*J. Plasma Physics* (2000), *vol.* 63, *part* 5, *pp.* 393–445. Printed in the United Kingdom 
© 2000 Cambridge University Press

# Nonlinear and three-wave resonant interactions in magnetohydrodynamics

G. M. WEBB,[1] A. R. ZAKHARIAN,[1] M. BRIO[2] and G. P. ZANK[3]

[1]Lunar and Planetary Laboratory, University of Arizona, Tucson, AZ 85721, USA

[2]Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA

[3]Bartol Research Institute, University of Delaware, Newark, DE 19716, USA

(Received 23 March 1999 and in revised form 18 February 2000)

**Abstract.** Hamiltonian and variational formulations of equations describing weakly nonlinear magnetohydrodynamic (MHD) wave interactions in one Cartesian space dimension are discussed. For wave propagation in uniform media, the wave interactions of interest consist of (a) three-wave resonant interactions in which high-frequency waves may evolve on long space and time scales if the wave phases satisfy the resonance conditions; (b) Burgers self-wave steepening for the magnetoacoustic waves, and (c) mean wave field effects, in which a particular wave interacts with the mean wave field of the other waves. The equations describe four types of resonant triads: slow–fast magnetoacoustic wave interaction, Alfvén–entropy wave interaction, Alfvén–magnetoacoustic wave interaction, and magnetoacoustic–entropy wave interaction. The formalism is restricted to coherent wave interactions. The equations are used to investigate the Alfvén-wave decay instability in which a large-amplitude forward propagating Alfvén wave decays owing to three-wave resonant interaction with a backward-propagating Alfvén wave and a forward-propagating slow magnetoacoustic wave. Exact solutions of the equations for Alfvén–entropy wave interactions are also discussed.

## 1. Introduction

Three-wave resonant interactions in magnetohydrodynamics (MHD) have been explored by a number of authors (e.g. Galeev and Oraevski 1963; Sagdeev and Galeev 1969; Akhiezer et al. 1975; Ali and Hunter 1998; Webb et al. 1999). The main aim of this paper is to investigate coherent three-wave resonant interactions of MHD waves in one Cartesian space dimension, using the formalism developed by Ali and Hunter (1998) and Webb et al. (1999). Webb et al. (1999) studied weakly nonlinear MHD wave interactions in a non-uniform large-scale background flow, with application to cosmic-ray-modified shocks. Ali and Hunter (1998) derived weakly nonlinear MHD wave interaction equations in one Cartesian space dimension, including the effects of weak dissipation due to viscosity and heat conduction, and the effects of Hall-current dispersion. In the ideal-MHD case, in which there are no cosmic-ray effects, and for a uniform background flow, the equations obtained by Webb et al. (1999) are essentially equivalent to the equations obtained by Ali and Hunter (1998), with zero viscosity, heat conduction, and Hall-current dispersion. The latter set of equations are coupled integro-differential evolution equations, in which

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

the integral terms describe three-wave resonant interactions. The equations also contain mean wave field interaction terms and Burgers self-wave steepening terms for the magnetoacoustic modes. Similar wave interaction equations were derived by Majda and Rosales (1984), who considered the resonant interaction of periodic weakly nonlinear sound waves and entropy waves in one Cartesian space dimension. Recent investigations of three-wave resonant interactions in plasma kinetic theory have been carried out by Vladimirov and Stenflo (1997) and Voitenko (1998a,b).

Analytical and numerical solutions of the three-wave resonant interaction equations for systems in which there are no self-wave steepening terms and for wave interactions in homogeneous media have been reviewed by Kaup et al. (1979). Exact solutions of the equations, including $N$-soliton solutions, travelling-wave solutions and space-independent solutions, in terms of Jacobian elliptic functions have been obtained. The equations can be solved by using the inverse scattering transform. The Jacobian elliptic function solutions are described in Sagdeev and Galeev (1969), where they are used to discuss parametric decay instabilities for Alfvén waves. Reiman (1979) considers three-wave resonant interactions in inhomogeneous media, for cases where the equations can be transformed to the homogeneous media equations.

Three- and four-wave resonant interactions in fluid dynamics and MHD are thought to be important in weak turbulence theory (see e.g. Kraichnan 1965; Sagdeev and Galeev 1969; Chin and Wentzel 1972; Davidson 1972; Cohen and Dewar 1974; Cohen 1975; Akhiezer et al. 1975; Skilling 1975a,b; Zakharov et al. 1992; Sridhar and Goldreich 1994; Goldreich and Sridhar 1995; Ng and Bhattacharjee 1996; Voitenko 1998b; Vladmirov and Stenflo 1997; Galtier et al. 2000). Sridhar and Goldreich (1994), Goldreich and Sridhar (1995), and Ng and Bhattacharjee (1996) consider incompressible, Alfvénic models. A weakly compressible turbulence model that is related to a reduced-MHD description has been investigated by Bhattacharjee et al. (1998). It is important to note that the waves in weak turbulence theory have random initial phases. The present work, and the results on resonant wave interactions by Ali and Hunter (1998) and Webb et al. (1999), assume that the waves have well-defined phases (coherent phases), and the random-phase approximation is not used.

Parametric instabilities of Alfvén waves have been discussed by a number of authors (e.g. Goldstein 1978; Wong and Goldstein 1986; Terasawa et al. 1986; Vinas and Goldstein 1991a,b; Jayanti and Hollweg 1993a,b; Hollweg 1994). Goldstein (1978) investigated the linear stability of the large-amplitude circularly polarized plane simple Alfvén wave in MHD. Similar analyses by Wong and Goldstein (1986), Terasawa et al. (1986), and Vinas and Goldstein (1991a,b) considered the stability of the large-amplitude circularly polarized Alfvén-like modes (i.e. the ion cyclotron and whistler waves) in a Hall-current plasma. These instabilities have been labelled as decay, beat, and modulational instabilities (see e.g. Jayanti and Hollweg 1993a,b; Hollweg 1994). One can view the MHD instabilities investigated by Goldstein (1978) and Jayanti and Hollweg (1993a,b) as being due to linear wave interactions in a non-uniform background flow, consisting of a large-amplitude nonlinear circularly polarized Alfvén wave. Thus these instabilities can be studied using the linear wave mixing equations derived by Webb et al. (1999), describing MHD wave interactions in a non-uniform background flow in one Cartesian space dimension. In this development, the linear waves interact with each other owing to gradients and the time dependence of the background flow. One can derive the dispersion equation

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

of Goldstein (1978) from the wave-mixing formalism of Webb et al. (1999). These processes are in general distinct from the three-wave resonant interactions of two Alfvén waves and a slow magnetoacoustic wave in a low-beta plasma studied by Galeev and Oraevski (1963) and Sagdeev and Galeev (1969), which can occur in a uniform background medium. In the case where the amplitude $A$ of the pump wave is small, and where the plasma beta $\beta$ is in the range $A \ll \beta < 1$, Jayanti and Hollweg (1993b) obtained the growth rate (to within a constant numerical factor) for the Alfvén- wave decay instability given in Sagdeev and Galeev (1969). Jayanti and Hollweg (1993b) also obtained a finite growth rate as $\beta \to 0$, which is different from that obtained by Sagdeev and Galeev (1969) (the growth rate in Sagdeev and Galeev (1969) diverges as $\beta^{-1/4}$ as $\beta \to 0$). The growth rate obtained in the present analysis is essentially of the same form as that obtained by Sagdeev and Galeev (1969). This suggests that a special analysis may be needed in order to obtain a correct treatment of three-wave resonant interactions in the limit $\beta \to 0$ (such an analysis is not carried out in the present paper). In the derivation of the three-wave resonant interaction equations in Ali and Hunter (1998) and Webb et al. (1999), a multiple-scales perturbation method is used, in which the resonant interaction equation for the $j$th wave mode arises from averaging over the fast time scale, while keeping the $j$th fast phase variable $\theta_j$ fixed. The linear stability analyses referred to above do not involve averaging over a fast time variable.

The three-wave resonant interaction equations for MHD waves are quite distinct from the Korteweg–de Vries (KdV) equation, modified Korteweg–de Vries (mKdV) equation, and derivative nonlinear Schrödinger (DNLS) equation, which describe the long-time evolution of a single wave mode due to the approximate balance between nonlinear wave steepening and Hall-current dispersion. The KdV equation describes the evolution of a weakly nonlinear magnetoacoustic wave propagating at an angle to the magnetic field, the mKdV equation describes an obliquely propagating Alfvén wave (see e.g. Taniuti and Wei 1968; Kakutani and Ono 1969), and the DNLS equation describes the evolution of a weakly nonlinear Alfvén wave propagating along the magnetic field (Rogister 1971; Mjølhus 1976; Mio et al. 1976; Spangler and Sheerin 1982). More complicated forms of the DNLS equation, and related equations, have been studied by Mjølhus and Wyller (1988), Verheest (1992), Khabibrakhmanov et al. (1992), Deconinck et al. (1993a,b), Brio (1989a,b), Hada (1993), Webb et al. (1995), Passot and Sulem (1995), and Medvedev and Diamond (1996). Medvedev and Diamond (1997) have discussed a noisy, kinetic nonlinear Schrödinger (KNLS) equation as a simple model of weakly compressible Alfvénic turbulence. Three- and four-wave resonant equations, on the other hand, describe the resonant interaction of multiple wave modes assumed to be present in the plasma. The waves only interact with each other if the resonance conditions on the wave phases are satisfied. In addition, the length and time scalings used in the resonant wave interaction equations are different than those used in the KdV, mKdV, and DNLS equations (the latter equations typically apply on much longer time scales than the resonant interaction equations).

It is important to note the physical limitations of using an MHD fluid-dynamical description, rather than a collisionless, kinetic plasma description. In particular, the MHD model does not incorporate Landau damping of magnetoacoustic modes due to wave–particle interactions (see e.g. Barnes 1966, 1979). The damping rates $\gamma_L = -\Im(\omega)/\Re(\omega)$ ($\omega$ is the wave frequency) of the magnetoacoustic waves in general increase with the plasma beta (see e.g. Fig. 4 of Barnes (1979), where $\gamma_L$ is plotted as

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

a function of the angle $\theta$ between the wave vector $\mathbf{k}$ and the background magnetic field $\mathbf{B}$ for the cases $\beta = 1$ and $\beta = 5$). The Landau damping for the fast-mode wave exhibits two peaks associated with stochastic heating of the thermal protons ($\theta \approx 10°$) and heating of the electrons ($\theta \approx 85 - 90°$) in $\beta \approx 1$ plasmas. For perpendicular propagation ($\theta = 90°$), there is no linear Landau damping of the fast mode. Further instabilties can arise in anisotropic plasmas with different parallel ($P_\parallel$) and perpendicular pressures ($P_\perp$) (e.g. the firehose and mirror instabilities: Barnes, 1979). The firehose instability occurs when $P_\parallel > P_\perp + B^2/\mu$, whereas the mirror instability occurs when $P_\perp > P_\parallel$. These instabilities are stabilized when the wavelength of the potentially unstable mode exceeds the gyroradius (Krall and Trivelpiece 1973, p. 496). There are also an infinite set of modes with zero real frequency in kinetic plasma theory, which are analogous to the MHD entropy waves, which are strongly damped (Barnes 1979).

Section 2 presents a synopsis of the wave interaction equations derived by Ali and Hunter (1998) and Webb et al. (1999). In one Cartesian space dimension, the MHD equations admit seven distinct eigenmodes, namely the backward and forward fast- and slow-mode magnetoacoustic waves, the backward and forward Alfvén waves, and the entropy wave. The evolution equation for each particular eigenmode contains transport terms describing the spatial transport of the wave along its characteristic, as well as Burgers self-wave steepening terms for the magnetoacoustic waves, plus wave interaction terms due to the other wave modes. Section 2 includes a discussion of the conditions for three-wave resonant interaction of periodic hyperbolic waves that occur in MHD, the physical meaning of the wave amplitudes for the different eigenmodes, and the form of the wave interaction coefficients (i.e. the Burgers self-wave interaction coefficients, the mean wave field interaction coefficients, and three-wave resonant interaction coefficients). The relationships between the wave interaction coefficients and the MHD conservation equations, and the MHD eigenvalues and eigenvectors, are delineated in Appendix A.

In Sec. 3, the wave interaction equations are transformed to a more symmetric form by rewriting the equations in terms of the variables $c_j = N_j^{1/2}$, where $N_j$ is the wave action of the $j$th-eigenmode ($N_j = E_{wj}/\omega_j$, where $E_{wj}$ is the wave energy density and $\omega_j$ is the wave frequency). In cases where the wave action density is not well defined (e.g. $\omega = 0$ for the entropy wave), the original wave amplitude is used. The $\{c_j\}$ are the natural Hamiltonian variables describing the waves (see e.g. Sagdeev and Galeev 1969). The wave interaction equations for the $\{c_j\}$ are used to derive quasiparticle number density conservation laws for the waves, which are related to the Manley–Rowe relations for three-wave resonant interactions. The quasiparticle number density conservation equations are also used to derive the canonical energy and momentum equations for the waves.

In Sec. 4, Lagrangian and Hamiltonian variational principles for the wave interaction equations are obtained for the different resonant wave triads, for the case where the waves are smooth and do not form shocks. The resonant wave triads involving: (a) magnetoacoustic–entropy wave interactions, (b) Alfvén–entropy wave interactions, (c) Alfvén–magnetoacoustic wave interactions, and (d) fast–slow magnetoacoustic wave interactions are considered in detail. The form of the wave interaction equations in the non-strictly hyperbolic cases, where the wave speeds of some of the eigenmodes may coincide, are also discussed.

In Sec. 5, the three-wave resonant interaction equations are used to investigate the Alfvén-wave decay instability studied by Sagdeev and Galeev (1969), which is

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

due to three-wave resonant interaction of two Alfvén waves and a slow magnetoacoustic wave in a low-beta plasma in which the gas sound speed $a$ is less than the Alfvén speed $b_x$. The character of the wave interactions in large-beta plasmas ($a > b_x$) is also discussed.

In Sec. 6, exact analytical solutions of the equations describing the resonant interaction of the backward and forward Alfvén waves with the entropy wave are obtained. The latter equations consist of two coupled linear integro-differential equations for the backward and forward Alfvén waves, in which the integral terms describe the resonant scattering of the Alfvén waves off an entropy-wave disturbance to produce the reverse Alfvén wave. Exact, periodic wave solutions are obtained by Fourier methods. Two different types of solutions are obtained: (a) solutions that depend on the fast phase variable $\theta$ and the slow time variable $\bar{t}$, and (b) solutions that depend on $\theta$, $\bar{t}$, and the long space variable $\bar{x}$. Illustrative examples of the different types of solutions are discussed.

Section 7 concludes with a summary and discussion.

## 2. Wave interaction equations

The basic MHD equations consist of the following: the mass continuity equation; the momentum equation for the MHD fluid, including the $\mathbf{J} \times \mathbf{B}$ force on the fluid, where $\mathbf{J} = \boldsymbol{\nabla} \times \mathbf{B}/\mu$ is the electric current, $\mathbf{B}$ is the magnetic induction, and $\mu$ is the magnetic permeability; Faraday's law relating $\mathbf{B}$ to the electric field $\mathbf{E} = -\mathbf{u} \times \mathbf{B}$, where $\mathbf{u}$ is the MHD fluid velocity; the entropy advection equation; and Gauss's law $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$. The system of equations is closed by an equation of state $p = p(\rho, S)$ for the gas pressure $p$ in terms of the gas density $\rho$ and entropy $S$.

The MHD eigenmodes consist of the fast and slow magnetoacoustic waves, the Alfvén wave, and the entropy wave. The fast- and slow-mode waves are compressive, and are analogous to sound waves in gas dynamics, whereas the Alfvén wave is analogous to the vorticity eigenmode in gas dynamics. Both the Alfvén and magnetoacoustic waves are characterized by zero entropy perturbations. The entropy wave has non-zero entropy perturbation, and is advected with the fluid. Entropy waves are also pressure-balance structures in which there are no gas pressure and magnetic field perturbations, but the entropy and density perturbations are non-zero. In ideal MHD, entropy waves are a consequence of non-isentropic initial conditions, but they can also be generated in non-uniform flows with a non-zero large-scale entropy gradient due to wave interactions with the background flow, and are also generated at shocks. For wave propagation parallel to $\mathbf{B}$ ($\mathbf{k} \parallel \mathbf{B}$), the Alfvén speed and one of the magnetoacoustic speeds coincide. Similarly, for wave propagation perpendicular to $\mathbf{B}$, the slow magnetoacoustic mode, the entropy wave, and the Alfvén wave all have zero phase speed relative to the fluid frame. These degeneracies in the MHD eigenmodes make the MHD equations non-strictly hyperbolic for wave propagation parallel and perpendicular to $\mathbf{B}$. For wave propagation perpendicular to $\mathbf{B}$, the slow-mode wave is also a pressure-balance structure in which there is zero total pressure perturbation: $\delta P_{\rm tot} = \delta p + \delta(B^2/2\mu) = 0$.

Using a multiple-scales analysis of the ideal-MHD equations about a uniform background state leads to weakly nonlinear wave interaction equations for the Alfvén, magnetoacoustic, and entropy waves of the form (Ali and Hunter 1998;

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

Webb et al. 1999):

$$\frac{\partial a_j}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}(\lambda_j a_j) + k_j \frac{\partial}{\partial \theta_j}\left( \tfrac{1}{2}\Gamma_{jjj} a_j^2 + {\sum_q}' \Gamma_{jjq}\langle a_q\rangle a_j + {\sum_{p<q}}' {\sum_q}' \hat{\Gamma}_{jpq}\langle a_q a_p\rangle_j \right) = 0, \qquad 1 \leqslant j \leqslant 7. \quad (2.1)$$

In (2.1), the $\{\lambda_j\}$ are the characteristic eigenvelocities for the MHD waves:

$$\left.\begin{array}{llll} \lambda_1 = u_x - c_f, & \lambda_2 = u_x - b_x, & \lambda_3 = u_x - c_s, & \lambda_4 = u_x, \\ \lambda_5 = u_x + c_s, & \lambda_6 = u_x + b_x, & \lambda_7 = u_x + c_f, & \end{array}\right\} \quad (2.2)$$

where $u_x$ is the background fluid velocity component along the $x$ axis; $b_x = B_x/(\mu\rho)^{1/2}$ is the Alfvén phase speed, and $c_f$ and $c_s$ are the fast and slow magnetoacoustic phase speeds. The phase velocities $\pm c_f$ and $\pm c_s$ satisfy the magnetoacoustic dispersion equation

$$c^4 - (a^2 + b^2)c^2 + a^2 b_x^2 = 0, \quad (2.3)$$

where $a = (\partial p/\partial\rho)^{1/2}$ is the gas sound speed and $\mathbf{b} = \mathbf{B}/(\mu\rho)^{1/2}$ is the Alfvén velocity.

In (2.1), the wave amplitudes $a_j = a_j(\bar{x}, \bar{t}, \theta_j)$ are functions of the slow variables $\bar{x}$ and $\bar{t}$ and the fast phase variable $\theta_j = (k_j\bar{x} - \omega_j\bar{t})/\epsilon$, where $\epsilon$ is a small parameter determining the magnitude of the wave perturbations, and $\omega_j$ and $k_j$ are the frequencies and wavenumbers of the waves with phase speed $\lambda_j = \omega_j/k_j$. The primes on the sums in (2.1) denote summations over the indices $p$ and $q$ with $p \neq j$ and $q \neq j$. The angular brackets in (2.1) (i.e. $\langle a_q\rangle$ and $\langle a_p a_q\rangle_j$) denote averages of $a_q$ and $a_p a_q$ over the fast time scale $\tau = t/\epsilon$ keeping the fast phase variable $\theta_j$ fixed (note that $\langle a_q\rangle$ does not depend on keeping $\theta_j$ fixed for $q \neq j$). The nonlinear wave interaction term $\frac{1}{2}\Gamma_{jjj}a_j^2$ in (2.1) represents Burgers self-wave interaction and steepening for the magnetoacoustic waves ($\Gamma_{jjj} \neq 0$ if $j = 1, 3, 5, 7$), the sum ${\sum_q}'\Gamma_{jjq}\langle a_q\rangle a_j$ represents the interaction of the $j$th wave mode with the mean wave field of the other waves, and the $\hat{\Gamma}_{jpq}\langle a_p a_q\rangle_j$ terms describe three-wave resonant wave interactions between the $p$th and $q$th wave modes to produce a sink or source of waves for the $j$th wave mode ($j \neq p \neq q$). In order for three-wave resonant interactions to occur, it is necessary that the wave interaction coefficients $\hat{\Gamma}_{jpq} \neq 0$, and for the resonance conditions

$$\omega_j = \mu_{jpq}\omega_q + \mu_{jqp}\omega_p, \qquad k_j = \mu_{jpq}k_q + \mu_{jqp}k_p, \quad (2.4)$$

to be satisfied, where the constants $\{\mu_{jpq}\}$ are given by

$$\mu_{jpq} = \frac{k_j(\lambda_j - \lambda_p)}{k_q(\lambda_q - \lambda_p)}. \quad (2.5)$$

For resonant interaction of periodic waves, the $\{\mu_{jpq}\}$ must be rational numbers. Equations (2.4) are equivalent to the equations $\theta_j = \mu_{jpq}\theta_q + \mu_{jqp}\theta_p$, for the wave phases.

If $\mathbf{B}_\perp = (0, B_y, B_z)^{\mathrm{T}} \neq 0$, the wave amplitudes $\{a_1, a_3, a_4, a_5, a_7\}$ for the magnetoacoustic and entropy waves are related to the wave density perturbations

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

$\{\rho_1, \rho_3, \rho_4, \rho_5, \rho_7\}$ by the equations

$$\rho_1 = \alpha_f a_1, \qquad \rho_3 = \alpha_s a_3, \qquad \rho_4 = a_4, \qquad \rho_5 = \alpha_s a_5, \qquad \rho_7 = \alpha_f a_7, \tag{2.6}$$

where

$$\alpha_f = \left(\frac{a^2 - c_s^2}{c_f^2 - c_s^2}\right)^{1/2}, \qquad \alpha_s = \left(\frac{c_f^2 - a^2}{c_f^2 - c_s^2}\right)^{1/2}. \tag{2.7}$$

The Alfvén-wave amplitudes $a_2$ and $a_6$ are related to the Alfvén-wave magnetic field perturbations $\mathbf{B}_2$ and $\mathbf{B}_6$ by the equations

$$\mathbf{B}_j = (0, \beta_z, -\beta_y)^{\mathrm{T}} a_j, \qquad j = 2, 6, \tag{2.8}$$

where $\boldsymbol{\beta}_\perp = \mathbf{B}_\perp / B_\perp$ and $\mathbf{B}_\perp = (0, B_y, B_z)^{\mathrm{T}}$. In the degenerate-eigenvalue case in which $B_\perp \to 0$ and $a > b_x$, the wave amplitudes of the slow-mode waves are related to the magnetic field perturbations by $\mathbf{B}_j = -a(\mu^{1/2}/\rho^{1/2})\boldsymbol{\beta}_\perp a_j$, $j = 3, 5$; but if $a < b_x$ then $\mathbf{B}_j = a(\mu^{1/2}/\rho^{1/2})\boldsymbol{\beta}_\perp a_j$, $j = 1, 7$ for the fast-mode waves.

The explicit form of the wave interaction coefficients $\Gamma_{jjq}$ and $\hat{\Gamma}_{jpq}$ in (2.1) are given below (see also Webb et al. 1999). A brief description of the derivation of (2.1) is given in Appendix A. The non-zero Burgers self-wave interaction coefficients $\{\Gamma_{jjj}\}$ in (2.1) are given by

$$\Gamma_{111} = -\frac{\alpha_f c_f}{\rho}\beta_f, \qquad \Gamma_{333} = -\frac{\alpha_s c_s}{\rho}\beta_s, \qquad \Gamma_{555} = -\Gamma_{333}, \qquad \Gamma_{777} = -\Gamma_{111}, \tag{2.9}$$

where

$$\beta_f = \tfrac{1}{2}[(\gamma+1)\alpha_f^2 + 3\alpha_s^2], \qquad \beta_s = \tfrac{1}{2}[(\gamma+1)\alpha_s^2 + 3\alpha_f^2], \tag{2.10}$$

and $\gamma = C_p/C_v$ is the adiabatic index of the gas (i.e. we assume $p = p_0(\rho/\rho_0)^\gamma \exp(S/C_v)$ as the equation of state of the gas). Note that the entropy- and Alfvén-wave self-wave interaction coefficients $\Gamma_{222} = \Gamma_{444} = \Gamma_{666} = 0$.

The non-zero mean wave interaction coefficients $\{\Gamma_{jjp} : p \neq j\}$ for $j \leqslant 4$ are

$$\left.\begin{aligned}
&\Gamma_{113} = -\frac{\alpha_s}{\rho}[(\beta_f - 2)c_f + c_s], \qquad \Gamma_{114} = \frac{c_f}{2\rho},\\
&\Gamma_{115} = -\frac{\alpha_s}{\rho}[(\beta_f - 2)c_f - c_s], \qquad \Gamma_{117} = -\frac{\alpha_f c_f(\beta_f - 2)}{\rho},\\
&\Gamma_{221} = \frac{\alpha_f(|b_x| - 2c_f)}{2\rho}, \qquad \Gamma_{223} = \frac{\alpha_s(|b_x| - 2c_s)}{2\rho},\\
&\Gamma_{224} = \frac{|b_x|}{2\rho}, \qquad \Gamma_{225} = \frac{\alpha_s(|b_x| + 2c_s)}{2\rho}, \qquad \Gamma_{227} = \frac{\alpha_f(|b_x| + 2c_f)}{2\rho},\\
&\Gamma_{331} = -\frac{\alpha_f}{\rho}[(\beta_s - 2)c_s + c_f], \qquad \Gamma_{334} = \frac{c_s}{2\rho},\\
&\Gamma_{335} = -\frac{\alpha_s c_s(\beta_s - 2)}{\rho}, \qquad \Gamma_{337} = -\frac{\alpha_f}{\rho}[(\beta_s - 2)c_s - c_f],\\
&\Gamma_{441} = -\frac{\alpha_f c_f}{\rho}, \qquad \Gamma_{443} = -\frac{\alpha_s c_s}{\rho},\\
&\Gamma_{445} = \frac{\alpha_s c_s}{\rho}, \qquad \Gamma_{447} = \frac{\alpha_f c_f}{\rho}.
\end{aligned}\right\} \tag{2.11}$$

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

For $j > 4$, the non-zero mean wave interaction coefficients may be obtained by using (2.11) and the symmetry relations

$$\Gamma_{jjp} = -\Gamma_{j'j'p'}, \qquad j' = 8 - j, \qquad p' = 8 - p, \tag{2.12}$$

where $1 \leqslant j \leqslant 7$ and $1 \leqslant p \leqslant 7$, associated with interchange of the backward- and forward-wave eigenvalues and eigenvectors.

The non-zero three-wave resonant interaction coefficients for $j < 4$ are given by the equations

$$\left.\begin{aligned}
&\hat{\Gamma}_{126} = -\frac{\alpha_f c_f}{2\mu a^2}, \qquad \hat{\Gamma}_{135} = -\frac{\alpha_f c_f(\beta_s - 1)}{\rho}, \qquad \hat{\Gamma}_{137} = -\frac{\alpha_s c_f(\beta_f - 1)}{\rho},\\
&\hat{\Gamma}_{147} = -\frac{c_f}{2\rho}, \qquad \hat{\Gamma}_{157} = -\frac{\alpha_s c_f(\beta_f - 1)}{\rho},\\
&\hat{\Gamma}_{216} = -\frac{\alpha_f |b_x|}{2\rho}, \qquad \hat{\Gamma}_{236} = -\frac{\alpha_s |b_x|}{2\rho}, \qquad \hat{\Gamma}_{246} = -\frac{|b_x|}{2\rho},\\
&\hat{\Gamma}_{256} = -\frac{\alpha_s |b_x|}{2\rho}, \qquad \hat{\Gamma}_{267} = -\frac{\alpha_f |b_x|}{2\rho},\\
&\hat{\Gamma}_{315} = -\frac{\alpha_f c_s(\beta_s - 1)}{\rho}, \qquad \hat{\Gamma}_{317} = -\frac{\alpha_s c_s(\beta_f - 1)}{\rho}, \qquad \hat{\Gamma}_{326} = -\frac{\alpha_s c_s}{2\mu a^2},\\
&\hat{\Gamma}_{345} = -\frac{c_s}{2\rho}, \qquad \hat{\Gamma}_{357} = -\frac{\alpha_f c_s(\beta_s - 1)}{\rho}.
\end{aligned}\right\} \tag{2.13}$$

The resonant wave interaction coefficients $\hat{\Gamma}_{4pq}$ in the entropy-wave evolution equation are all zero. The non-zero three-wave resonant interaction coefficients for $j > 4$ can be obtained by using (2.13) in conjunction with the symmetry relations

$$\hat{\Gamma}_{jpq} = -\hat{\Gamma}_{j'p'q'}, \qquad j' = 8 - j, \qquad p' = 8 - p, \qquad q' = 8 - q, \tag{2.14}$$

associated with interchanging the roles of the backward and forward waves. One can show that $\hat{\Gamma}_{jpq} = \hat{\Gamma}_{jqp}$ (see Webb et al. 1999). These symmetries, and other formulae related to the wave interaction coefficients (2.9)–(2.13) are described in more detail in Appendix A.

## 3. Hamiltonian variables and conservation laws

We consider wave propagation through a uniform background medium in which $\mathbf{u} = 0$ (or $\mathbf{u} = \text{const}$). In Sec. 3.1, we consider the form of the wave interaction equations (2.1) in terms of the natural Hamiltonian variables $\{c_j\}$ used by Sagdeev and Galeev (1969). Section 3.2 derives the Manley–Rowe relations, and quasiparticle number density conservation laws for the resonant wave triad $\{jpq\}$ for the strictly hyperbolic case ($\lambda_j \neq \lambda_p \neq \lambda_q$), and not involving the entropy wave. The quasiparticle number density equations are combined to yield the canonical energy and momentum equations for the waves. Section 3.3 considers the form of the Manley–Rowe relations and quasiparticle conservation laws for resonant triads involving

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

the entropy wave. The modification of the conservation laws for cases where there are shocks in the fast phase variables are also discussed.

### *3.1. Resonant interaction equations and Hamiltonian variables*

The natural Hamiltonian variables $\{c_j\}$ used by Sagdeev and Galeev (1969) are linearly related to the wave amplitudes $\{a_j\}$, and are given by the equations

$$c_j = N_j^{1/2}, \qquad N_j = \frac{E_{wj}}{\omega_j}, \qquad 1 \leqslant j \leqslant 7, \qquad j \neq 4, \tag{3.1}$$

and $c_4 = \rho_4$, where $\rho_4$ is the density perturbation of the entropy wave. In (3.1), $E_{wj}$ is the energy density of the $j$th-wave mode, $\omega_j$ is the wave frequency, and $N_j$ is the wave action or quasiparticle number density of the $j$th-wave mode. Using the eigenvector relations, one can show that $E_{wj} = a^2 a_j^2/\rho$ are the wave energy densities for the magnetoacoustic modes (i.e. $j = 1, 3, 5, 7$), and $E_{wj} = a_j^2/\mu$ are the energy densities for the Alfvén modes (i.e. $j = 2, 6$). The wave amplitudes $\{a_j\}$ are linearly related to the $\{c_j\}$ by equations of the form

$$a_j = \nu_j c_j, \qquad 1 \leqslant j \leqslant 7, \tag{3.2}$$

where

$$\nu_j = \begin{cases} (\rho\omega_j/a^2)^{1/2} & \text{for } j = 1, 3, 5, 7, \\ (\mu\omega_j)^{1/2} & \text{for } j = 2, 6, \end{cases} \tag{3.3}$$

and $\nu_4 = 1$.

In terms of the $\{c_j\}$, the wave interaction equations (2.1) become

$$\frac{\partial c_j}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}(\lambda_j c_j) + D_\theta \left( \tfrac{1}{2}\tilde{\Gamma}_{jjj} c_j^2 + {\sum_q}' \tilde{\Gamma}_{jjq} \langle c_q \rangle c_j + {\sum_{p<q}}' {\sum_q}' \tilde{\Gamma}_{jpq} \langle c_q c_p \rangle_j \right) = 0, \tag{3.4}$$

where $1 \leqslant j \leqslant 7$ and $\theta \equiv \theta_j$ in the $j$th equation in (3.4) (i.e. $D_\theta \equiv \partial/\partial\theta_j$ in the $j$th equation in (3.4)). The wave interaction coefficients $\tilde{\Gamma}_{jpq}$ in (3.4) are given by the equations

$$\left.\begin{aligned} &\tilde{\Gamma}_{jjj} = k_j \nu_j \Gamma_{jjj}, \qquad \tilde{\Gamma}_{jjq} = \nu_q k_j \Gamma_{jjq}, \\ &\tilde{\Gamma}_{jpq} = \hat{\Gamma}_{jpq} \frac{\nu_p \nu_q k_j}{\nu_j}, \qquad j \neq p \neq q. \end{aligned}\right\} \tag{3.5}$$

It turns out that, for a given resonant triad $(jpq)$, not involving the entropy wave, the resonant wave interaction coefficient $\tilde{\Gamma}_{jpq}$ is symmetric in $jpq$. The resonant interaction coefficient $\tilde{\Gamma}_{4pq}$ for the entropy-wave equation is zero.

The three-wave resonant interaction coefficients $\tilde{\Gamma}_{jpq}$ $(j \neq p \neq q)$ satisfy the symmetry relations

$$\left.\begin{aligned} &\tilde{\Gamma}_{jpq} = \tilde{\Gamma}_{pqj} = \tilde{\Gamma}_{qjp}, \qquad \tilde{\Gamma}_{jpq} = \tilde{\Gamma}_{jqp} \qquad (j \neq p \neq q \neq 4), \\ &\tilde{\Gamma}_{jp4} = \tilde{\Gamma}_{pj4}, \qquad \tilde{\Gamma}_{4jp} = 0 \qquad (j \neq p \neq 4). \end{aligned}\right\} \tag{3.6}$$

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

These symmetry relations are related to the total energy and momentum conservation equations for the waves (see Sec. 3.2).

Explicit formulae for the resonant interaction coefficients are

$$\left.\begin{aligned}
&\tilde{\Gamma}_{126} = \frac{\alpha_f}{2a\rho^{1/2}}\,(\omega_1\omega_2\omega_6)^{1/2}\,, \qquad \tilde{\Gamma}_{135} = \frac{\alpha_f(\beta_s-1)}{a\rho^{1/2}}\,(\omega_1\omega_3\omega_5)^{1/2}\,,\\
&\tilde{\Gamma}_{137} = \frac{\alpha_s(\beta_f-1)}{a\rho^{1/2}}\,(\omega_1\omega_3\omega_7)^{1/2}\,, \qquad \tilde{\Gamma}_{147} = \frac{(\omega_1\omega_7)^{1/2}}{2\rho},\\
&\tilde{\Gamma}_{157} = \frac{\alpha_s(\beta_f-1)}{a\rho^{1/2}}\,(\omega_1\omega_5\omega_7)^{1/2}\,, \qquad \tilde{\Gamma}_{236} = \frac{\alpha_s}{2a\rho^{1/2}}\,(\omega_2\omega_3\omega_6)^{1/2}\,,\\
&\tilde{\Gamma}_{246} = \tilde{\Gamma}_{624} = \frac{(\omega_2\omega_6)^{1/2}}{2\rho}, \qquad \tilde{\Gamma}_{426} = 0,\\
&\tilde{\Gamma}_{256} = \frac{\alpha_s}{2a\rho^{1/2}}\,(\omega_2\omega_5\omega_6)^{1/2}\,, \qquad \tilde{\Gamma}_{267} = \frac{\alpha_f}{2a\rho^{1/2}}\,(\omega_2\omega_6\omega_7)^{1/2}\,,\\
&\tilde{\Gamma}_{345} = \tilde{\Gamma}_{534} = \frac{(\omega_3\omega_5)^{1/2}}{2\rho}, \qquad \tilde{\Gamma}_{435} = \tilde{\Gamma}_{453} = 0,\\
&\tilde{\Gamma}_{357} = \frac{\alpha_f(\beta_s-1)}{a\rho^{1/2}}\,(\omega_3\omega_5\omega_7)^{1/2}\,.
\end{aligned}\right\} \tag{3.7}$$

The remaining resonant interaction coefficients $\tilde{\Gamma}_{jpq}$ can be obtained by using (2.14), (3.5), and the symmetry relations (3.6).

There are four main types of resonant wave triads: (a) magnetoacoustic–entropy wave interactions; (b) Alfvén–entropy wave interactions; (c) Alfvén–magnetoacoustic wave interactions; and (d) slow–fast magnetoacoustic wave interactions. A characterization of the possible resonant wave triads ($jpq$) for wave propagation in one Cartesian space dimension is given in Table 1. It is of interest to note that the resonant interactions in Table 1 always involve two waves of one type with a single wave of a different type. The resonant wave interactions described in Table 1 only apply to wave propagation in one Cartesian space dimension. More general types of wave interactions can occur in more than one space dimension.

The non-zero Burgers self-wave interaction coefficients in (3.4) and (3.5) are

$$\left.\begin{aligned}
&\tilde{\Gamma}_{111} = \frac{\alpha_f\beta_f}{a\rho^{1/2}}\omega_1^{3/2}, \qquad \tilde{\Gamma}_{777} = \frac{\alpha_f\beta_f}{a\rho^{1/2}}\omega_7^{3/2},\\
&\tilde{\Gamma}_{333} = \frac{\alpha_s\beta_s}{a\rho^{1/2}}\omega_3^{3/2}, \qquad \tilde{\Gamma}_{555} = \frac{\alpha_s\beta_s}{a\rho^{1/2}}\omega_5^{3/2}.
\end{aligned}\right\} \tag{3.8}$$

The Burgers coefficients for the Alfvén waves and the entropy wave are zero.

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

The mean wave field interaction coefficients $\tilde{\Gamma}_{jjp}$ for $j \leqslant 4$ are

$$\left.\begin{aligned}
&\tilde{\Gamma}_{113} = \frac{\alpha_s[(\beta_f-2)c_f + c_s]}{a\rho^{1/2}c_f}\omega_3^{1/2}\omega_1, \qquad \tilde{\Gamma}_{114} = -\frac{\omega_1}{2\rho},\\
&\tilde{\Gamma}_{115} = \frac{\alpha_s[(\beta_f-2)c_f - c_s]}{a\rho^{1/2}c_f}\omega_1\omega_5^{1/2}, \qquad \tilde{\Gamma}_{117} = \frac{\alpha_f(\beta_f-2)}{a\rho^{1/2}}\omega_1\omega_7^{1/2},\\
&\tilde{\Gamma}_{221} = \frac{\alpha_f(2c_f - b_x)}{2a\rho^{1/2}b_x}\omega_2\omega_1^{1/2}, \qquad \tilde{\Gamma}_{223} = \frac{\alpha_s(2c_s - b_x)}{2a\rho^{1/2}b_x}\omega_2\omega_3^{1/2},\\
&\tilde{\Gamma}_{224} = -\frac{\omega_2}{2\rho}, \qquad \tilde{\Gamma}_{225} = -\frac{\alpha_s(b_x + 2c_s)}{2a\rho^{1/2}b_x}\omega_2\omega_5^{1/2},\\
&\tilde{\Gamma}_{227} = -\frac{\alpha_f(b_x + 2c_f)}{2a\rho^{1/2}b_x}\omega_2\omega_7^{1/2}, \qquad \tilde{\Gamma}_{331} = \frac{\alpha_f[(\beta_s-2)c_s + c_f]}{a\rho^{1/2}c_s}\omega_3\omega_1^{1/2},\\
&\tilde{\Gamma}_{334} = -\frac{\omega_3}{2\rho}, \qquad \tilde{\Gamma}_{335} = \frac{\alpha_s(\beta_s-2)}{a\rho^{1/2}}\omega_3\omega_5^{1/2},\\
&\tilde{\Gamma}_{337} = \frac{\alpha_f[(\beta_s-2)c_s - c_f]}{a\rho^{1/2}c_s}\omega_3\omega_7^{1/2}, \qquad \tilde{\Gamma}_{441} = -\frac{\alpha_f c_f}{a\rho^{1/2}}k_4\omega_1^{1/2},\\
&\tilde{\Gamma}_{443} = -\frac{\alpha_s c_s}{a\rho^{1/2}}k_4\omega_3^{1/2}, \qquad \tilde{\Gamma}_{445} = \frac{\alpha_s c_s}{a\rho^{1/2}}k_4\omega_5^{1/2}, \qquad \tilde{\Gamma}_{447} = \frac{\alpha_f c_f}{a\rho^{1/2}}k_4\omega_7^{1/2}.
\end{aligned}\right\} \tag{3.9}$$

For $j > 4$, the mean wave field interaction coefficients may be determined by using the equations

$$\tilde{\Gamma}_{jjq} = -\frac{\nu_q k_j}{\nu_{q'}k_{j'}}\tilde{\Gamma}_{j'j'q'}, \qquad j' = 8 - j, \qquad q' = 8 - q, \tag{3.10}$$

in conjunction with (3.9).

*3.1.1. Alternative wave interaction equations.* An alternative form of the resonant wave interaction equations may be obtained by setting

$$c_j = \langle c_j\rangle + c_j', \qquad c_j' = C_{j,\theta}, \tag{3.11}$$

where $C_j$ is the wave potential for $c_j'$, and

$$\langle c_j\rangle = \frac{1}{2\pi}\int_0^{2\pi} c_j(\theta)\,d\theta, \tag{3.12}$$

is the mean wave amplitude. Note that $\langle c_j'\rangle = 0$. If there are no shocks in $\theta$ (i.e. the derivatives $c_{j,\theta}$ are smooth) then the wave interaction equations (3.4) may be split up into the equation system

$$\frac{\partial\langle c_j\rangle}{\partial\bar{t}} + \lambda_j\frac{\partial\langle c_j\rangle}{\partial\bar{x}} = 0 \tag{3.13}$$

for the mean wave amplitudes and the system

$$\frac{\partial c_j'}{\partial\bar{t}} + \lambda_j\frac{\partial c_j'}{\partial\bar{x}} + D_\theta\left(\tfrac{1}{2}\tilde{\Gamma}_{jjj}c_j'^{\,2} + \sum_q \tilde{\Gamma}_{jjq}\langle c_q\rangle c_j' + {\sum_{p<q}}'{\sum_q}'\tilde{\Gamma}_{jpq}\langle c_p'c_q'\rangle_j\right) = 0 \tag{3.14}$$

governing the evolution of the $\{c_j'\}$. Note that $\langle c_j\rangle$ depends on the long space and time variables $\bar{x}$ and $\bar{t}$, whereas $c_j' = c_j'(\bar{x},\bar{t},\theta)$ depends also on the fast phase variable $\theta$, where $\theta$ in the $j$th evolution equation (3.14) refers the phase of the $j$th-wave

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

**Table 1.** MHD resonant wave triads in one Cartesian space dimension.

| Label | Type | $jpq$ | $jpq$ |
|---|---|---|---|
| $a$ | 2 fast-mode, 1 entropy | 147 | |
| $a$ | 2 slow-mode, 1 entropy | 345 | |
| $b$ | 2 Alfvén, 1 entropy | 246 | |
| $c$ | 2 Alfvén, 1 fast-mode | 126 | 267 |
| $c$ | 2 Alfvén, 1 slow-mode | 236 | 256 |
| $d$ | 2 slow-mode, 1 fast-mode | 135 | 357 |
| $d$ | 2 fast-mode, 1 slow-mode | 137 | 157 |

mode. Equations (3.14) can also be written in terms of the wave potentials $\{C_j\}$. In particular, for a given resonant triad $\{jpq\}$, (3.14) can be written in the form

$$D_\theta \left( C_{j,\bar{t}} + \lambda_j C_{j,\bar{x}} + \tfrac{1}{2}\tilde{\Gamma}_{jjj} C_{j,\theta}^2 + \sum_{s=j,p,q} \tilde{\Gamma}_{jjs} \langle c_s \rangle C_{j,\theta} + \tilde{\Gamma}_{jpq} \langle c_p' c_q' \rangle_j \right) = 0. \quad (3.15)$$

The potential form (3.15) is useful in developing variational principles of the equations for a given resonant wave triad $\{jpq\}$.

On physical grounds, one might expect that (3.13) governing the mean wave amplitudes could be supplemented by ponderomotive force terms $Q_j$ on the right-hand side of (3.13), where $Q_j$ depends on terms proportional to $\partial/\partial\bar{x}(\langle {c_p'}^2 \rangle)$ and $\partial/\partial\bar{t}(\langle {c_p'}^2 \rangle)$ (the subscript $p$ denotes one of the wave modes of interest), whereby the mean wave field can be modified by the back-reaction of the wave pressure of the short-scale fluctuations. Such terms, although $O(\epsilon)$ times the terms on the left-hand side, could be of importance at late times. A similar situation arises, for example, in the theory of stellar winds (and, in particular, the solar wind) (see e.g. MacGregor and Charbonneau 1994), where Alfvén-wave ponderomotive forces are thought to play an important role in accelerating the background wind flow. In the present analysis, no ponderomotive forces will be included on the right-hand side of (3.13), but such ponderomotive forces will in general be present in a renormalized perturbation theory, in which the perturbation equations of different orders are resummed.

In the following subsections, we discuss the form of the Manley–Rowe relations for three-wave resonant interactions and the canonical energy and momentum equations for the waves. The possible resonant wave triads in MHD in one Cartesian space dimension are listed in Table 1.

### *3.2. The Manley–Rowe relations and conservation laws*

In this subsection, we obtain the Manley–Rowe relations for the resonant wave interaction equations (3.13) and (3.14). These relations concern conservation laws governing the change in the quasiparticle number densities of the waves during three-wave resonant interactions. We also discuss the canonical energy and momentum equations for the waves.

We consider resonant wave triads $\{jpq\}$ for the strictly hyperbolic case ($\lambda_j \neq \lambda_p \neq \lambda_q$) not involving the entropy wave. Triads involving the entropy wave are discussed in Sec. 3.3. The role of shocks in the fast phase variables $\{\theta_s : s = j, p, q\}$ on the results are also discussed.

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

The Manley–Rowe relations are relations between the quasiparticle number densities

$$\mathscr{N}_s = \int_0^{2\pi} {c'_s}^2 \, d\theta_s, \qquad s = j, p, q. \tag{3.16}$$

These relations are described in the following proposition.

**Proposition 1.** Consider the strictly hyperbolic resonant wave triad $\{jpq\}$ ($\lambda_j \neq \lambda_p \neq \lambda_q$), not involving the entropy wave, satisfying the resonance conditions

$$\theta_j = m\theta_p + n\theta_q, \tag{3.17}$$

for the wave phases, where $m$ and $n$ are integers. The wave quasiparticle number densities $\{\mathscr{N}_s : \ s = j, p, q\}$ satisfy the conservation laws

$$\frac{\partial}{\partial \bar{t}} \left( \mathscr{N}_p + m\mathscr{N}_j \right) + \frac{\partial}{\partial \bar{x}} \left( \lambda_p \mathscr{N}_p + \lambda_j m \mathscr{N}_j \right) + Q_{pj} = 0, \tag{3.18}$$

$$\frac{\partial}{\partial \bar{t}} \left( \mathscr{N}_q + n\mathscr{N}_j \right) + \frac{\partial}{\partial \bar{x}} \left( \lambda_q \mathscr{N}_q + \lambda_j n \mathscr{N}_j \right) + Q_{qj} = 0, \tag{3.19}$$

where the source terms $Q_{pj}$ and $Q_{qj}$ are due to possible shocks in the fast phase variables $\{\theta_s : \ s = j, p, q\}$ (the detailed forms for $Q_{pj}$ and $Q_{qj}$ are described in (3.28) et seq. below). In the absence of shocks (i.e. if $Q_{pj} = Q_{qj} = 0$), (3.18) and (3.19) are strict conservation laws. If there is no dependence of the waves on the long space variable $x$, and if $Q_{pj} = Q_{qj} = 0$, then (3.18) and (3.19) integrate to yield the classical Manley–Rowe relations

$$\mathscr{N}_p + m\mathscr{N}_j = M_0, \tag{3.20}$$

$$\mathscr{N}_q + n\mathscr{N}_j = M_1, \tag{3.21}$$

where $M_0$ and $M_1$ are constants. The Manley–Rowe relations (3.20) and (3.21) imply the constraints $\Delta\mathscr{N}_p = -m\Delta\mathscr{N}_j$ and $\Delta\mathscr{N}_q = -n\Delta\mathscr{N}_j$ for the changes $\{\Delta\mathscr{N}_s : \ s = j, p, q\}$ in the wave quasiparticle number densities due to three-wave resonant interactions. More generally, if $Q_{pj} = Q_{qj} = 0$, and if the waves depend on the long space variable $x$, then the Manley–Rowe relations take the form

$$\int_{-\infty}^{\infty} (\mathscr{N}_p + m\mathscr{N}_j) \, dx = \mathscr{M}_0, \tag{3.22}$$

$$\int_{-\infty}^{\infty} (\mathscr{N}_q + n\mathscr{N}_j) \, dx = \mathscr{M}_1, \tag{3.23}$$

where $\mathscr{M}_0$ and $\mathscr{M}_1$ are constants.

By using the resonance conditions

$$\omega_j = m\omega_p + n\omega_q, \qquad k_j = mk_p + nk_q, \tag{3.24}$$

the quasiparticle number density equations (3.18) and (3.19) can be combined to yield the canonical wave energy equation

$$\frac{\partial}{\partial \bar{t}} \left( \omega_p \mathscr{N}_p + \omega_q \mathscr{N}_q + \omega_j \mathscr{N}_j \right) + \frac{\partial}{\partial \bar{x}} \left( \lambda_p \omega_p \mathscr{N}_p + \lambda_q \omega_q \mathscr{N}_q + \lambda_j \omega_j \mathscr{N}_j \right) + \omega_p Q_{pj} + \omega_q Q_{qj} = 0. \tag{3.25}$$

In the absence of shocks, ($Q_{pj} = Q_{qj} = 0$), the canonical wave energy equation may

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

be written in the conservation form

$$\frac{\partial}{\partial \bar{t}}\left(\omega_p \mathscr{N}_p + \omega_q \mathscr{N}_q + \omega_j \mathscr{N}_j\right) + \frac{\partial}{\partial \bar{x}}\left(\lambda_p \omega_p \mathscr{N}_p + \lambda_q \omega_q \mathscr{N}_q + \lambda_j \omega_j \mathscr{N}_j\right) = 0. \quad (3.26)$$

Equation (3.26) is the generalization of the wave action equation (see e.g. Dewar 1970; Whitham 1974) appropriate to the case where there are multiple waves present. Similarly, using the resonance conditions (3.24) for the wavenumbers $k_j$, $k_p$, and $k_q$, and the quasiparticle conservation laws (3.18) and (3.19), yields the canonical momentum equation for the waves:

$$\frac{\partial}{\partial \bar{t}}\left(k_p \mathscr{N}_p + k_q \mathscr{N}_q + k_j \mathscr{N}_j\right) + \frac{\partial}{\partial \bar{x}}\left(\lambda_p k_p \mathscr{N}_p + \lambda_q k_q \mathscr{N}_q + \lambda_j k_j \mathscr{N}_j\right) + k_p Q_{pj} + k_q Q_{qj} = 0. \quad (3.27)$$

In the absence of shocks, $Q_{pj} = Q_{qj} = 0$, and (3.27) then becomes a conservation law for the waves.

In the presence of shocks, the source term $Q_{pj}$ in (3.18) has the form

$$\begin{aligned} Q_{pj} = \sum_l \left[\tfrac{2}{3}\tilde{\Gamma}_{ppp} {c'_p}^3 + \sum_{\sigma=j,p,q} \tilde{\Gamma}_{pp\sigma}\langle c_\sigma\rangle {c'_p}^2\right]_{\theta^+_{pl}}^{\theta^-_{pl}} \\ + m\sum_\nu \left[\tfrac{2}{3}\tilde{\Gamma}_{jjj} {c'_j}^3 + \sum_{\sigma=j,p,q} \tilde{\Gamma}_{jj\sigma}\langle c_\sigma\rangle {c'_j}^2\right]_{\theta^+_{j\nu}}^{\theta^-_{j\nu}} \\ + 2m\tilde{\Gamma}_{jpq}\int_0^{2\pi} \sum_\nu \left[c_q(\theta_q)c_p(\theta_p)c_j(\theta_j)\right]_{\theta^+_{j\nu}}^{\theta^-_{j\nu}} d\theta_q. \end{aligned} \quad (3.28)$$

In (3.28), the indices $l$ and $\nu$ refer to shocks in $\theta_p$ and $\theta_j$, located at $\theta = \theta_{pl}$ and at $\theta = \theta_{j\nu}$, where the indices $l$ and $\nu$ label the shocks. The notation

$$[S]_{\theta^+_{pl}}^{\theta^-_{pl}} = S\left(\theta_{pl} - \epsilon\right) - S\left(\theta_{pl} + \epsilon\right) \quad (3.29)$$

denotes the jump in the quantity $S$ in crossing the shock, and $\theta^-_{pl} = \theta_{pl} - \epsilon$ and $\theta^+_{pl} = \theta_{pl} + \epsilon$, denote the values of $\theta_p$ on either side of the shock ($0 < \epsilon \ll 1$). The quantity $Q_{qj}$ in (3.19) is given by a formula of the same form as (3.28), except that the replacements $p \to q$, $q \to p$, and $m \to n$ are made in (3.28).

The quasiparticle number density equations (3.18) and (3.19) may be obtained by first multiplying (3.14) by $2c'_j$, integrating with respect to $\theta \equiv \theta_j$ from $\theta = 0$ to $\theta = 2\pi$, to obtain the equation

$$\begin{aligned} \frac{\partial \mathscr{N}_j}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}\left(\lambda_j \mathscr{N}_j\right) + 2\tilde{\Gamma}_{jpq}\int_0^{2\pi} c'_j(\theta) D_\theta(\langle c'_p c'_q\rangle_j)\, d\theta \\ + \sum_\nu \left[\tfrac{2}{3}\tilde{\Gamma}_{jjj} {c'_j}^3 + \sum_{\sigma=j,p,q} \tilde{\Gamma}_{jj\sigma}\langle c_\sigma\rangle {c'_j}^2\right]_{\theta^+_{j\nu}}^{\theta^-_{j\nu}} = 0. \end{aligned} \quad (3.30)$$

Similar equations may be readily established for $\mathscr{N}_p$ and $\mathscr{N}_q$. To establish the

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

quasiparticle number density equation (3.18), it is necessary to use the identity

$$I_{pqj} + mI_{jpq} = m \sum_{\nu} \int_0^{2\pi} d\theta_q \left[ c_j'(\theta_j) c_p'(\theta_p) c_q'(\theta_q) \right]_{\theta_{j\nu}^+}^{\theta_{j\nu}^-}, \tag{3.31}$$

where

$$I_{pqj} = \int_0^{2\pi} c_p'(\theta) D_\theta \left( \langle c_q' c_j' \rangle_p \right) d\theta \tag{3.32}$$

is an integral arising from the three-wave resonant interactions. In the derivation of (3.31), use has been made of the resonance conditions (3.17), and the sums over $\nu$ in (3.30) and (3.31) are due to shocks in $\theta_j$.

### *3.3. Conservation laws for triads involving the entropy wave*

In this section, we consider the form of the quasiparticle number density conservation laws and the Manley–Rowe relations for resonant wave triads involving the entropy wave. This case requires special consideration, because the resonant interaction coefficient $\tilde{\Gamma}_{4pq} = 0$ in the entropy-wave evolution equation in (3.14) for resonant wave triads $\{4pq\}$. To make the analysis concrete, we consider explicitly the resonant wave triad $\{417\}$ involving the entropy wave, and the backward and forward fast magnetoacoustic waves, for which the resonance conditions are of the form

$$\theta_4 = m\theta_7 + n\theta_1, \qquad k_4 = mk_7 + nk_1, \qquad \omega_4 = m\omega_7 + n\omega_1, \tag{3.33}$$

where $m$ and $n$ are integers. The other resonant wave triads involving the entropy wave listed in Table 1 can be treated in a similar fashion. The wave dispersion relations $\{\omega_s = \lambda_s k_s : \ s = 1, 4, 7\}$, with $\lambda_1 = -c_f$, $\lambda_4 = 0$, and $\lambda_7 = c_f$ combined with the the resonance conditions (3.33) require that

$$k_1 = \frac{k_4}{2n}, \qquad k_7 = \frac{k_4}{2m}, \tag{3.34}$$

in order that the resonance conditions be satisfied.

**Proposition 2.** Consider the resonant wave triad $\{417\}$ involving the entropy wave and the backward and forward fast magnetoacoustic waves, satisfying the resonance conditions (3.33) and (3.34). In this case, the wave quasiparticle number densities $\{\mathcal{N}_s : \ s = 4, 1, 7\}$ satisfy the continuity equations

$$\frac{\partial}{\partial \bar{t}} \left( n\mathcal{N}_7 - m\mathcal{N}_1 \right) + \frac{\partial}{\partial \bar{x}} \left( \lambda_7 n\mathcal{N}_7 - \lambda_1 m\mathcal{N}_1 \right) + Q_{71} = 0, \tag{3.35}$$

$$\frac{\partial \mathcal{N}_4}{\partial \bar{t}} + Q_4 = 0, \tag{3.36}$$

where the source terms $Q_{71}$ and $Q_4$ are due to possible shocks in the fast phase variables $\{\theta_s : \ s = 1, 4, 7\}$. In the absence of shocks, and for wave interactions independent of the long space variable $x$, (3.35) and (3.36) yield the Manley–Rowe relations

$$n\mathcal{N}_7 - m\mathcal{N}_1 = M_0, \qquad \mathcal{N}_4 = M_1, \tag{3.37}$$

where $M_0$ and $M_1$ are constants. The relations (3.37) show that $\Delta\mathcal{N}_4 = 0$ and $n\Delta\mathcal{N}_7 = m\Delta\mathcal{N}_1$ are the changes in the quasiparticle number densities for resonant interactions between the entropy wave and the backward and forward fast magnetoacoustic waves.

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

In the presence of shocks, the source term $Q_{71}$ in (3.35) is given by

$$Q_{71} = n\sum_{\nu}\left[\tfrac{2}{3}\tilde{\Gamma}_{777}c_7'^3 + \sum_{s=1,4,7}\tilde{\Gamma}_{77s}\langle c_s\rangle c_7'^2\right]_{\theta^+_{7\nu}}^{\theta^-_{7\nu}}$$
$$-m\sum_{\mu}\left[\tfrac{2}{3}\tilde{\Gamma}_{111}c_1'^3 + \sum_{s=1,4,7}\tilde{\Gamma}_{11s}\langle c_s\rangle c_1'^2\right]_{\theta^+_{1\mu}}^{\theta^-_{1\mu}}. \tag{3.38}$$

The corresponding source term $Q_4$ in (3.36) due to shocks is

$$Q_4 = \sum_{\chi}\sum_{s=1,4,7}\tilde{\Gamma}_{44s}\langle c_s\rangle\left[c_4'^2\right]_{\theta^+_{4\chi}}^{\theta^-_{4\chi}}. \tag{3.39}$$

The indices $\nu$, $\mu$, and $\chi$ in (3.38) and (3.39) label the shocks involved.

Taking into account the resonance conditions (3.33) and (3.34), the quasiparticle number equation (3.35) can be recast in the form of the canonical energy equation

$$\frac{\partial}{\partial\bar{t}}\left(\omega_1\mathscr{N}_1 + \omega_7\mathscr{N}_7\right) + \frac{\partial}{\partial\bar{x}}\left(\lambda_1\omega_1\mathscr{N}_1 + \lambda_7\omega_7\mathscr{N}_7\right) + \frac{\omega_7}{n}Q_{71} = 0, \tag{3.40}$$

for the backward and forward fast magnetoacoustic waves. It is also straightforward to write down the canonical momentum equation for the waves analogous to (3.27).

The derivation of (3.35)–(3.40) is similar to the derivation of the results in Proposition 1. It is useful to note the identity

$$I_{714} = \frac{m}{n}I_{174} \tag{3.41}$$

in the derivation of (3.35), where the resonant wave integral $I_{pqj}$ is defined in (3.32).

## 4. Lagrangian and Hamiltonian variational formulations

In the previous section, the three-wave resonant interaction equations were written in terms of the natural Hamiltonian variables of (3.1)–(3.3). In terms of these variables, the resonant interaction equations have a more symmetric form (i.e. the resonant interaction coefficients $\tilde{\Gamma}_{jpq}$ ($j \neq p \neq q$) are symmetric in the indices $j$, $p$, and $q$, except for the entropy-wave case, where $\tilde{\Gamma}_{4pq} = 0$: see (3.7)). The use of Hamiltonian variables also facilitates the derivation of the quasiparticle number density conservation laws and the canonical wave energy equation (Secs 3.2 and 3.3). In this section, we use the Hamiltonian variables $\{c_j\}$ of (3.1)–(3.3) to derive Lagrangian and Hamiltonian variational principles for the three-wave resonant interaction equations (3.13) and (3.14).

Section 4.1 considers resonant wave triads involving magnetoacoustic waves and the entropy wave. As part of the discussion, we consider the form of the equations in the gas-dynamical limit in which the magnetic field $B \to 0$. In this case, the equations reduce to integro-differential Burgers equations obtained by Majda and Rosales (1984), describing the three-wave resonant interaction of the backward and forward sound waves and the entropy wave in adiabatic gas dynamics. The present analysis generalizes the Hamiltonian formulation of the Majda–Rosales equations obtained by Webb et al. (1997), where only the lowest-order resonant wave interactions were considered in any detail. Variational principles are also obtained for the resonant wave triads involving (a) Alfvén waves and the entropy wave (Sec. 4.2)

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

and (b) isentropic resonant wave interactions, not involving the entropy wave (Sec. 4.3). Section 4.3 considers the strictly hyperbolic case where all the wave speeds are distinct. This includes the cases of Alfvén–magnetoacoustic wave interactions and fast–slow magnetoacoustic wave interactions. The non-strictly hyperbolic cases where there are coincident eigenvalues are discussed in Sec. 4.4.

### *4.1. Fast magnetoacoustic–entropy wave interactions*

In this section, we consider equations describing three-wave resonant interactions between the entropy wave ($c_4$) and the backward and forward fast magnetoacoustic waves ($c_1$ and $c_7$). Similar equations can also be obtained describing the resonant interaction of the slow magnetoacoustic waves with the entropy wave. We assume that there are no shocks in the fast phase variables $\{\theta_s : s = 1, 4, 7\}$. The variational formulations of (3.13) and (3.14) are considerably simpler for the case where the waves have zero means or constant means (i.e. $\langle c_j \rangle = \text{const}$). We obtain variational principles for (3.13) and (3.14) for the general case where the mean wave amplitudes satisfy the evolution equations (3.13), as well as the simpler case when the $\langle c_j \rangle$ are constants. The Majda–Rosales (1984) equations describing the resonant interaction of sound waves and entropy waves in adiabatic gas dynamics are obtained by letting $B \to 0$.

As in Sec. 3.3, we consider the resonant wave triad $\{417\}$ involving the entropy wave and the backward and forward fast mode waves, satisfying the resonance conditions (3.33) and (3.34), namely

$$\theta_4 = m\theta_7 + n\theta_1, \qquad k_4 = mk_7 + nk_1, \qquad \omega_4 = m\omega_7 + n\omega_1. \tag{4.1}$$

Use of the dispersion equations for the waves in (4.1) requires that

$$k_1 = \frac{k_4}{2n}, \qquad k_7 = \frac{k_4}{2m}, \tag{4.2}$$

in order that the resonance conditions be satisfied. From (3.13), the wave means $\langle c_j \rangle$ satisfy the equations

$$\frac{\partial \langle c_j \rangle}{\partial \bar{t}} + \lambda_j \frac{\partial \langle c_j \rangle}{\partial \bar{x}} = 0, \qquad j = 1, 4, 7. \tag{4.3}$$

From (3.14), the wave amplitudes $\{c_j'\}$ satisfy the equations

$$F_1 \equiv \frac{\partial c_1'}{\partial \bar{t}} + \lambda_1 \frac{\partial c_1'}{\partial \bar{x}} + D_\theta \left( \tfrac{1}{2} \tilde{\Gamma}_{111} {c_1'}^2 + \sum_{s=1,4,7} \tilde{\Gamma}_{11s} \langle c_s \rangle c_1' + \tilde{\Gamma}_{147} \langle c_4' c_7' \rangle_1 \right) = 0, \tag{4.4}$$

$$F_4 \equiv \frac{\partial c_4'}{\partial \bar{t}} + D_\theta (\tilde{\Gamma}_{441} \langle c_1 \rangle c_4' + \tilde{\Gamma}_{447} \langle c_7 \rangle c_4') = 0, \tag{4.5}$$

$$F_7 \equiv \frac{\partial c_7'}{\partial \bar{t}} + \lambda_7 \frac{\partial c_7'}{\partial \bar{x}} + D_\theta \left( \tfrac{1}{2} \tilde{\Gamma}_{777} {c_7'}^2 + \sum_{s=1,4,7} \tilde{\Gamma}_{77s} \langle c_s \rangle c_7' + \tilde{\Gamma}_{147} \langle c_4' c_1' \rangle_7 \right) = 0. \tag{4.6}$$

The $\{F_j : j = 1, 4, 7\}$ denote the left-hand sides of (4.4)–(4.6). Equations (4.4)–(4.6) can also be written in the potential form (3.15) by noting that $c_j' = C_{j,\theta}$. Variational formulations of (4.3)–(4.6) are given in Sec. 4.1.1, in which (4.3) and (4.5) are treated as constraint equations by using Lagrange multipliers. The Manley–Rowe relations and the canonical wave energy equation for this system were discussed in Sec. 3.3.

*4.1.1. Lagrangian and Hamiltonian variational principles.* The main results in this section on the variational formulation of the resonant wave interaction equations

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

(4.3)–(4.6) are stated in the form of propositions. We first provide a Lagrangian variational principle for the equations, in which the entropy-wave evolution equation (4.5) and the mean wave field equations (4.3) are treated as constraint equations (Proposition 3). This is followed by a Hamiltonian variational principle (Proposition 4), in which the Lagrange multiplier evolution equations are an integral part of the Hamiltonian structure. We then discuss the simpler variational equations that result when the means are constants (Proposition 5). A discussion of the special case of the resonant interaction of sound waves and entropy waves (Majda and Rosales 1984; Webb et al. 1997) is given in Sec. 4.1.2.

**Proposition 3.** Equations (4.3)–(4.6) describing the three-wave resonant interactions of the fast magnetoacoustic waves ($c_1$ and $c_7$) and the entropy wave ($c_4$) may be obtained from the Euler–Lagrange equations of the constrained variational functional

$$J = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \int_0^{2\pi m} d\theta \; \mathscr{L}, \tag{4.7}$$

where

$$\begin{aligned}\mathscr{L} = -\Bigg( & \tfrac{1}{2} C_{1,\theta} C_{1,t} + \tfrac{1}{2} C_{7,\theta} C_{7,t} + \tfrac{1}{2}\lambda_1 C_{1,x} C_{1,\theta} + \tfrac{1}{2}\lambda_7 C_{7,x} C_{7,\theta} \\ & + \tfrac{1}{6}\tilde{\Gamma}_{111} C_{1,\theta}^3 + \tfrac{1}{6}\tilde{\Gamma}_{777} C_{7,\theta}^3 + \tilde{\Gamma}_{147}\langle c_4' c_7'\rangle c_1' \\ & + \tfrac{1}{2}\sum_{j=1,4,7} \tilde{\Gamma}_{11j}\langle c_j\rangle C_{1,\theta}^2 + \tfrac{1}{2}\sum_{j=1,4,7} \tilde{\Gamma}_{77j}\langle c_j\rangle C_{7,\theta}^2 \Bigg) \\ & + \sum_{j=1,7} \mu_j \left( \frac{\partial \langle c_j\rangle}{\partial \bar{t}} + \lambda_j \frac{\partial \langle c_j\rangle}{\partial \bar{x}} \right) \\ & + \sigma_4 [c_{4,t}' + (\tilde{\Gamma}_{441}\langle c_1\rangle + \tilde{\Gamma}_{447}\langle c_7\rangle) c_{4,\theta}'],\end{aligned} \tag{4.8}$$

and the $\{\mu_j : j = 1, 7\}$ and $\sigma_4$ are Lagrange multipliers. The mean wave amplitude

$$\langle c_4\rangle = f(\bar{x}) \tag{4.9}$$

is given by the initial conditions. The Lagrange multiplier terms in (4.8) ensure that the mean wave field equations (4.3) and the entropy-wave equation (4.5) are satisfied at a critical point of the variational functional $J$. The Euler–Lagrange equations for the constrained functional described by (4.7) and (4.8) consist of the evolution equations (4.3) for the mean wave amplitudes $\{\langle c_j\rangle : j = 1, 7\}$, the evolution equations (4.4)–(4.6) for the $\{c_j' : j = 1, 4, 7\}$, plus auxilary evolution equations for the Lagrange multipliers $\{\mu_j : j = 1, 7\}$ and $\sigma_4$. In particular, the Euler–Lagrange equations:

$$\frac{\delta J}{\delta \mu_j} \equiv \frac{\partial \langle c_j\rangle}{\partial \bar{t}} + \lambda_j \frac{\partial \langle c_j\rangle}{\partial \bar{x}} = 0, \qquad j = 1, 7, \tag{4.10}$$

yield the mean wave amplitude equations (4.3). The Euler–Lagrange equations:

$$\frac{\delta J}{\delta C_1} \equiv F_1 = 0, \qquad \frac{\delta J}{\delta \sigma_4} \equiv F_4 = 0, \qquad \frac{\delta J}{\delta C_7} \equiv F_7 = 0 \tag{4.11}$$

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

yield (4.4)–(4.6) for the $\{c_j' : j = 1, 4, 7\}$). The remaining Euler–Lagrange equations for the functional (4.7) are

$$\frac{\delta J}{\delta \langle c_1 \rangle} = -\left( \frac{\partial \mu_1}{\partial \bar{t}} + \lambda_1 \frac{\partial \mu_1}{\partial \bar{x}} + \tfrac{1}{2}\tilde{\Gamma}_{111} c_1'^2 + \tfrac{1}{2}\tilde{\Gamma}_{771} c_7'^2 - \sigma_4 \tilde{\Gamma}_{441} c_{4,\theta}' \right) = 0, \quad (4.12)$$

$$\frac{\delta J}{\delta \langle c_7 \rangle} = -\left( \frac{\partial \mu_7}{\partial \bar{t}} + \lambda_7 \frac{\partial \mu_7}{\partial \bar{x}} + \tfrac{1}{2}\tilde{\Gamma}_{117} c_1'^2 + \tfrac{1}{2}\tilde{\Gamma}_{777} c_7'^2 - \sigma_4 \tilde{\Gamma}_{447} c_{4,\theta}' \right) = 0, \quad (4.13)$$

$$\frac{\delta J}{\delta C_4} = D_\theta \left[ \frac{\partial \sigma_4}{\partial \bar{t}} + (\tilde{\Gamma}_{441} \langle c_1 \rangle + \tilde{\Gamma}_{447} \langle c_7 \rangle) \frac{\partial \sigma_4}{\partial \theta} + \tilde{\Gamma}_{147} \langle c_1' c_7' \rangle_4 \right] = 0. \quad (4.14)$$

Equations (4.12)–(4.14) are evolution equations for the Lagrange multipliers. The Euler–Lagrange equations (4.12) and (4.13) for the $\{\mu_j : j = 1, 7\}$ depend on mean wave field and Burgers interaction coefficients including entropy-wave effects ($\propto c_{4,\theta}'$), whereas the evolution equation (4.14) for $\sigma_4$ involves both mean wave field effects and three-wave resonant interaction effects.

In the derivation of (4.10)–(4.14), use has been made of the resonance condition $\theta_4 = m\theta_7 + n\theta_1$ for the wave phases, and the assumption that the wave amplitudes $\{c_j'\}$ are $2\pi$-periodic in $\theta$ has been used. Note that $\langle c_1' c_7' \rangle_4$ has two periods $T_m = 2\pi m$ and $T_n = 2\pi n$ in $\theta$, $\langle c_4' c_7' \rangle_1$ has a period of $T = 2\pi/n$ in $\theta$, and $\langle c_4' c_1' \rangle_7$ has a period of $T = 2\pi/m$ in $\theta$. In the determination of the variational derivatives of the functional (4.7), it is useful to note the identities

$$I(\langle c_4' c_7' \rangle_1, c_1') = I(\langle c_7' c_1' \rangle_4, c_4') = I(\langle c_1' c_4' \rangle_7, c_7'), \quad (4.15)$$

where

$$I(f, g) = \int_0^{2\pi m} f(\xi) g(\xi)\, d\xi, \quad (4.16)$$

denotes the inner product of two functions $f(\xi)$ and $g(\xi)$ with period $2\pi m$ in $\xi$.

**Proposition 4.** The variational equations (4.10)–(4.14) describing the three-wave resonant interactions of the fast magnetoacoustic waves ($c_1$ and $c_7$) and the entropy wave ($c_4$) can be cast in the Hamiltonian form

$$\frac{\partial c_1'}{\partial \bar{t}} = -D_\theta \left( \frac{\delta H}{\delta c_1'} \right), \qquad \frac{\partial c_7'}{\partial \bar{t}} = -D_\theta \left( \frac{\delta H}{\delta c_7'} \right), \quad (4.17)$$

$$\frac{\partial c_4'}{\partial \bar{t}} = \frac{\delta H}{\delta \sigma_4}, \qquad \frac{\partial \sigma_4}{\partial \bar{t}} = -\frac{\delta H}{\delta c_4'}, \quad (4.18)$$

$$\frac{\partial \langle c_j \rangle}{\partial \bar{t}} = \frac{\delta H}{\delta \mu_j}, \qquad \frac{\partial \mu_j}{\partial \bar{t}} = -\frac{\delta H}{\delta \langle c_j \rangle}, \qquad j = 1, 7, \quad (4.19)$$

where the Hamiltonian functional has the form

$$H = \int_{-\infty}^{\infty} dx \int_0^{2\pi m} d\theta \left[ \tfrac{1}{2}\lambda_1 (D_\theta^{-1} D_x c_1') c_1' + \tfrac{1}{2}\lambda_7 (D_\theta^{-1} D_x c_7') c_7' + \tfrac{1}{6}\tilde{\Gamma}_{111} c_1'^3 \right.$$
$$+ \tfrac{1}{6}\tilde{\Gamma}_{777} c_7'^3 + \tilde{\Gamma}_{147} \langle c_4' c_7' \rangle_1 c_1' + \tfrac{1}{2} \sum_{j=1,4,7} \tilde{\Gamma}_{11j} \langle c_j \rangle c_1'^2 + \tfrac{1}{2} \sum_{j=1,4,7} \tilde{\Gamma}_{77j} \langle c_j \rangle c_7'^2$$
$$\left. - \mu_1 \lambda_1 \frac{\partial \langle c_1 \rangle}{\partial \bar{x}} - \mu_7 \lambda_7 \frac{\partial \langle c_7 \rangle}{\partial \bar{x}} - (\tilde{\Gamma}_{441} \langle c_1 \rangle + \tilde{\Gamma}_{447} \langle c_7 \rangle) \sigma_4 c_{4,\theta}' \right]. \quad (4.20)$$

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

The variational equations (4.17)–(4.19) in Proposition 4 may be established by straightforward determination of the variational derivatives of $H$. Equations (4.17) are equivalent to the evolution equations (4.4) and (4.6) for $c_1'$ and $c_7'$, the evolution equation (4.18) for $c_4'$ is equivalent to (4.5), and the remaining evolution equations for the Lagrange multipliers $\{\mu_j : j = 1, 7\}$ and $\sigma_4$ are equivalent to (4.12)–(4.14) obtained in the Lagrangian variational formulation of Proposition 3.

Introducing the Poisson bracket $\{F, G\}$ for functionals $F$ and $G$, defined by

$$\{F,G\} = \int_{-\infty}^{\infty} dx \int_0^{2\pi m} d\theta \left[ -\sum_{j=1,7} \frac{\delta F}{\delta c_j'} D_\theta \left( \frac{\delta G}{\delta c_j'} \right) + \frac{\delta F}{\delta c_4'} \frac{\delta G}{\delta \sigma_4} - \frac{\delta F}{\delta \sigma_4} \frac{\delta G}{\delta c_4'} + \sum_{j=1,7} \left( \frac{\delta F}{\delta \langle c_j \rangle} \frac{\delta G}{\delta \mu_j} - \frac{\delta F}{\delta \mu_j} \frac{\delta G}{\delta \langle c_j \rangle} \right) \right], \tag{4.21}$$

it follows that

$$F_t = \{F, H\}, \tag{4.22}$$

where $F$ is any functional of the variables $\boldsymbol{\Psi} = (c_1', c_4', c_7', \sigma_4, \langle c_1 \rangle, \langle c_7 \rangle, \mu_1, \mu_7)^{\mathrm{T}}$. In particular, choosing $F$ to be each of the variables in $\boldsymbol{\Psi}$, one obtains the evolution equations (4.17)–(4.19). Since $\{H, H\} = 0$, it follows that $\partial H / \partial \bar{t} = 0$.

Simpler versions of the variational principles result if the wave means $\{\langle c_j \rangle : j = 1, 4, 7\}$ are constants, and if $c_{4,\bar{t}}' = 0$. From (4.5), $c_{4,\bar{t}}' = 0$ if

$$\tilde{\Gamma}_{441} \langle c_1 \rangle + \tilde{\Gamma}_{447} \langle c_7 \rangle \equiv \frac{\alpha_f c_f}{\rho} k_4 \left( \langle \rho_7 \rangle - \langle \rho_1 \rangle \right) \equiv \frac{\alpha_f}{\rho} k_4 \left( \langle u_{x7} \rangle + \langle u_{x1} \rangle \right) = 0, \tag{4.23}$$

where the $\rho_j$ and $u_{xj}$ $(j = 1, 7)$, denote the density and $x$-velocity fluctuations of the backward and forward fast magnetoacoustic waves.

**Proposition 5.** If the conditions (4.23) hold then $c_1'$ and $c_7'$ satisfy the Hamiltonian equations

$$\frac{\partial c_1'}{\partial \bar{t}} = -D_\theta \left( \frac{\delta H}{\delta c_1'} \right), \qquad \frac{\partial c_7'}{\partial \bar{t}} = -D_\theta \left( \frac{\delta H}{\delta c_7'} \right), \tag{4.24}$$

where the Hamiltonian functional is given by

$$H = \int_{-\infty}^{\infty} dx \int_0^{2\pi m} d\theta \left[ \tfrac{1}{2} \lambda_1 (D_\theta^{-1} D_x c_1') c_1' + \tfrac{1}{2} \lambda_7 (D_\theta^{-1} D_x c_7') c_7' + \tfrac{1}{6} \tilde{\Gamma}_{111} {c_1'}^3 + \tfrac{1}{6} \tilde{\Gamma}_{777} {c_7'}^3 + \tilde{\Gamma}_{147} \langle c_4' c_7' \rangle_1 c_1' + \tfrac{1}{2} \sum_{j=1,4,7} \tilde{\Gamma}_{11j} \langle c_j \rangle {c_1'}^2 + \tfrac{1}{2} \sum_{j=1,4,7} \tilde{\Gamma}_{77j} \langle c_j \rangle {c_7'}^2 \right]. \tag{4.25}$$

In this case, the entropy-wave amplitude $c_4' \equiv c_4'(\bar{x}, \theta)$ is a given function of $\bar{x}$ and $\theta$ determined by the initial conditions.

*4.1.2. Resonant interaction of sound waves and entropy waves.* Majda and Rosales (1984) considered the resonant interaction of periodic sound waves and entropy waves in one Cartesian space dimension, for the special case where the waves have zero means. In the limit as $B \to 0$, the backward and forward fast magnetoacoustic waves degenerate into the backward and forward sound waves. Hence the Majda–Rosales equations result from taking the limit as $B \to 0$ in the equation system (4.3)–(4.6).

In the limit as $B \to 0$, one obtains

$$c_s = 0, \qquad c_f = a, \qquad b_x = 0, \qquad \lambda_1 = -a, \qquad \lambda_4 = 0, \qquad \lambda_7 = a. \tag{4.26}$$

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

As $B \to 0$, the parameters $\alpha_f$, $\alpha_s$, $\beta_f$, and $\beta_s$ (see (2.7) and (2.10)), take the values

$$\alpha_f = 1, \qquad \alpha_s = 0, \qquad \beta_f = \tfrac{1}{2}(\gamma+1), \qquad \beta_s = \tfrac{3}{2}. \tag{4.27}$$

In the gas-dynamical limit as $B \to 0$, the interaction coefficients in the evolution equations (4.3)–(4.6) for the $\{c_j' : j = 1, 4, 7\}$ may be obtained by using the parameter values (4.27), and the eigenvalues (4.26) for the wave speeds, in (3.7)–(3.10). If the waves have zero means then the Hamiltonian (4.25) describing the interaction of the backward and forward sound waves and the entropy wave reduces to

$$H = \int_{-\infty}^{\infty} dx \int_0^{2\pi m} d\theta \left[ \tfrac{1}{2} a (D_\theta^{-1} D_x c_7') c_7' - \tfrac{1}{2} a (D_\theta^{-1} D_x c_1') c_1' + \frac{\gamma+1}{12 a \rho^{1/2}} (\omega_1^{3/2} c_1'^3 + \omega_7^{3/2} c_7'^3) + \frac{(\omega_1 \omega_7)^{1/2}}{2\rho} \langle c_4' c_7' \rangle_1 c_1' \right], \tag{4.28}$$

The backward and forward sound-wave equations in this case can be cast in the Hamiltonian form (4.24), and correspond to the equations studied by Majda and Rosales (1984), Majda et al. (1988), and Webb et al. (1997).

### *4.2. Resonant interaction of Alfvén and entropy waves*

Consider the resonant interaction of the backward and forward Alfvén waves ($a_2$ and $a_6$) with the entropy wave ($a_4$). The resonance conditions for periodic waves are

$$\omega_4 = m\omega_6 + n\omega_2, \qquad k_4 = mk_6 + nk_2, \tag{4.29}$$

where $m$ and $n$ are rational numbers (in the present analysis, $m$ and $n$ are taken to be integers). Using the wave dispersion equations $\omega_2 = -k_2 b_x$, $\omega_4 = 0$, and $\omega_6 = k_6 b_x$, the conditions (4.29) are satisfied if

$$k_6 = \frac{k_4}{2m}, \qquad k_2 = \frac{k_4}{2n}. \tag{4.30}$$

The interaction equations for the Alfvén waves reduce to

$$\frac{\partial a_2}{\partial \bar{t}} + \lambda_2 \frac{\partial a_2}{\partial \bar{x}} + D_\theta \left( -\frac{\omega_2}{2\rho} \langle a_4 \rangle a_2 + \frac{\omega_2}{2\rho} \langle a_4 a_6 \rangle_2 \right) = 0, \tag{4.31}$$

$$\frac{\partial a_6}{\partial \bar{t}} + \lambda_6 \frac{\partial a_6}{\partial \bar{x}} + D_\theta \left( -\frac{\omega_6}{2\rho} \langle a_4 \rangle a_6 + \frac{\omega_6}{2\rho} \langle a_4 a_2 \rangle_6 \right) = 0, \tag{4.32}$$

where $\lambda_2 = -b_x$ and $\lambda_6 = b_x$. The entropy-wave evolution equation $\partial a_4 / \partial \bar{t} = 0$ implies that $a_4 \equiv a_4(\bar{x}, \theta)$ is given by the initial conditions, and the mean value $\langle a_4 \rangle = \langle a_4 \rangle(\bar{x})$.

In terms of the variables $\{c_j' : j = 2, 4, 6\}$ and $\{\langle c_j \rangle : j = 2, 4, 6\}$, the system of equations (4.31)–(4.32) reduces to

$$\frac{\partial \langle c_j \rangle}{\partial \bar{t}} + \lambda_j \frac{\partial \langle c_j \rangle}{\partial \bar{x}} = 0, \qquad j = 2, 6, \tag{4.33}$$

$$\frac{\partial c_2'}{\partial \bar{t}} + \lambda_2 \frac{\partial c_2'}{\partial \bar{x}} + D_\theta \left[ -\frac{\omega_2}{2\rho} \langle c_4 \rangle c_2' + \frac{(\omega_2 \omega_6)^{1/2}}{2\rho} \langle c_4' c_6' \rangle_2 \right] = 0, \tag{4.34}$$

$$\frac{\partial c_6'}{\partial \bar{t}} + \lambda_6 \frac{\partial c_6'}{\partial \bar{x}} + D_\theta \left[ -\frac{\omega_6}{2\rho} \langle c_4 \rangle c_6' + \frac{(\omega_2 \omega_6)^{1/2}}{2\rho} \langle c_4' c_2' \rangle_6 \right] = 0. \tag{4.35}$$

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

In (4.33)–(4.35), $\langle c_4\rangle = f(\bar{x})$, where $f$ is an arbitrary function of $\bar{x}$. Note that $f(\bar{x})$ and $c_4' = c_4'(\bar{x},\theta)$ are determined by the initial conditions.

**Proposition 6.** The Alfvén- and entropy-wave resonant interaction equations (4.33)–(4.35) can be cast in the Hamiltonian form

$$\frac{\partial c_2'}{\partial \bar{t}} = -D_\theta\left(\frac{\delta H}{\delta c_2'}\right), \qquad \frac{\partial c_6'}{\partial \bar{t}} = -D_\theta\left(\frac{\delta H}{\delta c_6'}\right), \tag{4.36}$$

$$\frac{\partial \langle c_j\rangle}{\partial \bar{t}} = \frac{\delta H}{\delta \mu_j}, \qquad \frac{\partial \mu_j}{\partial \bar{t}} = -\frac{\delta H}{\delta \langle c_j\rangle}, \qquad j = 2, 6, \tag{4.37}$$

where

$$H = \int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta \left[\tfrac{1}{2}\lambda_2 (D_\theta^{-1} D_x c_2') c_2' + \tfrac{1}{2}\lambda_6 (D_\theta^{-1} D_x c_6') c_6' + \frac{(\omega_2\omega_6)^{1/2}}{2\rho}\langle c_4' c_6'\rangle_2 c_2' \right.$$
$$\left. - \frac{\langle c_4\rangle}{4\rho}(\omega_2 c_2'^{\,2} + \omega_6 c_6'^{\,2}) - \mu_2\lambda_2 \frac{\partial \langle c_2\rangle}{\partial \bar{x}} - \mu_6\lambda_6 \frac{\partial \langle c_6\rangle}{\partial \bar{x}}\right] \tag{4.38}$$

is the Hamiltonian. In the variational principle (4.36)–(4.38), the entropy-wave amplitude $c_4' = c_4'(\bar{x},\theta)$ and $\langle c_4\rangle$ are given by the initial conditions.

It is of interest to note that the Alfvén–entropy wave interaction equations are linear coupled evolution equations for a given kernel $c_4'(\bar{x},\theta)$, in which the resonant integral terms $\langle c_4' c_6'\rangle_2$ and $\langle c_4' c_2'\rangle_6$ couple the backward and forward Alfvén waves. Explicit, analytical solutions of (4.31)–(4.32) are studied in Sec. 6.

### *4.3. Isentropic, resonant wave interactions*

The variational formulations of the resonant wave interaction equations for resonant wave triads $\{jpq\}$ not involving the entropy wave, and for the strictly hyperbolic case ($\lambda_j \neq \lambda_p \neq \lambda_q$), are developed below. In these cases, the three-wave resonant interaction coefficient $\tilde{\Gamma}_{jpq}$ is invariant under any permutation of $\{jpq\}$. The different resonant wave triads $\{jpq\}$ not involving the entropy wave are listed in Table 1. The strictly hyperbolic case is dealt with in Sec. 4.3.1. Non-strictly hyperbolic cases are discussed in Sec. 4.4.

*4.3.1. The strictly hyperbolic case.* If there are no shocks in $\theta$, the wave interaction equations corresponding to the resonant wave triad $\{jpq\}$ ($j \neq p \neq q$), may be written in the form

$$\frac{\partial \langle c_s\rangle}{\partial \bar{t}} + \lambda_s \frac{\partial \langle c_s\rangle}{\partial \bar{x}} = 0, \tag{4.39}$$

$$F_s \equiv \frac{\partial c_s'}{\partial \bar{t}} + \lambda_s \frac{\partial c_s'}{\partial \bar{x}} + D_\theta\left(\tfrac{1}{2}\tilde{\Gamma}_{sss} c_s'^{\,2} + \sum_{\sigma=j,p,q} \tilde{\Gamma}_{ss\sigma}\langle c_\sigma\rangle c_s' + \tilde{\Gamma}_{s\alpha\beta}\langle c_\alpha' c_\beta'\rangle_s\right) = 0, \tag{4.40}$$

where $s = j, p, q$, $c_s' \equiv C_{s,\theta}$, and $\{s\alpha\beta\}$ is a permutation of $\{jpq\}$. From (2.4) and (2.5), the resonance conditions

$$\omega_j = m\omega_q + n\omega_p, \qquad k_j = mk_q + nk_p \tag{4.41}$$

are satisfied if

$$k_q = \frac{k_j(\lambda_j - \lambda_p)}{(\lambda_q - \lambda_p)m}, \qquad k_p = \frac{k_j(\lambda_j - \lambda_q)}{(\lambda_p - \lambda_q)n}, \tag{4.42}$$

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

where $\{\omega_s = k_s\lambda_s : \ s = j,p,q\}$ are the wave dispersion equations. The variational formulations of the resonant interaction equations (4.39) and (4.40) are discussed below.

**Proposition 7.** The wave interaction equations (4.39) and (4.40) for the hyperbolic isentropic resonant wave triad $\{jpq\}$ arise as critical-point conditions of the variational functional

$$J = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \int_{0}^{2\pi m} d\theta \ \mathscr{L}, \tag{4.43}$$

where

$$\begin{aligned}\mathscr{L} = -\Bigg[ & \tfrac{1}{2} \sum_{s=j,p,q} \left(C_{s,\bar{t}} + \lambda_s C_{s,\bar{x}}\right) C_{s,\theta} + \tfrac{1}{6} \sum_{s=j,p,q} \tilde{\Gamma}_{sss} C_{s,\theta}^3 \\ & + \tilde{\Gamma}_{jpq} \langle c_p' c_q' \rangle_j c_j' + \tfrac{1}{2} \sum_{s=j,p,q} \sum_{\sigma=j,p,q} \tilde{\Gamma}_{ss\sigma} \langle c_\sigma \rangle C_{s,\theta}^2 \Bigg] \\ & + \sum_{s=j,p,q} \mu_s \left( \frac{\partial \langle c_s \rangle}{\partial \bar{t}} + \lambda_s \frac{\partial \langle c_s \rangle}{\partial \bar{x}} \right),\end{aligned} \tag{4.44}$$

and the $\{\mu_s : \ s = j,p,q\}$ are Lagrange multipliers. In particular, the Euler–Lagrange equations

$$\frac{\delta J}{\delta \mu_s} = \frac{\partial \langle c_s \rangle}{\partial \bar{t}} + \lambda_s \frac{\partial \langle c_s \rangle}{\partial \bar{x}} = 0, \qquad s = j,p,q, \tag{4.45}$$

yield the mean wave amplitude equations (4.39). The Euler–Lagrange equations

$$\frac{\delta J}{\delta C_s} \equiv F_s = 0, \qquad s = j,p,q, \tag{4.46}$$

yields the evolution equations (4.40) for the $\{c_s' : \ s = j,p,q\}$, where $c_s' = C_{s,\theta}$. The remaining Euler–Lagrange equations

$$\frac{\delta J}{\delta \langle c_s \rangle} = -\left( \frac{\partial \mu_s}{\partial \bar{t}} + \lambda_s \frac{\partial \mu_s}{\partial \bar{x}} + \tfrac{1}{2} \sum_{\sigma=j,p,q} \tilde{\Gamma}_{\sigma\sigma s} {c_\sigma'}^2 \right) = 0 \tag{4.47}$$

yield evolution equations for the Lagrange multipliers $\{\mu_s : \ s = j,p,q\}$.

The proof of Proposition 7 follows the general ideas used in Proposition 3 concerning the resonant interaction of the fast magnetoacoustic waves ($c_1$ and $c_7$) and the entropy wave ($c_4$), discussed in Sec. 4.1. In the determination of the variational derivatives of the resonant wave interaction term in (4.43) and (4.44), it is useful to note the identities

$$I(\langle c_p' c_q' \rangle_j, c_j') = I(\langle c_q' c_j' \rangle_p, c_p') = I(\langle c_j' c_p' \rangle_q, c_q'), \tag{4.48}$$

where the inner product $I(f,g)$ of two functions $f(\xi)$ and $g(\xi)$ of period $2\pi m$ is defined in (4.16).

The Hamiltonian formulation of the variational equations (4.45)–(4.47) is given in the following proposition.

**Proposition 8.** The variational equations (4.45)–(4.47) describing isentropic resonant wave interactions of the hyperbolic resonant wave triad $\{jpq\}$ may be cast

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

in the Hamiltonian form

$$\frac{\partial c_s'}{\partial \bar{t}} = -D_\theta \left( \frac{\delta H}{\delta c_s'} \right), \qquad s = j, p, q, \tag{4.49}$$

$$\frac{\partial \langle c_s \rangle}{\partial \bar{t}} = \frac{\delta H}{\delta \mu_s}, \qquad \frac{\partial \mu_s}{\partial \bar{t}} = -\frac{\delta H}{\delta \langle c_s \rangle}, \qquad s = j, p, q, \tag{4.50}$$

where

$$H = \int_{-\infty}^{\infty} dx \int_0^{2\pi m} d\theta \left[ \tfrac{1}{2} \sum_{s=j,p,q} \lambda_s (D_\theta^{-1} D_x c_s') c_s' + \tfrac{1}{6} \sum_{s=j,p,q} \tilde{\Gamma}_{sss} c_s'^3 + \tilde{\Gamma}_{jpq} \langle c_p' c_q' \rangle_j c_j' \right.$$
$$\left. + \tfrac{1}{2} \sum_{s=j,p,q} \sum_{\sigma=j,p,q} \tilde{\Gamma}_{ss\sigma} \langle c_\sigma \rangle c_s'^2 - \sum_{s=j,p,q} \mu_s \lambda_s \frac{\partial \langle c_s \rangle}{\partial \bar{x}} \right] \tag{4.51}$$

is the Hamiltonian variational functional. Introducing the Poisson bracket for functionals $\{F, G\}$:

$$\{F, G\} = \int_{-\infty}^{\infty} dx \int_0^{2\pi m} d\theta \sum_{s=j,p,q} \left[ -\frac{\delta F}{\delta c_s'} D_\theta \left( \frac{\delta G}{\delta c_s'} \right) + \frac{\delta F}{\delta \langle c_s \rangle} \frac{\delta G}{\delta \mu_s} - \frac{\delta F}{\delta \mu_s} \frac{\delta G}{\delta \langle c_s \rangle} \right], \tag{4.52}$$

one obtains the equation

$$F_t = \{F, H\} \tag{4.53}$$

for the time derivative of a functional $F$ dependent on the variables $\{c_s', \langle c_s \rangle, \mu_s : s = j, p, q\}$. In particular, the evolution equations (4.49) and (4.50) may be written in the Poisson-bracket form

$$c_{s,t}' = \{c_s', H\}, \qquad \langle c_s \rangle_t = \{\langle c_s \rangle, H\}, \qquad \mu_{s,t} = \{\mu_s, H\}, \qquad s = j, p, q. \tag{4.54}$$

The equations for $\{\langle c_s \rangle, c_s' : s = j, p, q\}$ are equivalent to (4.39) and (4.40).

From Table 1, there are eight resonant triads $\{jpq\}$ involving Alfvén waves and magnetoacoustic waves that satisfy the variational formulations of Propositions 7 and 8.

### *4.4. Non-strictly hyperbolic resonant wave interactions*

Non-strictly hyperbolic wave interactions occur when one or more of the wave eigenvelocities (eigenvalues) coincide. There are two cases to consider, namely when the waves have their **k** vectors either (a) parallel to **B** (**k** $\parallel$ **B**) or (b) **k** perpendicular to **B** (**k** $\perp$ **B**). In the case **k** $\parallel$ **B**, one of the magnetoacoustic wave speeds ($c_f$ or $c_s$) coincides with the Alfvén speed $b_x$, and the other magnetoacoustic speed equals the sound speed. In case (b), with **k** $\perp$ **B**, the only non-zero wave speed is the fast magnetoacoustic speed $c_f$ (the Alfvén-, slow-magnetoacoustic and entropy-wave eigenspeeds are all zero in this limit). Representative three-wave resonant interactions for case (a) are discussed below. Similar results can also be obtained for case (b).

*4.4.1. Resonant wave interactions with* **k** $\parallel$ **B**. Consider the interaction of the different MHD wave modes for the case $\mathbf{B} = (B_x, 0, 0)^{\mathrm{T}}$ and $b_x > a$. The MHD eigenvalues in this case are

$$\lambda_1 = \lambda_2 = -b_x, \qquad \lambda_3 = -a, \qquad \lambda_4 = 0, \qquad \lambda_5 = a, \qquad \lambda_6 = \lambda_7 = b_x. \tag{4.55}$$

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

As an example of resonant wave interactions, consider the resonant wave triad $\{jpq\} = \{256\}$, involving the backward and forward Alfvén waves ($a_2$ and $a_6$) and the forward slow-mode wave ($a_5$). Because of the degeneracy of the eigenvelocities, the fast-mode waves ($a_1$ and $a_7$) can also participate in the resonant wave interactions if $k_1 = k_2$ and $k_6 = k_7$. We consider the case of resonant, periodic waves, in which the wave phases $\{\theta_j : \ j = 1, 2, 5, 6, 7\}$ are related by

$$\theta_1 = \theta_2, \qquad \theta_6 = \theta_7, \qquad \theta_2 = m\theta_6 + n\theta_5, \tag{4.56}$$

where $m$ and $n$ are integers. From (2.4), (2.5), and (4.56), the resonance conditions are satisfied if

$$k_5 = \frac{2b_x k_2}{(b_x - a)n}, \qquad k_6 = -\frac{(b_x + a)k_2}{(b_x - a)m}, \tag{4.57}$$

$$k_1 = k_2, \qquad k_6 = k_7. \tag{4.58}$$

In the following analysis, we assume that both the resonance conditions (4.57) and the auxilary conditions (4.58) on the wavenumbers are satisfied.

Using the results of Sec. 2, it is straightforward to write down the wave interaction equations in terms of the $\{a_j : \ j = 1, 2, 5, 6, 7\}$. Alternatively, for the case where there are no shocks in $\theta$, (3.13) and (3.14) yield the equation system

$$\frac{\partial \langle c_j \rangle}{\partial \bar{t}} + \lambda_j \frac{\partial \langle c_j \rangle}{\partial \bar{x}} = 0, \qquad j = 1, 2, 5, 6, 7, \tag{4.59}$$

$$\frac{\partial c_1'}{\partial \bar{t}} + \lambda_1 \frac{\partial c_1'}{\partial \bar{x}} + D_\theta(\tilde{\Gamma}_{115}\langle c_5 \rangle c_1' + \tilde{\Gamma}_{256}\langle c_5' c_7' \rangle_2) = 0, \tag{4.60}$$

$$\frac{\partial c_2'}{\partial \bar{t}} + \lambda_2 \frac{\partial c_2'}{\partial \bar{x}} + D_\theta(\tilde{\Gamma}_{225}\langle c_5 \rangle c_2' + \tilde{\Gamma}_{256}\langle c_5' c_6' \rangle_2) = 0, \tag{4.61}$$

$$\frac{\partial c_5'}{\partial \bar{t}} + \lambda_5 \frac{\partial c_5'}{\partial \bar{x}} + D_\theta[\tfrac{1}{2}\tilde{\Gamma}_{555} c_5'^{\,2} + \tilde{\Gamma}_{555}\langle c_5 \rangle c_5' + \tilde{\Gamma}_{256}\left(\langle c_1' c_7' \rangle_5 + \langle c_2' c_6' \rangle_5\right)] = 0, \tag{4.62}$$

$$\frac{\partial c_6'}{\partial \bar{t}} + \lambda_6 \frac{\partial c_6'}{\partial \bar{x}} + D_\theta(\tilde{\Gamma}_{665}\langle c_5 \rangle c_6' + \tilde{\Gamma}_{256}\langle c_5' c_2' \rangle_6) = 0, \tag{4.63}$$

$$\frac{\partial c_7'}{\partial \bar{t}} + \lambda_7 \frac{\partial c_7'}{\partial \bar{x}} + D_\theta(\tilde{\Gamma}_{775}\langle c_5 \rangle c_7' + \tilde{\Gamma}_{256}\langle c_5' c_1' \rangle_6) = 0 \tag{4.64}$$

for the wave means $\{\langle c_j \rangle\}$ and $c_j' = C_{j,\theta}$. The wave interaction coefficients in (4.60)–(4.64) are given by

$$\left.\begin{aligned} \tilde{\Gamma}_{256} &= \frac{(\omega_2\omega_5\omega_6)^{1/2}}{2a\rho^{1/2}}, \qquad \tilde{\Gamma}_{555} = \frac{(\gamma+1)}{2a\rho^{1/2}}\omega_5^{3/2}, \\ \tilde{\Gamma}_{115} &= \tilde{\Gamma}_{225} = -\frac{b_x + 2a}{2a\rho^{1/2}b_x}\omega_2\omega_5^{1/2}, \\ \tilde{\Gamma}_{665} &= \tilde{\Gamma}_{775} = \frac{2a - b_x}{2a\rho^{1/2}b_x}\omega_6\omega_5^{1/2}. \end{aligned}\right\} \tag{4.65}$$

Note that there is only a single resonant wave interaction coefficient $\tilde{\Gamma}_{256}$ in (4.60)–(4.64).

By using the MHD eigenvector relations ((2.6) et seq.), one can show that the magnetic field perturbations due to the fast magnetoacoustic waves and the Alfvén

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

waves may be written in the forms

$$\mathbf{B}_j = \begin{cases} \dfrac{a\mu^{1/2}}{\rho^{1/2}} a_j \boldsymbol{\beta}_\perp \equiv (\mu\omega_j)^{1/2} c_j \boldsymbol{\beta}_\perp \quad \text{if } j = 1, 7, \\ (\boldsymbol{\beta}_\perp \times \mathbf{e}_x) a_j \equiv (\mu\omega_j)^{1/2} c_j (\boldsymbol{\beta}_\perp \times \mathbf{e}_x), \quad \text{if } j = 2, 6, \end{cases} \tag{4.66}$$

where $\boldsymbol{\beta}_\perp = (0, B_y, B_z)^{\mathrm{T}}/B_\perp$. In the present case, we must consider the limit as $B_\perp = (B_y^2 + B_z^2)^{1/2} \to 0$. Thus the total magnetic field perturbation due to the backward fast-mode wave and the backward Alfvén wave ($\mathbf{B}_-$), and that due to the forward fast-mode and Alfvén waves ($\mathbf{B}_+$), are given by

$$\mathbf{B}_- = (\mu\omega_2)^{1/2}(c_1 \boldsymbol{\beta}_\perp + c_2 \boldsymbol{\beta}_\perp \times \mathbf{e}_x), \qquad \mathbf{B}_+ = (\mu\omega_6)^{1/2}(c_7 \boldsymbol{\beta}_\perp + c_6 \boldsymbol{\beta}_\perp \times \mathbf{e}_x), \tag{4.67}$$

It is evident from (4.67) that the Alfvén and fast magnetoacoustic magnetic field perturbations are mutually orthogonal (similarly, one can show that the slow-mode magnetic field perturbations are perpendicular to the Alfvén-wave magnetic field perturbations). Using the results (4.67), and noting that $\tilde{\Gamma}_{115} = \tilde{\Gamma}_{225}$ and $\tilde{\Gamma}_{665} = \tilde{\Gamma}_{775}$, (4.60)–(4.64) can be written in the more compact, symmetric form

$$\frac{\partial \mathbf{w}_2'}{\partial \bar{t}} + \lambda_2 \frac{\partial \mathbf{w}_2'}{\partial \bar{x}} + D_\theta(\tilde{\Gamma}_{225}\langle c_5 \rangle \mathbf{w}_2' + \tilde{\Gamma}_{256}\langle c_5' \mathbf{w}_6' \rangle_2) = 0, \tag{4.68}$$

$$\frac{\partial c_5'}{\partial \bar{t}} + \lambda_5 \frac{\partial c_5'}{\partial \bar{x}} + D_\theta(\tfrac{1}{2}\tilde{\Gamma}_{555} {c_5'}^2 + \tilde{\Gamma}_{555}\langle c_5 \rangle c_5' + \tilde{\Gamma}_{256}\langle \mathbf{w}_2' \cdot \mathbf{w}_6' \rangle_5) = 0, \tag{4.69}$$

$$\frac{\partial \mathbf{w}_6'}{\partial \bar{t}} + \lambda_6 \frac{\partial \mathbf{w}_6'}{\partial \bar{x}} + D_\theta(\tilde{\Gamma}_{665}\langle c_5 \rangle \mathbf{w}_6' + \tilde{\Gamma}_{256}\langle c_5' \mathbf{w}_2' \rangle_6) = 0, \tag{4.70}$$

where

$$\mathbf{w}_2' = (\mu\omega_2)^{-1/2} \mathbf{B}_-', \qquad \mathbf{w}_6' = (\mu\omega_6)^{-1/2} \mathbf{B}_+'. \tag{4.71}$$

A Hamiltonian variational formulation of (4.59)–(4.64) is given below:

**Proposition 9.** Equations (4.59)–(4.64) describing the resonant interaction of the fast magnetoacoustic waves ($c_1$ and $c_7$), the Alfvén waves ($c_2$ and $c_6$), and the forward slow-mode wave ($c_5$), applicable to the non-strictly hyperbolic case where the Alfvén and fast-mode wave speeds coincide, and in which the wave vectors $\mathbf{k} \parallel \mathbf{B}$, can be written in the Hamiltonian form

$$\frac{\partial c_s'}{\partial \bar{t}} = -D_\theta\left(\frac{\delta H}{\delta c_s'}\right), \qquad \frac{\partial \langle c_s \rangle}{\partial \bar{t}} = \frac{\delta H}{\delta \mu_s}, \qquad \frac{\partial \mu_s}{\partial \bar{t}} = -\frac{\delta H}{\delta \langle c_s \rangle}, \qquad s = 1, 2, 5, 6, 7, \tag{4.72}$$

where

$$H = \int_{-\infty}^{\infty} dx \int_0^{2\pi m} d\theta \left[ \tfrac{1}{2} \sum_s \lambda_s (D_\theta^{-1} D_x c_s') c_s' + \tfrac{1}{6}\tilde{\Gamma}_{555} {c_5'}^3 \right.$$
$$\left. + \tilde{\Gamma}_{256}\left(\langle c_1' c_7' \rangle_5 + \langle c_2' c_6' \rangle_5\right) c_5' + \tfrac{1}{2} \sum_s \tilde{\Gamma}_{ss5}\langle c_5 \rangle {c_s'}^2 - \sum_s \mu_s \lambda_s \frac{\partial \langle c_s \rangle}{\partial \bar{x}} \right] \tag{4.73}$$

is the Hamiltonian functional, and the sums over $s$ are for $s = 1, 2, 5, 6, 7$. It is also possible to write down the corresponding Lagrangian variational principle, in which the Lagrange multipliers $\{\mu_s\}$ ensure that the mean wave amplitude equations

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

(4.59) are satisfied. The evolution equations for the Lagrange multipliers $\{\mu_s\}$ are

$$\frac{\partial \mu_s}{\partial \bar{t}} + \lambda_s \frac{\partial \mu_s}{\partial \bar{x}} + \tfrac{1}{2}\delta^s{}_5 \sum_\sigma \tilde{\Gamma}_{\sigma\sigma s} c'^{\,2}_\sigma = 0, \qquad s = 1, 2, 5, 6, 7, \tag{4.74}$$

where the sum over $\sigma$ in (4.74) is over the values $\sigma = 1, 2, 5, 6, 7$.

One of the main differences between the non-strictly hyperbolic case in (4.72)–(4.74) and the strictly hyperbolic Hamiltonian (4.51) is the more complicated form of the resonant interaction terms in the non-strictly hyperbolic case. Further examples of equations describing resonant wave interactions for the non-strictly hyperbolic case where $\mathbf{k} \perp \mathbf{B}$ can also be obtained, but these equations will not be considered here.

## 5. Alfvén–magnetoacoustic wave interactions

As an example of resonant wave interactions, consider the interaction of the backward and forward Alfvén waves with the forward slow-mode wave. A similar example has been investigated by Galeev and Oraevski (1963) and Sagdeev and Galeev (1969) in their study of the Alfvén-wave decay instability due to three-wave resonant interactions in a low-beta plasma. We consider the strictly hyperbolic case when all the wave speeds are distinct in Sec. 5.1. Section 5.2 discusses the degenerate case $\mathbf{B} = (B_x, 0, 0)^{\mathrm{T}}$, in which the Alfvén and one of the magnetoacoustic wave speeds coincide.

### *5.1. The strictly hyperbolic case*

Consider the resonant triad $\{256\}$ consisting of the backward Alfvén wave ($\lambda_2 = -b_x$), forward Alfvén wave ($\lambda_6 = b_x$), and forward slow magnetoacoustic wave ($\lambda_5 = c_s$). The resonance conditions (4.41) and (4.42) are

$$\omega_2 = m\omega_6 + n\omega_5, \qquad k_2 = mk_6 + nk_5. \tag{5.1}$$

The resonance conditions (5.1) are satisfied if

$$k_2 = -\frac{(b_x - c_s)m}{b_x + c_s} k_6, \qquad k_5 = -\frac{2b_x m}{(b_x + c_s)n} k_6. \tag{5.2}$$

In particular, for the case $m = -1$ and $n = 1$, we obtain

$$k_2 = \frac{b_x - c_s}{b_x + c_s} k_6, \qquad k_5 = \frac{2b_x}{b_x + c_s} k_6, \qquad \theta_5 = \theta_2 + \theta_6, \tag{5.3}$$

The phase resonance condition $\theta_5 = \theta_2 + \theta_6$ corresponds to the collision of a backward Alfvén wave with a forward Alfvén wave, resulting in the generation of a forward slow-mode wave. Choosing $k_6 > 0$ in (5.3) implies that $k_2 > 0$ and $k_5 > 0$, which in turn implies that

$$\omega_2 < 0, \qquad \omega_5 > 0 \qquad \text{and} \qquad \omega_6 > 0. \tag{5.4}$$

Using (5.4), (3.2), and (3.3), it follows that the Hamiltonian variables $c_5$ and $c_6$ are real, but $c_2$ is purely imaginary. The resonance condition $\omega_5 = \omega_2 + \omega_6$ coupled with (5.4) implies that $\omega_6 > \omega_5 > 0 > \omega_2$ (note that $\omega_6 - \omega_5 = |\omega_2|$ and $\omega_6 > |\omega_2|$). Thus the frequency of the forward Alfvén wave, $\omega_6$, is greater in magnitude than both $|\omega_5|$ and $|\omega_2|$, which Sagdeev and Galeev (1969) argue is a necessary condition for the decay of the forward Alfvén wave.

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

From (3.13) and (3.14), the wave interaction equations for the resonant triad $\{256\}$ are

$$\left(\frac{\partial}{\partial \bar{t}} + \lambda_j \frac{\partial}{\partial \bar{x}}\right)\langle c_j\rangle = 0, \qquad j = 2,5,6, \tag{5.5}$$

$$\frac{\partial c_2'}{\partial \bar{t}} - b_x \frac{\partial c_2'}{\partial \bar{x}} + D_\theta(\tilde{\Gamma}_{225}\langle c_5\rangle c_2' + \tilde{\Gamma}_{256}\langle c_5' c_6'\rangle_2) = 0, \tag{5.6}$$

$$\frac{\partial c_5'}{\partial \bar{t}} + c_s \frac{\partial c_5'}{\partial \bar{x}} + D_\theta(\tfrac{1}{2}\tilde{\Gamma}_{555} {c_5'}^2 + \tilde{\Gamma}_{555}\langle c_5\rangle c_5' + \tilde{\Gamma}_{256}\langle c_2' c_6'\rangle_5) = 0, \tag{5.7}$$

$$\frac{\partial c_6'}{\partial \bar{t}} + b_x \frac{\partial c_6'}{\partial \bar{x}} + D_\theta(\tilde{\Gamma}_{665}\langle c_5\rangle c_6' + \tilde{\Gamma}_{256}\langle c_5' c_2'\rangle_6) = 0, \tag{5.8}$$

where the wave interaction coefficients $\tilde{\Gamma}_{256}$, $\tilde{\Gamma}_{555}$, $\tilde{\Gamma}_{225}$, and $\tilde{\Gamma}_{665}$ are given by

$$\tilde{\Gamma}_{256} = \frac{\alpha_s}{2a\rho^{1/2}}(\omega_2\omega_5\omega_6)^{1/2} = iH, \qquad \tilde{\Gamma}_{555} = \frac{\alpha_s\beta_s}{a\rho^{1/2}}\omega_5^{3/2},$$

$$\tilde{\Gamma}_{225} = -\frac{\alpha_s(b_x + 2c_s)}{2a\rho^{1/2}b_x}\omega_2\omega_5^{1/2}, \qquad \tilde{\Gamma}_{665} = \frac{\alpha_s(2c_s - b_x)}{2a\rho^{1/2}b_x}\omega_6\omega_5^{1/2}. \tag{5.9}$$

Using the Fourier series representation

$$c_j' = \sum_{n=-\infty}^{\infty} c_j^n(\bar{x},\bar{t})\exp(in\theta), \qquad j = 2,5,6, \tag{5.10}$$

(note that $c_j^0 = 0$, since $\langle c_j'\rangle = 0$), (5.6)–(5.8) yield the Fourier equations

$$\frac{\partial c_2^n}{\partial \bar{t}} - b_x \frac{\partial c_2^n}{\partial \bar{x}} + in\tilde{\Gamma}_{225}\langle c_5\rangle c_2^n - nHc_5^n c_6^{-n} = 0, \tag{5.11}$$

$$\frac{\partial c_5^n}{\partial \bar{t}} + c_s \frac{\partial c_5^n}{\partial \bar{x}} + in\left(\tilde{\Gamma}_{555}\langle c_5\rangle c_5^n + \tfrac{1}{2}\tilde{\Gamma}_{555}\sum_{q=-\infty}^{\infty} c_5^q c_5^{n-q}\right) - nHc_2^n c_6^n = 0, \tag{5.12}$$

$$\frac{\partial c_6^n}{\partial \bar{t}} + b_x \frac{\partial c_6^n}{\partial \bar{x}} + in\tilde{\Gamma}_{665}\langle c_5\rangle c_6^n - nHc_5^n c_2^{-n} = 0. \tag{5.13}$$

Equations (5.11)–(5.13) are used below to investigate the Alfvén-wave decay instability.

The above equations show that the waves are coupled to each other by the resonant wave interactions. The Burgers self-wave steepening term involving $D_\theta(\frac{1}{2}\tilde{\Gamma}_{555}{c_5'}^2)$ in (5.7), and the corresponding term in (5.12) for the slow-mode wave, considerably complicate the nature of the wave interactions, because they result in the generation of higher-order harmonics. However, if one considers the initial-value problem for (5.5)–(5.8), in which $|c_5'| \ll |c_6'|$, and $|c_2'| \ll |c_6'|$ at time $t = 0$, then one can neglect the role of the Burgers term, at least at early times. It is worth noting that the ratio of the Burgers and resonant interaction coefficients,

$$\Upsilon = \frac{\tilde{\Gamma}_{555}}{|\tilde{\Gamma}_{256}|} = 4\beta_s\left(\frac{c_s^2}{b_x^2 - c_s^2}\right)^{1/2}, \tag{5.14}$$

is roughly proportional to the plasma beta. Hence Burgers self-wave steepening for the slow magnetoacoustic wave will be small in the low-beta limit $0 < \beta \ll 1$. Assuming $\langle c_5\rangle$ = const, and that only the first-harmonic components are present

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

in the expansion (5.10), and neglecting the Burgers terms, the system (5.11)–(5.13) reduces to the simpler system

$$\left(\frac{\partial}{\partial \bar{t}} - b_x \frac{\partial}{\partial \bar{x}} + i\tilde{\Gamma}_{225}\langle c_5\rangle\right) c_2^1 - H c_5^1 c_6^{-1} = 0, \tag{5.15}$$

$$\left(\frac{\partial}{\partial \bar{t}} + c_s \frac{\partial}{\partial \bar{x}} + i\tilde{\Gamma}_{555}\langle c_5\rangle\right) c_5^1 - H c_2^1 c_6^1 = 0, \tag{5.16}$$

$$\left(\frac{\partial}{\partial \bar{t}} + b_x \frac{\partial}{\partial \bar{x}} + i\tilde{\Gamma}_{665}\langle c_5\rangle\right) c_6^1 - H c_5^1 c_2^{-1} = 0, \tag{5.17}$$

where $c_5^{-1} = (c_5^1)^*$, $c_6^{-1} = (c_6^1)^*$, and $c_2^{-1} = -(c_2^1)^*$. The system (5.15)–(5.17) with $\langle c_5\rangle = 0$ is a version of the three-wave resonant interaction equations studied by Kaup et al. (1979) and others. These equations admit a large number of exact solutions, and are solvable by the inverse scattering transform.

As part of a study of the decay instability, plane-wave solutions of (5.15)–(5.17) are considered below. To obtain the solutions, we introduce the wave amplitudes

$$\tilde{c}_j^1 = c_j^1 \exp(i\tilde{\Gamma}_{jj5}\langle c_5\rangle \bar{t}), \qquad j = 2, 5, 6. \tag{5.18}$$

In terms of the $\tilde{c}_j^1$, (5.15)–(5.17), reduce to the more symmetric equation system

$$\left(\frac{\partial}{\partial \bar{t}} - b_x \frac{\partial}{\partial \bar{x}}\right) \tilde{c}_2^1 - \tilde{H}\tilde{c}_5^1 (\tilde{c}_6^1)^* = 0, \tag{5.19}$$

$$\left(\frac{\partial}{\partial \bar{t}} + c_s \frac{\partial}{\partial \bar{x}}\right) \tilde{c}_5^1 - \tilde{H}^* \tilde{c}_2^1 \tilde{c}_6^1 = 0, \tag{5.20}$$

$$\left(\frac{\partial}{\partial \bar{t}} + b_x \frac{\partial}{\partial \bar{x}}\right) \tilde{c}_6^1 + \tilde{H}\tilde{c}_5^1 (\tilde{c}_2^1)^* = 0, \tag{5.21}$$

where

$$\tilde{H} = H \exp(i\Phi\bar{t}), \qquad \Phi = (\tilde{\Gamma}_{225} + \tilde{\Gamma}_{665} - \tilde{\Gamma}_{555})\langle c_5\rangle, \tag{5.22}$$

defines a modified form of the three-wave resonant interaction coefficient $H$.

*5.1.1. Exact plane-wave solutions.* Equations (5.19)–(5.21) admit plane-wave solutions of the form

$$\tilde{c}_j^1 = \hat{c}_j^1(\bar{t}) \exp[i(K_j \bar{x} - \nu_j \bar{t})], \qquad j = 2, 5, 6, \tag{5.23}$$

where the $\nu_j$ and $K_j$ satisfy the resonance conditions

$$K_5 = K_2 + K_6, \tag{5.24}$$

$$\nu_5 = \nu_2 + \nu_6 + \Phi, \tag{5.25}$$

where for exact solutions it is necessary that the frequencies $\{\nu_j : \ j = 2, 5, 6\}$ be all real.

The resonance conditions (5.24) and (5.25) are similar to the fast phase resonance conditions (5.1). Note that if $\langle c_5\rangle = 0$ then $\Phi = 0$. If $\langle c_5\rangle \neq 0$ then, in general, $\Phi \neq 0$, and the frequency-matching condition is modified owing to a non-zero mean wave field $\langle c_5\rangle$. The complex amplitudes $\hat{c}_j^1$ must satisfy the equations

$$\frac{d\hat{c}_2^1}{d\bar{t}} - i(\nu_2 + K_2 b_x)\hat{c}_2^1 - H\hat{c}_5^1(\hat{c}_6^1)^* = 0, \tag{5.26}$$

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

$$\frac{d\hat{c}_5^1}{d\bar{t}} - i(\nu_5 - K_5 c_s)\hat{c}_5^1 - H\hat{c}_2^1\hat{c}_6^1 = 0, \tag{5.27}$$

$$\frac{d\hat{c}_6^1}{d\bar{t}} - i(\nu_6 - K_6 b_x)\hat{c}_6^1 + H\hat{c}_5^1(\hat{c}_2^1)^* = 0. \tag{5.28}$$

In general, one can obtain a self-consistent solution of (5.24)–(5.28) in which the $\{\hat{c}_j^1 : \ j = 2, 5, 6\}$ are constants by specifying five parameters. In the present analysis, we specify $\hat{c}_2^1$, $\hat{c}_6^1$, $K_2$, and $K_6$, associated with the backward and forward Alfvén waves, and $\langle c_5 \rangle$.

One can also obtain exact solutions of (5.26)–(5.28) in which the $\hat{c}_j^1$ are functions of $\bar{t}$ in terms of Jacobian elliptic functions for the case where $\nu_j = \lambda_j K_j$ ($j = 2, 5, 6$) and $\langle c_5 \rangle = 0$ (Sagdeev and Galeev 1969).

Assuming that $\{|\hat{c}_j^1| \neq 0 : \ j = 2, 5, 6\}$ are constants, (5.26)–(5.28) can be combined to yield the equations

$$(\nu_2 + K_2 b_x)(\nu_5 - K_5 c_s) + H^2|\hat{c}_6^1|^2 = 0, \tag{5.29}$$

$$(\nu_6 - K_6 b_x)(\nu_5 - K_5 c_s) - H^2|\hat{c}_2^1|^2 = 0 \tag{5.30}$$

(these equations can be obtained from the $2 \times 2$ determinants in (5.26)–(5.28)). From (5.29) and (5.30), we obtain

$$\nu_6 - K_6 b_x = -(\nu_2 + K_2 b_x)\eta, \tag{5.31}$$

where

$$\eta = |\hat{c}_2^1|^2/|\hat{c}_6^1|^2 \tag{5.32}$$

measures the ratio of the amplitudes of the backward and forward Alfvén waves. Using the resonance conditions (5.24) and (5.25), and (5.31), in (5.29) yields the dispersion equation

$$(1 - \eta)\nu_2^2 + [K_2 b_x(1 - \eta) + \zeta]\nu_2 + K_2 b_x \zeta + H^2|\hat{c}_6^1|^2 = 0 \tag{5.33}$$

for $\nu_2$, where

$$\zeta = \Phi + K_6(b_x - c_s) - K_2(c_s + b_x\eta). \tag{5.34}$$

Equation (5.33) has solutions $\nu_2 = \nu_2^{\pm}$ of the form

$$\nu_2^{\pm} = \frac{K_2(c_s - b_x + 2\eta b_x) - \alpha \pm (b_x + c_s)[(K_2 - K_2^-)(K_2 - K_2^+)]^{1/2}}{2(1 - \eta)}, \tag{5.35}$$

where

$$\alpha = \Phi + K_6(b_x - c_s), \tag{5.36}$$

$$K_2^{\pm} = \frac{\alpha \pm 2H|\hat{c}_6^1|(1 - \eta)^{1/2}}{b_x + c_s}. \tag{5.37}$$

The solutions for $\nu_2^{\pm}$ in (5.35) are real if $K_2 > K_2^+$ or if $K_2 < K_2^-$, and yield exact solutions of the three-wave resonant interaction equations (5.19)–(5.21). Once $\nu_2$ is determined from (5.35), the solutions for $\nu_6$ and $\nu_5$ can then be determined from (5.31) and (5.25).

*5.1.2. Linear instability growth rate estimates.* If $|\hat{c}_2| \ll |\hat{c}_6|$ and $|\hat{c}_5| \ll |\hat{c}_6|$, and the $\{\hat{c}_j^1\}$ are constants, then (5.21) may be linearized by dropping the resonant interaction term in (5.21) in a first approximation. In this limit, an approximate solution of (5.21) is

$$\tilde{c}_6^1 \approx \hat{c}_6^1 \exp[i(K_6\bar{x} - \nu_6\bar{t})], \qquad \nu_6 = K_6 b_x. \tag{5.38}$$

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

Using this approximate solution in (5.19) and (5.20) then yields a linear system of equations for $\tilde{c}_2^1$ and $\tilde{c}_5^1$. This linear system of equations has plane-wave solutions of the form (5.23) for $\tilde{c}_2^1$ and $\tilde{c}_5^1$. Assuming that $\hat{c}_2^1$ and $\hat{c}_5^1$ are constants, the solutions for the frequencies $\nu_2$ and $\nu_5$ can be determined from (5.35) and (5.25) provided that $0 < \eta \ll 1$ (note that $\nu_2$ and $\nu_5$ are now allowed to be complex). Thus the solutions for $\nu_2^{\pm}$ in (5.35) are complex if $K_2$ lies in the wavenumber range

$$K_2^- < K_2 < K_2^+. \tag{5.39}$$

In the unstable wavenumber regime (5.39), the growth rates $\{\Gamma_j = \Im(\nu_j^+) : \ j = 2, 5\}$ are given by

$$\Gamma_2 = \tfrac{1}{2}(b_x + c_s)[(K_2^+ - K_2)(K_2 - K_2^-)]^{1/2}, \qquad \Gamma_5 = \Gamma_2, \tag{5.40}$$

where it is understood that $0 < \eta \ll 1$.

Equations (5.40) yield results for the decay instability similar to those obtained by Sagdeev and Galeev (1969). Thus, for example, if $K_6 = \langle c_5 \rangle = 0$ then $\alpha = \Phi = 0$, and the waves are unstable for

$$|K_2| < K_c \equiv \frac{2H|\hat{c}_6^1|}{b_x + c_s}. \tag{5.41}$$

The maximum growth rate for the backward Alfvén wave occurs when $K_2 = K_{2m} = \frac{1}{2}(K_2^+ + K_2^-)$, and has the value

$$\Gamma_{2m} = H|\hat{c}_6^1|, \qquad \text{and} \qquad K_{2m} = \frac{\alpha}{b_x + c_s}. \tag{5.42}$$

The decay behaviour of the large-amplitude forward Alfvén wave in this approximation can be obtained by setting

$$\tilde{c}_6^1 \approx [\hat{c}_6^1 + \delta\hat{c}_6^1(\bar{t})] \exp[i(K_6\bar{x} - \nu_6\bar{t})] \tag{5.43}$$

in (5.21) to obtain

$$\delta\hat{c}_6^1 = -\frac{H\hat{c}_5^1(\hat{c}_2^1)^*[\exp(2\Gamma_2\bar{t}) - 1]}{2\Gamma_2}, \tag{5.44}$$

where $\Gamma_2 = \Im(\nu_2^+)$ and the superscript $*$ in (5.44) denotes complex conjugation. Using the dispersion equation (5.33) and (5.27) and assuming that $0 < \eta \ll 1$, (5.27) yields the result

$$\hat{c}_5^1 = i\frac{H\hat{c}_2^1\hat{c}_6^1(\nu_2^- + \zeta)}{H^2|\hat{c}_6^1|^2} \tag{5.45}$$

for $\hat{c}_5^1$. In the limit as $\eta \to 0$, (5.45) implies that $|\hat{c}_5^1| = |\hat{c}_2^1|$. The solution (5.43) for $\tilde{c}_6^1$ may be used to show that the Manley–Rowe relations

$$|\tilde{c}_6^1|^2 + |\tilde{c}_2^1|^2 = M_0, \qquad |\tilde{c}_6^1|^2 + |\tilde{c}_5^1|^2 = M_1 \tag{5.46}$$

are satisfied to $O(\eta)$ if $\Gamma_2 \approx H|\hat{c}_6^1|$.

It is useful to express the results (5.35)–(5.42) in a more physical form. To this end, first note from (3.2) and (3.3) that

$$|\hat{c}_6^1| = \frac{\delta B_6}{2(\mu\omega_6)^{1/2}}, \tag{5.47}$$

where $\delta B_6$ is the maximum magnetic field perturbation for the forward Alfvén wave.

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

Using the resonance relations (5.3), and noting that $H = |\tilde{\Gamma}_{256}|$, we find

$$K_c = k_6 \frac{\alpha_s b_x b}{2^{1/2}(b_x + c_s)^2} \left[\frac{c_s(b_x - c_s)}{a^2}\right]^{1/2} \frac{\delta B_6}{B_0} \tag{5.48}$$

for the critical wavenumber $K_c$ in (5.41). Simlarly, from (5.42), the maximum growth rate $\Gamma_{2m}$ of the slow-mode wave is given by the formula

$$\Gamma_{2m} = H|\hat{c}_6^1| = \omega_6 \frac{\alpha_s b}{2^{3/2}(b_x + c_s)} \left[\frac{c_s(b_x - c_s)}{a^2}\right]^{1/2} \frac{\delta B_6}{B_0}. \tag{5.49}$$

In the limiting case of waves propagating parallel to $\mathbf{B}$, in a low-beta plasma, with $0 < \beta \equiv a^2/b^2 < 1$ (note that this definition of $\beta$ differs from the usual definition of the plasma beta as $\beta_p = p/(B^2/2\mu)$; in fact $\beta = \frac{1}{2}\gamma_g\beta_p$, where $\gamma_g$ is the adiabatic index of the gas), (5.49) yields the formula

$$\Gamma_{2m} = \frac{\omega_6}{2^{3/2}} \frac{(1 - \beta^{1/2})^{1/2}}{1 + \beta^{1/2}} \beta^{-1/4} \frac{\delta B_6}{B_0}. \tag{5.50}$$

For $\beta \ll 1$, (5.50) yields

$$\Gamma_{2m} \approx \frac{\omega_6}{2^{3/2}} \beta^{-1/4} \frac{\delta B_6}{B_0}. \tag{5.51}$$

This is similar to the growth rate $\Gamma_2^{(SG)}$ for the forward sound wave in the limit $\beta \to 0$ given by Sagdeev and Galeev (1969), namely

$$\Gamma_2^{(SG)} \approx 2^{1/2} \omega_6 \beta^{-1/4} \frac{\delta B_\perp^{(SG)}}{B_0}. \tag{5.52}$$

The difference between the growth rates (5.51) and (5.52) is due in part to the fact that Sagdeev and Galeev (1969) use a resonant interaction coefficient $\tilde{\Gamma}_{256}^{(SG)} = 2\tilde{\Gamma}_{256}$ (see also Chin and Wentzel (1972), who use the same resonant interaction coefficient value that we use). An extra factor of two arises from the fact that $\delta B_\perp^{(SG)} = 2\delta B_6$. This accounts for the factor-of-four difference between the numerical coefficients $2^{-3/2}$ and $2^{1/2}$ in the growth rates in (5.51) and (5.52).

The growth rate (5.50) only applies for parallel-propagating waves (note that the case of parallel propagation is not strictly hyperbolic: further discussion of this point is given in Sec. 5.2). More generally, the growth rate (5.49) for waves travelling at an angle $\Theta$ to $\mathbf{B}$ can be expressed in the form

$$\Gamma_{2m} = \frac{k_6 b \cos\Theta}{2^{3/2}} \left[\frac{g(g - \beta)}{g^2 - \beta\cos^2\Theta}\right]^{1/2} \frac{[1 - (\beta/g)^{1/2}]^{1/2}}{[1 + (\beta/g)^{1/2}]\beta^{1/4} g^{1/4}} \frac{\delta B_6}{B_0}, \tag{5.53}$$

where

$$g(\beta, \Theta) = \frac{c_f^2}{b^2} = \tfrac{1}{2}\{1 + \beta + [(1 + \beta)^2 - 4\beta\cos^2\Theta]^{1/2}\} \tag{5.54}$$

is the square of the ratio of the fast-mode speed $c_f$ to the total Alfvén speed $b = B_0/(\mu\rho)^{1/2}$. One can also show, using (5.48), that the waves are unstable for $|K_2| < K_c$, where

$$K_c = \frac{k_6}{2^{1/2}} \left[\frac{g(g - \beta)}{g^2 - \beta\cos^2\Theta}\right]^{1/2} \frac{[1 - (\beta/g)^{1/2}]^{1/2}}{[1 + (\beta/g)^{1/2}]^2 \beta^{1/4} g^{1/4}} \frac{\delta B_6}{B_0}. \tag{5.55}$$

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

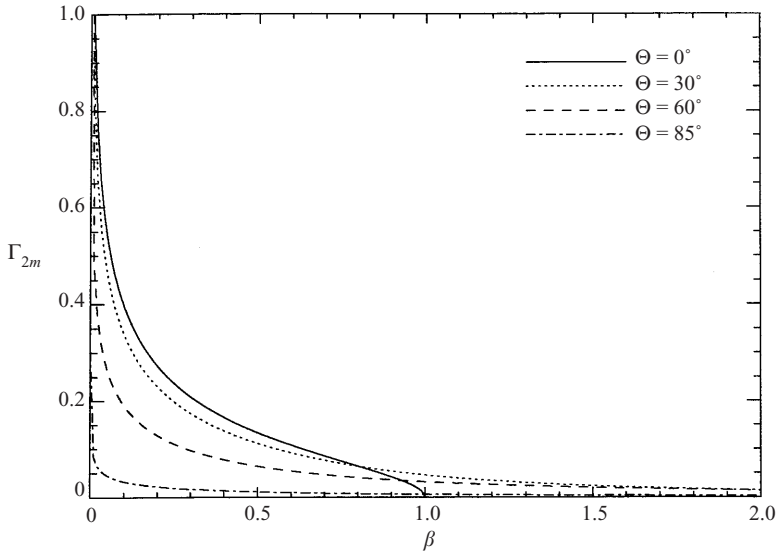



**Figure 1.** The maximum growth rate $\Gamma_{2m}$, (5.53), for the backward Alfvén wave ($c_2$) due to three-wave resonant interaction. The waves involved in the interaction are the backward Alfvén wave (wave 2), the forward Alfvén wave (wave 6), and the forward slow-mode wave (wave 5). $\Gamma_{2m}$ is shown as a function of the plasma beta ($\beta = a^2/b^2$) for a range of propagation angles $\Theta$ of the waves relative to the background magnetic field **B**.

Plots of the normalized growth rates $\Gamma_{2m}/(k_6 b \delta B_6/B_0)$ in (5.53) as functions of $\beta$ and for a range of $\Theta$ ($\Theta = 0°$, $30°$, $60°$, and $85°$) are shown in Fig. 1. The main points to note are the divergence of the growth rate as $\beta \to 0$ like $\beta^{-1/4}$ and the different behaviour of $\Gamma_{2m}$ for the special case $\Theta = 0$. In the $\Theta = 0$ case, there is only a non-zero growth rate for $0 < \beta < 1$, whereas there is a non-zero growth rate for any positive $\beta$ for the $\Theta \neq 0$ cases. The growth rate tends to zero as $\Theta \to 90°$, owing to the $\cos\Theta$ factor in (5.53). The growth rate decreases at very large $\beta$ like $\beta^{-3/2}$.

The normalized cutoff wavenumber $K_c/(k_6 \delta B_6/B_0)$ in (5.55) is shown in Fig. 2 as a function of $\beta$ for a range of propagation angles $\Theta$. For the case $\Theta = 0$, $\beta$ is restricted to the range $0 < \beta < 1$. Since the unstable waves have wavenumbers in the range $0 < |K_2| < K_c$, the figure shows that even relatively short-wavelength waves are unstable at small $\beta \ll 1$, but only very long waves are unstable at large $\beta$.

The dependence of the normalized growth rate $\Gamma_2/(k_6 b\, \delta B_6/B_0)$ of (5.40) on the wavenumber $K_2/k_6$ (i.e. on setting $k_6 b\, \delta B_6/B_0 = 1$ and $k_6 = 1$ in (5.40)) for the case $\Theta = 30°$ and a range of betas ($\beta = 0.05$, 0.2, 0.5, and 1.0) is illustrated in Fig. 3. The growth rate is maximal at $K_2 = 0$ and $\Gamma_2 \to 0$ as $K_2 \to K_c$.

The illustrative calculations in Figs 1–3 pertain to the case where the pump Alfvén wave has $K_6 = 0$. Further calculations could also be carried out illustrating the effects of varying $K_6 \neq 0$ and $\langle c_5 \rangle \neq 0$.

### *5.2. The degenerate case* $\mathbf{k} \parallel \mathbf{B}$

Consider the three-wave resonant interaction of the forward slow-mode wave ($c_5$) with the backward and forward Alfvén waves ($c_2$ and $c_6$) in the limit that $\mathbf{B}_\perp =$

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

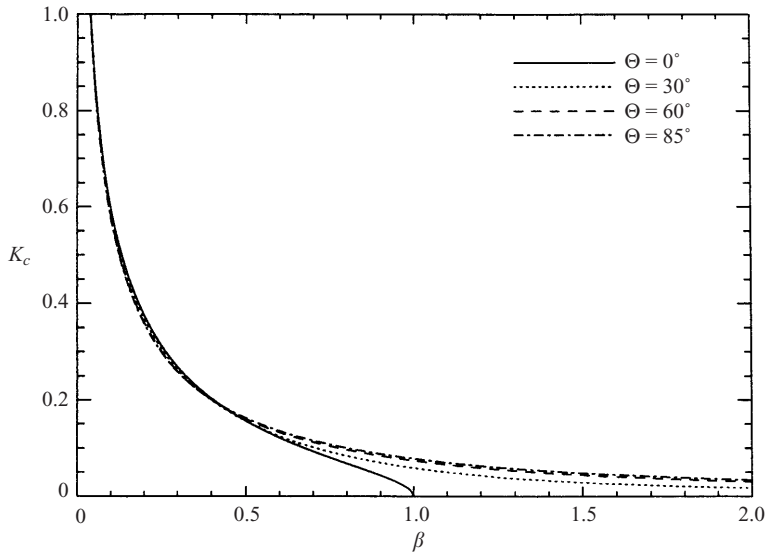



**Figure 2.** The cutoff wavenumber $K_c$, (5.55), associated with the resonant wave triad {256} described in Fig. 1. The small-amplitude forward slow-mode wave ($c_5$) and backward Alfvén wave ($c_2$) are unstable and exhibit wave growth for $|K_2| < K_c$, where $K_2$ is the envelope wavenumber for the backward Alfvén wave. The waves are stable for $|K_2| > K_c$.

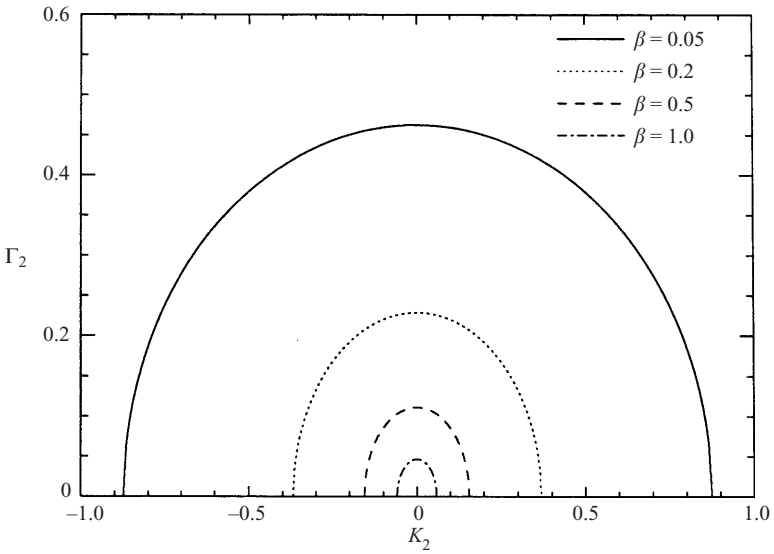



**Figure 3.** The linear growth rate $\Gamma_2$, (5.40), for the small-amplitude backward propagating Alfvén wave ($c_2$) due to three-wave resonant interaction involving the resonant wave triad {256} described in Fig. 1. The figure shows the variation of $\Gamma_2$ with the envelope wavenumber $K_2$ of the backward Alfvén wave for $\Theta = 30^\circ$, and a range of plasma betas ($\beta = 0.05$, 0.2, 0.5 and 1).

$(0, B_y, B_z)^{\mathrm{T}} \to 0$, in a low-beta plasma in which $b_x > a$. As discussed in Sec. 4.4, the fast-mode waves can then also participate in the wave interactions if $k_1 = k_2$ and $k_7 = k_6$. In this limit, $\lambda_1 = \lambda_2 = -b_x$ and $\lambda_7 = \lambda_6 = b_x$ are the degenerate eigenvalues for the fast-mode and Alfvén waves. The resonant wave interactions in

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

this case are given by (4.68)–(4.70), namely

$$\frac{\partial \mathbf{w}_2'}{\partial \bar{t}} - b_x \frac{\partial \mathbf{w}_2'}{\partial \bar{x}} + D_\theta(\tilde{\Gamma}_{225}\langle c_5\rangle \mathbf{w}_2' + \tilde{\Gamma}_{256}\langle c_5' \mathbf{w}_6'\rangle_2) = 0, \tag{5.56}$$

$$\frac{\partial c_5'}{\partial \bar{t}} + c_s \frac{\partial c_5'}{\partial \bar{x}} + D_\theta(\tfrac{1}{2}\tilde{\Gamma}_{555} c_5'^{\,2} + \tilde{\Gamma}_{555}\langle c_5\rangle c_5' + \tilde{\Gamma}_{256}\langle \mathbf{w}_2' \cdot \mathbf{w}_6'\rangle_5) = 0, \tag{5.57}$$

$$\frac{\partial \mathbf{w}_6'}{\partial \bar{t}} + b_x \frac{\partial \mathbf{w}_6'}{\partial \bar{x}} + D_\theta(\tilde{\Gamma}_{665}\langle c_5\rangle \mathbf{w}_6' + \tilde{\Gamma}_{256}\langle c_5' \mathbf{w}_2'\rangle_6) = 0, \tag{5.58}$$

where

$$\mathbf{w}_2' = c_1' \mathbf{e}_1 + c_2' \mathbf{e}_2, \qquad \mathbf{w}_6' = c_7' \mathbf{e}_1 + c_6' \mathbf{e}_2. \tag{5.59}$$

Equations (5.56)–(5.58) have the same mathematical form as the strictly hyperbolic case in (5.6)–(5.8) if the replacements

$$c_2' \to \mathbf{w}_2' \qquad \text{and} \qquad c_6' \to \mathbf{w}_6' \tag{5.60}$$

are made in (5.6)–(5.8). This implies that the growth-rate formulae (5.40) and the dispersion equation (5.33) for three-wave interactions also apply in the present case, provided that the substitutions

$$\left.\begin{array}{ll} \delta B_6 \to (\delta B_6^2 + \delta B_7^2)^{1/2}, & |\hat{c}_6^1| \to (|\hat{c}_6^1|^2 + |\hat{c}_7^1|^2)^{1/2}, \\ \delta B_2 \to (\delta B_2^2 + \delta B_1^2)^{1/2}, & |\hat{c}_2^1| \to (|\hat{c}_2^1|^2 + |\hat{c}_1^1|^2)^{1/2} \end{array}\right\} \tag{5.61}$$

are made in the strictly hyperbolic equations.

Whereas the strictly hyperbolic equations apply to linearly polarized waves, the non-strictly hyperbolic case allows for the possibility of circularly polarized waves or elliptically polarized waves for $\mathbf{w}_2'$ and $\mathbf{w}_6'$. To make this point more obvious, we first note from the MHD eigenvector relations that

$$\left.\begin{array}{l} \delta\mathbf{B}_{12} = \delta\mathbf{B}_1 + \delta\mathbf{B}_2 = (\mu\omega_2)^{1/2}\mathbf{w}_2', \qquad \delta\mathbf{u}_{12} = b_x \dfrac{\delta\mathbf{B}_{12}}{B_0}, \\ \delta\mathbf{E}_{12} = \delta\mathbf{E}_1 + \delta\mathbf{E}_2 = (\mu\omega_2)^{1/2} b_x (c_2'\mathbf{e}_1 - c_1'\mathbf{e}_2) \end{array}\right\} \tag{5.62}$$

give the magnetic field ($\delta\mathbf{B}_{12}$), velocity ($\delta\mathbf{u}_{12}$), and electric field ($\delta\mathbf{E}_{12}$) perturbations of the backward Alfvén and fast-mode waves, where

$$\mathbf{e}_1 = (0, \beta_y, \beta_z)^{\mathrm{T}}, \qquad \mathbf{e}_2 = (0, \beta_z, -\beta_y)^{\mathrm{T}}, \tag{5.63}$$

are mutually orthonormal vectors associated with the transverse magnetic field (note that $\boldsymbol{\beta}_\perp = (0, B_y, B_z)^{\mathrm{T}}/B_\perp$). For parallel propagation, we consider the limit as $B_\perp = (B_y^2 + B_z^2)^{1/2} \to 0$ in (5.63). Formulae similar to (5.62) apply for the perturbations $\delta\mathbf{B}_{67} = \delta\mathbf{B}_6 + \delta\mathbf{B}_7$, $\delta\mathbf{u}_{67}$, and $\delta\mathbf{E}_{67} = \delta\mathbf{E}_6 + \delta\mathbf{E}_7$ for the forward, degenerate Alfvén, and fast-mode waves by replacing $b_x \to -b_x$, $c_1' \to c_7'$, and $c_2' \to c_6'$ in (5.62). Without loss of generality, we can choose coordinates so that $\mathbf{e}_1 = (0, 0, 1)^{\mathrm{T}}$ and $\mathbf{e}_2 = (0, 1, 0)^{\mathrm{T}}$, corresponding to the choice $\beta_y = 0$ and $\beta_z = 1$ in (5.63). The polarization vector of the waves is conventionally defined in terms of the electric field perturbations $\delta\mathbf{E}_{12}$ and $\delta\mathbf{E}_{67}$ of the waves. The slow-mode wave in the present case is a sound wave with zero electric and magnetic field perturbations. Proceeding as in Sec. 5.1, we neglect Burgers self-wave steepening, and assume that only the first harmonics of the fast phase oscillations are present. The time-independent

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

equations analogous to (5.26)–(5.28) are

$$i(\nu_2 + K_2 b_x)\hat{\mathbf{w}}_2^1 + H\hat{c}_5^1(\hat{\mathbf{w}}_6^1)^* = 0, \tag{5.64}$$

$$i(\nu_5 - K_5 c_s)\hat{c}_5^1 + H\hat{\mathbf{w}}_2^1 \cdot \hat{\mathbf{w}}_6^1 = 0, \tag{5.65}$$

$$i(\nu_6 - K_6 b_x)\hat{\mathbf{w}}_6^1 - H\hat{c}_5^1(\hat{\mathbf{w}}_2^1)^* = 0, \tag{5.66}$$

where

$$\hat{\mathbf{w}}_2^1 = \hat{c}_1^1\mathbf{e}_1 + \hat{c}_2^1\mathbf{e}_2, \qquad \hat{\mathbf{w}}_6^1 = \hat{c}_7^1\mathbf{e}_1 + \hat{c}_6^1\mathbf{e}_2, \tag{5.67}$$

and $\hat{c}_5^1$ are the constant wave amplitudes analogous to (5.23). For circularly polarized backward-propagating degenerate Alfvén and fast-mode waves,

$$\hat{c}_1^1 = i\sigma\hat{c}_2^1, \qquad \sigma = \pm 1. \tag{5.68}$$

The algebraic constraint equations (5.64)–(5.66) require that

$$\hat{c}_7^1 = -i\sigma\hat{c}_6^1. \tag{5.69}$$

Using the results (5.68) and (5.69), we obtain

$$\left.\begin{aligned} \delta\hat{\mathbf{E}}_{12} &= (\mu\omega_2)^{1/2} b_x \hat{c}_2^1(\mathbf{e}_1 - i\sigma\mathbf{e}_2), \\ \delta\hat{\mathbf{E}}_{67} &= -(\mu\omega_6)^{1/2} b_x \hat{c}_6^1(\mathbf{e}_1 + i\sigma\mathbf{e}_2) \end{aligned}\right\} \tag{5.70}$$

for the electric field perturbations of the degenerate backward- and forward-propagating waves. Assuming that $\Re(\nu_2)\Re(\nu_6) < 0$, (5.70) imply that the degenerate backward- and forward-propagating waves have the same sense of rotation about $\mathbf{B}$. Thus, there are restrictions on the polarization of the waves in order that three-wave resonant interactions should occur.

## 6. Resonant, Alfvén–entropy-wave solutions

In this section, we obtain exact analytical solutions of (4.31) and (4.32) describing the resonant interaction of the backward and forward Alfvén waves ($a_2$ and $a_6$) with the entropy wave ($a_4$). The analysis is restricted to the simplest resonant interaction case when $m = 1$ and $n = -1$, so that the relationship between the wave phases is

$$\theta_4 = \theta_6 - \theta_2. \tag{6.1}$$

From (4.30), the resonance conditions (6.1) are satisfied if

$$k_6 = \tfrac{1}{2}k_4, \qquad k_2 = -\tfrac{1}{2}k_4. \tag{6.2}$$

The conditions (6.2) imply that the wavelengths of the waves $\{l_j : \ j = 2, 4, 6\}$ are related by $l_6 = 2l_4$ and $l_2 = 2l_4$, where $l_j = 2\pi/|k_j|$. Thus the wavelengths of the backward and forward Alfvén waves ($l_2$ and $l_6$) are twice that of the entropy wave ($l_4$). For $m = 1$ and $n = -1$, the frequencies $\omega_2$ and $\omega_6$ of the backward and forward Alfvén waves are the same, and the entropy-wave frequency $\omega_4 = 0$. Thus

$$\omega_2 = \omega_6 = \Omega, \qquad \omega_4 = 0, \tag{6.3}$$

are the conditions on the wave frequencies in the $m = 1$ and $n = -1$ resonant case.

Assuming that the mean wave amplitude for the entropy wave $\langle a_4\rangle = \text{const}$, the Alfvén-wave equations (4.31) and (4.32) for $m = 1$ and $n = -1$ can be written in

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

the form

$$\frac{d\bar{a}_2}{dt_2} + \tfrac{1}{2}\Omega D_{\tilde{\theta}}\left(\langle \bar{a}_4 \bar{a}_6 \rangle_2\right) = 0, \tag{6.4}$$

$$\frac{d\bar{a}_6}{dt_6} + \tfrac{1}{2}\Omega D_{\tilde{\theta}}\left(\langle \bar{a}_4 \bar{a}_2 \rangle_6\right) = 0, \tag{6.5}$$

where

$$\bar{a}_2 = \frac{a_2}{B}, \qquad \bar{a}_6 = \frac{a_6}{B}, \qquad \bar{a}_4 = \frac{\rho_4}{\rho} \equiv \bar{\rho}_4 \tag{6.6}$$

are dimensionless wave amplitudes,

$$\frac{d}{dt_j} = \frac{\partial}{\partial \bar{t}} + \lambda_j \frac{\partial}{\partial \bar{x}}, \qquad j = 2, 6, \tag{6.7}$$

are time derivatives along the Alfvén-wave characteristics (note that $\lambda_2 = -b_x$ and $\lambda_6 = b_x$), and

$$\tilde{\theta} = \theta + \tfrac{1}{2}\langle \bar{a}_4 \rangle \Omega \bar{t} \tag{6.8}$$

is a modified form of the wave phase variable $\theta$.

The resonant wave interaction terms in (6.4) and (6.5) involve the terms

$$\begin{aligned} \langle \bar{a}_4 \bar{a}_6 \rangle_2 &= \frac{1}{2\pi} \int_0^{2\pi} \bar{a}_4(\xi - \tilde{\theta}) \bar{a}_6(\xi)\, d\xi, \\ \langle \bar{a}_4 \bar{a}_2 \rangle_6 &= \frac{1}{2\pi} \int_0^{2\pi} \bar{a}_4(\tilde{\theta} - \xi) \bar{a}_2(\xi)\, d\xi. \end{aligned} \tag{6.9}$$

Using (6.9), (6.4) and (6.5) can be written in the forms

$$\frac{d\bar{a}_2}{dt_2} - \frac{\Omega}{4\pi} \int_0^{2\pi} K(\xi - \tilde{\theta}) \bar{a}_6(\xi)\, d\xi = 0, \tag{6.10}$$

$$\frac{d\bar{a}_6}{dt_6} + \frac{\Omega}{4\pi} \int_0^{2\pi} K(\tilde{\theta} - \xi) \bar{a}_2(\xi)\, d\xi = 0, \tag{6.11}$$

where

$$K(\theta) = \frac{\partial \bar{a}_4(\theta)}{\partial \theta} \tag{6.12}$$

is the derivative of the entropy-wave density profile. Note that in (6.9)–(6.12), we are using the modified phase variables $\tilde{\theta}_2$ and $\tilde{\theta}_6$ to describe the phase dependence of the Alfvén waves, but the non-modified phase variable $\theta_4$ is used to describe the entropy wave. This use of different phase variables for the Alfvén and entropy waves preserves the resonance condition (6.1) for the wave phases in (6.1) (i.e. $\theta_4 = \tilde{\theta}_6 - \tilde{\theta}_2$), and simplifies the solution derivation in the case where $\langle \bar{\rho}_4 \rangle \neq 0$. It is of interest to note that the resonant wave interaction terms in (6.10) and (6.11) may be expressed in terms of the integral operators

$$\left.\begin{aligned} \hat{K}[\bar{a}_2] &= \frac{1}{2\pi} \int_0^{2\pi} K(\tilde{\theta} - \xi) \bar{a}_2(\xi)\, d\xi, \\ \hat{K}^{\dagger}[\bar{a}_6] &= \frac{1}{2\pi} \int_0^{2\pi} K(\xi - \tilde{\theta}) \bar{a}_6(\xi)\, d\xi. \end{aligned}\right\} \tag{6.13}$$

The operator $\hat{K}^{\dagger}$ in (6.13) is the adjoint of the operator $\hat{K}$. Equations (6.10) and (6.11) are similar in form to the equations obtained by Majda and Rosales (1984)

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

describing the resonant interaction of sound waves and entropy waves in one Cartesian space dimension, except that (6.10) and (6.11) do not contain nonlinear Burgers self-wave steepening terms, which are present in the resonant sound wave equations. In a convenient abuse of notation, the overbars on the $\{\bar{a}_j\}$ will be dropped in the following analysis.

To find solutions of the linear resonant wave interaction equations (6.10) and (6.11), which are $2\pi$-periodic in $\tilde{\theta}$, we introduce the Fourier series representations

$$\left.\begin{aligned} a_j &= \tfrac{1}{2}a_j^0 + \sum_{n=1}^{\infty} a_j^n \cos(n\tilde{\theta}) + b_j^n \sin(n\tilde{\theta}), \qquad j = 2, 6, \\ a_4 &= \tfrac{1}{2}a_4^0 + \sum_{n=1}^{\infty} a_4^n \cos(n\theta) + b_4^n \sin(n\theta) \end{aligned}\right\} \tag{6.14}$$

for the wave amplitudes, where the Fourier coefficients $a_j^n$ and $b_j^n$ in general depend on the long space and slow time variables $\bar{x}$ and $\bar{t}$. The kernel $K(\theta)$ in (6.10) and (6.11) has the Fourier expansion

$$K(\theta) = \sum_{n=1}^{\infty} d_n \cos(n\theta) + e_n \sin(n\theta), \tag{6.15}$$

where

$$d_n = nb_4^n, \qquad e_n = -na_4^n, \qquad n \geqslant 1, \tag{6.16}$$

are the expressions for the $\{d_n\}$ and $\{e_n\}$ in terms of the entropy-wave Fourier coefficients. We also use the complex Fourier series representations

$$a_j = Z_j^0 + \sum_{n=1}^{\infty} Z_j^n \exp(in\tilde{\theta}) + Z_j^{n*} \exp(-in\tilde{\theta}), \qquad j = 2, 6, \tag{6.17}$$

where

$$Z_j^n = \tfrac{1}{2}(a_j^n - i\, b_j^n), \qquad Z_j^0 = \tfrac{1}{2}a_j^0, \qquad j = 2, 6, \tag{6.18}$$

and $Z_j^{n*} = \frac{1}{2}(a_j^n + ib_j^n)$ is the complex conjugate of $Z_j^n$.

Substitution of the Fourier series expansions (6.17) and (6.15) in (6.10) and (6.11) yields the equations

$$\frac{dZ_2^0}{dt_2} = 0, \qquad \frac{dZ_6^0}{dt_6} = 0, \tag{6.19}$$

$$\frac{dZ_2^n}{dt_2} = \tfrac{1}{4}\Omega(d_n + ie_n)Z_6^n, \qquad \frac{dZ_6^n}{dt_6} = -\tfrac{1}{4}\Omega(d_n - ie_n)Z_2^n, \qquad n \geqslant 1, \tag{6.20}$$

for the complex Fourier coefficients $\{Z_j^n : j = 2, 6\}$. From (6.19), the general solutions for $Z_2^0$ and $Z_6^0$ are

$$Z_2^0 = Z_2^0(\bar{x} + b_x\bar{t}), \qquad Z_6^0 = Z_6^0(\bar{x} - b_x\bar{t}), \tag{6.21}$$

where $Z_2^0$ and $Z_6^0$ are arbitrary functions of $\bar{x}+b_x\bar{t}$ and $\bar{x}-b_x\bar{t}$ respectively. Equations (6.20) can be combined to yield the Klein–Gordon equations

$$\left(\frac{\partial^2}{\partial\bar{t}^2} - b_x^2\frac{\partial^2}{\partial\bar{x}^2} + \Omega_n^2\right) Z_j^n = 0, \qquad j = 2, 6, \qquad n \geqslant 1, \tag{6.22}$$

where

$$\Omega_n = \tfrac{1}{4}\Omega(d_n^2 + e_n^2)^{1/2} \equiv \tfrac{1}{4}n\Omega[(b_4^n)^2 + (a_4^n)^2]^{1/2} \qquad (n \geqslant 1). \tag{6.23}$$

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

It follows from (6.22) and (6.23) that the natural oscillation frequencies of the system are determined by the Fourier coefficients $a_4^n$ and $b_4^n$ of the entropy wave. It is of interest to note that similar Klein–Gordon equations were obtained by Hollweg (1996) in a study of wave mixing of Alfvén waves in the solar wind, in which the waves are reflected by large-scale gradients in the background flow. This is clearly a different physical process than the three-wave resonant wave interactions being investigated here.

Solutions of (6.19) and (6.20) that are independent of $\bar{x}$ are obtained in Sec. 6.1 below. Solutions of (6.19) and (6.20) that depend on both $\bar{x}$ and $\bar{t}$ are obtained in Sec. 6.2.

### *6.1. Alfvén-wave solutions independent of $\bar{x}$*

If the $\{Z_j^n : \ j = 2, 6\}$ are independent of $\bar{x}$ then (6.20) reduce to

$$\frac{dZ_2^n}{d\bar{t}} = \tfrac{1}{4}\Omega(d_n + ie_n)Z_6^n, \qquad \frac{dZ_6^n}{d\bar{t}} = -\tfrac{1}{4}\Omega(d_n - ie_n)Z_2^n, \qquad n \geqslant 1, \tag{6.24}$$

Similarly, either from the Klein–Gordon equations (6.22) or from (6.24), we obtain the second-order differential equations

$$\frac{d^2Z_j^n}{d\bar{t}^2} + \Omega_n^2 Z_j^n = 0, \qquad j = 2, 6, \qquad n \geqslant 1. \tag{6.25}$$

The general solution of (6.25) for $Z_2^n$ for $n \geqslant 1$ is

$$Z_2^n = \alpha_n \exp[i(\Omega_n\bar{t} + \mu_n)] + \beta_n \exp[-i(\Omega_n\bar{t} + \delta_n)], \tag{6.26}$$

where $\alpha_n$, $\beta_n$, $\mu_n$, and $\delta_n$ are real integration constants. From (6.24) and (6.26), the solution for $Z_6^n$ for $n \geqslant 1$ is

$$Z_6^n = \exp[i(\tfrac{1}{2}\pi - \epsilon_n)]\left\{\alpha_n \exp[i(\Omega_n\bar{t} + \mu_n)] - \beta_n \exp[-i(\Omega_n\bar{t} + \delta_n)]\right\}, \tag{6.27}$$

where

$$\exp(i\epsilon_n) = \frac{d_n + ie_n}{(d_n^2 + e_n^2)^{1/2}} \equiv \frac{b_4^n - ia_4^n}{[(a_4^n)^2 + b_4^n)^2]^{1/2}}. \tag{6.28}$$

For the sake of simplicity, we consider solutions in which $Z_j^0 \equiv \frac{1}{2}a_j^0$ for $j = 2, 6$ are constants. This implies that the mean wave amplitudes $\langle a_j\rangle$ for the Alfvén waves are constants.

Using (6.18), the real Fourier coefficients $\{a_j^n\}$ and $\{b_j^n\}$ are given by the equations

$$a_j^n = 2\Re(Z_j^n), \qquad b_j^n = -2\Im(Z_j^n), \qquad n \geqslant 1, \qquad j = 2, 6. \tag{6.29}$$

Using (6.26) and (6.27) in the Fourier series (6.17) yields the solutions

$$a_2 = \tfrac{1}{2}a_2^0 + 2\sum_{n=1}^{\infty}[\alpha_n \cos(n\tilde{\theta} + \Omega_n\bar{t} + \mu_n) + \beta_n \cos(n\tilde{\theta} - \Omega_n\bar{t} - \delta_n)], \tag{6.30}$$

$$a_6 = \tfrac{1}{2}a_6^0 + 2\sum_{n=1}^{\infty}[\beta_n \sin(n\tilde{\theta} - \Omega_n\bar{t} - \delta_n - \epsilon_n) - \alpha_n \sin(n\tilde{\theta} + \Omega_n\bar{t} + \mu_n - \epsilon_n)] \tag{6.31}$$

as the Fourier solutions for $a_2$ and $a_6$.

The solutions (6.30) and (6.31) are general solutions of the resonant interaction equations (6.10) and (6.11) that depend only on $\theta$ and $\bar{t}$ (note $\tilde{\theta} = \theta + \frac{1}{2}\langle\bar{a}_4\rangle\Omega\bar{t}$ from

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

(6.8)). It is clear from (6.30) and (6.31) that the resonant interaction equations (6.10) and (6.11) admit travelling-wave solutions consisting of one Fourier component of the form

$$a_2 = \tfrac{1}{2}a_2^0 + 2\beta_n \cos(n\tilde{\theta} - \mathbf{\Omega}_n \bar{t} - \delta_n), \tag{6.32}$$

$$a_6 = \tfrac{1}{2}a_6^0 + 2\beta_n \cos(n\tilde{\theta} - \mathbf{\Omega}_n \bar{t} - \delta_n - \tfrac{1}{2}\pi - \epsilon_n), \tag{6.33}$$

In the solutions (6.32) and (6.33), there is a phase shift of $\Delta\phi = -\frac{1}{2}\pi - \epsilon_n$ between the forward Alfvén wave $a_6$ and the backward Alfvén wave $a_2$, where $\tan\epsilon_n = -a_4^n/b_4^n$, and

$$a_4 = \tfrac{1}{2}a_4^0 + a_4^n \cos(n\theta) + b_4^n \sin(n\theta), \tag{6.34}$$

is the solution ansatz for the entropy wave.

A further interesting solution from (6.30) and (6.31) is obtained by setting

$$\beta_n = \alpha_n, \qquad \delta_n = -\mu_n, \qquad a_6^0 = 0. \tag{6.35}$$

In this case, one can show that $a_6 = 0$, but $a_2 \neq 0$ and $a_4 \neq 0$, at time $t = 0$. The general solutions (6.30) and (6.31), with the parameter constraints (6.35), reduce to

$$a_2 = \tfrac{1}{2}a_2^0 + 4\sum_{n=1}^{\infty} \alpha_n \cos(n\tilde{\theta} + \mu_n)\cos(\mathbf{\Omega}_n \bar{t}), \tag{6.36}$$

$$a_6 = -4\sum_{n=1}^{\infty} \alpha_n \cos(n\tilde{\theta} + \mu_n - \epsilon_n)\sin(\mathbf{\Omega}_n \bar{t}). \tag{6.37}$$

Thus the solutions (6.36) and (6.37) describe the generation of the forward Alfvén wave ($a_6$) from the resonant collision of the backward Alfvén wave ($a_2$) with the entropy wave ($a_4$).

### *6.2. Alfvén-wave solutions dependent on $\bar{x}$ and $\bar{t}$*

For solutions dependent on both $\bar{x}$ and $\bar{t}$, the solutions for $a_2$ and $a_6$ are given by the complex Fourier series solution (6.17), in which the complex Fourier coefficients $\{Z_j^n : j = 2, 6\}$ satisfy the coupled wave evolution equations (6.20) and the Klein–Gordon equations (6.22). By using Fourier methods, it is straightforward to obtain solutions of (6.22) satisfying the singular initial conditions

$$Z_2^n(\bar{x}, 0) = N_2^n \delta(\bar{x}), \qquad Z_6^n(\bar{x}, 0) = N_6^n \delta(\bar{x}), \tag{6.38}$$

where $\{N_j^n : j = 2, 6\}$ are constants, and $|Z_j^n| \to 0$ as $|\bar{x}| \to \infty$. The solutions of (6.20) satisfying the initial conditions (6.38) are

$$\begin{aligned} Z_2^n = {} & \frac{N_2^n}{b_x}\delta\left(\bar{t} + \frac{\bar{x}}{b_x}\right) - \frac{N_2^n \mathbf{\Omega}_n}{2b_x}\left|\frac{\bar{t} - \bar{x}/b_x}{\bar{t} + \bar{x}/b_x}\right|^{1/2} J_1(z) H\left(\bar{t} - \frac{|\bar{x}|}{b_x}\right) \\ & + \frac{N_6^n \mathbf{\Omega}_n}{2b_x}\exp(i\epsilon_n) J_0(z) H\left(\bar{t} - \frac{|\bar{x}|}{b_x}\right), \end{aligned} \tag{6.39}$$

$$\begin{aligned} Z_6^n = {} & \frac{N_6^n}{b_x}\delta\left(\bar{t} - \frac{\bar{x}}{b_x}\right) - \frac{N_6^n \mathbf{\Omega}_n}{2b_x}\left|\frac{\bar{t} + \bar{x}/b_x}{\bar{t} - \bar{x}/b_x}\right|^{1/2} J_1(z) H\left(\bar{t} - \frac{|\bar{x}|}{b_x}\right) \\ & - \frac{N_2^n \mathbf{\Omega}_n}{2b_x}\exp(-i\epsilon_n) J_0(z) H\left(\bar{t} - \frac{|\bar{x}|}{b_x}\right), \end{aligned} \tag{6.40}$$

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

where

$$z = \Omega_n \left( \bar{t}^2 - \frac{\bar{x}^2}{b_x^2} \right)^{1/2}. \tag{6.41}$$

The solutions (6.39) and (6.40) for $\{Z_j^n : \ j = 2, 6\}$ consist of Dirac delta distribution terms in which the disturbance travels at the Alfvén speed, followed by an oscillatory wake trapped between the backward and forward characteristics, with $-b_x\bar{t} < \bar{x} < b_x\bar{t}$. In particular, if $N_6^n = 0$, the solution for $Z_2^n$ consists of a Dirac delta function pulse $N_2^n\delta(\bar{t}+\bar{x}/b_x)/b_x$ moving on the backward Alfvén-wave characteristic $\bar{x} = -b_x\bar{t}$, plus a $J_1(z)$ Bessel-function term produced by wave–wave interactions. The solution for $Z_6^n(\bar{x}, \bar{t})$ consists of a $J_0(z)$ Bessel-function wake generated by resonant wave–wave interactions. The solutions (6.39) and (6.40) are Green-function-type solutions of (6.20). Derivations of the solutions (6.39) and (6.40) are given in Appendix B. More general solutions of the initial-value problem for (6.20) obtained by convolution of the Green-function solutions (6.39) and (6.40) with the initial data are derived in Appendix B.

### *6.3. Illustrative examples*

In this section, we provide illustrative examples of three-wave resonant interactions between the backward and forward Alfvén waves and the entropy wave derived in Secs 6.1 and 6.2. We first consider solutions for the Alfvén waves ($a_2$ and $a_6$) that are independent of the long space variable $\bar{x}$, discussed in (6.30)–(6.37). We then consider solutions that depend on both the long space variable $\bar{x}$ and the long time variable $\bar{t}$, based on the Green-function solutions (6.39) and (6.40) and the complex Fourier expansions (6.17) for $a_2$ and $a_6$.

*6.3.1. Solutions independent of $\bar{x}$.* Consider the interaction of the backward and forward Alfvén waves with the entropy wave, for the case where the entropy-wave profile (6.34) is of the form

$$a_4 = \sin\theta. \tag{6.42}$$

Choosing

$$\delta_1 = 0, \qquad a_2^0 = a_6^0 = 0, \qquad \beta_1 = \tfrac{1}{2}, \tag{6.43}$$

the Alfvén-wave solutions (6.32) and (6.33) for $a_2$ and $a_6$ become

$$a_2 = \cos(\theta - \Omega_1\bar{t}), \qquad a_6 = \sin(\theta - \Omega_1\bar{t}), \tag{6.44}$$

where, from (6.23),

$$\Omega_1 = \tfrac{1}{4}\Omega, \qquad \omega_2 = \omega_6 = \Omega. \tag{6.45}$$

Figure 4 illustrates the Alfvén-wave solutions (6.44) at two time instants $\bar{t} = t_0 = 0$ and $\bar{t} = t_1 = \pi/(2\Omega_1) \equiv 2\pi/\Omega$ (Figs 4b,c) for the entropy-wave profile (6.42) (Fig. 4a). The Alfvén-wave solutions (6.44) consist of travelling waves in $(\theta, \bar{t})$ space. Note that the total wave energy density

$$E_w = \frac{a_2^2 + a_6^2}{\mu} = \mu^{-1} \tag{6.46}$$

of the Alfvén waves is constant.

One can also obtain travelling-wave solutions of (6.30) and (6.31) for $a_2(\theta, \bar{t})$ and $a_6(\theta, \bar{t})$ consisting of more than one Fourier component. For example, consider

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

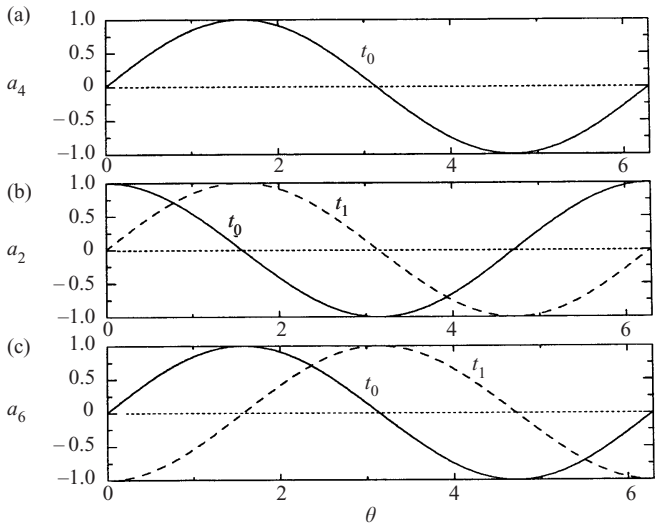



**Figure 4.** The three-wave resonant interaction of the backward and forward Alfvén waves ($a_2$ and $a_6$) with the entropy wave ($a_4$) for the solution (6.42)–(6.45). (a) shows the entropy-wave profile $a_4(\theta) = \sin\theta$, whereas (b) and (c) show the travelling Alfvén-wave solutions $a_2 = \cos(\theta - \Omega_1\bar{t})$ and $a_6 = \sin(\theta - \Omega_1\bar{t})$ at the two time instants $t_0 = 0$ and $t_1 = 2\pi/\Omega$, where $\Omega = \frac{1}{4}\Omega_1$. Note that $\omega_2 = \omega_6 = \Omega$, $\omega_4 = 0$, $k_2 = -\frac{1}{4}k_4$, and $k_6 = \frac{1}{4}k_4$ give the resonant wave frequencies and wavenumbers respectively.

solutions in which

$$a_4 = \sin(p\theta) + \sin(q\theta), \qquad (p \neq q) \tag{6.47}$$

(where $p$ and $q$ are integers). By choosing

$$\beta_p = \beta_q = \tfrac{1}{2}, \qquad a_2^0 = a_4^0 = a_6^0 = 0, \qquad \alpha_n = \delta_n = 0, \tag{6.48}$$

(6.30) and (6.31) yield the solutions

$$\left.\begin{aligned} a_2 &= \cos(p\theta - \Omega_p\bar{t}) + \cos(q\theta - \Omega_q\bar{t}), \\ a_6 &= \sin(p\theta - \Omega_p\bar{t}) + \sin(q\theta - \Omega_q\bar{t}), \end{aligned}\right\} \tag{6.49}$$

where

$$\Omega_p = \tfrac{1}{4}p\Omega, \qquad \Omega_q = \tfrac{1}{4}q\Omega. \tag{6.50}$$

Since

$$p\theta - \Omega_p\bar{t} \equiv p(\theta - \tfrac{1}{4}\Omega\bar{t}), \qquad q\theta - \Omega_q\bar{t} \equiv q(\theta - \tfrac{1}{4}\Omega\bar{t}), \tag{6.51}$$

the solutions (6.49) consist of travelling-wave solutions, dependent on the variable $\theta - \frac{1}{4}\Omega\bar{t}$. The solutions (6.47)–(6.49) can also be written in the form

$$\left.\begin{aligned} a_4 &= 2\sin[\tfrac{1}{2}(p+q)\theta]\cos[\tfrac{1}{2}(p-q)\theta], \\ a_2 &= 2\cos[\tfrac{1}{2}(p+q)\theta - \tfrac{1}{2}(\Omega_p+\Omega_q)\bar{t}]\cos[\tfrac{1}{2}(p-q)\theta - \tfrac{1}{2}(\Omega_p-\Omega_q)\bar{t}], \\ a_6 &= 2\sin[\tfrac{1}{2}(p+q)\theta - \tfrac{1}{2}(\Omega_p+\Omega_q)\bar{t}]\cos[\tfrac{1}{2}(p-q)\theta - \tfrac{1}{2}(\Omega_p-\Omega_q)\bar{t}]. \end{aligned}\right\} \tag{6.52}$$

From (6.52), the wave profiles consist of a waveform with period

$$T_\theta^{(1)} = \frac{2\pi}{p-q} \tag{6.53}$$

in $\theta$, which forms the envelope of a higher-frequency component with period $T_\theta^{(2)} =$

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

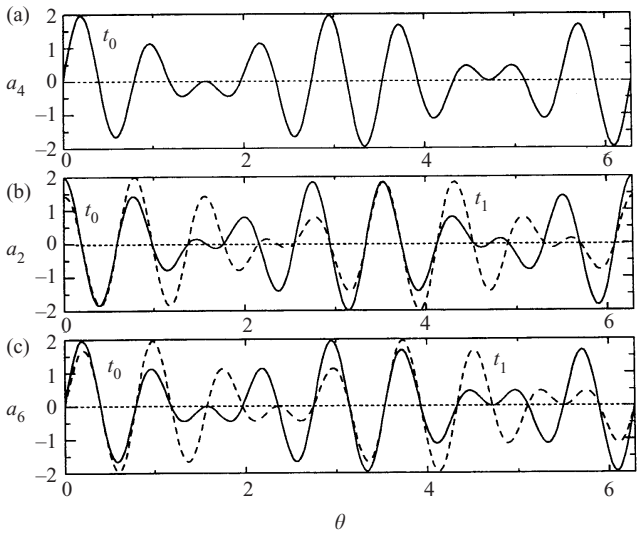



**Figure 5.** The solution (6.47)–(6.52) for three-wave resonant interaction of the backward and forward Alfvén waves ($a_2$ and $a_6$) with the entropy wave ($a_4$). (a) shows the entropy-wave profile $a_4(\theta) = \sin(p\theta)+\sin(q\theta)$, with $p = 9$ and $q = 7$. The solutions for $a_2$ and $a_6$, shown in (b) and (c), are travelling-wave solutions in the variable $\theta - \frac{1}{4}\Omega\bar{t}$, and the resonance conditions (6.1) and (6.2) are the same as in Figure 4. The solutions for $a_2$ and $a_6$ are shown at times $t_0 = 0$ and $t_1 = \pi/\Omega$.

$2\pi/(p+q)$. The solutions (6.52) are illustrated in Fig. 5 for the case with $p = 9$ and $q = 7$. Figure 5(a) shows the entropy-wave profile, whereas Figs 5(b,c) show the travelling-wave solutions for the Alfvén waves at two times $t_0 = 0$ and $t_1 = \pi/\Omega$.

The solutions (6.30) and (6.31) are travelling-wave solutions only if $\Omega_n/n$ is independent of $n$. A non-travelling-wave form is obtained, for example, by choosing an entropy-wave profile of the form

$$a_4 = \sin(p\theta) + 2\sin(q\theta). \tag{6.54}$$

Because $a_4^p = a_4^q = 0$, the frequencies $\Omega_p$ and $\Omega_q$ have the forms

$$\Omega_p = \tfrac{1}{4}p\Omega, \qquad \Omega_q = \tfrac{1}{2}q\Omega, \tag{6.55}$$

and hence $\Omega_p/p \neq \Omega_q/q$. By appropriate choice of the integration constants, (6.30) and (6.31) yield the solutions

$$\left.\begin{aligned} a_2 &= \cos(p\theta - \Omega_p\bar{t}) + \cos(q\theta - \Omega_q\bar{t}), \\ a_6 &= \sin(p\theta - \Omega_p\bar{t}) + \sin(q\theta - \Omega_q\bar{t}) \end{aligned}\right\} \tag{6.56}$$

for the backward and forward Alfvén waves. However, because the phase speeds $\Omega_p/p = \frac{1}{4}\Omega$ and $\Omega_q/q = \frac{1}{2}\Omega$ of the two harmonic wave components in (6.56) in $(\theta, \bar{t})$ space are not equal, the resulting solutions are not travelling waves. The non-travelling-wave solutions (6.54)–(6.56) are illustrated in Fig. 6 for the case $p = 9$ and $q = 7$, at the two time instants $t_0 = 0$ and $t_1 = 2\pi/(5\Omega)$.

As an illustrative example of solutions of the form (6.36) and (6.37) for $a_2$ and $a_6$, consider solutions with the entropy-wave profile

$$a_4 = \cos\theta. \tag{6.57}$$

Since $a_4^1 = 1$ and $b_4^1 = 0$, (6.28) implies that $\epsilon_1 = -\frac{1}{2}\pi$. Choosing $\delta_1 = \mu_1 = 0$, $a_6^0 = 0$,

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

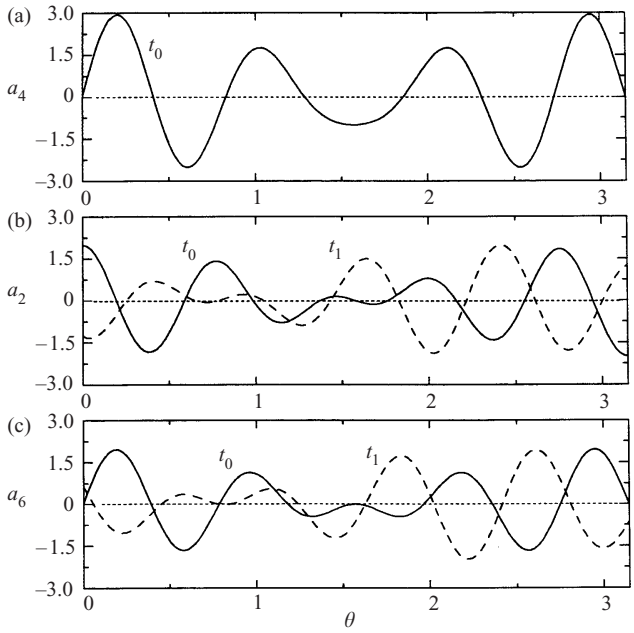



**Figure 6.** The solution (6.54)–(6.56) describing the resonant interaction of the backward and forward Alfvén waves ($a_2$ and $a_6$, shown in (b) and (c)) with the entropy wave ($a_4$, shown in (a)). This solution is not a travelling-wave solution. The parameters $p = 9$ and $q = 7$. The Alfvén-wave profiles $a_2$ and $a_6$ are shown at times $t_0 = 0$ and $t_1 = \pi/[2(\Omega_q - \Omega_p)] \equiv 2\pi/(5\Omega)$.

and $\alpha_1 = \frac{1}{4}$, (6.36) and (6.37) yield the solutions

$$a_2 = \cos\theta \cos(\Omega_1 \bar{t}), \qquad a_6 = \sin\theta \sin(\Omega_1 \bar{t}) \tag{6.58}$$

for the Alfvén waves. Note that at time $\bar{t} = 0$, $a_4 = a_2 = \cos\theta$, but $a_6 = 0$. Thus the solution (6.58) for the forward Alfvén wave is generated by resonant interaction of the backward Alfvén wave with the entropy wave. In the solutions (6.58), $\Omega_1 = \frac{1}{4}\Omega$, where $\omega_2 = \omega_6 \equiv \Omega$ are the frequencies of the fast scale oscillations. Figure 7 shows the time evolution of the solutions (6.58) for $a_2$ and $a_6$ for the three time instants $t_0 = 0$, $\bar{t} = t_1 = \pi/(4\Omega_1)$ and, $\bar{t} = t_2 = \pi/(2\Omega_1)$. Initially, $a_2 \neq 0$ and $a_6 = 0$ at time $\bar{t} = 0$. Note that the solutions simply oscillate in time and do not move along the $\theta$ axis (i.e. the solutions are 'breather'-type solutions). As $\bar{t}$ increases, the forward Alfvén wave is generated by the resonant interactions, and at time $t_2 = \pi/(2\Omega_1)$, the backward Alfvén wave $a_2 \equiv 0$, and only the forward Alfvén wave is present. At still later times, the backward wave is regenerated by the wave interactions.

*6.3.2. Solutions dependent on $\bar{x}$ and $\bar{t}$.* As an example where $a_2(\theta, \bar{x}, \bar{t})$ and $a_6(\theta, \bar{x}, \bar{t})$ depend on both $\bar{x}$ and $\bar{t}$ as well as $\theta$, consider solutions of (6.10) and (6.11) corresponding to the entropy-wave profile

$$a_4 = \sin\theta, \qquad K(\theta) = \cos\theta, \tag{6.59}$$

and satisfying the initial conditions

$$a_2(\theta, \bar{x}, 0) = 2\cos\theta\ \delta(\bar{x}), \qquad a_6(\theta, \bar{x}, 0) = 0. \tag{6.60}$$

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

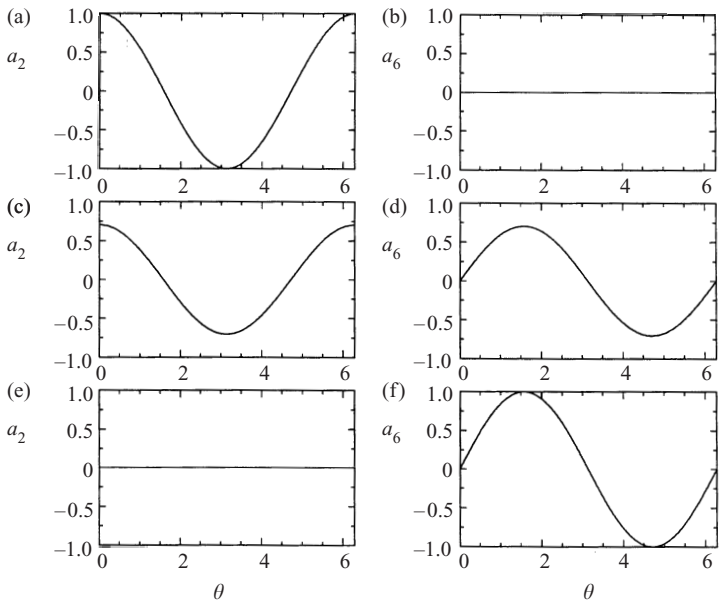



**Figure 7.** The three-wave resonant interaction, breather-type solutions (6.57)–(6.58) involving the Alfvén waves ($a_2$ and $a_6$) and the entropy wave $a_4$. At time $t = 0$ (a,b), $a_2 = a_4 = \cos\theta$ and $a_6 = 0$. (c) and (d) show the Alfvén-wave profiles at time $t_1 = \pi/\Omega$, while (e) and (f) show the profiles at $t_2 = 2\pi/\Omega$. (a–d) illustrate the generation of the forward Alfvén wave $a_6$ from the resonant interaction of the backward Alfvén wave with the entropy wave.

From (6.17), (6.39), and (6.40), the required solutions for $a_2$ and $a_6$ may be written in the form

$$a_2 = 2\cos\theta\ Z_2, \qquad a_6 = 2\cos\theta\ Z_6, \tag{6.61}$$

where

$$Z_2 = \delta(\bar{x} + b_x\bar{t}) - \frac{\Omega_1}{2b_x}\left|\frac{\bar{t} - \bar{x}/b_x}{\bar{t} + \bar{x}/b_x}\right|^{1/2} J_1(z)H\left(\bar{t} - \frac{|\bar{x}|}{b_x}\right), \tag{6.62}$$

$$Z_6 = -\frac{\Omega_1}{2b_x}J_0(z)H\left(\bar{t} - \frac{|\bar{x}|}{b_x}\right), \tag{6.63}$$

$z = \Omega_1(\bar{t}^2 - \bar{x}^2/b_x^2)^{1/2}$ and $\Omega_1 = \frac{1}{4}\Omega$. Since $a_6 = 0$ initially at $t = 0$, the solutions describe the generation of the forward Alfvén wave $a_6$ due to resonant interaction of the backward Alfvén wave with the entropy wave, for a case where the long space dependence of the initial data plays an important role. More realistic solutions with non-singular initial data can be obtained by using the convolution-type solutions in Appendix B.

The evolution of the solutions (6.61) for the backward and forward Alfvén waves are illustrated in Fig. 8. The figure shows the spatial profiles of $Z_2(\bar{x}, \bar{t})$ and $Z_6(\bar{x}, \bar{t})$ for three time instants $\bar{t} = 0.5$, 5, and 15. The parameters $b_x = \Omega_1 = 1$. At early times, $\Omega_1\bar{t} \ll 1$, $|z| \ll 1$, and, using the approximations for the Bessel functions for small $z$ (Abramowitz and Stegun 1965, p. 360, formula 9.1.7) in (6.62) and (6.63)

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

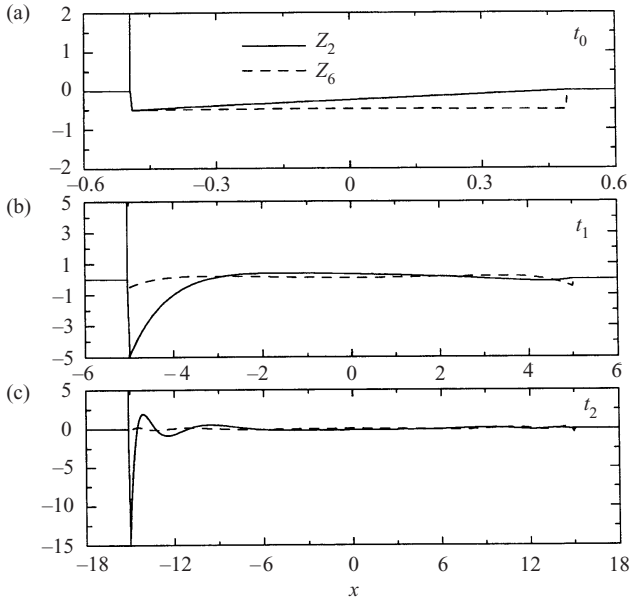



**Figure 8.** The long space ($\bar{x}$) and time ($\bar{t}$) behaviour of the Green-function-type-solution (6.59)–(6.63) for three-wave resonant interaction of the backward and forward Alfvén waves ($a_2$ and $a_6$) and entropy wave $a_4 = \cos\theta$. The figure shows the behaviour $Z_2(\bar{x},\bar{t})$ and $Z_6(\bar{x},\bar{t})$, where $a_2 = 2\cos\theta Z_2(\bar{x},\bar{t})$ and $a_6 = 2\cos\theta Z_6(\bar{x},\bar{t})$, at three time instants $\bar{t} = 0.5, 5$, and 15. Note that $a_6 = 0$ and $a_2 = 2\cos\theta\delta(\bar{x})$ at time $t = 0$. The solutions are restricted to the region between the backward and forward Alfvén-wave characteristics $|\bar{x}| < b_x\bar{t}$.

yields the approximate solutions

$$Z_2 \approx \delta(\bar{x} + b_x\bar{t}) - \frac{\Omega_1^2}{2b_x}\left(\bar{t} - \frac{\bar{x}}{b_x}\right) H\left(\bar{t} - \frac{|\bar{x}|}{b_x}\right), \tag{6.64}$$

$$Z_6 \approx -\frac{\Omega_1}{2b_x} H\left(\bar{t} - \frac{|\bar{x}|}{b_x}\right). \tag{6.65}$$

Thus the solution for $Z_2$ at early times (Fig. 8a) consists of a positive delta-function pulse $\delta(\bar{x} + b_x\bar{t})$ on the backward Alfvén-wave characteristic, followed by a negative, sawtooth-shaped wake limited to the region $|\bar{x}| < b_x\bar{t}$ between the characteristics. The generated wave $Z_6$ at short times consists of a negative rectangular-shaped pulse with $Z_6 \approx -\Omega_1/(2b_x)$ in the region $|\bar{x}| < b_x\bar{t}$. The approximate solutions (6.64) and (6.65) also apply in the vicinity of the backward Alfvén-wave characteristic $\bar{x} = -b_x\bar{t}$. In particular, (6.64) shows that for $\bar{x} \approx -b_x\bar{t}$ ($\bar{x} > -b_x\bar{t}$), $Z_2$ is linear in $\bar{x}$, and the depth of the depression of the $Z_2$ wake increases linearly with time. The damped, oscillatory character of the $Z_2$ and $Z_6$ wakes at large times $\bar{t} \gg |\bar{x}|/b_x$ (Fig. 8c) is characteristic of the behaviour of the Bessel functions $J_0(z)$ and $J_1(z)$ for large $z$ (Abramowitz and Stegun 1965 p. 364, formula 9.2.1).

## 7. Summary and concluding remarks

One of the aims of this paper was to obtain Lagrangian and Hamiltonian variational formulations of equations describing weakly nonlinear three-wave resonant interactions in magnetohydrodynamics in one Cartesian space dimension, which

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

were derived by Ali and Hunter (1998) and Webb et al. (1999). Similar resonant interaction equations for sound waves and entropy waves in one Cartesian space dimension were derived by Majda and Rosales (1984).

The wave interaction equations were reduced to a more symmetric form by the introduction of new wave amplitude variables $\{c_j\}$ given by the equations

$$c_j = N_j^{1/2}, \qquad N_j = E_{wj}/\omega_j, \tag{7.1}$$

where $N_j$ is the wave action density, $E_{wj}$ is the wave energy density, and $\omega_j$ is the frequency of the $j$th wave mode (see also Sagdeev and Galeev 1969). In cases where $N_j$ is ill defined (e.g. for entropy waves with $\omega_j = 0$), the original perturbation wave amplitudes $\{a_j\}$ used by Webb et al. (1999) are used. In the new variables, the resonant wave interaction coefficients $\tilde{\Gamma}_{jpq}$ for a given resonant wave triad $\{jpq\}$ are symmetric with respect to any permutation of $\{jpq\}$ provided that $j \neq p \neq q \neq 4$ (see (3.6)). If the resonant interactions involve the entropy wave (i.e. $j = 4$) then $\tilde{\Gamma}_{pq4} = \tilde{\Gamma}_{qp4}$ and $\tilde{\Gamma}_{4pq} = 0$, $p \neq q \neq 4$ (see (3.6)). The symmetry relations (3.6) for the wave interaction coefficients are related to the quasiparticle number density conservation laws, the Manley–Rowe relations, and the canonical energy and momentum equations for the waves (Sec. 3.2).

The different possible resonant wave triads $\{jpq\}$ are listed in Table 1. The resonant triads in Table 1 always involve two waves of one type (i.e. the backward and forward waves for a particular mode) with a single wave of a different type. In resonant interactions between the entropy wave and magnetoacoustic waves, the magnetoacoustic wave equations contain integral terms representing the resonant interaction of a magnetoacoustic wave with the entropy wave to produce the reverse magnetoacoustic wave. On the other hand, the entropy wave is unaffected by resonant interaction of the magnetoacoustic waves, since the relevant wave interaction coefficient, $\tilde{\Gamma}_{4pq}$, is zero. In the variational formulations of the equations (Sec. 4), the entropy-wave equation and the mean wave field equations are treated as constraint equations by using Lagrange multipliers. In the limit as $B \to 0$, the magnetoacoustic–entropy wave interaction equations reduce to the Majda–Rosales (1984) equations describing the resonant interaction of the backward and forward sound waves with the entropy wave. Similar equations describe the resonant interaction of the backward and forward Alfvén waves with the entropy wave. However, the Alfvén–entropy wave interaction equations are linear, and contain no nonlinear Burgers self-wave steepening terms, which are present in the magnetoacoustic–entropy wave equations.

For resonant interactions not involving the entropy wave (i.e. Alfvén–magnetoacoustic wave interactions or fast–slow magnetoacoustic wave interactions), each wave equation contains three-wave resonant interaction terms due to the other two waves. The resonant interaction coefficient in each of the three wave evolution equations has the same value if the $\{c_j\}$ are used as dependent variables, since the interaction coefficients $\{\tilde{\Gamma}_{jpq}\}$ are invariant with respect to permutations of $\{jpq\}$.

Non-strictly hyperbolic wave interactions occur when one or more of the wave eigen-velocities (eigenvalues) coincide. This occurs when the wave vectors $\mathbf{k}$ are either parallel to $\mathbf{B}$ ($\mathbf{k} \parallel \mathbf{B}$) or perpendicular to $\mathbf{B}$ ($\mathbf{k} \perp \mathbf{B}$). In the case $\mathbf{k} \parallel \mathbf{B}$, one of the magnetoacoustic speeds is equal to the Alfvén speed. For $\mathbf{k} \perp \mathbf{B}$, all of the eigenvalues are zero, except for the fast magnetoacoustic waves. Because the waves with the same phase velocity have phases that are linear multiples of each other,

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

it follows that more than three waves will in general be involved in the resonant interactions. This leads to more resonant wave interaction terms in the Lagrangian and Hamiltonian variational functionals describing the waves than in the strictly hyperbolic case, where all the eigenvalues are distinct.

In Sec. 5, the three-wave resonant interaction of the backward and forward Alfvén waves with the forward slow-mode wave was used to study the Alfvén-wave decay instability. The case where a large-amplitude forward-propagating Alfvén wave interacts with a much smaller-amplitude backward-propagating Alfvén wave and a small-amplitude forward-propagating slow-mode wave was studied. The dependence of the linear growth rates for the small amplitude waves on the plasma beta, propagation angle of the waves relative to the background magnetic field **B**, and the envelope wavenumbers of the waves was determined (Figs 1–3). For wave propagation oblique to the field, a non-zero growth rate is obtained for the small-amplitude waves at all plasma betas ($\beta > 0$). However, for parallel-propagating waves, it is necessary that $0 < \beta < 1$ to obtain a non-zero growth rate. The largest growth rates are obtained for small plasma beta. For the degenerate case of wave propagation parallel to **B** (**k** $\parallel$ **B**) in a low-beta plasma, the Alfvén and fast magnetoacoustic waves have the same phase speed, and both these waves can resonantly interact with the slow-mode wave (which is a sound wave in this limit). The analysis showed that the degenerate backward circularly polarized Alfvén–fast-mode wave must have the same sense of rotation of the electric field as the corresponding degenerate forward-propagating circularly polarized Alfvén wave (Section 5.2). The linear stability analysis was carried out by neglecting the Burgers self-wave steepening for the slow magnetoacoustic wave. This is valid at early times for the case where the amplitude of the slow-mode wave is sufficiently small.

Exact solutions of the Alfvén–entropy wave interaction equations were obtained in Sec. 6 by using Fourier methods. The equations reduce to two coupled linear integro-differential equations for the backward and forward Alfvén waves in which the integral terms describe the resonant interaction between an Alfvén wave and the entropy wave to produce the reverse Alfvén wave. The solutions that depend only on the fast phase variable $\theta$ and the slow time variable $\bar{t}$ exhibit a variety of wave forms, including travelling-wave solutions (Figs 4 and 5), non-travelling-wave solutions (Fig. 6), and breather-type solutions that simply oscillate in time (Fig. 7).

Green-function-type solutions of the wave interaction equations were obtained that depend on $\theta$, $\bar{t}$, and $\bar{x}$: (6.17), (6.39), and (6.40). The $(\bar{x}, \bar{t})$-dependent part of the solutions satisfies the coupled first-order wave equations (6.20), in which the coupling coefficients for the backward and forward Alfvén waves depend on the entropy-wave Fourier coefficients. The solutions in Fig. 8 consist of a Dirac delta-function pulse for the backward Alfvén wave travelling along the backward-wave characteristic $\bar{x} = -b_x\bar{t}$, plus wakes for the backward and forward Alfvén waves created by wave–wave interactions, which are limited to the region between the characteristics (i.e. $-b_x\bar{t} < \bar{x} < b_x\bar{t}$, where $b_x$ is the Alfvén-wave phase speed). The coupled first-order wave equations (6.20) can be combined to yield Klein–Gordon-type equations (6.22) for the wave amplitudes. Similar Klein–Gordon-type equations were obtained by Hollweg (1996) in a discussion of the interaction of backward- and forward-propagating Alfvén waves in the solar wind due to large-scale gradients in the background flow. However, the wave interactions in the present paper are due to three-wave resonant interactions of the Alfvén waves with the entropy wave, which is a different physical mechanism than that considered by Hollweg (1996).

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

Only a limited number of resonant wave interactions involving (a) the resonant interaction of Alfvén waves and the entropy wave (Sec. 6) and (b) the three-wave resonant interaction between the backward and forward Alfvén waves and the forward slow magnetoacoustic wave (Sec. 5) have been investigated in detail in the present paper. It is of interest to investigate solutions for other resonant triads, either by analytical or numerical means. In particular, a study of the effect of Burgers self-wave steepening of the slow magnetoacoustic wave on the Alfvén-wave decay instability discussed in Sec. 5 is of interest. The equations corresponding to the resonant interaction of two magnetoacoustic waves of the same type with the entropy wave are similar to the equations obtained by Majda and Rosales (1984) for the interaction of the backward and forward sound wave with an entropy wave. A further possible extension of the present work is an investigation of MHD wave interactions in two or more space dimensions (see Hunter et al. (1986) for the resonant interaction of sound waves, entropy waves, and vorticity eigenmodes in adiabatic gas dynamics). The relationship between the resonant interaction equations investigated here and weak turbulence equations involving three-wave resonant interactions remains as a subject for further investigation.

*Acknowledgements*

The work of G.M.W. was supported in part by NASA Grant NAG5-5180. G.P.Z. is suppported in part at the BRI by NSF–DOE Award ATM-9713223, and in part by an NSF Young Investigator Award ATM 9357861.

## Appendix A

In this appendix, we describe in more detail the formulae (2.9)–(2.13) for the wave interaction coefficients. In particular, we show the relationships between the coefficients and the MHD equations.

The MHD equations in one Cartesian space dimension can be written in the conservative form

$$\frac{\partial \Psi^\alpha}{\partial t} + \frac{\partial F^\alpha}{\partial x} = 0, \qquad 1 \leqslant \alpha \leqslant 7, \tag{A 1}$$

where

$$\begin{aligned} &\boldsymbol{\Psi} = \left(\rho, M_x, M_y, M_z, B_y, B_z, \sigma\right)^{\mathrm{T}}, \\ &\mathbf{M} = \rho \mathbf{u}, \qquad \sigma = \rho S, \end{aligned} \tag{A 2}$$

defines the state vector $\boldsymbol{\Psi}$. In (A 2), $\mathbf{M} = \rho\mathbf{u}$ is the mass flux and $\sigma = \rho S$ is the conserved entropy density. Equations (A 1) can also be written in the matrix form

$$\frac{\partial \Psi^\alpha}{\partial t} + \mathscr{A}^{\alpha\beta} \frac{\partial \Psi^\beta}{\partial x} = 0, \qquad 1 \leqslant \alpha \leqslant 7, \tag{A 3}$$

where

$$\mathscr{A}^{\alpha\beta} = \frac{\partial F^\alpha}{\partial \Psi^\beta}, \qquad 1 \leqslant \alpha \leqslant 7, \qquad 1 \leqslant \beta \leqslant 7. \tag{A 4}$$

The detailed form of the MHD fluxes $\{F^\alpha\}$ and the matrix $\mathscr{A}^{\alpha\beta}$ are given in Webb et al. (1999).

Let $\{\mathbf{R}_s\}$ and $\{\mathbf{L}_s\}$ denote the right and left eigenvectors of the matrix $\mathscr{A}$ corresponding to the eigenvalue $\lambda_s$, i.e.

$$(\mathscr{A} - \lambda_s \mathbf{I}) \cdot \mathbf{R}_s = 0, \qquad \mathbf{L}_s \cdot (\mathscr{A} - \lambda_s \mathbf{I}) = 0, \tag{A 5}$$

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

where the eigenvalues $\lambda = \lambda_s$ satisfy the eigenvalue equation

$$\det(\mathscr{A} - \lambda \mathbf{I}) = 0. \tag{A 6}$$

The eigenvectors are normalized so that

$$\mathbf{L}_s \cdot \mathbf{R}_j = \delta^s{}_j, \qquad 1 \leqslant s \leqslant 7, \qquad 1 \leqslant j \leqslant 7. \tag{A 7}$$

The formulae (2.9)–(2.13) can be obtained from the formulae (see Webb et al. 1999)

$$\Gamma_{jjj} = \left(\mathbf{R}_j \cdot \boldsymbol{\nabla}_\Psi\right) \lambda_j, \tag{A 8}$$

$$\Gamma_{jjq} = \left(\mathbf{R}_j \cdot \boldsymbol{\nabla}_\Psi\right) \lambda_q, \qquad q \neq j, \tag{A 9}$$

$$\hat{\Gamma}_{jpq} = \mathbf{L}_j \cdot \left(\mathbf{R}_q \cdot \boldsymbol{\nabla}_\Psi \mathscr{A} \cdot \mathbf{R}_p\right) \equiv L_j^\beta \frac{\partial^2 F^\beta}{\partial \Psi^\alpha \partial \Psi^\gamma} R_q^\alpha R_p^\gamma, \tag{A 10}$$

where $j \neq p \neq q$ in (A 10). Equations (A 8)–(A 10) are formal expressions for the Burgers self-wave interaction coefficients $\{\Gamma_{jjj}\}$, the mean wave field interaction coefficients $\{\Gamma_{jjq} : \ q \neq j\}$, and the three-wave resonant interaction coefficients $\{\hat{\Gamma}_{jpq}\}$ respectively. The formulae (2.9)–(2.13) follow from (A 8)–(A 10) by using the results for the eigenvectors $\{\mathbf{L}_s\}$ and $\{\mathbf{R}_s\}$, the eigenvalues $\{\lambda_s\}$, the fluxes $F^\alpha$, and the matrix $\mathscr{A}$ given in Webb et al. (1999).

## Appendix B

In this appendix, we outline the derivation of the Green-function-type solutions (6.39) and (6.40) of the coupled unidirectional wave equations (6.20) for $Z_2^n$ and $Z_6^n$, which are associated with the Klein–Gordon equations (6.22). We also present the general solution of the initial-value problem for (6.20), by convolving the Green-function solutions with the initial-value data.

To solve (6.20), satisfying the singular initial conditions (6.38), we introduce the Laplace transforms

$$\bar{Z}_j^n(\bar{x}, s) = \int_0^\infty \exp(-s\bar{t}) Z_j^n(\bar{x}, \bar{t}) \, d\bar{t}. \tag{B 1}$$

The Laplace-transform versions of (6.20) can be written in the matrix form

$$\frac{d}{d\bar{x}} \begin{pmatrix} \bar{Z}_2^n \\ \bar{Z}_6^n \end{pmatrix} + \mathbf{A} \cdot \begin{pmatrix} \bar{Z}_2^n \\ \bar{Z}_6^n \end{pmatrix} = \frac{1}{b_x} \begin{pmatrix} -Z_2^n(\bar{x}, 0) \\ Z_6^n(\bar{x}, 0) \end{pmatrix}, \tag{B 2}$$

where

$$\mathbf{A} = \frac{1}{b_x} \begin{pmatrix} -s & \Omega_n \exp(i\epsilon_n) \\ \Omega_n \exp(-i\epsilon_n) & s \end{pmatrix}, \tag{B 3}$$

where $\Omega_n$ and $\exp(i\epsilon_n)$ are defined in (6.23) and (6.28). The initial values $Z_2^n(\bar{x}, 0)$ and $Z_6^n(\bar{x}, 0)$ are given in (6.38).

The solutions of (B 2), satisfying the initial conditions (6.38) and satisfying the boundary conditions $|Z_j^n| \to 0$ $(j = 2, 6)$ as $|\bar{x}| \to \infty$, are

$$\bar{Z}_2^n = \frac{\exp(-\Delta^{1/2}|\bar{x}|/b_x)}{2b_x\Delta^{1/2}} [N_2^n(s - \Delta^{1/2}) + N_6^n \Omega_n \exp(i\epsilon_n)] + \frac{N_2^n}{b_x} \exp\left(-\frac{\Delta^{1/2}|\bar{x}|}{b_x}\right) H(-\bar{x}), \tag{B 4}$$

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

$$\bar{Z}_6^n = \frac{\exp(-\Delta^{1/2}|\bar{x}|/b_x)}{2b_x\Delta^{1/2}}[-N_2^n\boldsymbol{\Omega}_n\exp(-i\epsilon_n) + N_6^n(s-\Delta^{1/2})] + \frac{N_6^n}{b_x}\exp\left(-\frac{\Delta^{1/2}|\bar{x}|}{b_x}\right)H(\bar{x}), \tag{B 5}$$

where

$$\Delta = s^2 + \boldsymbol{\Omega}_n^2 \tag{B 6}$$

and $H(x)$ is the Heaviside step function.

Using the inverse Laplace transforms

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp(pt)\{\exp(-bp) - \exp[-b(p^2+\alpha^2)^{1/2}]\}\,dp = \frac{\alpha b}{(t^2-b^2)^{1/2}}J_1\big(\alpha(t^2-b^2)^{1/2}\big)H(t-b), \tag{B 7}$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp[pt - b(p^2+\alpha^2)^{1/2}]\left[1 - \frac{p}{(p^2+\alpha^2)^{1/2}}\right]dp = \alpha\left(\frac{t-b}{t+b}\right)^{1/2}J_1\big(\alpha(t^2-b^2)^{1/2}\big)H(t-b), \tag{B 8}$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\exp[pt - b(p^2+\alpha^2)^{1/2}]}{(p^2+\alpha^2)^{1/2}}\,dp = J_0\big(\alpha(t^2-b^2)^{1/2}\big)H(t-b), \tag{B 9}$$

(Erdelyi et al. 1954, Vol.1, p. 248, formulae 23, 25, and 24), and noting that

$$\delta(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp(pt)\,dp, \tag{B 10}$$

is a representation of the Dirac delta distribution, the inverse Laplace transforms of (B 4) and (B 5) yield the solutions (6.39) and (6.40) for the $\{Z_j^n :\ j = 2, 6\}$.

More general solutions of (6.20), satisfying the initial conditions

$$Z_2^n(\bar{x},0) = N_2^n(\bar{x}), \qquad Z_6^n(\bar{x},0) = N_6^n(\bar{x}), \tag{B 11}$$

may be written in the form

$$\begin{pmatrix} Z_2^n \\ Z_6^n \end{pmatrix} = \int_{-\infty}^{\infty} N_2^n(x')\mathbf{G}_2^n(\bar{x}-x',t) + N_6^n(x')\mathbf{G}_6^n(\bar{x}-x',t)\,dx', \tag{B 12}$$

where the vector Green functions $\mathbf{G}_2^n$ and $\mathbf{G}_6^n$ have the forms

$$\mathbf{G}_2^n = \big(G_{22}^n, G_{26}^n\big)^{\mathrm{T}}, \qquad \mathbf{G}_6^n = \big(G_{62}^n, G_{66}^n\big)^{\mathrm{T}}. \tag{B 13}$$

The components of $\mathbf{G}_2^n$ and $\mathbf{G}_6^n$ in (B 13) are given by the formulae

$$\left.\begin{aligned}
G_{22}^n(\bar{x},t) &= \frac{1}{b_x}\delta\left(\bar{t}+\frac{\bar{x}}{b_x}\right) - \frac{\boldsymbol{\Omega}_n}{2b_x}\left|\frac{\bar{t}-\bar{x}/b_x}{\bar{t}+\bar{x}/b_x}\right|^{1/2}J_1(z)H\left(\bar{t}-\frac{|\bar{x}|}{b_x}\right),\\
G_{26}^n(\bar{x},\bar{t}) &= -\frac{\boldsymbol{\Omega}_n}{2b_x}\exp(-i\epsilon_n)J_0(z)H\left(\bar{t}-\frac{|\bar{x}|}{b_x}\right),\\
G_{62}(\bar{x},\bar{t}) &= \frac{\boldsymbol{\Omega}_n}{2b_x}\exp(i\epsilon_n)J_0(z)H\left(\bar{t}-\frac{|\bar{x}|}{b_x}\right),\\
G_{66}^n(\bar{x},t) &= \frac{1}{b_x}\delta\left(\bar{t}-\frac{\bar{x}}{b_x}\right) - \frac{\boldsymbol{\Omega}_n}{2b_x}\left|\frac{\bar{t}+\bar{x}/b_x}{\bar{t}-\bar{x}/b_x}\right|^{1/2}J_1(z)H\left(\bar{t}-\frac{|\bar{x}|}{b_x}\right),
\end{aligned}\right\} \tag{B 14}$$

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

where $z$ is defined in (6.41). In physical applications of (B 12), the initial data should be chosen to be non-singular, since the initial data $N_2^n(\bar{x})$ and $N_6^n(\bar{x})$ describe the long-space-scale behaviour. In spite of this, the Green-function solutions (B 14) (or (6.39) and (6.40)) are still useful in illustrating the role of the long space scale in the solutions.

## References


Abramowitz, M. and Stegun, I. A. 1965 *Handbook of Mathematical Functions*. Dover, New York.

Akhiezer, A. I., Akhiezer, I. A., Polovin, R. V., Sitenko, A. G. and Stepanov, K. N. 1975 *Plasma Electrodynamics*, Vol. 2, p. 108. Pergamon, Oxford.

Ali, G. and Hunter, J. K. 1998 *Wave Motion* **27**, 257.

Barnes, A. 1966 *Phys. Fluids* **9**, 1483.

Barnes, A. 1979 In: *Solar System Plasma Physics*, Vol. 1 (ed. E. N. Parker, C.F. Kennel and L.J. Lanzerotti), p. 250. North-Holland, Amsterdam.

Bhattercharjee, A., Ng, C. S. and Spangler, S. R. 1998 *Astrophys. J.* **494**, 409.

Brio, M. 1989a *Notes Numer. Fluid Mech.* **24**, 43.

Brio, M. 1989b *Contemp. Maths* **100**, 55.

Chin, C. and Wentzel, D. G. 1972 *Astrophys. Space Sci.* **16**, 465.

Cohen, R. H. 1975 *J. Geophys. Res.* **80**, 3678.

Cohen, R. H. and Dewar, R. L. 1974 *J. Geophys. Res.* **79**, 4174.

Davidson, R. C. 1972 *Methods in Nonlinear Plasma Theory*. Academic, New York.

Deconinck, B., Meuris, P. and Verheest, F. 1993a *J. Plasma Phys.* **50**, 445.

Deconinck, B., Meuris, P. and Verheest, F. 1993b *J. Plasma Phys.* **50**, 457.

Dewar, R. L. 1970 *Phys. Fluids* **13**, 2710.

Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, G. 1954 *Tables of Integral Transforms*, Vol. 1. McGraw-Hill, New York.

Galeev, A. A. and Oraevski, V. N. 1963 *Soviet Phys. Dokl.* **7**, 988.

Galtier, S., Nazarenko, S.V., Newell, A.C. and Pouquet, A. 2000 *J. Plasma Phys.* **63**, 447.

Goldreich, P. and Sridhar, S. 1995 *Astrophys. J.* **438**, 763.

Goldstein, M. L. 1978 *Astrophys. J.* **219**, 700.

Hada, T. 1993 *Geophys. Res. Lett.* **20**, 2415.

Hollweg, J. V. 1994 *J. Geophys. Res.* **99**, 23431.

Hollweg, J. V. 1996 In: *Solar Wind 8* (ed. D. Winterhalter, J. T. Gosling, S. R. Habbal, W.S. Kurth and M. Neugebauer). *AIP Conf. Proc.* **382**, 327.

Hunter, J. K., Majda, A. and Rosales, R. 1986 *Stud. Appl. Maths* **75**, 187.

Jayanti, V. and Hollweg, J. V. 1993a *J. Geophys. Res.* **98**, 13247.

Jayanti, V. and Hollweg, J. V. 1993b *J. Geophys. Res.* **98**, 19049.

Kakutani, T. and Ono, H. 1969 *J. Phys. Soc. Jpn* **26**, 1305.

Kaup, D. J., Reiman, A. and Bers, A. *Rev. Mod. Phys.* **51**, 275.

Khabibrakhmanov, I. Kh., Galinsky, V. L. and Verheest, F. 1992 *Phys. Fluids* **B4**, 2538.

Kraichnan, R. H. 1965 *Phys. Fluids* **8**, 1385.

Krall, N. A. and Trivelpiece, A. W. 1973 *Principles of Plasma Physics*. McGraw-Hill, New York (reprinted 1986 San Francisco Press).

MacGregor, K. B. and Charbonneau, P. 1994 *Astrophys. J.* **430**, 387.

Majda, A. and Rosales, R. 1984 *Stud. Appl. Maths* **71**, 149.

Majda A., Rosales, R. and Schonbeck, M. 1988 *Stud. Appl. Maths* **79**, 205.

Medvedev, M. V. and Diamond, P. H. 1996 *Phys. Plasmas* **3**, 863.

Medvedev, M. V. and Diamond, P. H. 1997 *Phys. Rev.* **E56**, R2371.

Mio, K., Ogino, T. Minami, K. and Takeda, S. 1976 *J. Phys. Soc. Jpn* **41**, 265.

Mjølhus, E. 1976 *J. Plasma Phys.* **16**, 321.

Mjølhus, E. and Wyller, J. 1988 *J. Plasma Phys.* **40**, 299.


https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press

Ng, C. S. and Bhattacharjee, A. 1996 *Astrophys. J.* **465**, 845.
Passot, T. and Sulem, P. L. 1995 In: *Small Scale Structures in Three-Dimensional Hydrodynamic and Magnetohydrodynamic Turbulence* (ed. M. Meneguzzi, A. Pouquet and P. L. Sulem), p. 386. Lecture Notes in Physics, Vol. 462, Springer-Verlag, Berlin.
Reiman, A. 1979 *Rev. Mod. Phys.* **51**, 311.
Rogister, A. 1971 *Phys. Fluids* **14**, 2733.
Sagdeev, R. Z. and Galeev, A. A. 1969 *Nonlinear Plasma Theory*. Benjamin, New York.
Skilling, J. 1975a *Mon. Not. R. Astron. Soc.* **173**, 245.
Skilling, J. 1975b *Mon. Not. R. Astron. Soc.* **173**, 255.
Spangler, S. R. and Sheerin, J. P. 1982 *J. Plasma Phys.* **27**, 193.
Sridhar, S. and Goldreich, P. 1994 *Astrophys. J.* **432**, 612.
Taniuti, T. and Wei, C. C. 1968 *J. Phys. Soc. Jpn* **24**, 941.
Terasawa, T., Hoshino, M., Sakai, J.-I. and Hada, T. 1986 *J. Geophys. Res.* **91**, 4171.
Verheest, F. 1992 *J. Plasma Phys.* **47**, 25.
Vinas, A. F. and Goldstein, M. L. 1991a *J. Plasma Phys.* **46**, 107.
Vinas, A. F. and Goldstein, M. L. 1991b *J. Plasma Phys.* **46**, 129.
Vladimirov, S. V. and Stenflo, L. 1997 *Phys. Plasmas* **4**, 1249.
Voitenko, Yu. M. 1998a *J. Plasma Phys.* **60**, 497.
Voitenko, Yu. M. 1998b *J. Plasma Phys.* **60**, 515.
Webb, G. M., Brio, M. and Zank, G. P. 1995 *J. Plasma Phys.* **54**, 201.
Webb, G. M., Zakharian, A., Brio, M., and Zank, G. P. 1997 *J. Phys. A: Math. Gen.* **30**, 4227.
Webb, G. M., Zakharian, A., Brio, M. and Zank, G. P. 1999 *J. Plasma Phys.* **61**, 295.
Whitham, G. B. 1974 *Linear and Nonlinear Waves*. Wiley, New York.
Wong, H. K. and Goldstein, M. L. 1986 *J. Geophys. Res.* **91**, 5617.
Zakharov, V. E., L'vov, V. S. and Falkovich, G. 1992 *Kolmogorov Spectra of Turbulence I. Wave Turbulence*. Springer-Verlag, New York.

https://doi.org/10.1017/S0022377800008370 Published online by Cambridge University Press