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Some remarks on random zeta functions

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Abstract. We answer in the negative a question of Viviane Baladi: in contrast to its deterministic or averaged versions, the random zeta function has no meromorphic extensions, hence no poles which would describe the Lyapunov exponents of the corresponding random transfer operator. In fact we show that the sequence of traces of the random iterates of this operator does not determine almost surely these exponents.

1. *Introduction*

Given a dynamical system $f : X \to X$ with a weight $g : X \to [0, \infty)$ (usually the inverse of the Jacobian of T w.r.t. some reference measure on X), the *Ruelle zeta function* is

$$\zeta(z) := \exp - \sum_{n \geq 1} \frac{z^n}{n} \sum_{x \in X : T^n x = x} g^{(n)}(x) \tag{1}$$

where $g^{(n)}(x) = g(x) \cdots g(T^{n-1}x)$. This is a natural generalization of the Artin–Mazur zeta function [**2**] obtained by taking $g \equiv 1$.

In good cases, equation (1) converges on some disk to a holomorphic function which can be extended to a meromorphic function on a larger disk $|z| < R$ and the poles of that extension have a dynamical interpretation:

$$\begin{aligned} &\text{the poles of } \zeta \text{ are exactly the inverses of the eigenvalues in } |z| > R^{-1} \\ &\text{of the transfer operator } \mathcal{L} \text{ acting on a suitable function space.} \end{aligned} \tag{2}$$

The *transfer operator* $\mathcal{L}$ is the key object in the study of the ergodic and statistical properties of (T, X, m). It is defined by

$$(\mathcal{L}\phi)(x) = \sum_{y \in f^{-1}(x)} g(y) \cdot \phi(y).$$

See [**6**] for background reading on this subject.

We shall be concerned with *random* dynamical systems, i.e. instead of iterating a fixed map $f : X \to X$, we consider a family of maps f_ω, $\omega \in \Omega$, $(\Omega, \mathbb{P})$ being a probability

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space, together with an automorphism $T : (\Omega, \mathbb{P}) \to (\Omega, \mathbb{P})$ and we study the *random orbits*, i.e. the sequences

$$x,\ f_\omega(x),\ f_\omega^2(x),\ f_\omega^3(x)\dots$$

where $f_\omega^n = f_{T^{n-1}\omega} \circ \cdots \circ f_\omega$. The weight function is also allowed to depend on ω, i.e. it becomes a function $g_\omega(x)$. We set $g_\omega^{(n)}(x) = g_{T^{n-1}\omega}(f_\omega^{n-1}(x)) \cdots g_\omega(x)$.

A large part of the theory extends to the random setting, see [**1, 3, 7–11**] and the references therein. In particular, Ruelle [**12**], considering the case of an i.i.d. perturbation of a given dynamical system, proved that if one defines an *averaged* zeta function

$$\zeta_{\mathrm{av}}(z) := \exp - \sum_{n\geq 1} \frac{z^n}{n} \int_\Omega \sum_{x\in X : f_\omega^n x = x} g_\omega^{(n)}(x)\, d\mathbb{P}(\omega)$$

and an averaged transfer operator

$$(\mathcal{L}_{\mathrm{av}}\phi)(x) = \int_\Omega \sum_{y\in f_\omega^{-1}(x)} g_\omega(y)\cdot\phi(y)\, d\mathbb{P}(\omega),$$

then, in a good setting, one still has the relationship equation (2).

Now, in the i.i.d. case, $T : (\Omega, P) \to (\Omega, P)$ is ergodic and the moduli of the eigenvalues of $\mathcal{L}_{\mathrm{av}}$ are exactly the Lyapunov exponents of the cocyle $\mathcal{L}_\omega$,

$$(\mathcal{L}_\omega\phi)(x) = \sum_{y\in f_\omega^{-1}(x)} g_\omega(y)\cdot\phi(y).$$

In particular, the moduli of the eigenvalues of $\mathcal{L}_{\mathrm{av}}$ can be determined almost surely by looking at a *fixed* ω. Baladi [**4**] defined the *random zeta function*

$$\zeta_\omega(z) := \exp - \sum_{n\geq 1} \frac{z^n}{n} \sum_{x\in X : f_\omega^n x = x} g_\omega^{(n)}(x)$$

and asked whether the relationship equation (2) still holds in the random (non-averaged) form.

We answer Baladi's question in the *negative*, by considering a natural and simple example she suggested [**5**]. We prove the following for this class of example.

- Almost surely, the random zeta function cannot be extended beyond its disk of holomorphy. Thus it does not define poles at all.
- Except for the largest one, the Lyapunov exponents cannot be read off almost surely from the sequence of the coefficients in the series defining $\zeta_\omega(z)$.

The simple setting is as follows. Let $X = \{1, 2\}^{\mathbb{Z}}$, $f_\omega = f : X \to X$ be the left shift $f((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$ and $g_\omega((x_n)_{n\in\mathbb{Z}}) = A_\omega(x_0, x_1)$, with $A_\omega \in M_2(\mathbb{R})$ a random matrix.

Now, the random transfer operator $\mathcal{L}_\omega$ is just the random matrix A_ω and the random zeta function is

$$\zeta_\omega(z) = \exp - \sum_{n\geq 1} \frac{z^n}{n} \operatorname{tr}(A_\omega^n)$$

with $A_\omega^n = A_{T^{n-1}\omega} \cdots A_\omega$.

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We shall prove the first point above by only assuming that A_ω is a 'non-degenerate' i.i.d. random perturbation of a hyperbolic matrix.

For the second point we assume additionally that the matrices are diagonal and that the law of the perturbation is smooth enough.

To make precise statements, we need a few definitions.

1.1. *Definitions and results.* Let (Ω_0, P) be a probability space and define a random matrix M, i.e. a measurable map $M : (\Omega_0, P) \to M_2(\mathbb{R})$ such that:

- for any non-trivial polynomial Q in the coefficients of M, $P(Q(M) = 0) = 0$;
- $P(\|M\| \leq 1) = 1$.

Let $\mathbb{P} = P^{\mathbb{Z}}$, let T be the left shift on $\Omega = \Omega_0^{\mathbb{Z}}$ and let $M : \Omega \to M_2(\mathbb{R})$ be defined by $M(\omega) = M(\omega_0)$. The sequence $M, M \circ T, \ldots$ defined on $(\Omega, \mathbb{P})$ is the i.i.d. process generated by (Ω_0, P, M).

Fix $\epsilon > 0$. For $\omega \in \Omega$, let $A(\omega) = \Lambda(\mathrm{Id} + \epsilon M(\omega)) \left(\begin{smallmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{smallmatrix}\right)$, where Λ is the unique positive number such that

$$\limsup_{n\to\infty} \|A_\omega^n\|^{1/n} = 1, \quad \mathbb{P}\text{-a.s. in } \omega, \tag{3}$$

with A_ω^n the random product $A(T^{n-1}\omega) \circ \cdots \circ A(\omega)$ ($A^0 = \mathrm{Id}$).

THEOREM 1. *Assume that $\epsilon > 0$ is small. Then $\mathbb{P}$-almost surely, the series $\exp - \sum_{k=0}^{n} (z^n/n) \operatorname{tr}(A_\omega^n)$ converges in the unit disk to a function $\zeta_\omega(z)$ and the unit circle is its natural boundary: one cannot analytically extend this function to any connected open set strictly containing the unit disk.*

COROLLARY 1. *Assume that $\epsilon > 0$ is small. Then $\mathbb{P}$-almost surely, $\zeta_\omega(z)$ has no meromorphic extension beyond the unit disk.*

For our second result, let $0 < \lambda_1, \lambda_2 < 1$ and $0 < \epsilon < 1/2$ and assume that the random matrix M is $\left(\begin{smallmatrix} m_{11} & 0 \\ 0 & m_{22} \end{smallmatrix}\right)$ where $a = \log(1 + \epsilon m_{11})$ and $b = \log(1 + \epsilon m_{22})$ are random variables such that the law of $(a, b) \in \mathbb{R}^2$ is $Q := q(x, y)\, dx\, dy$ where q is Lipschitz and satisfies $\log q \in L^1(Q)$.

Define $A_\omega(\lambda)$ as A_ω except that $\frac{1}{3}$ is replaced with λ.

Let $T_i : \Omega \to \mathbb{C}^{\mathbb{N}}$ be defined by $T_i(\omega) = (\operatorname{tr}(A_\omega^n(\lambda_i)))_{n\in\mathbb{N}}$.

THEOREM 2. *If $\Omega_1, \Omega_2 \subset \Omega$ have full measure then*

$$T_1(\Omega_1) \cap T_2(\Omega_2) \neq \emptyset.$$

COROLLARY 2. *There does not exist a function $F : \mathbb{C}^{\mathbb{N}} \to \mathbb{R}$ such that, for a.e. ω and all λ in a set with at least two points,*

$$F((\operatorname{tr}(A_\omega^n(\lambda)))_{n\in\mathbb{N}}) = \lambda.$$

We believe these results to be much more general and, in particular, not to be restricted to the i.i.d. case. To support this belief, we point out that according to the Pólya–Carlson Theorem [**13**, Theorem 5.2, p. 381], if A_ω is an arbitrary (thus not necessarily i.i.d. or

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even mixing) process of random matrices with integer coefficients, then ζ_ω satisfies the conclusion of Theorem 1 except if it is rational, clearly an exceptional case.

Specifically, we make the following conjecture.

CONJECTURE. *Let A be a random matrix process. Assume that it is not 'exceptional' (in particular, not periodic). Then the conclusions of both Theorems 1 and 2 hold.*

2. *Proof of Theorem 1*

2.1. *Strategy of proof.* Observe first that $|\operatorname{tr} A^n| \le 2\|A^n\|$, therefore the (random) radius of convergence, $\liminf_{n\to\infty} |n^{-1}\operatorname{tr}(A^n)|^{-1/n}$, of the series defining $\zeta_\omega(z)$ is at least 1, proving the (trivial) part of the theorem.

To prove the interesting part, we proceed by contradiction. We assume that there is a positive probability that $\zeta_\omega(z)$ can be extended analytically to some connected open set U_ω containing strictly the unit disk (no regularity is assumed for the map $\omega \mapsto U_\omega$).

CLAIM 1. *Under this assumption, there exist $\Omega_* \subset \Omega$ with positive measure m_*, a point β_* on the unit circle, constants $C_* < \infty$ and $\kappa_* > 0$ such that, for all $\omega \in \Omega_*$, for all $k \ge 0$*

$$\frac{1}{2^k}\left|\sum_{\ell=0}^{k}\binom{k}{\ell}\beta_*^\ell \operatorname{tr}(A_\omega^\ell)\right| \le C_* e^{-\kappa_* k}. \tag{4}$$

We defer the proof of this claim to the next subsection. For now, let us explain how we are going to use it to bring about the contradiction. The idea is that the randomness of the matrices A, together with the hyperbolicity, implies that the sequence $(\operatorname{tr}(A^n))_{n\ge 0}$ is essentially stationary and non-constant. This will contradict the very rigid constraint of equation (4). To show this, we shall consider a linear combination of the series ζ_ω for ω's differing only in one of the matrices $A_{T^n\omega}$, $n \ge 0$, in order to deal with just one term in equation (4).

To define these alternate ω's, we shall need six independent copies of the i.i.d. process $(\Omega, \mathbb{P}, M)$. Set $\tilde\Omega = \Omega^6 = \Omega^{\{0,1,\dots,5\}}$, $\tilde{\mathbb{P}} = \mathbb{P}^6$ and $\tilde\pi(\tilde\omega) = \tilde\omega^5 \in \Omega$.

Let $\tilde\omega \in \tilde\Omega_* := \tilde\pi^{-1}(\Omega_*)$ and n be a large integer to be specified later.

The alternate ω's, ω and $\omega^0, \dots, \omega^4$ will be obtained from $\tilde\omega \in \tilde\Omega$ by setting $\omega = \tilde\pi(\tilde\omega)$, $\omega^0 = \tilde\pi(\Phi_n^0(\tilde\omega))$ and $\omega^j = \tilde\pi(\Phi_{n+1}^j(\tilde\omega))$ for $j = 1, \dots, 4$ where

$$\Phi_n^i(\tilde\omega) = \tilde\alpha \quad \text{with } \tilde\alpha_k^j = \begin{cases} \tilde\omega_k^j, & \text{if } k \ne n, \\ \tilde\omega_k^{\sigma_i j}, & \text{if } k = n, \end{cases}$$

σ_i being the permutation on $\{0, 1, 2, 3, 4, 5\}$ that exchanges i and 5 and leaves the rest fixed.

We shall find coefficients $c_1, \dots, c_4$ such that

$$\sum_{i=1}^{4} c_i \operatorname{tr}(A_{\omega^i}^k - A_\omega^k) - \operatorname{tr}(A_{\omega^0}^k - A_\omega^k) = 0$$

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for all $k \neq n+1$. Therefore, using equation (4) with $k = 2(n+1)$, we shall get, provided $\omega, \omega^0, \ldots, \omega^4 \in \Omega_*$,

$$|\mathrm{tr}(A_{\omega^0}^{n+1} - A_{\omega}^{n+1})| \leq 10 \max_i |c_i| \cdot C_* e^{-\kappa_*(n+1)}$$

from which we shall obtain a contradiction.

Let us see immediately how the condition $\omega, \omega^0, \ldots, \omega^4 \in \Omega_*$ can be satisfied.

LEMMA 3. *For $\tilde{\mathbb{P}}$-almost all $\tilde{\omega} \in \tilde{\Omega}_*$, for all sequences of integers $n_k \nearrow \infty$, there exists a subsequence m_k such that, for all large k,*

$$\Phi_{m_k}^0(\tilde{\omega}), \Phi_{m_k+1}^0(\tilde{\omega}), \ldots, \Phi_{m_k+1}^4(\tilde{\omega}) \in \tilde{\Omega}_*.$$

Proof. Observe first that each Φ_n^i is an automorphism of $(\tilde{\Omega}, \tilde{\mathbb{P}})$ because of the product structure of $\tilde{\mathbb{P}}$.

Now, let $k \geq 1$. Fix $\delta_k > 0$ such that the δ_k-neighborhood $B(\Omega_*, \delta_k)$ satisfies $\mathbb{P}(B(\Omega_*, \delta_k) \setminus \Omega_*) < 1/2^k$. Define m_k to be the smallest n_j, $j \geq 1$, such that, for all $p \geq n_j$, $d(\tilde{\omega}, \Phi_p^i(\tilde{\omega})) < \delta_k$. Observe that

$$\begin{aligned} &\sum_{k\geq 1} \tilde{\mathbb{P}}(\{\tilde{\omega} \in \tilde{\Omega}_* : \exists i\, \Phi_{m_k}^i(\tilde{\omega}) \notin \tilde{\Omega}_*\}) \\ &\qquad \leq \sum_{k\geq 1} \tilde{\mathbb{P}}(\{\tilde{\omega} \in \tilde{\Omega}_* : \exists i\, \Phi_{m_k}^i(\tilde{\omega}) \in B(\tilde{\Omega}_*, \delta_k) \setminus \tilde{\Omega}_*\}) < \infty. \end{aligned} \tag{5}$$

By the Borel–Cantelli Lemma, this means that for $\tilde{\mathbb{P}}$-almost all $\tilde{\omega} \in \tilde{\Omega}_*$, for all k large enough, $\Phi_{m_k}^i(\tilde{\omega}) \in \tilde{\Omega}_*$ for $i = 0, \ldots, 4$. But this is the statement of the lemma. □

2.2. *Proof of Claim 1.* To begin with, we observe that $\zeta_\omega(z)$ can be extended to some connected domain strictly containing the unit disk $D_1 \subset \mathbb{C}$ iff $Z_\omega(z) := \sum_{n\geq 1} (z^n/n)\,\mathrm{tr}(A_\omega^n)$ also can. Obviously, if Z_ω can be extended then just use $\zeta_\omega = \exp \circ Z_\omega$ to extend ζ_ω to the same domain.

Conversely, assume that ζ_ω can be extended to some open connected domain $U \supsetneq D_1$. ζ_ω has only finitely many zeros in any compact subset of U. Therefore there exists β on the unit circle S^1 and $\epsilon > 0$ such that ζ_ω extends to an analytic, non-vanishing function on $D_1 \cup B(\beta, \epsilon)$. This set being simply connected, there exists a unique analytic logarithm of ζ_ω on this set taking the value 0 at $z = 0$. But on D_1 this logarithm must coincide with Z_ω. Therefore we do have a non-trivial extension of Z_ω. The equivalence is proved.

To continue, we prove that one can find $\beta_* \in S^1$ and $\epsilon_* > 0$ such that Z_ω extends to $D_1 \cup B(\beta_*, \epsilon_*)$ with positive probability. Indeed, let $\beta_1, \beta_2, \ldots$ be a dense sequence in S^1 and consider the measurable subsets

$$\Omega(n) = \{\omega \in \Omega : Z_\omega \text{ can be analytically extended to } D_1 \cup B(\beta_n, 1/n)\}.$$

The assumption implies that the measure of $\bigcup_{n\geq 1} \Omega(n)$ is strictly positive. As this is a countable union, there exists some n such that $\mathbb{P}(\Omega(n)) > 0$, i.e. Z_ω can be extended to $D_1 \cup B(\beta_*, \epsilon_*)$ (setting $\beta_* = \beta_n$ and $\epsilon_* = 1/n$).

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To conclude, we follow [**13**, pp. 305–306]. Let $Y(z) = Z(-\beta_* z/(1+z))$. As $z \mapsto -\beta_* z/(1+z)$ maps D_1 univalently to the half-plane $\operatorname{Re} z > -1/2$ with S^1 mapped to $\operatorname{Re} z = -1/2$, Y is analytic on the disk $D_{1/2}$ and on the boundary of this disk only $w = -1/2$ can be singular. This point $-1/2$ is the image of β_*, therefore it is *not* singular. Y having no singular point on $|z| = 1/2$, the radius of convergence of Y is strictly greater than $1/2$. Thus,

$$\frac{1}{2^n}|b_n| \le C_* e^{-\kappa_* n} \tag{6}$$

for some $C_* < \infty$ and $\kappa_* > 0$ if we write $Y(z) = \sum_{n\ge 0} b_n z^n$. Finally,

$$\begin{aligned}
Y(z) &= \sum_{n\ge 1} \frac{\operatorname{tr}(A^n)}{n} \left(\beta_* \frac{-z}{1+z}\right)^n \\
&= \sum_{n\ge 1} \frac{\beta_*^n \operatorname{tr}(A^n)}{n} (-z)^n (1+z) \sum_{r\ge 0} \binom{n+r}{r} (-1)^r z^r \\
&= \sum_{r\ge 0} \sum_{n\ge 1} \frac{\beta_*^n \operatorname{tr}(A^n)}{n} (-z)^n (1+z) \binom{n+r}{r} (-1)^r z^r \\
&= \sum_{r\ge 0} \sum_{k\ge r+1} \frac{\beta_*^{k-r} \operatorname{tr}(A^{k-r})}{k-r} (-z)^n (1+z) \binom{k}{r} (-1)^r z^r \quad (k = n+r) \\
&= (1+z) \sum_{k\ge 1} \sum_{r=0}^{k-1} z^k (-1)^k \frac{\beta_*^{k-r} \operatorname{tr}(A^{k-r})}{k-r} \binom{k}{r} \\
&= (1+z) \sum_{k\ge 0} z^k (-1)^k \sum_{r=0}^{k-1} \frac{\beta_*^k \operatorname{tr}(A^k)}{k} \binom{k}{r}.
\end{aligned}$$

But, together with equation (6), this implies the claim. □

2.3. *Estimation of the traces.* Fix $0 < \epsilon < 1/100$ such that

$$|M(T^n\omega)_{11} - M(T^n\omega^0)_{11}| > 100\epsilon \tag{7}$$

for a set of integers n with density at least $99/100$, for almost all $\tilde{\omega}$. This is possible because of the non-degeneracy of the measure P and the independence between $M(T^n\omega)$ and $M(T^n\omega^0)$ (recall that $\omega^0 = \tilde{\pi}(\Phi_n^0(\tilde{\omega}))$ and $\omega = \tilde{\pi}(\tilde{\omega})$).

After discarding a subset with zero measure from $\Omega_* \ni \tilde{\omega}$, we can assume that, if $v := \binom{1}{0}$, for all large n

$$e^{-(\kappa_*/2)n} \le \|A_\omega^n . v\| \le e^{(\kappa_*/2)n} \tag{8}$$

and

$$A^n . v = \|A^n . v\| \begin{pmatrix} 1 \\ \pm 2\epsilon \end{pmatrix}$$

where $\pm 2\epsilon$ represents a number in $(-2\epsilon, 2\epsilon)$.

The first fact is just that the largest Lyapunov exponent of the cocycle A is zero (recall equation (3)). The second fact follows from the forward invariance under all matrices $A(\omega)$, $\omega \in \Omega$, of the cone $C_+ = \{(x, y) \in \mathbb{R}^2 : |y| \le 2\epsilon|x|\}$ which contains v.

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Define

$$w := \lim_{N\to\infty} \frac{(A_\omega^N)^{-1}.\binom{1}{0}}{\left\|(A_\omega^N)^{-1}.\binom{1}{0}\right\|}.$$

Observe that w depends on ω. By hyperbolicity this limit exists and satisfies

$$\frac{1}{4^n} \le \|A_\omega^n.w\| \le \frac{1}{2^n},$$
$$A_\omega^n.w = \|A_\omega^n.w\| \begin{pmatrix} \pm 2\epsilon \\ 1 \end{pmatrix}$$

for all $n \ge 0$.

This follows from the forward invariance under all matrices $A(\omega)^{-1}$, $\omega \in \Omega$, of the cone $C_- = \{(x, y) \in \mathbb{R}^2 : |x| \le 2\epsilon|y|\}$ which contains $\binom{0}{1}$.

(v, w) is a basis of $\mathbb{R}^2$. Let (v^*, w^*) be the dual basis (both vectors depend on ω). Remark that $v_* = (1 \pm 2\epsilon, \pm 2\epsilon)$ and $w^* = (\pm 2\epsilon, 1 \pm 2\epsilon)$.

We have, by definition,

$$\operatorname{tr}(A_\omega^n) = v^* \cdot A_\omega^n.v + w^* \cdot A_\omega^n.w.$$

Let us compute

$$\begin{aligned} A_{\omega^0}^{n+1}.v - A_\omega^{n+1}.v &= (A_{T^n\omega^0} - A_{T^n\omega}).A_\omega^n.v \\ &= \Lambda\epsilon(M_{T^n\omega^0} - M_{T^n\omega}).\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}.A_\omega^n.v. \end{aligned}$$

Now, writing $M_{T^n\omega^0} - M_{T^n\omega} = (m'_{ij})_{ij}$, we get as $A_\omega^n.v = \|A_\omega^n.v\|\binom{1}{\pm 2\epsilon}$,

$$A_{\omega^0}^{n+1}.v - A_\omega^{n+1}.v = \epsilon\Lambda\|A_\omega^n.v\| \begin{pmatrix} m'_{11} \pm \frac{2}{3}\epsilon m'_{12} \\ m'_{21} \pm \frac{2}{3}\epsilon m'_{22} \end{pmatrix}.$$

Likewise,

$$A_{\omega^0}^{n+1}.w - A_\omega^{n+1}.w = \epsilon\Lambda\|A_\omega^n.w\| \begin{pmatrix} \frac{1}{3}m'_{12} \pm 2\epsilon m'_{11} \\ \frac{1}{3}m'_{22} \pm 2\epsilon m'_{21} \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \operatorname{tr} A_{\omega^0}^{n+1} - \operatorname{tr} A_\omega^{n+1} &= (1 \pm 4\epsilon)\epsilon\Lambda\|A_\omega^n.v\|(m'_{11} \pm \epsilon) \pm (1 \pm 4\epsilon)\epsilon\Lambda\|A^n.w\| \\ &= \epsilon\Lambda\|A_\omega^n.v\|(m'_{11} \pm 6\epsilon) \pm \frac{2\epsilon\Lambda}{2^n} \\ &= \epsilon\Lambda\|A_\omega^n.v\|(m'_{11} \pm 7\epsilon) \end{aligned}$$

for all large n. In particular, for a set of integers n with density at least $99/100$, we have

$$|\operatorname{tr} A_{\omega^0}^{n+1} - \operatorname{tr} A_\omega^{n+1}| \ge \Lambda e^{-(\kappa_*/2)n} 93\epsilon^2 \tag{9}$$

using equation (7).

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2.4. *Conclusion of the proof of Theorem 1.* We would like to find random coefficients $c_1, \dots, c_4$ such that

$$\sum_{i=1}^{4} c_i \operatorname{tr}(A^k_{\omega^i} - A^k_\omega) = \operatorname{tr}(A^k_{\omega^0} - A^k_\omega).$$

As $A_{T^k\omega^i} = A_{T^k\omega}$ for $k \le n+1$, it is enough to have

$$\sum_{i=1}^{4} c_i (A_{T^{n+1}\omega^i} - A_{T^{n+1}\omega}) A_{T^n\omega} = (A^2_{T^n\omega^0} - A^2_{T^n\omega}). \tag{10}$$

CLAIM 2. *Maybe after increasing C_* there is a sequence of integers n with density at least 99/100, such that, for each n in this sequence, one can find $c_1, \dots, c_4 \in \mathbb{R}$ with $|c_i| \le C_*$ which satisfy equation (10).*

We first prove that the claim implies Theorem 1.

For a.e. $\tilde\omega \in \tilde\Omega_*$, there are arbitrarily large integers n taken in the subsequence given by the above claim and such that estimates (8) and (9) hold and $\omega^0, \dots, \omega^4 \in \Omega_*$.

Observe that, setting $t_k = \sum_{i=0}^{4} c_i \operatorname{tr}(A^k_{\omega^i} - A^k_\omega)$ with $c_0 = -1$, we have for $k = n+1$,

$$|t_{n+1}| = |\operatorname{tr}(A^{n+1}_{\omega^0} - A^{n+1}_\omega)| \ge 93\Lambda e^{-(\kappa_*/2)(n+1)}\epsilon^2.$$

On the other hand, as $\omega, \omega^0, \dots, \omega^4 \in \Omega_*$, equation (4) with $k = 2(n+1)$ gives

$$|t_{n+1}| \le 10 \max|c_i| \cdot C_*(n+1)e^{-\kappa_* \cdot 2(n+1)}.$$

Thus,

$$e^{-(\kappa_*/2)(n+1)}(n+1) \ge \frac{93\Lambda}{10C_*^2}\epsilon^2 > 0,$$

a contradiction for large n, proving the theorem.

We now prove the claim.

Proof of Claim 2. Observe that equation (10) is a linear equation in $c_1, \dots, c_4$. Each coefficient of that equation is a given quadratic polynomial, with coefficients ± 1, in the entries of the matrices:

$$M(T^n\omega),\ M(T^{n+1}\omega),\ M(T^n\omega^0),\ M(T^{n+1}\omega^i) \quad (i = 1, \dots, 4).$$

It follows firstly that the determinant

$$\Delta(M(T^n\omega), M(T^{n+1}\omega), M(T^n\omega^0), M(T^{n+1}\omega^i))$$

(which we will write as $\Delta(n)$ for short) associated to equation (10) is a fixed polynomial in the coefficients of the above-mentioned matrices.

Now, if $A(T^{n+1}\omega) = 0$, $A(T^n\omega) = \mathrm{Id}$ and if

$$A(T^{n+1}\omega^1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A(T^{n+1}\omega^2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$A(T^{n+1}\omega^3) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A(T^{n+1}\omega^4) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{11}$$

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then equation (10) obviously has a unique solution; therefore, $\Delta(n)$ is a non-trivial polynomial.

By the non-degeneracy assumption of the probability P and the product structure of $\mathbb{P}$, the non-trivial polynomial $\Delta(n)$ is almost surely non-zero. Therefore, there exists $C > 0$ such that $|\Delta(n)| > C$ for a set of integers n with density at least $99/100$.

On the other hand, observe that with the random matrix A being uniformly bounded, there exists a uniform bound C' on the Euclidean norm of the columns of the coefficients in equation (10).

Using Cramer's Formula for solving equation (10), we see that for these n the solution satisfies $|c_i| \leq C'^4/C \leq C_*$ by enlarging C_*.

This proves the claim. □

3. *Proof of Theorem 2*

Recall that $0 < \lambda_1 < \lambda_2 < 1, \epsilon > 0$ and that

$$A_\omega(\lambda) = \begin{pmatrix} e^{a(\omega)} & 0 \\ 0 & \lambda e^{b(\omega)} \end{pmatrix} \quad \text{and} \quad A^n_\omega(\lambda) = \begin{pmatrix} e^{a(\omega)+\cdots+a(T^{n-1}\omega)} & 0 \\ 0 & \lambda^n e^{b(\omega)+\cdots+b(T^{n-1}\omega)} \end{pmatrix}. \tag{12}$$

CLAIM 3. *There exists a measurable map $\tau : (\Omega, \mathbb{P}) \to (\Omega, \mathbb{P})$ such that $\tau_*\mathbb{P} \ll \mathbb{P}$ and*

$$\operatorname{tr}(A^n_\omega(\lambda_2)) = \operatorname{tr}(A(\lambda_1)^n_{\tau\omega})$$

for all $n \geq 0$ and $\mathbb{P}$-a.e. ω.

In particular, $\mathbb{P}(\tau(\Omega)) > 0$, implying Theorem 2. Indeed,

$$T_1(\Omega_1) \cap T_2(\Omega_2) \supset T_1(\Omega_1 \cap \tau\Omega_2) = T_2(\tau^{-1}\Omega_1 \cap \Omega_2).$$

We now prove the claim.

Proof of Claim 3. We assume, without further loss of generality, that $\Omega = \prod_{n\in\mathbb{N}} \mathbb{R} \times \mathbb{R}$ with $\mathbb{P} = \prod_{n\in\mathbb{N}} Q$ (recall that Q is the law of (a, b)).

Define $\tau((a_n, b_n)_{n\in\mathbb{N}}) =: (a'_n, b'_n)_{n\in\mathbb{N}}$ as follows. Let

$$\delta_n = \log(1 + (\lambda_2^n - \lambda_1^n)e^{b_0+\cdots+b_n-(a_0+\cdots+a_n)}) \quad \text{for all } n \geq 0$$

and $\delta_1 = 0$. Set

$$\begin{aligned} a'_n &= a_n - \delta_{n-1} + \delta_n - \delta_{n-1}, \\ b'_n &= b_n. \end{aligned}$$

Observe that

$$\begin{aligned} \operatorname{tr}(A^n_{\tau\omega}(\lambda_1)) &= e^{a'_0+\cdots+a'_n} + \lambda_1^n e^{b'_0+\cdots+b'_n} = e^{a_0+\cdots+a_n+\delta_n} + \lambda_1^n e^{b_0+\cdots+b_n} \\ &= e^{a_0+\cdots+a_n} + (\lambda_2^n - \lambda_1^n)e^{b_0+\cdots+b_n} + \lambda_1^n e^{b_0+\cdots+b_n} \\ &= \operatorname{tr}(A^n_\omega(\lambda_2)). \end{aligned}$$

Now,

$$\frac{d\tau_*\mathbb{P}}{d\mathbb{P}} = \prod_{n\in\mathbb{N}} \frac{q(a'_n, b'_n)}{q(a_n, b_n)}.$$

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We must prove that the above infinite product is convergent. We have

$$\log \frac{q(a_n', b_n')}{q(a_n, b_n)} \leq \log\left(1 + \frac{q(a_n', b_n') - q(a_n, b_n)}{q(a_n, b_n)}\right) \leq \frac{K|a_n' - a_n|}{q(a_n, b_n)}.$$

Using the law of the iterated logarithm [**14**, p. 48] to estimate the i.i.d. sums $a_0 + \cdots + a_n$, $b_0 + \cdots + b_n$, we get

$$a_n' - a_n = \mathcal{O}(\lambda_2^n e^{b_0+\cdots+b_n-(a_0+\cdots+a_n)}) = \mathcal{O}(\lambda_2^n e^{Cn^{0.6}}) = \mathcal{O}(\rho^n)$$

for any $\rho \in (\lambda_2, 1)$ (the $\mathcal{O}$-estimate above contains a factor depending on ω).

On the other hand, let $\rho' \in (\rho, 1)$ and compute

$$\sum_{n\geq 0} \mathbb{P}(\{\omega : q(T^n\omega) < \rho'^n\}) = \sum_{n\geq 0} \mathbb{P}(\{\omega : q(\omega) < \rho'^n\}) \leq \int_\Omega \left|\frac{\log q}{\log \rho'}\right| d\mathbb{P} < \infty$$

as $\log q \in L^1(Q)$ so that $\log q \in L^1(\mathbb{P})$. Therefore, by the Borel–Cantelli Lemma, $q(T^n\omega) \geq C(\omega)\rho'^n$ for all $n \geq 0$ and some $C(\omega) > 0$ a.e. (observe that $q > 0$ a.e. as $\log q \in L^1(\mathbb{P})$). Therefore,

$$\sum_{n\geq 0} \frac{K|a_n' - a_n|}{q(a_n, b_n)} \leq \frac{K}{C(\omega)} \sum_{n\geq 0} \left(\frac{\rho}{\rho'}\right)^n < \infty.$$

Thus, the infinite product converges and $0 < d\tau_*\mathbb{P}/d\mathbb{P} < \infty$.

This proves the claim and therefore the theorem. □

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