh}(\mathcal{U})=\lambda} E^{\lambda}_{\mathcal{UU}} \mid \lambda \vdash m\}$ forms a basis of the centre of $\bigoplus_{\lambda \vdash m} \mathrm{Mat}_{f_\lambda}(\mathbb{C})$, where the sum is taken over the standard tableaux of shape λ. Thus, by (4.1), we obtain another basis $\{\sum_{\mathrm{sh}(\mathcal{U})=\lambda} e^{\lambda}_{\mathcal{UU}} \mid \lambda \vdash m\}$ of $Z(\mathbb{C}[S_m])$. Set $I_\lambda = \sum_{\mathrm{sh}(\mathcal{U})=\lambda} e^{\lambda}_{\mathcal{UU}}$ for each $\lambda \vdash m$.

Now, we compute the transition matrix between the two bases $\{C_\mu \mid \mu \vdash m\}$ and $\{I_\lambda \mid \lambda \vdash m\}$ of $Z(\mathbb{C}[S_m])$.

For each $\lambda \vdash m$ and $\sigma \in S_m$, we fix the Young basis $\{v_{\mathcal{U}}\}$ on the irreducible representation space V_λ of S_m. Write $\pi^{\lambda}_{\mathcal{UV}}(\sigma)$ for the $(\mathcal{U}, \mathcal{V})$-entry of the matrix $\pi^\lambda(\sigma)$ associated with the representation (π^λ, V_λ). Since $C_\mu = \sum_{\mathrm{type}(\sigma)=\mu} \sigma$, it follows from Corollary 3.5 that

$$C_\mu = \sum_{\mathrm{type}(\sigma)=\mu} \sum_{\lambda,\mathcal{U},\mathcal{V}} \pi^{\lambda}_{\mathcal{UV}}(\sigma) e^{\lambda}_{\mathcal{UV}} = \sum_{\lambda,\mathcal{U},\mathcal{V}} \Big(\sum_{\mathrm{type}(\sigma)=\mu} \pi^{\lambda}_{\mathcal{UV}}(\sigma) \Big) e^{\lambda}_{\mathcal{UV}}, \tag{4.2}$$

where:

(1) λ runs over the partitions of m; and
(2) $\mathcal{U}$ and $\mathcal{V}$ are taken over the standard tableaux of shape λ.

Now, consider the algebra homomorphism $\widetilde{\pi}^\lambda : \mathbb{C}[S_m] \to \mathrm{End}(V_\lambda)$ associated with the irreducible representation (π^λ, V_λ). Since $C_\mu \in Z(\mathbb{C}[S_m])$, the map $\widetilde{\pi}^\lambda(C_\mu) : V_\lambda \to V_\lambda$ is an S_m-module homomorphism. So, Schur's lemma and the definition of C_μ yield

$$\widetilde{\pi}^\lambda(C_\mu) = a(\lambda, \mu)\, \mathrm{id}_{V_\lambda} = \sum_{\mathrm{type}(\sigma)=\mu} \pi^\lambda(\sigma) \tag{4.3}$$

for some $a(\lambda, \mu) \in \mathbb{C}$. Thus, from (4.3),

$$\sum_{\mathrm{type}(\sigma)=\mu} \pi^{\lambda}_{\mathcal{UV}}(\sigma) = a(\lambda, \mu)\, \delta_{\mathcal{UV}}.$$

Hence, (4.2) gives rise to

$$C_\mu = \sum_{\lambda,\mathcal{U},\mathcal{V}} a(\lambda, \mu)\, \delta_{\mathcal{UV}}\, e^{\lambda}_{\mathcal{UV}} = \sum_{\lambda,\mathcal{U}} a(\lambda, \mu)\, e^{\lambda}_{\mathcal{UU}}. \tag{4.4}$$

Taking the trace of both sides of (4.3),

$$\mathrm{tr}\, \widetilde{\pi}^\lambda(C_\mu) = \sum_{\mathrm{type}(\sigma)=\mu} \mathrm{tr}\, \pi^\lambda(\sigma) = a(\lambda, \mu)\, f_\lambda. \tag{4.5}$$

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From Section 2.5,

$$\sum_{\mathrm{type}(\sigma)=\mu} \mathrm{tr}\, \pi^{\lambda}(\sigma) = \sum_{\mathrm{type}(\sigma)=\mu} \chi_{\lambda}(\sigma) = n_{\mu} \chi_{\lambda}^{\mu}. \tag{4.6}$$

It follows from (4.5) and (4.6) that $a(\lambda, \mu) = (n_{\mu}/f_{\lambda})\chi_{\lambda}^{\mu}$, and (4.4) implies that

$$C_{\mu} = \sum_{\substack{\lambda \vdash m \\ \mathrm{sh}(\mathcal{U})=\lambda}} \frac{n_{\mu}}{f_{\lambda}} \chi_{\lambda}^{\mu}\, e_{\mathcal{U}\mathcal{U}}^{\lambda} = \sum_{\lambda \vdash m} \frac{n_{\mu}}{f_{\lambda}} \chi_{\lambda}^{\mu} \sum_{\mathrm{sh}(\mathcal{U})=\lambda} e_{\mathcal{U}\mathcal{U}}^{\lambda} = \sum_{\lambda \vdash m} \frac{n_{\mu}}{f_{\lambda}} \chi_{\lambda}^{\mu} I_{\lambda}. \tag{4.7}$$

This yields the following result.

THEOREM 4.1. *The transition matrix between the bases* $\{C_{\mu} \mid \mu \vdash m\}$ *and* $\{I_{\lambda} \mid \lambda \vdash m\}$ *of* $Z(\mathbb{C}[S_m])$ *is given by* $T = (T_{\lambda\mu})_{\lambda \vdash m, \mu \vdash m}$ *whose* (λ, μ)*-entry is* $T_{\lambda\mu} = (n_{\mu}/f_{\lambda})\chi_{\lambda}^{\mu}$.

4.3. Frobenius characteristic and Schur functions. Recall from Section 2.5 that the characteristic map is the linear isomorphism defined by

$$ch : Z(\mathbb{C}[S_m]) \longrightarrow \Lambda_n^m, \quad C_{\mu} \mapsto \frac{p_{\mu}}{z_{\mu}} \quad \text{for each } \mu \vdash m.$$

By the Frobenius character formula (see [1, Theorem 4.2.8] for details),

$$p_{\mu} = \sum_{\lambda \vdash m} \chi_{\lambda}^{\mu} S_{\lambda}.$$

Applying the characteristic map ch,

$$ch(C_{\mu}) = \sum_{\lambda \vdash m} \frac{\chi_{\lambda}^{\mu}}{z_{\mu}} S_{\lambda}. \tag{4.8}$$

However, from (4.1), $Z(\mathbb{C}[S_m]) \simeq \bigoplus_{\lambda \vdash m} \mathbb{C}\,\mathrm{id}_{V_{\lambda}}$. Motivated by (4.7), we define a linear map

$$\widetilde{ch} : Z(\mathbb{C}[S_m]) \longrightarrow \bigoplus_{\lambda \vdash m} \mathbb{C}\,\mathrm{id}_{V_{\lambda}}, \quad \widetilde{ch}\Big(\frac{1}{m!} C_{\mu}\Big) = \frac{1}{m!} \sum_{\lambda \vdash m} \frac{n_{\mu}}{f_{\lambda}} \chi_{\lambda}^{\mu}\, \mathrm{id}_{V_{\lambda}}.$$

Using the fact that $n_{\mu} = m!/z_{\mu}$, the linear map $\widetilde{ch}$ becomes

$$\widetilde{ch}\Big(\frac{1}{m!} C_{\mu}\Big) = \sum_{\lambda \vdash m} \frac{\chi_{\lambda}^{\mu}}{z_{\mu}} \frac{\mathrm{id}_{V_{\lambda}}}{f_{\lambda}}. \tag{4.9}$$

Recall that $\{S_{\lambda} \mid \lambda \vdash m\}$ forms a basis of Λ_n^m and identify Λ_n^m with $\bigoplus_{\lambda \vdash m} \mathbb{C}\,\mathrm{id}_{V_{\lambda}}$ via the correspondence $S_{\lambda} \mapsto \mathrm{id}_{V_{\lambda}}/f_{\lambda}$. Comparing (4.8) with (4.9), (4.9) can be understood as the (normalised) realisation of the characteristic map through the identity matrices. This realisation shows that, with respect to the characteristic map, the basis of Λ_n^m consisting of Schur functions is more natural than that consisting of power-sum symmetric functions.

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