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Conformally maximal polynomial-like dynamics and invariant harmonic measure

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Abstract. Let $f : V \to U$ be a (generalized) polynomial-like map. Suppose that harmonic measure $\omega = \omega(\cdot, \infty)$ on the Julia set J_f is equal to the measure of maximal entropy m for $f : J_f \hookleftarrow$. Then the dynamics (f, V, U) is called maximal. We are going to give a criterion for the dynamics to be conformally equivalent to a maximal one, that is to be conformally maximal. In the second part of this paper we construct an invariant 'harmonic' measure μ such that $d\mu/d\omega$ is Hölder for certain dynamics. This allows us to prove in this class of dynamical systems that $\omega \approx m$ is necessary and sufficient for (f, V, U) to be conformally maximal. In the particular case when f is expanding and J_f is a circle, our result becomes a theorem of Shub and Sullivan; so throughout the paper we are dealing with an analog of a theorem of Shub and Sullivan on 'wild' (e.g. totally disconnected) J_f and for certain non-expanding f. We also construct (under certain assumptions) invariant harmonic measure on J_f. In this respect, our work stems from one of the works of Carleson.

0. *Introduction and notation*

We will be considering triples (f, V, U), where U is a topological disc and $V = V_1 \cup \cdots \cup V_k$ is the union of topological discs whose closures are disjoint and are contained in U. Also, $f_i \stackrel{\text{def}}{=} f|V_i$ is an unbranched or branched covering $V_i \to U$ of degree d_i (so 'unbranched' means that $d_i = 1$). By $d = d_1 + \cdots + d_k$ we denote the degree of the map $f : V \to U$. These dynamical systems will be called generalized polynomial-like systems (GPL).

If $k = 1$, $d \geq 2$, we come to a class of polynomial-like systems (PL) introduced in [**DH**] and playing an important role in classification of polynomial dynamics.

Being GPL means to be quasi-conformally equivalent to a polynomial:

$$f \in \text{GPL} \Rightarrow \exists\, h \in qc(U) : f = h^{-1} \circ \text{poly} \circ h. \tag{0.1}$$

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This shows that many metric properties of $K_f \stackrel{\text{def}}{=} \bigcap_{n\geq 0} f^{-n}(U)$ and $J_f = \partial K_f$ are the same for a certain polynomial. This is just because metric properties such as 'zero area', 'absolute area zero', 'removability' or 'uniform perfectness', etc are qc-invariant.

Our paper is devoted to the study of the harmonic measure of J_f. However, the properties of harmonic measure are not usually qc-invariant. In particular, if (f, V, U) happens to be a polynomial system, then harmonic measure equals the measure of maximal entropy, but if (f, V, U) is a general polynomial-like system this may not be the case. This is the case which interests us in this paper.

Basically, in this paper it is shown that harmonic measure keeps enough information about the dynamics; for example if harmonic measure equals the measure of maximal entropy then the dynamics is polynomial (warning: this holds only under certain restrictions).

Now let us recall that for GPL its filled-in Julia set is always uniformly perfect [**E**], [**H**], [**MR**], that is (see [**P**])

$$\operatorname{cap}(K_f \cap B(x, r)) \geq c \cdot \operatorname{cap}(B(x, r)) \quad \forall\, x \in K_f,\ r \leq r_0.$$

In particular, Green's function G of $A_\infty(f) \stackrel{\text{def}}{=} \mathbb{C} \setminus K_f$ can be extended to a subharmonic function on $\mathbb{C}$ by putting $G(z) = 0$, $z \in K_f$. Taking Green's function G with a pole at ∞, we define harmonic measure on J_f as the Laplacian of $G : \omega \stackrel{\text{def}}{=} \omega_f \stackrel{\text{def}}{=} \Delta G$.

If μ is an arbitrary probability measure on J_f, then a general theorem [**Pa**] says that there exists a *Jacobian* $J_\mu = J_\mu(f)$ on a set of full μ measure. It means that there exists $Y \subset J$, $\mu(J \setminus Y) = 0$, and an integrable function J_μ such that for every $E \subset Y$ on which f is one-to-one onto $f(E)$, we have $\mu(f(E)) = \int_E J_\mu \, d\mu$. If $Y = J$ we say that μ *has a Jacobian.*

For any rational dynamics f, there exists on J_f a measure with constant Jacobian. Moreover, such a measure is unique and the Jacobian equals $d = \deg f$. This is a classical result of Lyubich [**Ly1**]. This measure is f-invariant and has the maximal possible entropy $\log d$. The symbol m stands for this measure. If f happens to be a polynomial, then harmonic measure ω has a constant Jacobian [**Br**] and so $m = \omega$.

This remark along with (0.1) allows us to claim that for a GPL f, there exists a unique measure with a constant Jacobian. In fact, if $f = h^{-1} \circ p \circ h$, $h \in qc(U)$, $p \in \text{Poly}_d$, then $m_f \stackrel{\text{def}}{=} (h^{-1})^* \omega_p$ will have Jacobian d. Clearly, such a measure is unique on J_f (in particular it does not depend on h) because there is only one constant Jacobian measure on J_p.

We call a GPL system (f, V, U), maximal if $m_f = \omega_f$.

We call a GPL system f, conformally maximal if it is conformally equivalent to a maximal system, that is

$$f = H^{-1} \circ g \circ H,$$

$H : U_f \to U_g$ is a conformal isomorphism and (g, V_g, U_g) is a maximal system. An obvious necessary condition for this is

$$\omega_f \approx m_f, \tag{0.2}$$

where the symbol '$\approx$' means that measures belong to the same class, i.e. they are mutually

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absolutely continuous. In [**LyV**] it was proven that (0.2) is also sufficient if (f, V, U) is a Cantor repeller.

The first part of this paper, §§1–3, concerns the general criterion of conformal maximality.

The second part of this paper is devoted to the discussion of sufficiency of (0.2) in general. This sufficiency is proven under certain assumptions and, in particular, for non-recurrent GPL systems f for which J_f is totally disconnected.

Commenting on the second part, let us mention that throughout we play with ideas from [**Ly1**], [**Ly2**], [**Po**], [**DPU**] and [**LyV**]. In the first part, we base our approach on [**Lo**] and [**LyV**].

We summarize some background material on potential theory in Appendix 1, and in Appendix 2 we give a brief account of the thermodynamic formalism to make the paper more accessible.

1. *Main results, basic ideas, and dynamics on the circle*

First we formulate a criterion for being conformally maximal and then we state a conditional result facing the equivalence of (0.2) and conformal maximality (Theorem 1.3). Theorem 1.4 shows that for non-recurrent dynamics there exists an invariant harmonic measure and its density with respect to ω is Hölder continuous. One of the corollaries of Theorem 1.4 is that (0.2) implies conformal maximality for non-recurrent dynamics. We conclude this section by considering a simple case $J_f = \mathbb{T}$ to illustrate the ideas of the proofs.

THEOREM 1.1. *Let (f, V, U) be a GPL system. Two assertions are equivalent:*

(1) *(f, V, U) is conformally maximal;*

(2) *there exists a non-negative subharmonic function τ on U, which is positive and harmonic in $U \setminus K_f$, vanishes on K_f and satisfies*

$$\tau(fz) = d\tau(z). \tag{Aut}$$

THEOREM 1.2. *Let (f, V, U) be a PL system. Then the assertions from Theorem 1.1 are equivalent to a third assertion:*

(3) *there exist a polynomial p, $\deg p = d$, and a conformal isomorphism $H : U_f \to H(U_f)$ such that*

$$f = H^{-1} \circ p \circ H. \tag{CoEq}$$

Remark. Let us mention that there are conformally maximal GPL systems which are not conformally polynomials (that is (CoEq) fails for them). The example is in [**LyV**].

Theorems 1.1 and 1.2 are proved in §2 and §3 respectively.

Our next goal is to show that (0.2) implies the existence of the automorphic harmonic function τ (see (Aut)). In this respect Theorem 1.1 serves as a step in proving the implication

$$\omega_f \approx m_f \Rightarrow f \text{ is conformally maximal.} \tag{1.1}$$

Strangely enough, (1.1) is an easy corollary of Theorem 1.1 if J_f is connected and a pretty tricky exercise in potential theory if J_f is a totally disconnected Julia set for any

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expanding f. Here we show this for non-recurrent f (and totally disconnected J_f; the case of general J_f is considered in [**BV4**]). Again, we base our approach on potential theory in $A_\infty(f) = \mathbb{C} \setminus J_f$. This domain was proved to be a uniformly John domain [**BV2**], [**BV3**].

Also, we try to remove the non-recurrency assumption. To do this we introduce a 'ratio condition' (R) on Green's function G of $A_\infty(f)$. Here it is:

$$\exists\, C, \varepsilon > 0\, \forall\, \xi_1, \xi_2 \in V \setminus K_f, \quad \left| \frac{G(f\xi_1)}{G(\xi_1)} : \frac{G(f\xi_2)}{G(\xi_2)} - 1 \right| \le C|\xi_1 - \xi_2|^\varepsilon. \tag{R}$$

Remark. (R) was proved for the case of Cantor repellers in [**LyV**]. For expanding f, it can be proven along the same lines. For totally disconnected J_f and non-recurrent f, the proof becomes somewhat elaborate (see [**BV4**]). It is probable (but not known) that one can get (R) beyond the scope of non-recurrency; for a non-renormalizable situation this is very plausible.

To see how powerful (R) is, we use a result of Grishin [**Gr**] stated as Lemma A.1.1 in Appendix 1.

In Lemma A.1.1 we put $u = G \circ f$, $v = G$ and thus $\mu_u = f_*\omega$, $\mu_v = \omega$. Then clearly

$$\mu_u(E) = \omega(f(E)) \tag{1.2}$$

for each Borel $E \subset J_f$ such that $f : E \to f(E)$ is injective.

For $x \in J_f$ denote

$$\varphi(x) = -\log \frac{G(fx)}{G(x)} \stackrel{\text{def}}{=} -\lim_{\substack{\xi \to x \\ \xi \in V \setminus K_f}} \log \frac{G(f\xi)}{G(\xi)}.$$

By (R), the limit always exists and represents a Hölder continuous function φ. Now the Lemma A.1.1 and (1.2) show

$$\omega(f(E)) = \int_E e^{-\varphi(x)}\, d\omega(x) \tag{1.3}$$

for any Borel set E such that $f : E \to f(E)$ is injective. So we have shown the following assertion.

ASSERTION 1. *Assume* (R).

Then harmonic measure has the Jacobian J_ω
such that $\log J_\omega$ *is Hölder continuous.* (H)

We will show later that (R) also implies that the topological pressure (see Appendix 2) $P(f, \varphi)$ vanishes:

$$P(f, \varphi) = 0. \tag{1.4}$$

To understand the ergodic properties of the harmonic measure, we consider the Perron–Frobenius–Ruelle operator $P_\varphi : C(J_f) \to C(J_f)$ (see Appendix 2). Almost periodicity of P_f would allow us to define $\psi_0 = \lim_{n\to\infty} P_\varphi^n \mathbf{1}$, and $\mu = \psi_0\, d\omega$ would be f-invariant

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harmonic measure with continuous density. Actually, almost periodicity is proved in [**DU**] under the condition

$$\varphi < 0 (= P(f, \varphi)). \tag{1.5}$$

Moreover, if (1.5) held, ψ_0 would even be Hölder continuous as was proved in [**DPU**]. Condition (1.5) takes the form

$$\frac{G(fx)}{G(x)} = \lim_{\xi \to x} \frac{G(f\xi)}{G(\xi)} > 1, \quad x \in J_f. \tag{1.6}$$

There is no reason for (1.6) to hold and the scheme of [**DU**], [**DPU**] is not applicable. Still, we prove the following assertion.

ASSERTION 2. *Let J_f be totally disconnected, let* (H) *hold, and let*

$$\varphi = -\log \frac{G(fx)}{G(x)}.$$

Then P_φ is almost periodic on $C(J)$ and, moreover, $\psi_0 = \lim P_\varphi^n \mathbf{1}$ is Hölder continuous.

In particular, there exists an invariant harmonic measure μ with Hölder density with respect to ω.

Remark. Let us note that in [**DU**], [**DPU**] and in Assertion 2 there are no restrictions on critical points of f. Carleson [**C**] was the first to construct invariant harmonic measure in a hyperbolic situation. We consider our work as stemming from [**C**].

THEOREM 1.3. *Let J_f be totally disconnected, and let* (H) *hold. There exists an invariant harmonic measure $\mu = \psi_0 \cdot \omega$ with Hölder continuous density ψ_0, $\psi_0 > 0$. If, in addition,* (R) *holds and $\omega_f \approx m_f$, then f is conformally maximal.*

THEOREM 1.4. *Let f be non-recurrent and let J_f be totally disconnected. Then* (R) *holds. In particular, there exists an invariant harmonic measure $\mu = \psi_0 \, d\omega$ with Hölder continuous density ψ_0.*

THEOREM 1.5. *Let f be non-recurrent and let J_f be totally disconnected. Also, let $\omega_f \approx m_f$. Then f is conformally maximal.*

Let us illustrate our methods by using a simple example of (F, A_2, A_1), where $A_2 \subset \bar{A}_2 \subset A_1$ are topological annuli containing the unit circle $\mathbb{T}$, and F is an unbranched d covering $F : A_2 \to A_1$, $F(\mathbb{T}) = \mathbb{T}$. Formally, this is not a GPL because we map annuli instead of discs. But the ideas are more conspicuous here because the Julia set $J_F = \mathbb{T}$ and harmonic measure $\omega = \ell =$ Lebesgue measure on $\mathbb{T}$ are so simple. In addition, F is automatically expanding on $\mathbb{T}$. In this situation we can apply some basic facts of the thermodynamic formalism (see Appendix 2).

Clearly, the measure $\omega = \ell | \mathbb{T}$ has a good Jacobian, that is $|F'(\xi)|, \xi \in \mathbb{T}$. The potential $\varphi = -\log |F'(\xi)|$ is Hölder continuous and even real analytic. Thus, by Lemma A.2.2 we have an invariant Lebesgue measure $\mu = \rho \, d\ell$ with a good density:

$$\log \rho \text{ is real analytic.} \tag{1.7}$$

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Lemma A.2.2 does not give real analyticity, but it follows from a result of Krzyzewsky [**K**]. We are grateful to the referee for the suggestion to use this result here.

To prove conformal maximality we let our main assumption (0.2) come into play. As $m_F \approx \ell$, we have $m_F \approx \mu$, which means $m_F = \mu = \rho\, d\ell$ by ergodicity. Now let us note that

$$F = h^{-1} \circ z^d \circ h$$

for a homeomorphism (actually quasi-symmetry) h. Clearly, h^* maps m_F to $m_{z^d} = \ell$. So $(h^{-1})^*\ell = m_F = \rho\, d\ell$, i.e. h^{-1} is absolutely continuous and

$$(h^{-1})' = \rho = \text{real analytic.}$$

So F and z^d are conjugated on $\mathbb{T}$ by a real analytic map $h|\mathbb{T}$. This map can be extended as a conjugation of F and z^d into a thin annulus around $\mathbb{T}$. From this thin annulus it extends automatically to A_1. Thus F is conformally maximal (even conformally equivalent to z^d).

At this point let us recall a theorem of Shub and Sullivan [**SS**] in that if two expanding Blascke products are absolutely continuously conjugate on the unit circle, then they are conjugate by a Möbius transformation.

Our situation is quite similar: $m_F \approx \omega_F = l$ implies that the conjugacy h is absolutely continuous and then we get the analyticity of h.

Remark. We would like to mention that part of the results of our paper, namely §7, proof of Theorem 1.5, can be considered as the analog of [**SS**] for 'cantor-like' limit sets.

We are going to show conformal maximality for the case of connected Julia sets. This is formulated in the following proposition.

PROPOSITION 1.6. *Let U and V be two topological discs, $\overline{U} \subset V$, and let $f : U \to V$ be a branched cover of degree d. Suppose that we have $\omega \approx m$. Then f is conformally maximal.*

Proof. Let us denote by K_f the filled Julia set of f, that is $K_f \stackrel{\text{def}}{=} \bigcap_{n\geq 0} f^{-n}(U)$.

Consider the Riemann mapping

$$R : \mathbb{C} \setminus K_f \to \mathbb{C} \setminus \mathbb{D},$$

where $\mathbb{D}$ is the unit disk. Consider the annuli $A_1 = R(U \setminus K_f)$, $A_2 = R(V \setminus K_f)$ and the map $F : A_1 \to A_2$, $F = R \circ f \circ R^{-1}$.

By the Schwarz reflection principle we can extend the analytic map F across the unit circle $\mathbb{T}$ so that now F is defined in the annulus containing $\mathbb{T}$. Moreover, $F(\mathbb{T}) = \mathbb{T}$ and we get that F is an expanding analytic map of the circle.

By conformality of the Riemann mapping we have that $(R^{-1})^*\omega_F = \omega_f$. Also, by the construction of the maximal entropy measure in [**Ly1**], [**FLM**] (or see §1.6), we have that $(R^{-1})^* m_F = m_f$. Therefore, we obtain that $m_F \approx \omega_F$.

As in the above consideration, it follows that F is conformally maximal (even conformally equivalent to z^d).

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This conjugacy to $z \to z^d$ may not be true for f. On the other hand, we have a 'one sided conjugacy' which is enough for conformal maximality. To see this, observe that without loss of generality we can now think that $F(z) = z^d$.

For the function $z \to z^d$, the function $\tau(z) = \log(|z|)$ satisfies property (Aut) from Theorem 1.1. The function $\tau_1 = \tau \circ R$ is the automorphic function corresponding to f. Applying Theorem 1.1 we are done. □

This was an illustration of how we are going to approach the proofs of Theorems 1.3–1.5. Instead of using Lebesgue measure to start we will use harmonic measure ω on J_f. *In many interesting cases J_f is disconnected.*

In spite of the disconnectedness, Assertion 1 above states that

$$(\mathrm{R}) \Rightarrow \log J_\omega \text{ is Hölder continuous on } J_f. \tag{1.8}$$

So, in our simple example (F, A_2, A_1) we need another approach which uses

$$\log \rho \text{ is Hölder continuous} \tag{1.9}$$

instead of the much more powerful (1.7). Starting with (1.9) we can proceed as follows. As $m_F = \mu = \rho\, d\ell$, we have $d = J_\mu = J_\ell(\rho \circ F)/\rho$. Let us use the notation $\gamma = -\log \rho$, $G_{\mathbb{T}} = \log|z|$. Then we can rewrite $d = |F'|(\rho \circ F)/\rho$ as

$$\log \frac{G_{\mathbb{T}}(F(\xi))}{G_{\mathbb{T}}(\xi)} - \log d = \gamma \circ F - \gamma, \quad \xi \in \mathbb{T}. \tag{1.10}$$

Actually, in a neighborhood of $\mathbb{T}$

$$\log \frac{G_{\mathbb{T}}(F(z))}{G_{\mathbb{T}}(z)} - \log d = (\gamma \circ F)(\xi) - \gamma(\xi) + O(|z - \xi|), \quad \xi \in \mathbb{T}. \tag{1.11}$$

Let us consider the following definition:

$$\hat{\tau}(z) = \lim_{n\to\infty} d^n G_{\mathbb{T}}(F^{-n}z) \Big/ \lim_{n\to\infty} d^n G_{\mathbb{T}}(F^{-n}z_0).$$

Here, by $\{F^{-n}\}$ we mean *any* sequence of compatible branches.

The homology equation (1.11) allows us to show that this definition is correct: $\hat{\tau}$ is really a function (see details in §7). But then it is harmonic in $A_1 \setminus \mathbb{T}$, vanishes on $\mathbb{T}$ and subharmonic in A_1. Also, it satisfies (Aut):

$$\hat{\tau}(Fz) = d\hat{\tau}(z).$$

Now the criterion of conformal maximality (Theorem 1.1) finishes the proof of conformal maximality of (F, A_2, A_1).

This second approach will be used throughout the paper.

Let us recall the method:

(1) Find out how good is the Jacobian of harmonic measure.

(2) Provided that its logarithm is Hölder, we prove that $\log(d\omega/dm)$ is Hölder (under (0.2)). Actually, we prove the existence of invariant harmonic measure μ with Hölder $\log(d\mu/d\omega)$ in this case.

(3) The fact that $\log(d\omega/dm)$ is Hölder, allows us to write a homologous equation which is like (1.16).

(4) Using the homologous equation, we define the automorphic harmonic $\hat{\tau}$.

(5) Theorem 1.1 then claims conformal maximality of our GPL system (f, V, U).

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2. *A criterion of conformal maximality*

Here we prove Theorem 1.1.

(2) $\Rightarrow$ (1). Let τ exist. Without loss of generality we assume that the period of τ over a simple closed contour $C \subset U \setminus V$ equals $2\pi d$:

$$(\Delta\tau)(K_f) = \int_C \frac{\partial \tau}{\partial n}\, ds = 2\pi d.$$

Moreover, by the automorphic property

$$\begin{aligned}\int_{\partial V_i} \frac{\partial \tau}{\partial n}\, ds &= (\Delta\tau)(V_i \cap K_f) = \frac{1}{d}\Delta(\tau \circ f)(V_i \cap K_f)\\ &= \frac{d_i}{d}(\Delta\tau)(f(V_i \cap K_f)) = \frac{d_i}{d}(\Delta\tau)(K_f) = 2\pi\, d_i,\end{aligned}$$

where

$$d_i = \deg(f|V_i).$$

Let us consider the harmonic conjugate $\tilde{\tau}$ of τ. By the above relation, we have that

$$\varphi(z) = e^{(\tau + i\tilde{\tau})(t)}$$

is a well-defined holomorphic function in $U \setminus V$.

Let us extend φ from $U \setminus V$ into $\mathbb{C} \setminus V$ as a C^∞ mapping with the only restrictions that

$$\varphi(z) = z^d, \quad z \sim \infty; \quad \varphi \neq 0.$$

Pull back the standard conformal structure σ_0 on $\varphi(\mathbb{C} \setminus V)$ and define

$$\sigma = \begin{cases} \varphi_* \sigma_0 & \text{on } \mathbb{C} \setminus V,\\ \sigma_0 & \text{on } V.\end{cases}$$

Clearly, $\sigma = \sigma_0$ on U (not only on V!).

The measurable Riemann mapping theorem now gives a quasi-conformal h, $h(z) \sim z$, $z \to \infty$, such that

$$h^* \sigma_0 = \sigma. \tag{2.1}$$

We define $g = h^{-1} \circ f \circ h$, $V_g = h^{-1}(V)$, $U_g = h^{-1}(U)$, and thus $h : U_g \to U$ is conformal. Let us check that g is a maximal system, i.e. that

$$\omega_g = m_g. \tag{2.2}$$

To this end, consider

$$\tau_g(z) \stackrel{\text{def}}{=} \frac{1}{d} \begin{cases} \log|\varphi \circ h|(z), & z \in \mathbb{C} \setminus V_g;\\ (\tau \circ h)(z), & z \in U_g.\end{cases}$$

First of all the definition is correct, as by construction $\tau = \log|\varphi|$ on $U \setminus V$. The second line of the definition represents clearly a harmonic function in $U_g \setminus K_g$ and a subharmonic function in U_g. But the first line also represents a harmonic function (now in $\mathbb{C} \setminus V_g$). This is because (see (2.1))

$$(\varphi \circ h)^* \sigma_0 = \sigma_0 \quad \text{on } \mathbb{C} \setminus V_g$$

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and $\varphi \neq 0$. Automatically $\tau_g|K_g = 0$ and $\tau_g(z) \sim \log|z|$, $z \to \infty$. Thus, τ_g is Green's function G_g of $A_\infty(g) = \mathbb{C} \setminus K_g$ with a pole at infinity. On the other hand, $G_g = \tau_g$ is automorphic:

$$G_g(gz) = dG_g(z), \quad z \in V_g, \tag{2.3}$$

because τ was assumed to satisfy (Aut).

Now (2.3) implies that $\omega_g = \Delta G_g$ has the following property:

$$\omega_g(g(E)) = \Delta(G_g \circ g)(E) = d\omega_g(E) \tag{2.4}$$

for any Borel $E \subset J_g$ such that $g : E \to g(E)$ is injective. But as was explained in §0, there exists a unique measure satisfying (2.4), that is the measure of maximal entropy m_g. So $\omega_g = m_g$ and we are done.

(1) $\Rightarrow$ (2). We modify the idea of [**Lo**]. Let us have a conformally maximal system. We need to prove the existence of an automorphic function τ. To do this we may assume that our system (g, V, U) is already maximal (clearly, the existence of an automorphic function is conformally invariant). Let $\{x_i(\xi)\}_{i=1}^d$ denote all g-preimages of $\xi \in J_g$ counting with multiplicity. Let $u \in C(J_g)$, and denote $\varphi_0 = \log d$. Then as the measure of maximal entropy m is $P^*_{\varphi_0}$ invariant, $(P^*_{\varphi_0} m)(u) = \int u\, dm$, that is

$$\int_{J_g} \sum_{i=1}^{d} u(x_i(\xi))\, dm(\xi) = d \int_{J_g} u\, dm.$$

By maximality we have

$$\int_{J_g} \sum_{i=1}^{d} u(x_i(\xi))\, d\omega(\xi) = d \int_{J_g} u\, d\omega. \tag{2.5}$$

We introduce the function

$$H_0 \stackrel{\text{def}}{=} G \circ g - d \cdot G, \tag{2.6}$$

where $G(z) = \int \log|z - \xi|\, d\omega(\xi) + c$ is Green's function. We are going to use (2.5) to prove that H_0 is harmonic in U.

Introducing the notation

$$\frac{\partial}{\partial z} G = \Phi \quad \text{and} \quad \frac{\partial}{\partial z} H_0 = A,$$

(2.6) gives

$$\Phi(g(z))g'(z) - d\Phi(z) = A(z).$$

Our purpose now is to show that $A \in \mathrm{Hol}(U)$.

Since

$$\Phi(z) = \int_{J_g} \frac{d\omega(\xi)}{\xi - z},$$

we have that

$$A(z) = \int_{J_g} \frac{g'(z)\, d\omega(\xi)}{\xi - g(z)} - d \int_{J_g} \frac{d\omega(\xi)}{\xi - z} \in \mathrm{Hol}(V \setminus J_g).$$

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We are going to use (2.5) to remove the singularities of A. Consider a contour C in $U \setminus V$. Then for every $n \geq 0$,

$$\begin{aligned}\int_C z^n A(z)\,dz &= \int_{J_g} d\omega(\xi) \int_C \frac{g'(z)z^n\,dz}{\xi - g(z)} - d\int_{J_g} d\omega(\xi)\int_C \frac{z^n\,dz}{\xi - z} \\ &= \int_{J_g} \sum_{i=1}^{d} (x_i(\xi))^n\,d\omega(\xi) - d\int_{J_g} \xi^n\,d\omega(\xi) = 0\end{aligned}$$

according to (2.5). This proves that A is holomorphic in V and it follows that H_0 is harmonic.

Two cases may occur: (a) $H_0 \equiv 0$ in V; (b) $\{z \in V : H_0(z) = 0\}$ is locally a finite union of real analytic curves. If the first case occurs we get G as our harmonic automorphic function. So let us consider (b).

Let N be a neighborhood of J_g, $\bar{N} \subset V$, put $\Gamma = \{z \in N : H_0(z) = 0\}$. Then $\Gamma = \bigcup_{i=1}^{n} \Gamma_i$, where each Γ_i is a real analytic arc. Clearly, Γ covers J_g (as $H_0|J_g = (G \circ g - G)|J_g = 0$). As J_g has no isolated points we can throw away those Γ_ℓ for which $\#(J_g \cap \Gamma_\ell) < \infty$. After this operation the rest of Γ_k will cover J_g. So let $J_g \subset \Gamma_0 = \bigcup_{i=1}^{m} \Gamma_i$ and let $\#(J_g \cap \Gamma_i) = \infty$, $i = 1, \ldots, m$. Now it is clear that $g^{-1}(\Gamma_0) \subset \Gamma_0$.

We call a cross-point any point of J_g which is an intersection of two different arcs Γ_i, $i = 1, \ldots, m$. If p_0 is a cross-point, then the set $g^{-n}(p_0)$ consists of cross-points (as $g^{-1}\Gamma_0 \subset \Gamma_0$). But the number of cross-points is obviously finite. So there are no cross-points at all.

Let $\mathcal{O}_0$ be a thin neighborhood of Γ_0 in which a holomorphic symmetry $z \to z^*$ with respect to Γ_0 is defined. In $\mathcal{O}_1 = g^{-1}\mathcal{O}_0$ we then get

$$gz^* = (gz)^*. \tag{2.7}$$

Let us put $\hat{G}(z) = G(z) + G(z^*)$, $\hat{H}_0(z) = H_0(z) + H_0(z^*)$. Then (2.6) and (2.7) give us

$$(\hat{G} \circ g - d \cdot \hat{G})(z) = \hat{H}_0(z), \quad z \in \mathcal{O}_1.$$

By definition, $\hat{H}_0 \equiv 0$ on Γ_0. However, this function is symmetric with respect to Γ_0 and so $\partial\hat{H}_0/\partial n \equiv 0$ on Γ_0. Thus, $\hat{H}_0 \equiv 0$ on $\mathcal{O}_1$ and we have a neighborhood $\mathcal{O}_1$ of J_g in which

$$\hat{G} \circ g = d \cdot \hat{G}. \tag{2.8}$$

Then a standard extension 'by means of equation' gives us $\hat{G}$ on the whole V with the same automorphic property (2.8).

Theorem 1.1 is completely proved. □

3. *Conformal equivalence to a polynomial*

Proof of Theorem 1.2. We need only to show that the existence of automorphic τ implies conformal equivalence of f to a certain polynomial dynamics, (2) ⇒ (3). This can be done exactly as in [**LyV**], so we allow ourselves to be brief. Let (f, V, U) be a polynomial-like system with automorphic τ. We use the Douady–Hubbard model to

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construct (F, A_2, A_1), where $A_2 \subset \bar{A}_2 \subset A_1$ are annuli containing $\mathbb{T}$ and $F : A_2 \to A_1$ is a regular covering of degree $d = \deg f$. The construction of F involves the fundamental annulus $A = U \setminus V$ for f. As $\tau(fz) = d \cdot \tau(z)$, $z \in \partial V$, we can put $T = \tau \circ \varphi$, where φ is a conformal isomorphism of A and $(A_1 \setminus A_2) \cap (\mathbb{C} \setminus \mathbb{D})$ conjugating f and F respectively on ∂A and $\partial(A_1 \setminus A_2)$. Then $T(Fz) = dT(z)$, $z \in \partial A_2 \cap (\mathbb{C} \setminus \mathbb{D})$, and T can be extended with this property to the whole of A_2.

Clearly $\Delta T \approx \ell|\mathbb{T}$. But $(\Delta T)(F(E)) = d(\Delta T)(E)$ for each Borel $E \subset \mathbb{T}$ with injective $F : E \to F(E)$. So $\ell \approx \Delta T = m_F$. We can apply the considerations of §1 to conclude that F is conformally equivalent to z^d. Now a theorem of Douady and Hubbard [**DH**] says that f is conformally equivalent to a polynomial. □

4. *J_f is totally disconnected and (H) ⇒ invariant harmonic measure has good density*
In §1 we defined

$$\varphi(x) = -\log \frac{G(fx)}{G(x)} \stackrel{\text{def}}{=} \lim_{\substack{\xi \to x \\ \xi \in V \setminus K_f}} -\log \frac{G(f\xi)}{G(\xi)}, \quad x \in J_f,$$

which exists due to (R). Recall that φ is Hölder and $e^{-\varphi}$ = the Jacobian of harmonic measure ω.

We will prove that P_φ is almost periodic. This would follow from general results (see Appendix 2) if we had

$$P(f, \varphi) > \sup_{x \in J_f} \varphi(x). \tag{4.1}$$

As we do not have (4.1), we need to do some work. We are first going to prove some simple consequences of two lemmas of Przytycki (see Appendix 2). We introduce the following notation. Let $\{V_t^n : t = 1, \dots, t_n\} = \tau_n$ be the family of components of $f^{-n}(U)$. Each V_t^n has multiplicity $m_t^n \geq 1$ and $\sum_{t=1}^{t_n} m_t^n = d^n$.

LEMMA 4.1. *Suppose J_f is totally disconnected. Let $\delta > 0$ be fixed. Then there exists $C' = C'_\delta$ such that for any n the multiplicity m_t^n is at most $C' e^{n\delta}$.*

Proof. Fix δ, and take C_δ and ε from Lemma A.2.4. Clearly, we can fix $n_0(\varepsilon)$ such that

$$\forall\, n \geq n_0\ \forall t,\ \operatorname{diam} V_t^n \leq \frac{\varepsilon}{2}. \tag{4.2}$$

Then

$$m_t^n \leq d^{n_0(\varepsilon)} \cdot C_\delta \cdot e^{n\delta} = C'_\delta e^{n\delta}.$$

□

LEMMA 4.2. *Suppose again that J_f is totally disconnected. Let $\delta > 0$ be fixed. Let*

$$\tau'_n = \{V_t^n : \operatorname{diam} V_t^n \geq e^{-n\delta}\}.$$

Then there exists $C'' = C''_\delta$ such that for every n

$$\operatorname{card} \tau'_n \leq C'' e^{8n\delta}.$$

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Proof. Fix δ. We will be using Lemma 4.1 with this δ and Lemma A.2.3 with $\delta_1 = \delta$ and $\delta_2 = 4\delta$. Fix V_t^n. Inside it there are at most $Ke^{n\delta_1} = Ke^{n\delta}$ groups of D. Each D in the same group is mapped onto the same $B \in U_n$. The cardinality of each group is thus at most $m_t^n \le C'e^{n\delta}$. If the diameter of each D lying inside V_t^n was at most $e^{-4n\delta}$, then we could estimate

$$\operatorname{diam} V_t^n \le Ke^{n\delta} \cdot C'e^{n\delta} \cdot e^{-4n\delta} < e^{-n\delta}$$

if n is large. Thus, if $V_t^n \in \tau_n'$ it contains D of diameter at least $e^{-4n\delta}$. Using claim 3 of Lemma A.2.3. we get

$$\operatorname{card} \tau_n' \le Ke^{8n\delta}. \qquad \square$$

After these general considerations, we pass to properties of the harmonic measure. First we prove the following lemma.

LEMMA 4.3. *Let J_f be disconnected. (Note: not necessarily totally disconnected!) Then there exists $\delta_0 > 0$ such that for every n and every t*

$$\omega(V_t^n) \le e^{-n\delta_0}.$$

Proof. We need some geometric notation. Let C be a contour in U surrounding V. Let us denote by C_t^n the preimage of C lying in V_t^n. Suppose that $f^{n-1}V_t^n = V_j$, $\hat{V} = V \setminus V_j$, $\hat{V}^n = f^{-n}(\hat{V}) \cap V_t^n$. We introduce three harmonic functions. Let

$$u^n(z) \stackrel{\text{def}}{=} \omega(V_t^n, \hat{V}^n, z), \quad \hat{v}^n(z) \stackrel{\text{def}}{=} \omega(A_\infty, \hat{V}^n, z) \quad \text{and} \quad v^n(z) \stackrel{\text{def}}{=} \omega(A_\infty, V_t^n \setminus \hat{V}^n, z).$$

We use the notation $A_\infty = \hat{\mathbb{C}} \setminus J_f$. Let $\hat{u}$ be a harmonic function in $U \setminus J_f$ which is 1 on $\hat{V} \cap J_f$ and 0 on ∂U. Then for $\xi \in V_t^n$, $\hat{v}^n(\xi) \ge \hat{u}^n(\xi) = (\hat{u} \circ f^n)(\xi)$. In particular,

$$\min_{\xi \in C_t^n} \hat{v}^n(\xi) \ge \min_{\xi \in C} \hat{u}(\xi) \ge \lambda > 0.$$

This means that $v^n(\xi) \le 1 - \lambda$ for $\xi \in C_t^n$. So

$$v^n(\xi) \le 1 - \lambda \quad \text{on } \partial V_t^n.$$

Now let us consider the nest $V^n \stackrel{\text{def}}{=} V_t^n \subset V^{n-1} \subset \cdots \subset V^0 \stackrel{\text{def}}{=} U$. Applying the consideration we have just applied to V^n to $V^{n-1}, V^{n-2}, \ldots, V'$, we get

$$v^n(\xi) \le (1 - \lambda)^n \quad \text{on } \partial U.$$

So $\omega(V_t^{n+1}) = v^n(\infty) \le (1 - \lambda)^n$ which proves the lemma. $\square$

Let $\omega(E, z)$ denote the harmonic measure of $E \subset J_f$ evaluated at $z \in A_\infty = \hat{\mathbb{C}} \setminus J_f$, and $\omega(E) = \omega(E, \infty)$.

Let $\hat{C}$ be a contour close to ∂U and surrounding it such that each component of $f^{-n}(\operatorname{int} \hat{C})$ contains only one V_t^n. Let v be a harmonic function which is 0 on $\hat{C}$ and 1 on J_f. Then for $\xi \in \partial V_t^n$,

$$\omega(J_f \cap V_t^n, \xi) \ge (v \circ f^n)(\xi)$$

by the maximum principle.

This implies that

$$\min_{\xi \in \partial V_t^n} \omega(J_f \cap V_t^n, \xi) \ge \min_{\xi \in \partial U} v(\xi) \ge \lambda > 0 \tag{4.3}$$

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LEMMA 4.4. *If $z \in V_t^n$ we have*

$$G(z) \le C \cdot n\omega(J \cap V_t^n). \tag{4.4}$$

Proof. Clearly,

$$\operatorname{dist}(C_t^n, \partial V_t^n) \ge C \cdot (\|f'\|_\infty)^{-n}. \tag{4.5}$$

Without loss of generality we can assume that $\operatorname{diam} U < 1$. If $z \in C_t^n$, $\xi \in \partial V_t^n$ the formula for $G(z, \xi)$ (see Appendix 1) implies that

$$G(z, \xi) \le \log \frac{1}{\operatorname{dist}(C_t^n, \partial V_t^n)} \le C' \cdot n. \tag{4.6}$$

Now the functions $\xi \to G(z, \xi)$ and $\omega(J_f \cap V_t^n, \xi)$ are harmonic in $\Omega_t^n = A_\infty \setminus V_t^n$ and vanish on $\partial\Omega_t^n \setminus \partial V_t^n$. On the other hand, if $\xi \in \partial V_t^n$, the estimates (4.6) and (4.3) imply that

$$G(z, \xi) \le \lambda^{-1} \cdot C' \cdot n \cdot \omega(J_f \cap V_t^n, \xi).$$

Thus, this estimate holds on the whole boundary of Ω_t^n and thus at $\xi = \infty$. This proves the lemma. □

The crucial estimate follows in the next lemma.

LEMMA 4.5. *There exist $\delta_1 > 0$ and $n_0 > 0$ such that for $n \ge n_0$, uniformly in $x \in J_f$,*

$$\sum_{\substack{y \in f^{-n}x \\ y \in V_t^n \in \tau_n'}} E_n(y) \le e^{-n \cdot \delta_1}.$$

Proof. Recall that $E_n(y) = e^{\varphi(y)+\varphi(fy)+\cdots+\varphi(f^{n-1}y)}$, where $\varphi = -\log J_\omega$. In particular, clearly

$$\begin{aligned} \forall\, y \in J_f \cap V_t^n, \quad E_n(y) &\le \max_{\xi \in \bar{V}_t^n} \frac{G(\xi)}{G(f^n\xi)} = \max_{\xi \in \partial V_t^n} \frac{G(\xi)}{G(f^n\xi)} \\ &\le K \max_{\xi \in \partial V_t^n} G(\xi), \end{aligned} \tag{4.7}$$

where $K = (1/\min_{\xi \in \partial U} G(\xi)) < \infty$. Now we use Lemmas 4.1–4.4 to continue:

$$\sum_{\substack{y \in V_t^n \in \tau_n' \\ y \in f^{-n}x}} E_n(y) \le K \cdot C_\delta e^{n\delta} n \sum_{V_t^n \in \tau_n'} \omega(J_f \cap V_t^n) \le K \cdot C_\delta e^{10n\delta} e^{-n\delta_0}.$$

If we choose δ sufficiently small, we obtain our claim. □

LEMMA 4.6. *Let $g_n = P_\varphi^n \mathbf{1}$. Then*

$$0 < C^{-1} \le g_n \le C < \infty.$$

Proof. One can easily check that (see Lemma A.2.2)

$$P_\varphi^* \omega = \omega.$$

So we have

$$\int_{J_f} g_n \, d\omega = 1. \tag{4.8}$$

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Fix an integer $n_0 > 0$, to be specified later, and denote for $x \in J_f$,

$$\hat{g}_n(x) = \sum \{E_n(y) : f^k V_t^n \notin \tau'_{n-k},\ 0 \le k \le n - n_0\}.$$

In other words, the sum is taken over branches such that they behave geometrically well from depth n to depth $n - n_0$.

Suppose now that for some K, for every $x \in J_f$, $m < n$,

$$g_m(x) \le K.$$

We will show that $g_n(x) \le K$. The proof differs essentially from the proof in [**Pr1**] because we cannot use the $P(f, \varphi) > \sup \varphi$ condition which was decisive in [**Pr1**]. Still,

$$\begin{aligned} g_n(x) - \hat{g}_n(x) &\le \sum_{k=0}^{n-n_0} \sum_{y \in f^{-(n-k)} x \in V_t^{n-k} \in \tau'_{n-k}} E_{n-k}(y) g_k(y) \\ &\le K \sum_{k=0}^{n-n_0} \sum_{y \in f^{-(n-k)} x \in V_t^{n-k} \in \tau'_{n-k}} E_{n-k}(y) \\ &\le K \cdot C_\delta \sum_{k=0}^{n-n_0} e^{-\delta(n-k)} \le \varepsilon(n_0) K. \end{aligned} \tag{4.9}$$

It is clear from the Hölder continuity of φ and from the choice of τ'_{n-k}, that

$$\frac{\hat{g}_n(z)}{\hat{g}_n(w)} \le C_1(n_0) \quad \forall\, z, w \in J_f. \tag{4.10}$$

Also, from (4.9) we have

$$1 - \varepsilon(n_0) K \le \int \hat{g}_n \, d\omega \le 1.$$

Now (4.9) and (4.10) show ($\varepsilon = \varepsilon(n_0)$)

$$\frac{1 - \varepsilon K}{C_1} \le \hat{g}_n(x) \le C_1. \tag{4.11}$$

Thus,

$$\frac{1 - \varepsilon K}{C_1} \le g_n(x) \le C_1 + \varepsilon K. \tag{4.12}$$

Choosing n_0 in such a way that $\varepsilon(n_0) \le \frac{1}{2}$, we see that $K = 2C_1(n_0)$ guarantees that $g_n(x) \le K$. With the bound K chosen, we can use the left inequality in (4.12) with $\varepsilon = \varepsilon(n_1)$, $C_1 = C_1(n_1)$. Choosing n_1 large enough, we see that $\hat{g}_n$ (and thus g_n) is bounded away from zero by a constant independent of n. □

LEMMA 4.7. *Let ψ be a continuous function on J_f bounded away from zero on J_f.*

(1) *Then the sequence $\{P_\varphi^n(\psi)\}_{n=0}^\infty$ consists of uniformly bounded away from zero and equicontinuous functions.*

(2) *If ψ is Hölder, then $P_\varphi^n(\psi)$ are uniformly Hölder.*

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Proof. (1) Let $\lambda = \min \psi > 0$. Let

$$\hat{g}_{n,\psi}(x) = \sum_{\substack{y \in f^{-n}x \\ y \in V_t^n \in \tau_n'}} \psi(y) E_n(y).$$

In Lemma 4.5 we proved that $\hat{g}_{n,\psi}(x) \le C_{\delta_1} e^{-n\delta_1}$. Combining with $P_\varphi^n(\psi) \ge \lambda \cdot C^{-1}$ established in Lemma 4.6, we can write

$$\hat{g}_{n,\psi}(x) \le C'_{\delta_1} e^{-n\delta_1} (P_\varphi^n \psi)(x).$$

This estimate is sufficient to carry out the proof of Theorem 1 of [**Pr1**].

(2) As in [**Pr1**], Lemma 3.4 of [**DPU**] (which is Theorem 5.1 below) finishes the proof. □

As Przytycki's paper is not widely available, we would like to make some comments. In the general non-expanding situation the results in [**DU**] and in [**Pr1**] do not give that $P_\varphi^n(\psi)$ is uniformly Hölder. However, Przytycki proved in [**Pr1**] that this follows from additional information concerning the growth of the size of preimages of a ball centered in J. This was proved in [**DPU**] (Lemma 3.4) and we also give a different proof in §5 (Theorem 5.1). The proof of Theorem 5.1 is due to the second author.

Remark 4.8. The Hölder continuity of $\varphi = -\log J_\omega$ can be weakened in the first claim above. The Dini condition $\int_0 \tau_\varphi(t)/t\, dt < \infty$ would be sufficient if τ_φ denotes the modulus of continuity of φ.

Now we are ready to prove the first claim of Theorem 1.3.

THEOREM 4.9. *Let J_f be totally disconnected. Let*

$$\varphi = -\log J_\omega \text{ be Hölder continuous.} \tag{H}$$

Then there exists invariant measure $\mu = \psi_0 \cdot \omega$ with Hölder continuous ψ_0, $\psi_0 > 0$.

Proof. We apply Lemma 4.7 to $\psi = \mathbf{1}$. Let $X = \overline{\text{convh}\{P_\varphi^n \mathbf{1}\}}$. By the first part of Lemma 4.7, X is a convex compact set in $C(J_f)$. Note that $\int h(x)\, d\omega(x) = 1$ for each $h \in X$. By the second part, each element of X is a Hölder function. Let $\psi_0 \in X$ be a fixed point of $P_\varphi : X \to X$. So ψ_0 is Hölder continuous and $\psi_0 \ge \lambda > 0$. Let us check that $\psi_0 \cdot \omega$ is f-invariant. Fix an arbitrary $h \in C(J_f)$. Then

$$\langle \psi_0 \cdot h, \omega \rangle = \langle P_\varphi(\psi_0) \cdot h, \omega \rangle = \langle P_\varphi(\psi_0 \cdot h \circ f), \omega \rangle = \langle \psi_0 \cdot h \circ f, \omega \rangle.$$

We can rewrite this as

$$\langle h \circ f, \psi_0 \cdot \omega \rangle = \langle h, \psi_0 \cdot \omega \rangle$$

and this is exactly the f-invariance of $\psi_0 \cdot \omega$. Now we are done. □

5. *Estimate of the size of preimages of the ball. An analog of Mañé's lemma*

Mañé proved in [**M**] that under certain conditions on the critical points' behavior, the size of components of preimages of a sufficiently small ball (centered at the Julia set of rational function) stays bounded. The conditions are as follows. There are no parabolic

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points and for each critical point c on the Julia set its orbit does not accumulate to c. We call the last condition *semi-hyperbolicity*.

We intend to prove the analog of Mañé's result which holds without the assumption of semi-hyperbolicity. For the sake of completeness we also sketch the proof of Mañé's original lemma.

We intend to work in the setting of GPL dynamics, and even in a slightly more general setting of holomorphic repellers (see the definition below). Because of the qc-invariance of the problem, this does neither restrict nor expand the generality in comparison with the case of rational dynamics.

So let U be a *finitely* connected domain with analytic boundary, let $V_1, \dots, V_k$ be *finitely* connected subdomains, $\overline{V_1} \subset U$, $i = 1, \dots, k$, and let $f_i : V_i \to U$ be branched or regular coverings of finite degree d_i. Denote $V = \bigcup_{i=1}^{k} V_i$, $f = (f_1, \dots, f_k)$, $f : V \to U$ and $d = d_1 + \cdots + d_k$. The triple (f, V, U) is called a holomorphic repeller. The symbol $|B|$ denotes the diameter of B.

For the sake of simplicity we assume that

$$\operatorname{int} K_f = \emptyset, \tag{EI}$$

where $K_f = \bigcap_{n\geq 0} f^{-n}(U)$. As always, $J_f = \partial K_f$ and thus equals K_f due to (EI).

The next estimate was crucial in proving the second assertion of Lemma 4.10 which amounts to Hölder continuity of $P_\varphi^n(\psi)$. In its turn, this Hölder continuity means the Hölder continuity of the density of invariant harmonic measure.

THEOREM 5.1. *Let (f, V, U) be a holomorphic repeller satisfying* (EI). *Then there exist numbers $A, L, D, r_0 > 0$, such that if $B_n(r)$ denotes any component of $f^{-n}(B(x, r))$, $x \in U$, $r \leq r_0$, then*

$$|B_n(r)| \leq r^{1/D} e^{Ln}. \tag{5.1}$$

This result was mentioned to the third author by Feliks Przytycki. Our proof here (belonging to the second author) is different (and may be shorter) from the proof of Denker, Przytycki and Urbanski which recently appeared in [**DPU**].

LEMMA 5.2. *Let (f, V, U) be as above. Then there exist A, B such that*

$$|B_n(e^{-An})| \leq e^{-Bn}. \tag{5.2}$$

Clearly, Lemma 5.2 implies Theorem 5.1. In fact, let $r = e^{-A'n}$, $A' \geq A$. Put $N = [(A'/A)n]$, then $r \leq e^{-AN}$ and $|B_N(e^{-AN})| \leq e^{-BN}$ by the lemma. On the other hand, if $e^T = \max_U |f'|$, we have

$$\begin{aligned} |B_n(r)| &\leq |B_n(e^{-AN})| \leq |f^{N-n} B_N(e^{-AN}| \leq e^{T(N-n)} e^{-BN} \\ &\leq e^{-B(N+1)/2} \leq (e^{-A'n})^{B/2A} = r^{B/2A}, \end{aligned}$$

if B is large with respect to T. Put $D = 2A/B$. If $r \geq e^{-An}$ the choice of $L \sim A/D$ proves (5.1).

Notation. A component of $f^{-n}(B(x, R))$ containing y will be denoted by $B_n^x(R, y)$. The letters x, y will often be omitted. We call the components $B_n(R), B_{n-1}(R), \dots, B(x, R)$ the *chain* if they are sent by f one to another.

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To prove Lemma 5.2 we use the following lemmas.

LEMMA 5.3. *Let (f, V, U) be as before. There exists K such that for any x the chain $B_n(e^{-Kn}), \dots, B(x, e^{-Kn})$ meets each critical point only once.*

LEMMA 5.4. *Let $\pi = \pi_2 \circ \pi_1$, where $\pi_1 : W \to V$, $\pi_2 : V \to B(0, R)$, $R > 2$, are branched coverings, π_1 with one critical point of multiplicity d_1 and π_2 with one critical point of multiplicity d_2. Denote by W_0 a component of $\pi^{-1}(B(0, 1))$. Then*

$$\operatorname{mod}(W \setminus W_0) \geq \frac{a}{d_1 d_2} \log R,$$

where a is a positive absolute constant.

LEMMA 5.5. *If $W_0 \subset W$ are simply connected domains, then $|W_0| \leq C_0 |W| e^{-c_1 \operatorname{mod}(W \setminus W_0)}$, where C_0, c_1 are absolute.*

Lemmas 5.4 and 5.5 are almost obvious. One just needs to notice that if a doubly connected domain Ω has a large modulus, then it contains a standard annulus whose modulus is also large enough.

Let $c_1, \dots, c_k$ be all our critical points and let $d_1, \dots, d_k$ be their multiplicities.

Now Lemma 5.3 implies Lemma 5.2. As the chain $B_n(e^{-Kn}), \dots, B(x, e^{-Kn})$ meets each critical point at most once, we apply Lemma 5.3 to claim that

$$\operatorname{mod}(B_n(e^{-Kn}) \setminus B_n(e^{-2Kn})) \geq \frac{aKn}{d_1 d_2 \cdots d_k},$$

and so by Lemma 5.5

$$|B_n(e^{-An})| = |B_n(e^{-2Kn})| \leq C \cdot C_0 e^{-c_1 aKn/d_1 d_2 \cdots d_k} \leq e^{-Bn}.$$

It remains to prove Lemma 5.3.

Proof. Use induction on n. If K is large enough, the '$n = 1$' claim holds. We know that $B_{n-1}(e^{-K(n-1)}), \dots, B(x, e^{-K(n-1)})$ contains each critical point at most once. So the '$k = n$' claim fails only if, say, $c_1 \in B_n(e^{-Kn})$ and simultaneously $c_1 \in B_{n_1}(e^{-Kn})$, $n > n_1$. By the '$k = n_1$' assertion the chain $B_{n_1}(e^{-Kn_1}), \dots, B(x, e^{-Kn_1})$ meets each critical point at most once. So by Lemma 5.4,

$$\operatorname{mod}(B_{n_1}(e^{-Kn_1}) \setminus B_{n_1}(e^{-Kn})) \geq \frac{aK}{d_1 d_2 \cdots d_k}(n - n_1).$$

In particular, by Lemma 5.5

$$|B_{n_1}(e^{-Kn})| \leq C \cdot C_0 \exp\left[-\frac{c_1 aK}{d_1 d_2 \cdots d_k}(n - n_1)\right] \leq \gamma^{n-n_1}, \tag{5.3}$$

where γ can be chosen as close to 0 as we wish by the choice of large K.

We have already mentioned that $c_1 \in B_{n_1}(e^{-Kn})$, and that $c_1 \in B_n(e^{-Kn}) \Rightarrow f^{(n-n_1)} c_1 \in B_{n_1}(e^{-Kn})$. Now (5.3) implies

$$|c_1 - f^{(n-n_1)} c_1| \leq \gamma^{n-n_1}.$$

But this is impossible if γ is small enough. See [**Pr2**] for example. Theorem 5.1 is completely proved. □

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The next proposition is a variant of Mañé's lemma [**M**] for one critical point.

PROPOSITION 5.6. *Let (f, V, U) be a holomorphic repeller with* (EI). *Let f be semi-hyperbolic on J_f and let c be a critical point of f on J_f. Then for any ε, there exists δ such that*

$$\forall\, x \in J_f,\ \forall\, n, \quad |B_n^x(\delta)| \le \varepsilon. \tag{5.4}$$

Proof. Let $\Delta = \operatorname{dist}(c, \{f^j c\}_{j\ge 1})$. We assume $\Delta < 1$. The Koebe distortion theorem and assumption (EI) readily imply the following assertion.

$$\left.\begin{array}{l}\text{There exists } \delta_k > 0 \text{ such that for any } r < \delta_k,\\ \text{if the chain } B_{n-1}^x(2r), \dots, B(x, 2r) \text{ does not meet } c,\\ \text{then } |B_j(r)| \le \Delta^k,\ j = 0, \dots, n.\end{array}\right\} \tag{5.5}$$

Now we claim that:

$$\left.\begin{array}{l}\text{There exists } \eta_k > 0 \text{ such that for any } r < \eta_k,\\ \text{if the chain } B_{n-1}(r), \dots, B(x, r) \text{ does not meet } c\\ \text{but } c \in B_n(r), \text{ then } |B_j(r)| \le \Delta^k,\ j = 0, \dots, n.\end{array}\right\} \tag{5.6}$$

Claim (5.6) is obvious for $n = 1$. Let it hold for $2, \dots, n-1$. Let $\ell_0 = \min\{m : c \in B_m(2r)\}$. If $\ell_0 \ge n$, then (5.5) implies (5.6) for n. So suppose $\ell_0 < n$. Choose $B(y, r) \subset B(x, 2r)$ in such a way that $f^{\ell_0}(c) \in B(y, r)$. Clearly

$$B(x, r) \cap B(y, r) \ne \emptyset. \tag{5.7}$$

The induction hypothesis for ℓ_0 shows that

$$|B_{\ell_0-1}^y(r)| \le \Delta^k$$

Also, (5.5) implies

$$|B_{\ell_0-1}^x(r)| \le \Delta^k.$$

But $B_{\ell_0-1}^x(r) \cap B_{\ell_0-1}^y(r) \ne \emptyset$ because of (5.7) and the fact that $f^{\ell_0-1} : B_{\ell_0-1}(2r) \to B(x, 2r)$ is univalent. We see that

$$|fc - f^{n-\ell_0+1}c| \le 2\Delta^k.$$

Both points are in $B_{\ell_0-1}(2r)$. So $|c - f^{n-\ell_0}c| \le c_f \Delta^{\alpha_f k}$. For $k \ge k_f$, this comes to a contradiction with the non-recurrency assumption. Claim (5.6) is proved.

To finish the proof of Proposition 5.6, let us fix a large k: $\Delta^k \le \varepsilon/2$ and let $r < \delta = \eta_k/2$. Let n be the first place where $c \in B_n(2r)$. Then

$$|B_n(2r)|, |B_{n-1}(2r)|, \dots, |B(x, 2r)| \le \Delta^k \le \frac{\varepsilon}{2}. \tag{5.8}$$

On the other hand, $B(c, 4\varepsilon) \cap \{f^j c\}_{j\ge 1} = \emptyset$ if ε is small enough and so all components of $f^{-j}(B(c, \varepsilon/2))$ are small enough for all $j \ge 0$ by (5.5):

$$\left|B_j^c\left(\frac{\varepsilon}{2}\right)\right| \le \varepsilon. \tag{5.9}$$

Now $B_m(r)$ lies in $B_m(2r)$ and if $m \le n$, (5.8) gives the required estimate. On the other hand, $B_m(r)$ lies in $B_j^c(\varepsilon/2)$ if $m > n$, and so (5.9) gives the required estimate. □

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6. *Non-recurrent dynamics*

We are going to list here the simplifications stemming from the assumption of non-recurrency:

$$\forall\, c \in \mathrm{Crit}(f) \cap J_f, \quad \inf_{n\geq 1} |c - f^n c| \geq \Delta > 0. \tag{NR}$$

For simplicity, we consider only the case of totally disconnected J_f.

The basic fact stemming from (NR) is the following lemma which basically follows from Proposition 5.6 and can be found in many references, e.g. in [**CJY**] or in [**BV2**]. The notation is in §4.

LEMMA 6.1. *Let (f, V, U) be a non-recurrent GPL system with totally disconnected J_f. Then there exist c and $\delta > 0$ such that*

$$\operatorname{diam} V_t^n \leq c e^{-n\delta}. \tag{6.1}$$

Another basic result implied by (NR) is somewhat more difficult. It can be found in [**BV3**].

THEOREM 6.2. *Let (f, V, U) be as before. Let G be Green's function of $\mathbb{C} \setminus J_f$. Then* (R) *holds:*

$$\exists\, c, \varepsilon > 0\, \forall\, \xi_1, \xi_2 \in V \setminus J_f, \quad \left| \frac{G(f\xi_1)}{G(\xi_1)} : \frac{G(f\xi_2)}{G(\xi_2)} - 1 \right| \leq c|\xi_1 - \xi_2|^{\varepsilon}. \tag{R}$$

Here is a corollary to Theorem 6.2.

THEOREM 6.3. *Let (f, V, U) be a GLP non-recurrent system with totally disconnected J_f. Then the Jacobian J_ω of harmonic measure ω is such that* $\log J_\omega$ *is Hölder continuous.*

Lemma 6.1 also has a very important corollary.

LEMMA 6.4. *Let f be as before. Let φ be a Hölder continuous function on J_f and let ν be a probabilistic measure on J_f with the property*

$$P_\varphi^* \nu = \nu.$$

Then P_φ is almost periodic on $C(J_f)$, and if ψ is Hölder continuous on J_f, then all functions $P_\varphi^n(\psi)$ are uniformly Hölder continuous. Moreover, there exists a Hölder continuous fixed point ψ_0 of P_φ such that $P_\varphi^n(\psi) \to \psi_0$. The measure $\psi_0 \cdot \nu$ is f-invariant and ergodic, and $\psi_0 > 0$ on J_f.

Proof. We can repeat the proof of Theorem 1 of [**Pr1**] using Lemma 6.1 systematically. For convenience, we recall the main steps in the proof of Theorem 1 in [**Pr1**].

The first step is to show that there exists $C > 0$ such that for any $x, y \in J$ and $n \in \mathbb{N}$ we have

$$\frac{P_\varphi^n(1)(x)}{P_\varphi^n(1)(y)} \leq C.$$

While in the general case this estimate is hard to get, in our semi-hyperbolic situation by Lemma 6.1 it becomes an easy exercise.

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The second step is to prove that we have the following formula for the topological pressure:

$$P(f,\varphi) = \lim_{n\to\infty} \frac{1}{n} \log P_\varphi^n(1)(x),$$

for any $x \in J$.

At this step Przytycki uses essentially the inequality $\varphi < P(f,\varphi)$ which we do not need as we are helped again by Lemma 6.1.

After these steps the statement of the lemma follows in a standard way. □

Now we can prove Theorem 1.4, which is the existence of f-invariant harmonic measure with Hölder continuous density.

First consider $\varphi \stackrel{\text{def}}{=} -\log J_\omega$. This is Hölder continuous by Theorem 6.3. Apply Lemma 6.4 to get f-invariant measure $\mu = \psi_0 \cdot \omega$. As $\psi_0 > 0$ and ψ_0 is Hölder continuous, obviously $\log \psi_0$ is Hölder continuous and Theorem 1.4 is proved. □

7. *The homology equation and conformal maximality*

Here we prove Theorem 1.5 and 'the conformal maximality' part of Theorem 1.3. The proof is the same. The difference is that (R) is assumed in Theorem 1.3 and we can forget about this assumption in Theorem 1.5, because under the conditions of non-recurrency and total disconnectedness (R) will hold (see [**BV3**] and [**BV4**]).

We would like to mention that this result was proved in full generality, simultaneously and independently by the first author and by A. Zdunik in April 1995.

We think that it is still useful to give the proof in our setting.

The setting is as follows. Given (R), we have already proved that

$$\varphi(x) \stackrel{\text{def}}{=} \log \frac{G(x)}{G(fx)} \stackrel{\text{def}}{=} \lim_{z\to x} \log \frac{G(z)}{G(fz)}$$

is Hölder continuous and there exists $\psi_0 = e^\gamma$ (the eigenvector of P_φ), with Hölder continuous γ and such that

$$\psi_0 \cdot \omega = m \tag{7.1}$$

Having in mind that $J_\omega = e^{-\varphi}$ and $J_m = d$, we rewrite (7.1) as follows ($\gamma = \log \psi_0$):

$$\varphi(x) + \log d = \gamma \circ f(x) - \gamma(x), \quad x \in J_f. \tag{7.2}$$

Having (R) in mind, we can also notice that

$$\left| \log \frac{G(fz)}{G(z)} - \log d + \gamma(f(x)) - \gamma(x) \right| \le c|z-x|^\varepsilon. \tag{7.3}$$

Now let q_i be a repelling fixed or periodic point of f on J_f with period ℓ. Let $B = B_{q_i}$ be such a small disc centered at q_i such that the components $W_{n\ell}$ of $f^{-n\ell}(B)$ containing q_i are contained in B, $n = 1, \ldots$, and let B be free from critical points of f^ℓ.

Let us construct a certain harmonic function on $\hat{B} = B \cup f^{-1}(B) \cup \cdots \cup f^{-(\ell-1)}(B)$. First on B. We denote by $g = g_{q_i}$ the inverse branch of $f^{-\ell} : B \to W_\ell \subset B$. Now (7.3) will give

$$\left| \frac{G(g^{n-1}z)}{d^\ell G(g^n z)} - 1 \right| \le Cq^n, \tag{7.4}$$

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for some $q < 1$.

Now (7.4) implies that the following limit represents a subharmonic function:

$$U_{q_i}(z) = \lim_{n\to\infty} d^{\ell n} G(g^n(z)), \quad z \in B \tag{7.5}$$

such that

$$U_{q_i}(z) \le CG(z) \tag{7.6}$$

and U_{q_i} is harmonic in $B \setminus J_f$.

Now we define U_{q_i} on $f^{-j}(B)$ as

$$U_{q_i}(z) = d^{-j} U_{q_i}(f^j z), \quad j = 1, \dots, \ell - 1. \tag{7.7}$$

Applying (7.3) and (7.2) and recalling that

$$\varphi(x) = \lim_{\substack{z\to x \\ z\in B\setminus J_f}} \log \frac{G(z)}{G(fz)},$$

we obtain

$$\lim_{\substack{z\to x \\ z\in B\setminus J_f}} \frac{G(z)}{U_{q_i}(z)} = e^{-\gamma(x)+\gamma(q_i)}, \quad x \in B \cap J_f. \tag{7.8}$$

Applying (7.2) again ($\hat{B} = B \cup f^{-1}(B) \cup \cdots \cup f^{-(\ell-1)}(B)$), we obtain

$$\lim_{\substack{z\to x \\ z\in \hat{B}\setminus J_f}} \frac{G(z)}{U_{q_i}(z)} = e^{-\gamma(x)+\gamma(q_i)}, \quad x \in \hat{B} \cap J_f. \tag{7.9}$$

Now let θ be any preperiodic point on J_f. Let $f^m\theta = q_i$, where q_i is periodic, and let B_θ be the component of $f^{-m}(B)$ containing θ, where $B = B_{q_i}$. On B_θ we can define U_θ as

$$U_\theta(z) = d^{-m} U_{q_i}(f^m z).$$

Then (7.2) and (7.3) show

$$\lim_{\substack{z\to x \\ z\in B_\theta\setminus J_f}} \frac{G(z)}{U_\theta(z)} = e^{-\gamma(x)+\gamma(q_i)}, \quad x \in B_\theta \cap J_f. \tag{7.10}$$

Put $V_\theta = e^{\gamma(q_i)} U_\theta$ on B_θ and $V_{q_i} = e^{\gamma(q_i)} U_{q_i}$ on $\hat{B}_{q_i}$. Let us try to show that these functions define one function on $\mathcal{O} = \bigcup_\theta B_\theta \bigcup_i \hat{B}_{q_i}$.

In fact, (7.9) and (7.10) show that if θ_1 and θ_2 are two periodic or preperiodic points, and $B_{\theta_1}, B_{\theta_2}$ $V_{\theta_1}, V_{\theta_2}$ are domains and functions constructed on them, then

$$|V_{\theta_1}(z) - V_{\theta_2}(z)| = o(G(z)), \quad z \in B_{\theta_1} \cap B_{\theta_2},\ z \to J_f. \tag{7.11}$$

Lemma A.1.2 and (7.11) imply that $\Delta V_{\theta_1} = \Delta V_{\theta_2}$ on $B_{\theta_1} \cap B_{\theta_2}$. Thus, $V_{\theta_1} - V_{\theta_2}$ is harmonic in $B_{\theta_1} \cap B_{\theta_2}$, and it also vanishes on $J_f \cap B_{\theta_1} \cap B_{\theta_2}$. Either $V_{\theta_1} \equiv V_{\theta_2}$ on $B_{\theta_1} \cap B_{\theta_2}$, or $J_f \cap B_{\theta_1} \cap B_{\theta_2}$ is covered by finitely many real analytic curves. Suppose J_f is covered by finitely many real analytic curves. As in the proof of Theorem 1.1, we conclude that we may only consider this situation when these curves are disjoint. Now

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let $*$ be a holomorphic symmetry with respect to these curves. Instead of V_θ we are going to work with

$$\hat{V}_\theta(z) \stackrel{\text{def}}{=} V_\theta(z) + V_\theta(z^*).$$

The advantage is that now $\hat{V}_{\theta_1} \equiv \hat{V}_{\theta_2}$ in $B_{\theta_1} \cap B_{\theta_2}$.

In any case we obtain a function τ on $\mathcal{O}$ such that $\tau|B_\theta = \hat{V}_\theta$ (or V_θ if the first possibility '$V_{\theta_1} = V_{\theta_2}$' always occurs).

Note that $fz^* = (fz)^*$. So regardless of the situation and of the branch of f^{-1},

$$\tau(z) = d\tau(f^{-1}z), \quad z \in \mathcal{O}. \tag{7.12}$$

It may happen that $\mathcal{O}$ does not contain the whole J_f. We are going to extend τ as follows.

Fix a certain repelling periodic point q_i. Let V_t^N be a component of $f^{-N}(U)$ which contains q_i and which is small enough to be inside $\mathcal{O}$ (recall that J_f is totally disconnected).

Let $z \in U$ not be a critical value of f^N and let Γ be a curve in U connecting z with a point $\xi \in V_t^N$. Let Γ be free from critical values of f^N. Let x, y be any two f^N-preimages of z lying in V_t^n. We use the branched covering $f^N : V_t^N \to U$ to lift Γ to two curves γ_x, γ_y starting at x, y respectively and lying in V_t^N. In a neighborhood W of Γ we can define τ^x, τ^y by

$$\tau^x(t) = d^N\tau(f_x^{-N}t), \quad \tau^y(t) = d^N\tau(f_y^{-N}t).$$

Here f_x^{-N}, f_y^{-N} are two branches of f^{-N} on W, mapping W onto neighborhoods of γ^x, γ^y respectively. Note that τ^x and τ^y coincide in a neighborhood of ξ. This is just because (7.12) holds. Being harmonic in W, these two functions coincide in W. Thus we define τ on U such that

$$\tau(f^N z) = d^N\tau(z), \quad \forall z \in V_t^N. \tag{7.13}$$

Also (7.2), (7.3) and (7.13) show that

$$\lim_{\substack{z \to x \\ z \in U \setminus J_f}} \frac{G(z)}{\tau(z)} = e^{-\gamma(x)}, \quad x \in J_f. \tag{7.14}$$

To build an automorphic function we first deduce from (7.14) and (7.3) that

$$\lim_{\substack{z \to x \\ z \in U \setminus J_f}} \frac{d\tau(z)}{\tau(fz)} = 1, \quad x \in J_f. \tag{7.15}$$

Thus, $\tau(fz) - d\tau(z)$ is harmonic in U and vanishes on J_f. There are two possibilities: either

$$\tau(fz) \equiv d\tau(z);$$

or J_f is covered by finitely many disjoint real analytic curves Γ and $H(z) = \tau(fz) - d\tau(z)$ vanishes on Γ.

Introducing a holomorphic symmetry $*$ with respect to Γ, we define $\hat{\tau}(z) = \tau(z) + \tau(z^*)$ and $\hat{H}(z) = H(z) + H(z^*)$. Clearly, $fz^* = (fz)^*$ and we see that

$$\hat{H}(z) = 0 \quad \text{on } \Gamma.$$

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By symmetry, $\partial\hat{H}/\partial n = 0$ on Γ and so $\hat{H} \equiv 0$. Thus,

$$\hat{\tau}(fz) = d\hat{\tau}(z).$$

An automorphic function has been built and Theorems 1.5 and 1.3 are completely proved. □

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Appendix 1. Background on potential theory

Any subharmonic function u on an open set U of $\mathbb{C}$ generates a positive σ-finite measure $\mu_u = \Delta u$, called its Riesz measure (here the Laplacian Δ is understood in the sense of distributions). This measure behaves naturally under conformal change of variables (so, actually it can be defined on any Riemann surface).

Let us assume that K is a compact in $\mathbb{C}$ such that $\mathbb{C} \setminus K$ is connected and that K is sufficiently thick (i.e. positive capacity). Then there exists a Green's function G with a pole at ∞: a non-negative subharmonic function on $\mathbb{C}$, harmonic on $\mathbb{C} \setminus K$, vanishing on K and having asymptotics $G(z) \approx \log|z|$ near ∞. Its Riesz measure is called the harmonic measure on K evaluated at ∞ and it is denoted by ω.

The harmonic measure is always supported on the boundary ∂K and if E is set intersecting ∂K, then we denote by $\omega(E) = \omega(\mathbb{C} \setminus K, E, \infty)$ the harmonic measure of $E \cup \partial K$. Knowing the harmonic measure ω, one can restore the Green function as its logarithmic potential:

$$G(z) = c + \int \log|z - t|\, d\omega(t),$$

where $c = 1/\operatorname{cap}(K)$.

One can consider Green's functions $G(\cdot, \xi)$ with an arbitrary pole $\xi \in \mathbb{C} \setminus K$ and the corresponding harmonic measures evaluated at ξ; our notation will be $\omega(E, \xi) = \omega(\mathbb{C} \setminus K, E, \xi)$. The relation between the Green's function and harmonic measure becomes

$$G(z, \xi) = \log\frac{1}{|z - \xi|} - \int \log\frac{1}{|z - t|}\, d\omega(t, \xi).$$

If v is any harmonic function on $\mathbb{C} \setminus K$ and continuous on the closure of $\mathbb{C} \setminus K$, we can obtain the value of v at any point $x \in \mathbb{C} \setminus K$ by the Poisson formula:

$$v(x) = \int v(t)\, d\omega(t, x).$$

Finally, we state two lemmas of Grishin [**Gr**].

LEMMA A.1.1. *Let V be a neighborhood of a compact K and let u, v be two subharmonic functions on V, vanishing on K and positive and harmonic in $V \setminus K$. Suppose that for*

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every $x \in \partial K$ there exists the limit

$$\rho(x) = \lim_{\substack{\xi \to x \\ \xi \in V \setminus K}} \frac{u(x)}{v(x)}$$

and that this limit is continuous. Then $d\mu_u = \rho\, d\mu_v$, where μ_u, μ_v denote the Riesz measures of u and v.

LEMMA A.1.2. *Let w_1, w_2 be two non-negative subharmonic functions on $\mathcal{O}$ with $w_1 \geq w_2$. Let*

$$J = \left\{ z \in \mathcal{O} : (w_1 - w_2)(z) = \liminf_{r \to 0} \int_0^{2\pi} (w_1 - w_2)(z + re^{i\theta})\, d\theta = 0 \right\}.$$

Then

$$\Delta w_1 \geq \Delta w_2 \quad \text{on } J.$$

Appendix 2. Background on thermodynamic formalism

We quote some basic facts of the Sinai–Bowen–Ruelle formalism for expanding endomorphisms. Let us assume for simplicity that our Julia set is the unit circle $\mathbb{T}$. The pressure of the function $\varphi \in C(\mathbb{T})$ is denoted by $P(F, \varphi)$ (see [**DGS**] for properties) and is defined by

$$P(f, \varphi) = \sup \left\{ h_\nu(F) + \int_{\mathbb{T}} \varphi\, d\nu : \nu \text{ is a } F\text{-invariant ergodic probability measure} \right\}.$$

Recall that $P_\varphi : C(\mathbb{T}) \to C(\mathbb{T})$ is given by $P_\varphi \psi(x) = \sum_{Fy=x} \psi(y) e^{\varphi(y)}$ and $\mathcal{L}_\varphi : C(\mathbb{T}) \to C(\mathbb{T})$ is defined by $\mathcal{L}_\varphi = P_{\varphi - P(F,\varphi)}$. Then $\mathcal{L}_\varphi$ is almost periodic, if for every $\psi \in C(\mathbb{T})$ the sequence of functions $\mathcal{L}_\varphi^n \psi$ is bounded and equicontinuous.

LEMMA A.2.1. *Suppose that F is expanding and φ is Hölder continuous. Then $\mathcal{L}_\varphi$ is almost periodic. There is a positive fixed point for $\mathcal{L}_\varphi$, namely a function ψ_0 such that $\mathcal{L}_\varphi \psi_0 = \psi_0$. Moreover, $\psi_0 = \lim \mathcal{L}_\varphi^n \mathbf{1}$ and it is Hölder continuous. There is a unique probability measure η on $\mathbb{T}$ such that $\mathcal{L}_{\psi_0}^* \eta = \eta$. We also have $\mathcal{L}_\varphi^n \psi \to \psi_0 \cdot \int_{\mathbb{T}} \psi\, d\eta$ uniformly as $n \to \infty$.*

The expanding case is similar to the case of a one-sided topological Markov chain and one can find the proof of the lemma in [**Bo**] or [**R**].

Given a measure ν with good Jacobian, Lemma A.2.1 allows us to construct an equivalent F-invariant measure μ with good density $d\mu/d\nu$.

LEMMA A.2.2. *Suppose that F is expanding and ν is a probability measure on $\mathbb{T}$ with Jacobian J_ν such that*

$$\varphi = -\log J_\nu \text{ is Hölder continuous.} \tag{A2.1}$$

Then there exists a unique F-invariant measure μ such that

$$\log \frac{d\mu}{d\nu} \text{ is Hölder continuous.} \tag{A2.2}$$

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Proof. Let $E_n(y) = \exp[\sum_{i=0}^{n-1} \varphi(f^i(y))]$. F is expanding so we can easily write (x_0 is any point on $\mathbb{T}$)

$$P(F,\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{F^n y = x_0} E_n(y).$$

Taking a sufficiently small neighborhood V of x_0 and using the Hölder continuity of φ, we see that

$$\frac{E_n(y)}{E_n(z)} \approx 1$$

for any two points y, z in the same component V_n of $f^{-n}(V)$. This and the fact that $\varphi = -\log J_\nu$ imply that $\sum_{F^n y = x_0} E_n(y) \le C \sum \nu(V_n)$, the sum being taken over all components of $f^{-n}(V)$. Now it is clear that $P(F,\varphi) \le 0$.

On the other hand, let us compute $P_\varphi^* \nu$. Fix a subset $E \subset \mathbb{T}$ on which F is injective. Then (we repeat reasoning on p. 116 of [**Pr1**])

$$\begin{aligned} P_\varphi^*(\nu)(E) &= \int P_\varphi(\chi_E)\, d\nu \int_{f(E)} e^{\varphi \circ (f|E)^{-1}}\, d\nu \\ &= \int_E e^{\varphi\circ(f|E)^{-1}\circ f} e^{-\varphi}\, d\nu = \nu(E). \end{aligned}$$

Decomposing $\mathbb{T}$ into a finite number of E_i on each of which F is injective, we get

$$P_\varphi^* \nu = \nu. \tag{A2.3}$$

Thus, the spectral radius of P_φ^* is at least 1. But $r(P_\varphi^*) = e^{P(F,\varphi)} \ge 1$. So $P(F,\varphi) \ge 0$ which shows $P(F,\varphi) = 0$.

Now Lemma A.2.1 can be applied to $\varphi = -\log J_\nu$ to obtain $\psi_0 = \lim_{n\to\infty} P_\varphi^n \mathbf{1}$ and η, $P_\varphi^* \eta = \eta$. By uniqueness, $\eta = \nu$. Clearly, $\mu \stackrel{\text{def}}{=} \psi_0\, d\nu$ is F-invariant and $\log \psi_0$ is Hölder continuous because ψ_0 is Hölder continuous, and $\psi_0 > 0$. □

For the general, non-expanding case a version of the theory has been developed by Denker and Urbanski in [**DE**]. Their result essentially says that the results of the previous two lemmas hold if we replace F by any rational map f and replace $\mathbb{T}$ by the Julia set $J = J_f$, but making an important additional assumption that

$$P(f,\varphi) > \sup_{x\in J_f} \varphi(x). \tag{A2.4}$$

A different proof of this result was given by Przytycki in [**Pr1**]. The difficulty in the general case is due to the fact that we may meet critical points as we take backward iterates of a small ball. The first step in overcoming this difficulty is to show that the bad event of meeting critical points does not happen very often. This is expressed by the following lemma (Lemma 1 in [**Pr1**]):

LEMMA A.2.3. *For every $\varepsilon, \delta_1, \delta_2 > 0$, there exists $K > 0$ such that for every $n \ge 0$ there exists U_n, a covering of U by discs, such that:*

(1) $\operatorname{card} U_n \le K e^{n\delta_1}$, *if for every $B \in U_n$ we denote by $\mathcal{B}_n$ the family of components of $f^{-n}(B)$ each repeated with its multiplicity; then*
(2) *for every $D \in \mathcal{B}_n$ and $0 \le i \le n$,* $\operatorname{diam} f^i D < \varepsilon$;
(3) $\operatorname{card}\{D \in \mathcal{B}_n : \operatorname{diam} D \ge e^{-n\delta_2}\} \le K e^{2n\delta_2}$.

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Remark. Actually, this type of assertion was invented by Lyubich [**Ly1**], [**Ly2**]. We would like to mention that now one can do slightly better. Actually, one may choose a covering U_n such that
$(1')$ $\operatorname{card} U_n \leq Kn^2$
holds. Just surround $a \in \operatorname{Crit}(f^n)$ by a disc of radius $r_n = e^{-Ln}$ with large n, in contrast to $r_n = e^{-e^{n\delta_1/2}}$ in [**Ly1**]. The rest of the construction is left without change. Estimates follow from Theorem 5.1 above or Lemma 3.4 of [**DPU**].

For $x, y \in K_f$ we define

$$\rho_n(x, y) = \max\{\operatorname{dist}(f^i(x), f^i(y)),\ i = 0, \ldots, n\}.$$

We write $B_n(x, \varepsilon)$ to denote the ball centered at x with radius ε in the metric ρ_n. We say x, y is (n, ε)-*close* if $y \in B_n(x, \varepsilon)$. We will also need the following lemma (Lemma 3 from [**Pr1**]).

LEMMA A.2.4. *For every $\delta > 0$ there exist $C_\delta, \varepsilon > 0$ such that for every $x_0 \in J_f$, every $n > 0$ and $y_0 \in f^{-n}(x_0)$, the number of points $z \in f^{-n}(x_0)$ which are (n, ε)-close to y_0 is less than $C_\delta e^{n\delta}$.*

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