ALGEBRAIZABLE WEAK LOGICS

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Abstract. We extend the framework of abstract algebraic logic to weak logics, namely, logical systems that are not necessarily closed under uniform substitution. We interpret weak logics by algebras expanded with an additional predicate, and we introduce a loose and strict version of algebraizability for weak logics. We study this framework by investigating the connection between the algebraizability of a weak logic and the algebraizability of its schematic fragment, and we then prove a version of Blok and Pigozzi's Isomorphism Theorem in our setting. We apply this framework to logics in team semantics and show that the classical versions of inquisitive and dependence logic are strictly algebraizable, while their intuitionistic versions are only loosely so.

§1. Introduction. At least since Tarski's 1936 article on logical notions, logic has often been understood as the subject studying those notions which are "invariant under all possible one-one transformations of the world onto itself" [29, p. 149]. Tarski's view was inspired by Felix Klein's 1872 *Erlanger Programm*, and expresses very neatly the philosophical idea of logic as the discipline with the most general character. While, for instance, metric geometry studies notions that are invariant under transformations that preserve distances, and topology studies notions invariant under continuous maps, logic deals with notions which are invariant under arbitrary bijections. Formally, this led to the technical definition of logics as *consequence relations which are additionally closed under uniform substitutions*.

As a matter of fact, Tarski's approach can be seen as a key step in the transition from the "symbolic" algebraic logic of the nineteenth century (exemplified by the works of Boole, De Morgan, Jevons, Peirce, etc.) to the contemporary field of abstract algebraic logic. Using the general notion of logic as a closure operator on the term algebra, Rasiowa made the first steps into the "mathematical" version of algebraic logic, and in particular she developed in *An Algebraic Approach to Non-Classical Logics* [27] a general theory of algebraization for implicative logics. Finally, the algebraic approach was put in its contemporary formulation by Blok and Pigozzi, who introduced in *Algebraizable Logics* [4] the notion of algebraizable logics and developed their general theory.

Received February 26, 2025.

2020 *Mathematics Subject Classification.* Primary 03G27, Secondary 03B50, 03B55, 03C05.

Key words and phrases. algebraizability, weak logics, uniform substitution, inquisitive logic, dependence logic.

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0022-4812/00/0000-0000
DOI:10.1017/jsl.2026.10180

https://doi.org/10.1017/jsl.2026.10180 Published online by Cambridge University Press

In recent years, however, there has been an increasing interest into systems which do possess a logical nature but fail nonetheless to be closed under uniform substitution in Tarski's original strong sense. The field of modal logic is particularly rich of such examples: Buss' pure provability logic [6], public announcement logic (PAL) [19, 20], and other epistemic logics are all examples of this behaviour. Furthermore, propositional logics based on team semantics, such as inquisitive [12] and dependence logic [30], also do not satisfy Tarski's requirement of closure under uniform substitution. We believe that this state of affairs is not a mistake requiring correction, but that it rather reflects the increasing plurality of logics and their aptness for applications. At the same time, however, the anomalous behaviour of these logical systems prevented so far a uniform abstract study, and did not allow for immediately applying facts and results from abstract algebraic logic, thus forcing scholars to reprove abstract results in these settings, or adapt standard techniques to their specific situation.

Motivated by these facts, we propose in this article a generalisation of the notion of algebraizable logic which breaks apart from Tarski's original view of logics as being invariant under *all* substitutions. In other words, we want to study consequence operators that are invariant under *some* substitutions, but not necessarily all. We introduce the notion of *weak logic*, generalising previous definitions of Ciardelli and Roelofsen [12] and Punčochář [24], as a consequence relation which is invariant under all substitutions which map atomic variables to atomic variables. In other words, we relax Tarski's constraint of invariance under arbitrary substitutions and we require that it holds only with respect to these so-called atomic substitutions (cf. Section 3 below).

We proceed in this article as follows. In Sections 3–6 we develop at length the key aspects of the theory of algebraizable weak logics. In Section 3 we define weak logics and introduce what we call expanded algebras as their corresponding algebraic notion. In Section 4 we introduce the notions of loose and strict algebraizability for weak logics and we show that the (loose or strict) equivalent algebraic semantics of a weak logic is unique, thus mirroring the classical result by Blok and Pigozzi for standard algebraizable logics. Then, in Section 5, we study the relation between a weak logic and its schematic fragment of consequences invariant under arbitrary substitutions. In particular, we provide a characterisation of the algebraizability of weak logics in terms of the algebraizability of their schematic fragment. Additionally, we introduce the notion of *standard companion* and generalise to this setting previous results from [1, 3]. Finally, in Section 6 we develop on these results and we prove a version of Blok and Pigozzi's Isomorphism Theorem for strictly algebraizable weak logics. These sections thus establish the fundamentals of the theory of algebraizable weak logics and show that several key results from the field of abstract algebraic logic carry on to the setting without uniform substitution. On a partially separate line of inquiry, in Section 7 we take a small excursus explaining how to adapt the usual matrix semantics of propositional logics to the setting of weak logics. We show that (similarly to the standard setting) every weak logic admits a matrix semantics, and we describe the class of reduced models of (strictly) algebraizable weak logics. We put our abstract framework to the test in Section 8, where we apply it to the specific case of inquisitive and dependence logics. In particular, we build on previous results from [3, 11, 26] to show that the classical version of inquisitive and dependence

https://doi.org/10.1017/jsl.2026.10180 Published online by Cambridge University Press

logic is strictly algebraizable, while their intuitionistic versions are only loosely so. To our eyes, this indicates a significant difference between the classical version of inquisitive and dependence logic—which can essentially be recasted as a theory over their schematic fragment—and their intuitionistic counterparts, which do not admit such reformulation.

Together with these results on inquisitive and dependence logic, we regard as the main contribution of the present work the fact that it provides a framework for reasoning about logical systems lacking uniform substitution. In particular, this work relates several algebraic studies on inquisitive logic [2, 3, 24, 25], dependence logic [26], and polyatomic logics [1], by showcasing all the algebraic semantics from these works as instances of what we call core semantics in the present article. Given the increasing popularity of logics without uniform substitution in the logic literature, we hope that our approach will be useful to researchers working in these areas also in the future.

§2. Preliminaries. We recall in this section some basic facts concerning logics, algebras, and model theory. We also fix some notation that we shall follow throughout the rest of the article. The following general context sets the framework of our work.

CONTEXT 2.1. Throughout this article we always let Var be a set of variables and we let $\mathcal{L}$ be an algebraic (i.e., purely functional) signature (unless we specify otherwise). We denote by $\mathsf{Fm}_{\mathcal{L}}$ both the set of first-order terms over Var in the signature $\mathcal{L}$ and the term algebra in the signature $\mathcal{L}$ over Var. We omit the index $\mathcal{L}$ when it is clear from the context. Notice that, since we are often dealing with propositional logical systems, we often refer to elements of $\mathsf{Fm}_{\mathcal{L}}$ also as *(propositional) formulas* in the language $\mathcal{L}$. These should not be confused with the first-order formulas in the signature $\mathcal{L}$.

Given the language $\mathcal{L}$, we recall the standard abstract Tarskian definition of (propositional) logic. We refer the reader to [16, Section 1] for more on consequence relations and logics, and for slight variations of these definitions.

DEFINITION 2.2. A *(finitary) consequence relation* is a relation $\vdash \subseteq \wp(\mathsf{Fm}) \times \mathsf{Fm}$ such that, for all $\Gamma \subseteq \mathsf{Fm}$:

(1) $\Gamma \vdash \phi$ for all $\phi \in \Gamma$;
(2) if $\Gamma \vdash \phi$ for all $\phi \in \Delta$, and $\Delta \vdash \psi$, then $\Gamma \vdash \psi$;
(3) if $\Gamma \vdash \phi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \phi$;
(4) if $\Gamma \vdash \phi$, then there is some $\Delta \subseteq \Gamma$ such that $|\Delta| < \aleph_0$ and $\Delta \vdash \phi$.

REMARK 2.3. Condition 2.2(3) already follows from Condition 2.2(1) together with Condition 2.2(2) (cf. also [16, p. 14]), but we include it to make explicit the fact that we always assume consequence relations (and logics) to be monotone. Differently, Condition 2.2(4) is not always required in the definition of consequence relation and it essentially restricts the focus to the finitary ones. This reflects our elementary approach to the subject, as Condition 2.2(4) allows us to translate propositional systems into first-order logic and to avoid the use of infinitary logical systems lacking compactness. We stress that this requirement is not necessary, and it is possible to study consequence relations and propositional logics where

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this condition fails. We refer the interested reader to [16] for discussions of Condition 2.2(4) and also for an (algebraic) treatment of non-finitary propositional logics.

DEFINITION 2.4. A *substitution* is an endomorphism $\sigma : \mathsf{Fm} \to \mathsf{Fm}$ of the $\mathcal{L}$-term algebra. We denote by $\mathsf{Subst}(\mathcal{L})$ the set of all substitutions in the language $\mathcal{L}$. If $x_1, \dots, x_n \in \mathsf{Var}$ and $\phi_1, \dots, \phi_n$ are arbitrary $\mathcal{L}$-formulas, we denote by $\Gamma[\phi_1 \dots \phi_n / x_1 \dots x_n]$ the result of simultaneously substituting each ϕ_i for all occurrences of x_i in the formulas in Γ.

DEFINITION 2.5 (Logic). A consequence relation $\vdash$ is closed under *uniform substitution* if $\Gamma \vdash \phi$ entails $\sigma[\Gamma] \vdash \sigma(\phi)$ for all substitutions $\sigma \in \mathsf{Subst}(\mathcal{L})$. A *(standard) logic* is a consequence relation $\vdash$ which is closed under uniform substitution.

EXAMPLE 2.6. Obvious examples of standard logics are the classical propositional logic `CPC` and the intuitionistic propositional logic `IPC`. Non-examples of logics in this sense are first-order logic, as it is not a consequence relation of the propositional term algebra, infinitary logics, and higher-order systems.

In the context of abstract algebraic logic, one is often interested in the *algebraic semantics* of a propositional logic $\vdash$. This is provided by algebras, i.e., first-order structures in some purely functional language $\mathcal{L}$. We first fix some notation concerning first-order models.

NOTATION 2.7. *Let $\mathcal{L}$ be a first-order language, not necessarily functional. We use Latin letters $A, B, \dots$ both to denote first-order $\mathcal{L}$-structures and their underlying domain. When confusion may arise, we also write $\mathsf{dom}(A)$ to refer to the underlying universe of A. For all function symbols $f \in \mathcal{L}$ and all relation symbols $R \in \mathcal{L}$, we write f^A and R^A for their interpretation in A. We use the same notations for symbols and their interpretation when it does not cause confusion. If $t(\bar{x})$ is a term in the language $\mathcal{L}$ (i.e., a propositional formula), we usually call its interpretation t^A a polynomial. If $X \subseteq A$, we write $\langle X \rangle$ for the substructure of A generated by X. The symbol $\models$ refers to the standard satisfaction symbol from first-order logic. We usually denote classes of structures by boldface font—both for arbitrary collections $(\mathbf{Q}, \mathbf{K}, \dots)$ and designated ones, e.g., the class of all Heyting algebras* **HA** *or the class of all Boolean algebras* **BA**.

We assume the reader is familiar with the usual constructions from model theory and universal algebra, and refer to [5, 9] for background. We recall in particular the following notions of maps, as we shall need them in the rest of the article.

DEFINITION 2.8. Let $h : A \to B$ be a function between two $\mathcal{L}$-structures A and B, for some first-order language $\mathcal{L}$. We define the following notions:

(1) we say that h is a *homomorphism* if for every function symbol $f \in \mathcal{L}$ we have

$$h(f(a_1, \dots, a_n)) = f(h(a_1), \dots, h(a_n))$$

and for every relation symbol $R \in \mathcal{L}$,

$$A \models R(a_1, \dots, a_n) \implies B \models R(h(a_1), \dots, h(a_n));$$

(2) we say that h is a *strong homomorphism* if it is a homomorphism and, additionally, we have that $R^B = h[R^A]$ for every relation symbol $R \in \mathcal{L}$;

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(3) we say that h is a *strict homomorphism* if it is a homomorphism and, for every relation symbol $R \in \mathcal{L}$,

$$A \models R(a_1, \dots, a_n) \iff B \models R(h(a_1), \dots, h(a_n));$$

(4) we say that h is an *embedding* if it is an injective strict homomorphism.

We write $A \leqslant B$ if A is a substructure of B, i.e., if the identity map $\mathrm{id} : A \to B$ is an embedding. We write $A \cong B$ if A is isomorphic to B. We say that B is a homomorphic image of A if there is a surjective homomorphism $h : A \twoheadrightarrow B$; we say that B is a strong (resp. strict) homomorphic image of A if h is a strong (resp. strict) homomorphism.

REMARK 2.9. We briefly explain the rationale behind the different versions of homomorphism. The notion of homomorphism from Definition 2.8 is standard from the literature in model theory and universal algebra (cf. [9, pp. 70–71] and [5, p. 203]). The notion of strong homomorphism comes from [9, p. 321] and is motivated by the following observation. Let A be an $\mathcal{L}$-structure and let $\mathcal{L}'$ consists of the functional symbols from $\mathcal{L}$. If θ is a congruence of the algebraic reduct of A we can consider the quotient $\mathcal{L}'$-structure A/θ and expand it to an $\mathcal{L}$-structure by letting $R^{A/\theta} = R^A/\theta$ for all relational symbols $R \in \mathcal{L}$. Then, the projection map induced by this quotient is a strong homomorphism. Finally, we take the notion of strict homomorphism from [7, 14], and we stress that strict homomorphisms correspond to those quotients which are additionally compatible with the relational part of the vocabulary from $\mathcal{L}$, in the sense that, if $(a_i, b_i) \in \theta$ for all $1 \leqslant i \leqslant n$, then for every relational symbol $R \in \mathcal{L}$ we have that $A \models R(a_1, \dots, a_n)$ if and only if $A \models R(b_1, \dots, b_n)$. In this article we will mostly be dealing with strong homomorphisms, but we will consider strict homomorphisms in Section 7.

NOTATION 2.10. *Let* $\mathbf{K}$ *be any class of first-order structures. We denote by* $\mathbb{I}(\mathbf{K})$ *its closure under isomorphic copies, by* $\mathbb{S}(\mathbf{K})$ *its closure under substructures, by* $\mathbb{P}(\mathbf{K})$ *its closure under (direct) products, and by* $\mathbb{P}_{\mathrm{U}}(\mathbf{K})$ *its closure under ultraproducts. Finally, we write* $\mathbb{H}(\mathbf{K})$ *for its closure under strong homomorphic images, and* $\mathbb{H}_{\mathrm{s}}(\mathbf{K})$ *for its closure under strict homomorphic images.*

DEFINITION 2.11. A class of algebras $\mathbf{K}$ is a *quasivariety* if it is closed under the operators $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$, $\mathbb{P}_{\mathrm{U}}$, i.e., if it is closed under isomorphic copies, subalgebras, products, and ultraproducts. A class of algebras $\mathbf{K}$ is a *variety* if it is closed under $\mathbb{H}$, $\mathbb{S}$, and $\mathbb{P}$, i.e., if it is closed under homomorphic images, subalgebras, and products. We denote by $\mathbb{Q}(\mathbf{K})$ and $\mathbb{V}(\mathbf{K})$ the quasivariety and the variety generated by $\mathbf{K}$, respectively.

Crucially, the closure of a class of structures under (some of) the operators $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$, $\mathbb{P}_{\mathrm{U}}$, $\mathbb{H}$ is related to conditions pertaining its axiomatisability. Most famously, it is a fundamental result by Keisler and Shelah that a class of structures $\mathbf{K}$ is elementary (i.e., first-order axiomatisable) if and only if $\mathbf{K}$ is closed under ultraproducts. In this work we are concerned with less general definability conditions. We introduce them by focusing on three special subclasses of first-order formulas.

NOTATION 2.12. *We write* $\bar{x}$ *as a shorthand for a sequence of variables* $(x_0, \dots, x_n)$. *Also, as we often deal with equational classes of structures, we abide to the usual convention from universal algebra to distinguish the syntactical equality symbol,*

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written $\approx$*, from the semantical equality symbol, which we denote by* $=$*. To help the reader to distinguish when we talk of first-order formulas from when we deal with propositional ones, we use lowercase Greek symbols* $\phi, \psi, \ldots$ *for the latter and uppercase Greek symbols* $\Phi, \Psi, \ldots$ *for the former.*

DEFINITION 2.13. Let $\mathcal{L}$ be an arbitrary first-order signature, not necessarily functional. Then we define the following types of formulas:

(1) an *equation* is a formula of the form $\varepsilon \approx \delta$, where ε and δ are two $\mathcal{L}$-terms;
(2) a *quasiequation* is a formula of the form $\bigwedge_{i \leqslant n}(\varepsilon_i \approx \delta_i) \to \varepsilon \approx \delta$, for some $n < \omega$, where all ε_i, δ_i and ε, δ are $\mathcal{L}$-terms;
(3) a *basic Horn formula* is a formula of the form $\bigvee_{1 \leqslant i \leqslant n} \Psi_i$, where every Ψ_i is either an atomic or a negated atomic formula, and at most one Ψ_i is atomic;
(4) a basic Horn formula is *strict* if exactly one of its disjuncts is atomic;
(5) a *universal Horn formula* is a formula of the form

$$\forall x_1 \ldots \forall x_m \Big(\bigwedge_{1 \leqslant i \leqslant \ell} \Phi_i(\bar{x}) \Big)$$

for some $m, \ell < \omega$, and where each Φ_i for $1 \leqslant i \leqslant m$ is a basic Horn formula;
(6) a universal Horn formulas is *strict* if all the basic Horn formulas Φ_i occurring in it are strict.

The following classical results relate the closure under the operators $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$, $\mathbb{P}_{\mathrm{U}}$, $\mathbb{H}$ with different definability conditions. We refer the reader to [5, I: Theorem 2.23, Theorem 2.25; II: Theorem 11.9] for their proofs. Facts (1) and (2) are also known, respectively, as Birkhoff's Theorem and Maltsev's Theorem.

FACT 2.14. *Let* $\mathcal{L}$ *be an algebraic signature and* $\mathbf{K}$ *a class of* $\mathcal{L}$*-algebras, then*:

(1) $\mathbf{K}$ *is a variety if and only if it is axiomatised by equations*;
(2) $\mathbf{K}$ *is a quasivariety if and only if it is axiomatised by quasiequations.*

Moreover, if $\mathcal{L}$ *is any first-order signature and* $\mathbf{K}$ *is a class of* $\mathcal{L}$*-structures, then*:

(3) $\mathbf{K}$ *is closed under* $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$, *and* $\mathbb{P}_{\mathrm{U}}$ *if and only if it is axiomatised by universal Horn sentences.*

In the light of the previous results, it is convenient to fix some shorthand notation to talk about (fragments of) first-order theories, and their classes of models.

NOTATION 2.15. *Let* $\mathcal{L}$ *be an arbitrary first-order language. If* T *is a set of first-order sentences in* $\mathcal{L}$*, we write* $\mathrm{Mod}(T)$ *for the class of all* $\mathcal{L}$*-structures* A *such that* $A \models T$*. We write* $\mathrm{Mod}(\mathcal{L})$ *for the class of all* $\mathcal{L}$*-structures. Notice that, if* T *is a set of formulas* (*e.g., a set of equations or quasiequations*)*, then* $\mathrm{Mod}(T)$ *is the class of structures which models the sentences* $\forall x_1 \ldots \forall x_n(\Phi(\bar{x}))$*, for every* $\Phi \in T$*. On the other hand, let* $\mathbf{K}$ *be a class of* $\mathcal{L}$*-structures. Then we write* $\mathrm{Th}_{\mathrm{fol}}(\mathbf{K})$ *for the set of all* $\mathcal{L}$*-sentences true in* $\mathbf{K}$; $\mathrm{Th}_{\mathrm{h}}(\mathbf{K})$ *for the set of all universal Horn sentences true in* $\mathbf{K}$; $\mathrm{Th}_{\mathrm{qe}}(\mathbf{K})$ *for the set of all quasiequations true in* $\mathbf{K}$; *and, finally,* $\mathrm{Th}_{\mathrm{e}}(\mathbf{K})$ *for the set of all equations true in* $\mathbf{K}$*. We also write* $\mathsf{Eq}_{\mathcal{L}}$ *for the set of all equations in* $\mathcal{L}$*, and* $\mathsf{QEq}_{\mathcal{L}}$ *for the set of all quasiequations in* $\mathcal{L}$*—we omit the index* $\mathcal{L}$ *when it is clear from the context.*

Finally, we conclude this preliminary section by recalling what is the propositional consequence relation induced by a class of algebras.

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NOTATION 2.16. *Recall that* $\mathsf{Fm}_{\mathcal{L}}$ *is the term algebra in the language* $\mathcal{L}$. *A (propositional) assignment is a homomorphism* $h : \mathsf{Fm}_{\mathcal{L}} \to A$, *where* A *is an* $\mathcal{L}$*-structure. We denote the family of all assignments into* A *as* $\mathsf{Hom}(\mathsf{Fm}, A)$. *If* $h : \mathsf{Fm}_{\mathcal{L}} \to A$ *is an assignment and* $\alpha \approx \beta$ *an equation, then we often write simply* $h(\alpha \approx \beta)$ *for the formula* $h(\alpha) \approx h(\beta)$.

DEFINITION 2.17. Let $\mathbf{K}$ be a class of $\mathcal{L}$-algebras and let $\Theta \cup \{\varepsilon \approx \delta\} \subseteq \mathsf{Eq}$ a set of equations, then the *equational consequence relative to* $\mathbf{K}$ is defined as follows:

$$\Theta \models_{\mathbf{K}} \varepsilon \approx \delta \iff \text{for all } A \in \mathbf{K},\ h \in \mathsf{Hom}(\mathsf{Fm}, A),\\ \text{if } h(\varepsilon_i) = h(\delta_i) \text{ for all } \varepsilon_i \approx \delta_i \in \Theta,\ \text{then } h(\varepsilon) = h(\delta).$$

And we write $\models_{\mathbf{K}} \bigwedge_{i\leqslant n} \varepsilon_i \approx \delta_i \to \varepsilon \approx \delta$ if $\bigwedge_{i\leqslant n} \varepsilon_i \approx \delta_i \models_{\mathbf{K}} \varepsilon \approx \delta$. We often write $A \models \bigwedge_{i\leqslant n} \varepsilon_i \approx \delta_i \to \varepsilon \approx \delta$ in place of $\models_{\{A\}} \bigwedge_{i\leqslant n} \varepsilon_i \approx \delta_i \to \varepsilon \approx \delta$. The related notions for equations are defined analogously.

REMARK 2.18. Notice that, in Definition 2.17, we can assume without loss of generality that $\mathbf{K}$ is a quasivariety, since by Fact 2.14 the validity of quasiequations is preserved under the operations $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$, $\mathbb{P}_{\mathrm{U}}$. Then, if the set of formulas Θ is finite, we have that $\Theta \models_{\mathbf{K}} \varepsilon \approx \delta$ is equivalent to $\mathbf{K} \models \forall x_1 \ldots \forall x_n (\bigwedge \Theta \to \varepsilon \approx \delta)$. This shows that, when the set of premises Θ is finite, then the consequence relation $\models_{\mathbf{K}}$ can be encoded by the consequence relation from first-order logic. Notice that this ultimately justifies the notational conventions from Definition 2.17 above. If the relation $\models_{\mathbf{K}}$ is not finitary this does not need to be the case: one needs to work with so-called *generalised quasivarieties* instead, and replace the consequence relation from first-order logic by the consequence relation of some suitable infinitary logic. Given our present interest in finitary logical systems we will not expand on this issue, and we simply refer the interested reader to [16].

§3. Weak logics and expanded algebras.

In this section we introduce *weak logics* as a generalisation of propositional logical systems and we provide several examples of them. Alongside, we define *expanded algebras* and *core semantics* to provide an algebraic interpretation to these logics.

3.1. Weak logics.

We start by introducing the notion of weak logical systems, which is the key object of interest of the present work. As we are interested in logical systems which are not necessarily closed under uniform substitution, we firstly identify the restricted class of atomic substitutions.

DEFINITION 3.1. An *atomic substitution* is a substitution $\sigma \in \mathsf{Subst}(\mathcal{L})$ such that $\sigma[\mathsf{Var}] \subseteq \mathsf{Var}$. We denote by $\mathsf{At}(\mathcal{L})$ the set of all *atomic substitutions* in $\mathcal{L}$.

DEFINITION 3.2 (Weak logic). A *weak logic* is a (finitary) consequence relation $\Vdash$ such that, for all atomic substitutions $\sigma \in \mathsf{At}(\mathcal{L})$, $\Gamma \Vdash \phi$ entails $\sigma[\Gamma] \Vdash \sigma(\phi)$.

REMARK 3.3. A weak logic is thus a consequence relation which is closed under atomic substitution. Intuitively, this principle reifies the least prerequisite a consequence relation must satisfy in order to be characterizable as a logic: the validity of the consequences in a weak logic can depend on the logical complexity of its formulas, but not on the specific variables that occur in them. Philosophically, this can be interpreted as a weakening of the Bolzanian–Tarskian notion of logicality.

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Obviously, standard logics are weak logics. More poignantly, there are several examples of weak logics which are not standard logics and that have been extensively studied in the literature. Their existence and recognition constitutes the main motivation behind our interest for this class of consequence relations and for the abstract results of this article.

EXAMPLE 3.4. PAL [19] is an example of a modal logic that is not closed under uniform substitution [20]. However, it can be shown that PAL is closed under atomic substitution [20, Section 2.1] and it is therefore a weak logic. Introducing the proper syntax and semantics of PAL is out of scope of this article, but we mention the following example from [20] to provide the reader with some intuition why uniform substitution fails. Given a set of agents A, the language of PAL extends the basic modal language with operators K_i, for all $i \in A$, and $\langle\phi\rangle$ for any formula ϕ. The sentence $K_i\phi$ should be read as "agent i knows that ϕ" and $\langle\phi\rangle\psi$ as "after the truthful announcement of ϕ to all agents, ψ holds." Let the atoms of the language stand for *facts*—that is, sentences that can be truly uttered at any time. Consider then the principle:

$$p \to \langle p\rangle p \quad \text{(if } p \text{ is true, } p \text{ remains true after a truthful announcement).} \qquad (\star)$$

The schema $(\star)$ is valid for *facts*, but in general does not hold if we substitute p with a sentence talking about the epistemic state of an agent. Let ϕ be the sentence *"Ljubljana became the capital of an independent Slovenia in 1991, and agent j does not know this,"* with translation $c \wedge \neg K_j c$. Now substituting ϕ for p in $(\star)$ gives us a Moorean sentence—after truthfully announcing ϕ, agent j learns that *"Ljubljana became the capital of an independent Slovenia in 1991,"* and thus the conclusion $\langle\phi\rangle\phi$ is no longer truthful.

EXAMPLE 3.5. Logics based on team semantics, such as inquisitive and dependence logics [11, 12, 30], offer a rich supply of examples of weak logics. In Section 8 we will focus particularly on InqB, InqB$^{\otimes}$, InqI, and InqI$^{\otimes}$, namely, the classical and the intuitionistic versions of inquisitive and dependence logic. However, already now we can provide a conceptual motivation why InqB is not closed under uniform substitution. One of the main goals of InqB is to serve as a basis for a uniform treatment of both truth-conditional statements and questions in natural language. To that end, the intended semantics of InqB must establish when a piece of information *supports* a statement or *settles* a question rather than their truth conditions. We call the evidence an *information state* and represent it as a set of possible worlds.

Let p be an arbitrary statement without inquisitive content, e.g., *"It is raining in Glasgow."* Assume that p holds in the possible worlds a and b, i.e., the information state $\{a, b\}$ supports p (see Figure 1). We form the polar question $?p$—*"Is it raining in Glasgow?"*, and model it as the set of alternatives $\{a, b\}$ and $\{c, d\}$. Let's check the validity of *double negation elimination* (DNE)—$\neg\neg q \to q$; we interpret negation as the complement of the union of alternatives. Thus any information state supporting $\neg\neg p$ will support the statement p as well (Figure 1(c)). However, this is not the case for questions—e.g., the state $\{b, d\}$ supports $\neg\neg ?p$, but does not settle $?p$ as the possible worlds b and d do not agree on a same answer. Hence we can conclude that the schema DNE is valid only for statements without inquisitive content, i.e., for propositional atoms.

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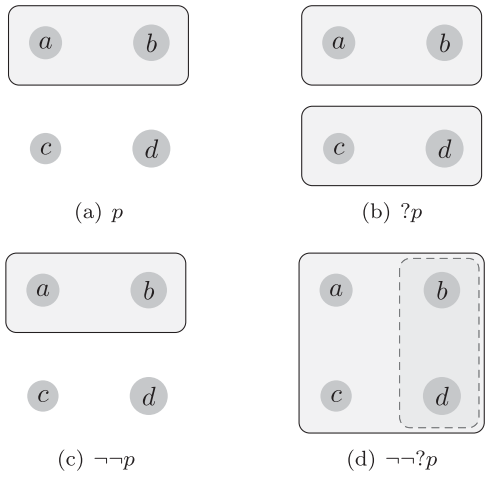


FIGURE 1. Double-negation elimination for statements and polar questions.

Actually, InqB is a concrete example of a wider class of weak logics—a *double negation atoms logic* or DNA-*logic*. A DNA-*logic* (also *negative variant* of an intermediate logic [10, 22]) is a set of formulas $\mathtt{L}^{\neg} = \{\phi[\neg p_0, \dots, \neg p_n / p_0, \dots, p_n] : \phi \in \mathtt{L}\}$, where L is an intermediate logic, namely, a logic comprised between IPC and CPC. It can be proved (see, e.g., [10, Proposition 3.2.15]) that DNA-logics are closed under atomic substitutions. However, for any DNA-logic $\mathtt{L} \neq \mathtt{CPC}$ it is the case that $\neg\neg p \to p \in \mathtt{L}^{\neg}$, but $(\neg\neg(p \vee \neg p) \to p \vee \neg p) \notin \mathtt{L}^{\neg}$, showing that DNA-logics are not standard logics. We also notice that DNA-logics can be further generalised to χ-logics, defined in [26], which provide yet another non-trivial example of weak logics.

We briefly mention the following natural notions, although we will not use them in the rest of the article. If $\Vdash$ is a weak logic we know that it is at least closed under all atomic substitutions $\sigma \in \mathsf{At}(\mathcal{L})$, but in general there could be more substitutions for which the logic $\Vdash$ is closed. We call such substitutions *admissible* for $\Vdash$.

DEFINITION 3.6 (Admissible substitutions). Let $\Vdash$ be a weak logic. The set of *admissible substitutions* $\mathsf{AS}(\Vdash)$ is the set of all substitutions σ such that, for all sets of formulas $\Gamma \cup \{\phi\} \subseteq \mathsf{Fm}$, $\Gamma \Vdash \phi$ entails $\sigma[\Gamma] \Vdash \sigma(\phi)$.

REMARK 3.7. As noticed above, we immediately have that $\mathsf{At}(\mathcal{L}) \subseteq \mathsf{AS}(\Vdash)$. However, in stark contrast with the set of atomic substitutions, determining the set of admissible substitutions of a weak logic is in principle much harder. An example of such a characterization can be given for the case of inquisitive logic InqB: one can in fact verify that $\sigma \in \mathsf{AS}(\mathtt{InqB})$ if and only if σ is a classical substitution, namely, if for all $p \in \mathsf{Var}$, $\sigma(p) \equiv_{\mathtt{InqB}} \psi$, where ψ is a disjunction-free formula.

Even if in weak logics we cannot freely substitute formulas in place of variables, we often want to consider the subset of formulas for which this is possible. We refer to this subset as the core of a logic.

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DEFINITION 3.8 (Core of a logic). The *core* of a weak logic $\Vdash$ is the set $\mathsf{core}(\Vdash)$ of all formulas $\psi \in \mathsf{Fm}$ such that for all sets of formulas $\Gamma \cup \{\phi\}$ we have that

$$\Gamma \Vdash \phi \implies \Gamma[\psi/x] \Vdash \phi[\psi/x],$$

where $x \in \mathsf{Var}$ is any atomic variable.

REMARK 3.9. Equivalently, we can say that ψ is a core formula of $\Vdash$ if and only if for all $x \in \mathsf{Var}$ the substitution σ such that $\sigma{\restriction}\mathsf{Var} \setminus \{x\} = \mathsf{id}_{\mathsf{Var}}$ and $\sigma(x) = \psi$ is admissible. Clearly, we always have that $\mathsf{Var} \subseteq \mathsf{core}(\Vdash)$.

3.2. Expanded algebras. In order to make sense of weak logics from an algebraic perspective, we need to refine the usual algebraic semantics from abstract algebraic logic in order to handle the failure of uniform substitution. To this end, we introduce *expanded algebras* as the expansion of standard algebras by an extra predicate symbol.

DEFINITION 3.10 (Expanded algebra). Let A be an $\mathcal{L}$-algebra and P a unary predicate, an *expanded algebra* is a structure in the language $\mathcal{L} \cup \{P\}$. We denote P^A also by $\mathsf{core}(A)$ and we refer to it as the *core* of the expanded algebra A.

REMARK 3.11. Essentially, expanded algebras are first-order structures with exactly one predicate symbol of arity 1, and arbitrary many functional symbols. Since the relational part of the language consists of only one predicate, we always assume without loss of generality that it consists of the same symbol, so that we always regard any two expanded $\mathcal{L}$-algebras as structures in the same vocabulary. When the signature $\mathcal{L}$ is clear from the context we talk simply of algebras and expanded algebras. Notice that algebras augmented by a single predicate symbol often appear in abstract algebraic logic as *matrices* (cf. [16, Chapter 4]). The key difference between expanded algebras and matrices is in the role played by the additional predicate symbol. In fact, in the context of expanded algebras the predicate identifies the *core* of the algebra, which we use to restrict the possible interpretation of the atomic formulas (cf. 3.17 below). Differently, in the context of logical matrices, the predicate rather identifies a designated "truth set." We will consider in Section 7 a matrix-like approach to weak logics which unifies these two points of views.

Since expanded algebras are first-order structures, we can apply the definition of maps from Definition 2.8 in their setting. We recall that a strong homomorphism $h : A \to B$ of two expanded algebras is an $\mathcal{L}$-algebra homomorphism such that $h[\mathsf{core}(A)] = \mathsf{core}(B)$. We recall from Notation 2.10 that $\mathbb{H}(\mathbf{K})$ indicates the closure of $\mathbf{K}$ under images by strong homomorphisms. We can then extend the notions of quasivarieties and varieties to the setting of expanded algebras.

DEFINITION 3.12. A class of expanded algebras $\mathbf{K}$ is a *quasivariety* if it is closed under $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$, and $\mathbb{P}_{\mathrm{U}}$. A class of expanded algebras $\mathbf{K}$ is a *variety* if it is closed under $\mathbb{H}$, $\mathbb{S}$, and $\mathbb{P}$. We denote by $\mathbb{Q}(\mathbf{K})$ the quasivariety generated by $\mathbf{K}$ and by $\mathbb{V}(\mathbf{K})$ the variety generated by $\mathbf{K}$.

NOTATION 3.13. *Let* $\mathbf{K}$ *be a class of expanded* $\mathcal{L}$*-algebras, then we write* $\mathbf{K}{\restriction}\mathcal{L}$ *for the class of its* $\mathcal{L}$*-reducts. If it is clear from the context, we also write* $\mathbf{K}$ *for* $\mathbf{K}{\restriction}\mathcal{L}$.

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REMARK 3.14. We notice the following: if $\mathbf{K}$ is a quasivariety of expanded $\mathcal{L}$-algebras, then its $\mathcal{L}$-reducts $\mathbf{K}\restriction\mathcal{L}$ form a quasivariety of $\mathcal{L}$-algebras. However, if $\mathbf{K}$ is a quasivariety of $\mathcal{L}$-algebras, then it is not the case that an arbitrary expansion $\mathbf{K}'$ of the algebras in $\mathbf{K}$ gives rise to a quasivariety of expanded algebras. To obtain a quasivariety of expanded algebras we need to consider the generated quasivariety $\mathbb{Q}(\mathbf{K}')$. We shall consider in Section 4.3 later some cases when this additional step is not necessary, namely, the case when a quasivariety of algebras determines uniquely a quasivariety of expanded algebras.

Crucially, expanded algebras allow us to define a more fine-grained consequence relation than the one we introduced in Definition 2.17. The key idea is to use the core of the algebra to restrict the possible valuation of the atomic variables of the language. To our knowledge, the idea of restricting the possible valuations of the atomic variables to a specific subset of an algebra first appeared in [2].

DEFINITION 3.15. The *expanded term algebra* in the signature $\mathcal{L}$ is the structure $\mathsf{Fm}_{\mathcal{L}}$ augmented with $\mathsf{core}(\mathsf{Fm}_{\mathcal{L}}) = \mathsf{Var}$. We often write $(\mathsf{Fm}, \mathsf{Var})$ and omit the index $\mathcal{L}$ and its signature operations when the language is clear from the context. A homomorphism from $\mathsf{Fm}_{\mathcal{L}}$ to an expanded $\mathcal{L}$-algebra A is a *core assignment*, i.e., it is a homomorphism $h : \mathsf{Fm}_{\mathcal{L}} \to A$ such that $h(x) \in \mathsf{core}(A)$ for all $x \in \mathsf{Var}$. We write $\mathsf{Hom}^c(\mathsf{Fm}_{\mathcal{L}}, A)$ for the set of core assignments from $\mathsf{Fm}_{\mathcal{L}}$ to A.

REMARK 3.16. We notice that, alternatively, one could also consider the expansion of the term algebra Fm with the core defined by letting $\mathsf{core}(\mathsf{Fm}_{\mathcal{L}}) = \mathsf{core}(\Vdash)$. This means that the endomorphisms of the term algebra are all the admissible substitutions of $\Vdash$, and not only the atomic substitutions. As this makes only for a minor generalisation of our results, we stick to the former definition and always consider the atomic formulas as the underlying core of the term algebra.

DEFINITION 3.17 (Core semantics). Let $\mathbf{K}$ be a class of expanded algebras and $\Theta \cup \{\varepsilon \approx \delta\}$ a set of equations, then the *equational core-consequence relation relative to* $\mathbf{K}$ is defined as follows:

$$\Theta \models^c_{\mathbf{K}} \varepsilon \approx \delta \iff \text{for all } A \in \mathbf{K},\ h \in \mathsf{Hom}^c(\mathsf{Fm}, A),\\ \text{if } h(\varepsilon_i) = h(\delta_i) \text{ for all } \varepsilon_i \approx \delta_i \in \Theta,\ \text{then } h(\varepsilon) = h(\delta).$$

We then write $\models^c_{\mathbf{K}} \bigwedge_{i\leqslant n} \varepsilon_i \approx \delta_i \to \varepsilon \approx \delta$ if $\bigwedge_{i\leqslant n} \varepsilon_i \approx \delta_i \models^c_{\mathbf{K}} \varepsilon \approx \delta$. We usually write $A \models^c \bigwedge_{i\leqslant n} \varepsilon_i \approx \delta_i \to \varepsilon \approx \delta$ in place of $\models^c_{\{A\}} \bigwedge_{i\leqslant n} \varepsilon_i \approx \delta_i \to \varepsilon \approx \delta$. The related notions for equations are defined analogously.

REMARK 3.18. Crucially, if Θ is a finite set of equations $\{\varepsilon_i \approx \delta_i : i\leqslant n\}$, then we have that $\bigwedge_{i\leqslant n} \varepsilon_i \approx \delta_i \models^c_{\mathbf{K}} \varepsilon \approx \delta$ holds if and only if

$$\mathbf{K} \models \forall x_0, \dots, \forall x_m \Big(\bigwedge_{i\leqslant m} \mathsf{core}(x_i) \wedge \bigwedge_{i\leqslant n} \varepsilon_i \approx \delta_i \to \varepsilon \approx \delta \Big),$$

where all variables from $\Theta \cup \{\varepsilon \approx \delta\}$ are from $x_0, \dots, x_m$. In other words, when Θ is finite the core semantics over a class of expanded algebra can be encoded in terms of the standard first-order consequence relation $\models$, exactly as in the case of the relation $\models_{\mathbf{K}}$. As we stressed in Remark 2.18, if the relation $\models^c_{\mathbf{K}}$ is not finitary, then

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one could provide a similar translation into a suitable infinitary logic and replace quasivarieties by generalised quasivarieties.

By Definition 3.17, in core semantics atomic variables are always assigned to core elements and, as a result of this feature, arbitrary formulas are always interpreted inside the subalgebra generated by the core. This motivates the interest in core-generated structures and in quasivarieties which are generated by these structures.

DEFINITION 3.19. An expanded algebra A is *core-generated* if $A = \langle \mathsf{core}(A) \rangle$. If $\mathbf{Q}$ is a class of algebras, we write $\mathbf{Q}_{\mathrm{CG}}$ for its subclass of core-generated structures. A quasivariety $\mathbf{Q}$ of expanded algebras is *core-generated* if it is generated by its subclass of core-generated expanded algebras, i.e., $\mathbf{Q} = \mathbb{Q}(\mathbf{Q}_{\mathrm{CG}})$. Similarly, a variety $\mathbf{V}$ of expanded algebras is *core-generated* if $\mathbf{V} = \mathbb{V}(\mathbf{V}_{\mathrm{CG}})$.

The next proposition shows that unrestricted substitutions interact nicely with core-generated quasivarieties in core semantics. In a sense, the next result shows that core semantics in core-generated classes of structures approximates as much as possible the standard consequence relations defined over quasivarieties.

LEMMA 3.20. *Let A be a core-generated expanded algebra and let $h \in \mathsf{Hom}(\mathsf{Fm}, A)$, then there are a core assignment $g \in \mathsf{Hom}^c(\mathsf{Fm}, A)$ and a substitution σ such that $h(\phi) = g(\sigma(\phi))$ for all $\phi \in \mathsf{Fm}$.*

PROOF. Let $(x_i)_{i<\omega}$ be an enumeration of Var. Since A is core-generated, we have that for all $x_i \in \mathsf{Var}$ there is a polynomial t_i such that $h(x_i) = t_i(a^i_0, \dots, a^i_{n_i})$ with $a^i_j \in \mathsf{core}(A)$ for all $j \leqslant n_i$. Let now $\{y^i_j : i < \omega,\ j \leqslant n_i\}$ be another enumeration of Var and define $g \in \mathsf{Hom}^c(\mathsf{Fm}, A)$ such that $g(y^i_j) = a^i_j$ for all $i < \omega$, $j \leqslant n_i$. In particular, this means that for all formulas $\psi(x_0, \dots, x_m) \in \mathsf{Fm}$ we have

$$h(\psi(x_0, \dots x_m)) = g(\psi(t_0(y^0_0, \dots, y^0_{n_0}), \dots, t_m(y^m_0, \dots, y^m_{n_m}))).$$

By construction we have that g is a core assignment. Let σ be the substitution defined by letting $\sigma(x_i) = t_i(y^i_0, \dots, y^i_{n_i})$ for all $x_i \in \mathsf{Var}$, then from the display above we derive that $h(\phi) = g(\sigma(\phi))$ for all $\phi \in \mathsf{Fm}$. ⊣

PROPOSITION 3.21. *Let $\mathbf{Q}$ be a core-generated quasivariety of expanded algebras, then $\sigma(\Theta) \models^c_{\mathbf{Q}} \sigma(\varepsilon \approx \delta)$ for all $\sigma \in \mathsf{Subst}(\mathcal{L})$ holds if and only if $\Theta \models_{\mathbf{Q}} \varepsilon \approx \delta$.*

PROOF. We consider the left-to-right direction. Suppose that $\Theta \not\models_{\mathbf{Q}} \varepsilon \approx \delta$. Since $\mathbf{Q}$ is core-generated, there is a core-generated expanded algebra $A \in \mathbf{Q}$ and some $h \in \mathsf{Hom}(\mathsf{Fm}, A)$ such that $A \models h(\Theta)$ and $A \not\models h(\varepsilon) \approx h(\delta)$ (recall Definition 2.17). From Lemma 3.20 it follows that there is a core assignment $g \in \mathsf{Hom}^c(\mathsf{Fm}, A)$ and a substitution σ such that $h(\phi) = g(\sigma(\phi))$ for all $\phi \in \mathsf{Fm}$. Then it follows that $A \models g(\sigma[\Theta])$ but $A \not\models g(\sigma(\varepsilon \approx \delta))$, whence $\sigma(\Theta) \not\models^c_{\mathbf{Q}} \sigma(\varepsilon \approx \delta)$.

We consider the right-to-left direction. Suppose that $\sigma(\Theta) \not\models^c_{\mathbf{Q}} \sigma(\varepsilon \approx \delta)$, thus we can find an algebra $A \in \mathbf{Q}$ and an assignment $h \in \mathsf{Hom}^c(\mathsf{Fm}, A)$ such that $A \models h(\sigma[\Theta])$ and $A \not\models h(\sigma(\varepsilon \approx \delta))$. Then we have that $h \circ \sigma \in \mathsf{Hom}(\mathsf{Fm}, A)$, whence $\Theta \not\models_{\mathbf{Q}} \varepsilon \approx \delta$. ⊣

Since core semantics only looks at the substructure $\langle \mathsf{core}(A) \rangle$ of an expanded algebra A, it make sense to consider extensions of A that preserve the core.

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DEFINITION 3.22. We say that B is a *core superalgebra* of A if $B \in \mathrm{Mod}(\mathcal{L})$, $A \leqslant B$ (i.e., A is a subalgebra of B) and $\mathsf{core}(A) = \mathsf{core}(B)$. If $\mathbf{K}$ is class of expanded algebras, we write $\mathbb{C}(\mathbf{K})$ for the class of all core superalgebras of elements of $\mathbf{K}$.

The next lemma and the following proposition establish some fundamental facts about the relation $\models^c$. Importantly, they show that the validity of quasiequations is preserved both under the operators $\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{P}_{\mathrm{U}}$, and also under the core superstructure operator $\mathbb{C}$ from Definition 3.22. Additionally, we also show that equations are preserved under strong homomorphisms. The following lemma is essentially a rephrasing of Remark 3.18.

LEMMA 3.23. *Let* $\mathbf{K}$ *be a class of expanded algebras, let* $\bigwedge_{i\leqslant n} \varepsilon_i \approx \delta_i \to \varepsilon \approx \delta$ *be a quasiequation, and let* V *be all the variables occurring in it, then*

$$\{\varepsilon_i \approx \delta_i : i \leqslant n\} \models^c_{\mathbf{K}} \varepsilon \approx \delta \iff \{\varepsilon_i \approx \delta_i : i \leqslant n\} \cup \bigcup_{x \in V} \mathsf{core}(x) \models_{\mathbf{K}} \varepsilon \approx \delta.$$

PROOF. This follows immediately from the fact that an assignment $h \in \mathsf{Hom}(\mathsf{Fm}, A)$ is a core assignment if and only if $h(x) \in \mathsf{core}(A)$ for all $x \in \mathsf{Var}$. ⊣

PROPOSITION 3.24. *Let* $\mathbf{K}$ *be a class of expanded algebras, then the following hold*:

(1) *let* $\bigwedge_{i\leqslant n} \varepsilon_i \approx \delta_i \to \varepsilon \approx \delta$ *be a quasi-equation, then for all* $\mathbb{O} \in \{\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{P}_{\mathrm{U}}, \mathbb{C}\}$ *we have that*

$$\{\varepsilon_i \approx \delta_i : i \leqslant n\} \models^c_{\mathbf{K}} \varepsilon \approx \delta \implies \{\varepsilon_i \approx \delta_i : i \leqslant n\} \models^c_{\mathbb{O}(\mathbf{K})} \varepsilon \approx \delta;$$

(2) *if* $\models^c_{\mathbf{K}} \varepsilon \approx \delta$, *then* $\models^c_{\mathbb{H}(\mathbf{K})} \varepsilon \approx \delta$.

PROOF. For Claim (1) recall that the validity of universal Horn formulas is preserved by the operators $\mathbb{O} \in \{\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{P}_{\mathrm{U}}\}$. Then Lemma 3.23 implies that the validity of formulas in core semantics is preserved by $\mathbb{O} \in \{\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{P}_{\mathrm{U}}\}$. Preservation of validity by $\mathbb{C}$ follows immediately by the definitions of core semantics and core superstructure. Claim (2) is immediate by the definition of core semantics and the fact that, if $B \in \mathbb{H}(\mathbf{K})$, then there is $A \in \mathbf{K}$ and a surjective homomorphism $h : A \to B$ such that $\mathsf{core}(B) = h[\mathsf{core}(A)]$. ⊣

We conclude this section by showing a version of Maltsev's Theorem for the setting of core-generated quasivarieties, i.e., we prove that every core-generated quasivariety is axiomatised by its validities under core semantics.

DEFINITION 3.25. We define the following notions.

(1) For any set T of quasiequations (or equations), we let $\mathrm{Mod}^c(T)$ be the class of expanded algebras A such that $A \models^c T$ and $\mathrm{Mod}^c_{\mathrm{CG}}(T)$ for its subclass of core-generated models.
(2) For any class $\mathbf{K}$ of expanded algebras, we denote by $\mathrm{Th}^c_{\mathrm{qe}}(\mathbf{K})$ the set of all quasiequations true in $\mathbf{K}$ under core semantics, and we denote by $\mathrm{Th}^c_{\mathrm{e}}(\mathbf{K})$ the set of all equations true in $\mathbf{K}$ under core semantics.

The following proposition is an immediate corollary of Maltsev's Theorem (cf. Fact 2.14) and Proposition 3.21.

PROPOSITION 3.26. *Let* $\mathbf{Q}$ *be a quasivariety of expanded algebras and let* A *be a core-generated expanded algebra. Then* $A \in \mathbf{Q}_{\mathrm{CG}}$ *if and only if* $A \models^c \mathrm{Th}^c_{\mathrm{qe}}(\mathbf{Q})$.

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PROOF. The direction from left-to-right follows immediately by the definition of $\mathrm{Th}^c_{\mathrm{qe}}(\mathbf{Q})$. We consider the direction from right-to-left. Let A be core-generated and suppose $A \notin \mathbf{Q}_{\mathrm{CG}}$. It follows that $A \notin \mathbf{Q}$ and so by Maltsev's Theorem $A \not\models \mathrm{Th}_{\mathrm{qe}}(\mathbf{Q})$. Let $\bigwedge_{i\leqslant n} \varepsilon_i \approx \delta_i \to \alpha \approx \beta \in \mathrm{Th}_{\mathrm{qe}}(\mathbf{Q})$ be such that $A \not\models \bigwedge_{i\leqslant n} \varepsilon_i \approx \delta_i \to \alpha \approx \beta$. By Proposition 3.21 it follows that there is a substitution σ such that $A \not\models^c \sigma(\bigwedge_{i\leqslant n} \varepsilon_i \approx \delta_i \to \alpha \approx \beta)$. Now, since the relation $\models_{\mathbf{Q}}$ is closed under uniform substitution, it follows in particular that $\sigma(\bigwedge_{i\leqslant n} \varepsilon_i \approx \delta_i \to \alpha \approx \beta) \in \mathrm{Th}_{\mathrm{qe}}(\mathbf{Q})$. Moreover, since obviously $\mathrm{Th}_{\mathrm{qe}}(\mathbf{Q}) \subseteq \mathrm{Th}^c_{\mathrm{qe}}(\mathbf{Q})$, we obtain that $\sigma(\bigwedge_{i\leqslant n} \varepsilon_i \approx \delta_i \to \alpha \approx \beta) \in \mathrm{Th}^c_{\mathrm{qe}}(\mathbf{Q})$. We conclude that $A \not\models^c \mathrm{Th}^c_{\mathrm{qe}}(\mathbf{Q})$. ⊣

§4. Algebraizability of weak logics. In this section we use quasivarieties of expanded algebras to provide three different notions of algebraizability for the setting of weak logics, which we, respectively, call *loose algebraizability*, *strict algebraizability,* and *fixed-point algebraizability*. We proceed as follows: firstly, we review the notion of algebraizability for the setting of standard logics in Section 4.1, then in Section 4.2 we introduce the loose notion of algebraizability for weak logics, in Section 4.3 we consider the strict version of algebraizability for weak logics, and finally in Section 4.4 we focus on the very specific notion of fixed-point algebraizability.

4.1. Algebraizability of standard logics. We first review in this section the notion of algebraizability for the setting of standard logics. Recall that we denote by Fm and Eq, respectively, the set of formulas and equations in the signature $\mathcal{L}$. We shall refer to two maps, called transformers [16], that allow us to translate formulas into equations and *vice versa*.

DEFINITION 4.1. A pair of *transformers* in the language $\mathcal{L}$ is a pair of functions $\tau : \mathsf{Fm} \to \wp(\mathsf{Eq})$ and $\Delta : \mathsf{Eq} \to \wp(\mathsf{Fm})$. We say that τ and Δ are *structural* if for all substitutions $\sigma \in \mathsf{Subst}(\mathcal{L})$, $\tau(\sigma(\phi)) = \sigma(\tau(\phi))$ and $\sigma(\Delta(\varepsilon,\delta)) = \Delta(\sigma(\varepsilon,\delta))$, where we let $\sigma(\varepsilon,\delta) = (\sigma(\varepsilon),\sigma(\delta))$. We say that the transformers τ, Δ are *finitary* if for all $\phi \in \mathsf{Fm}$ and $\varepsilon \approx \delta \in \mathsf{Eq}$, $|\tau(\phi)| < \aleph_0$ and $|\Delta(\varepsilon,\delta)| < \aleph_0$. For any set of formulas $\Gamma \subseteq \mathsf{Fm}$, we let $\tau(\Gamma) = \bigcup_{\phi\in\Gamma} \tau(\phi)$ and for all set of equations $\Theta \subseteq \mathsf{Eq}$ we let $\Delta(\Theta) = \bigcup_{\varepsilon\approx\delta\in\Theta} \Delta(\varepsilon,\delta)$.

The following notion of algebraizability was introduced by Blok and Pigozzi in their seminal article [4]. We write $\Gamma_0 \equiv_{\mathbf{K}} \Gamma_1$ as a shorthand for $\Gamma_0 \vDash_{\mathbf{K}} \Gamma_1$ and $\Gamma_1 \vDash_{\mathbf{K}} \Gamma_0$.

DEFINITION 4.2 (Algebraizability). A (standard) logic $\vdash$ is *algebraizable* if there are a quasivariety of (standard) algebras $\mathbf{Q}$ and structural transformers $\tau : \mathsf{Fm} \to \wp(\mathsf{Eq})$ and $\Delta : \mathsf{Eq} \to \wp(\mathsf{Fm})$ such that

$$\Gamma \vdash \phi \iff \tau[\Gamma] \models_{\mathbf{Q}} \tau(\phi) \tag{A1}$$

$$\Delta[\Theta] \vdash \Delta(\varepsilon,\delta) \iff \Theta \models_{\mathbf{Q}} \varepsilon \approx \delta \tag{A2}$$

$$\phi \dashv\vdash \Delta[\tau(\phi)] \tag{A3}$$

$$\varepsilon \approx \delta \equiv_{\mathbf{Q}} \tau[\Delta(\varepsilon,\delta)]. \tag{A4}$$

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If these conditions are met, we then say that $\mathbf{Q}$ is an *equivalent algebraic semantics* of $\vdash$, and that $(\mathbf{Q}, \tau, \Delta)$ witnesses the algebraizability of $\vdash$.

REMARK 4.3. The restriction to quasivarieties of algebras stems from the fact that we are exclusively considering finitary consequence relations $\vdash$. See [16, Section 3] for an extension of this definition to generalised quasivarieties and non-finitary logics.

We recall the following two important facts about algebraizability. We refer the reader to [16, Proposition 3.12, Theorem 3.37] for a proof of these results.

FACT 4.4. *Let $\vdash$ be a standard logic, then*:

(1) *if Conditions 4.2($A1$) and 4.2($A4$), or 4.2($A2$) and 4.2($A3$) hold, then $\vdash$ is algebraized by $(\mathbf{Q}, \tau, \Delta)$;*
(2) *if $\vdash$ is algebraizable, then there are two finitary transformers τ, Δ witnessing this fact.*

It is a key property of algebraizability that the equivalent algebraic semantics of a standard logic is unique. See [16] for a proof of this result.

FACT 4.5. *If the tuples $(\mathbf{Q}_0, \tau_0, \Delta_0)$ and $(\mathbf{Q}_1, \tau_1, \Delta_1)$ both witness the algebraizability of a standard logic $\vdash$, then*:

(1) $\mathbf{Q}_0 = \mathbf{Q}_1$;
(2) $\Delta_0(\varepsilon, \delta) \dashv\vdash \Delta_1(\varepsilon, \delta)$, *for all* $\varepsilon \approx \delta \in \mathsf{Eq}$;
(3) $\tau_0(\phi) \equiv_{\mathbf{Q}_i} \tau_1(\phi)$ *with* $i \in \{0, 1\}$, *for all* $\phi \in \mathsf{Fm}$.

4.2. Loose algebraizability of weak logics. By using the core consequence relation $\models^c$ in place of the standard one $\models$, we can provide a first version of the notion of algebraizability in the setting of weak logics. Obviously, we write $\Gamma_0 \equiv^c_{\mathbf{K}} \Gamma_1$ as a shorthand for $\Gamma_0 \models^c_{\mathbf{K}} \Gamma_1$ and $\Gamma_1 \models^c_{\mathbf{K}} \Gamma_0$.

DEFINITION 4.6. A weak logic $\Vdash$ is *loosely algebraizable* if there is a core-generated quasivariety of expanded algebras $\mathbf{Q}$ and two structural transformers $\tau : \mathsf{Fm} \to \wp(\mathsf{Eq})$ and $\Delta : \mathsf{Eq} \to \wp(\mathsf{Fm})$ such that

$$\Gamma \Vdash \phi \iff \tau[\Gamma] \models^c_{\mathbf{Q}} \tau(\phi) \tag{W1}$$

$$\Delta[\Theta] \Vdash \Delta(\varepsilon, \delta) \iff \Theta \models^c_{\mathbf{Q}} \varepsilon \approx \delta \tag{W2}$$

$$\phi \dashv\Vdash \Delta[\tau(\phi)] \tag{W3}$$

$$\varepsilon \approx \delta \equiv^c_{\mathbf{Q}} \tau[\Delta(\varepsilon, \delta)]. \tag{W4}$$

If these conditions are met, we then say that $\mathbf{Q}$ is a *loose algebraic semantics* of $\Vdash$, and that the tuple $(\mathbf{Q}, \tau, \Delta)$ witnesses the loose algebraizability of $\Vdash$.

To show this definition is indeed robust, we adapt the proof of the uniqueness of the equivalent algebraic semantic of standard logics [16, Theorem 3.17] to the case of loosely algebraizable weak logics. In the light of the following theorem, we often refer to the tuple $(\mathbf{Q}, \tau, \Delta)$ witnessing the loose algebraizability of the weak logic $\Vdash$ as *the equivalent loose algebraic semantics* of $\Vdash$.

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THEOREM 4.7. *If both the tuples* $(\mathbf{Q}_0, \tau_0, \Delta_0)$ *and* $(\mathbf{Q}_1, \tau_1, \Delta_1)$ *witness the algebraizability of* $\Vdash$, *then*:

(1) $\mathbf{Q}_0 = \mathbf{Q}_1$;
(2) $\Delta_0(\varepsilon, \delta) \dashv\Vdash \Delta_1(\varepsilon, \delta)$, *for all* $\varepsilon \approx \delta \in \mathsf{Eq}$;
(3) $\tau_0(\phi) \equiv^c_{\mathbf{Q}_i} \tau_1(\phi)$ *with* $i \in \{0, 1\}$, *for all* $\phi \in \mathsf{Fm}$.

PROOF. Notice that the two witnesses of algebraizability give rise to two different consequence relations, which we shall denote by $\models^0_{\mathbf{Q}_0}$ and $\models^1_{\mathbf{Q}_1}$.

Clause (2): $\Delta_0(\varepsilon, \delta) \dashv\Vdash \Delta_1(\varepsilon, \delta)$, for all $\varepsilon \approx \delta \in \mathsf{Eq}$. We prove $\Delta_0(\varepsilon, \delta) \Vdash \Delta_1(\varepsilon, \delta)$. Let $\phi \in \Delta_1(\varepsilon, \delta)$, then we clearly have that

$$\tau_0(\phi(\varepsilon, \varepsilon)), \phi(\varepsilon, \varepsilon) \approx \phi(\varepsilon, \delta) \models^0_{\mathbf{Q}_0} \tau_0(\phi(\varepsilon, \delta)).$$

By 4.6(W2) it follows:

$$\Delta_0(\tau_0(\phi(\varepsilon, \varepsilon))), \Delta_0(\phi(\varepsilon, \varepsilon), \phi(\varepsilon, \delta)) \Vdash \Delta_0(\tau_0(\phi(\varepsilon, \delta))),$$

hence, by 4.6(W3),

$$\phi(\varepsilon, \varepsilon), \Delta_0(\phi(\varepsilon, \varepsilon), \phi(\varepsilon, \delta)) \Vdash \phi(\varepsilon, \delta). \tag{a}$$

Now, we have that $\varnothing \models^1_{\mathbf{Q}_1} \varepsilon \approx \varepsilon$, hence by $\phi \in \Delta_1(\varepsilon, \varepsilon)$ and 4.6(W2), we obtain

$$\varnothing \Vdash \phi(\varepsilon, \varepsilon). \tag{b}$$

Moreover, it also follows that $\varepsilon \approx \delta \models^0_{\mathbf{Q}_0} \phi(\varepsilon, \varepsilon) \approx \phi(\varepsilon, \delta)$, hence by 4.6(W2),

$$\Delta_0(\varepsilon, \delta) \Vdash \Delta_0(\phi(\varepsilon, \varepsilon), \phi(\varepsilon, \delta)). \tag{c}$$

Finally, by (a)–(c), it follows that $\Delta_0(\varepsilon, \delta) \Vdash \phi(\varepsilon, \delta)$, hence $\Delta_0(\varepsilon, \delta) \Vdash \Delta_1(\varepsilon, \delta)$. The converse direction is proven analogously.

Clause (1): $\mathbf{Q}_0 = \mathbf{Q}_1$. We first prove that $\mathbf{Q}_0$ and $\mathbf{Q}_1$ satisfy the same quasiequations under core semantics. We show only that $\mathrm{Th}^c_{\mathrm{qe}}(\mathbf{Q}_0) \subseteq \mathrm{Th}^c_{\mathrm{qe}}(\mathbf{Q}_1)$, as the other direction follows analogously. Let $\bigwedge_{i \leqslant n} \varepsilon_i \approx \delta_i \to \varepsilon \approx \delta \in \mathrm{Th}^c_{\mathrm{qe}}(\mathbf{Q}_0)$, then it follows that $\bigwedge_{i \leqslant n} \varepsilon_i \approx \delta_i \models^0_{\mathbf{Q}_0} \varepsilon \approx \delta$ and this yields that $\bigcup_{i \leqslant n} \Delta_0(\varepsilon_i, \delta_i) \Vdash \Delta_0(\varepsilon, \delta)$ by 4.6(W2). By point (3) above, it follows that $\bigcup_{i \leqslant n} \Delta_1(\varepsilon_i, \delta_i) \Vdash \Delta_1(\varepsilon, \delta)$, hence by 4.6(W2) we get $\bigwedge_{i \leqslant n} \varepsilon_i \approx \delta_i \models^1_{\mathbf{Q}_1} \varepsilon \approx \delta$. The latter entails $\bigwedge_{i \leqslant n} \varepsilon_i \approx \delta_i \to \varepsilon \approx \delta \in \mathrm{Th}^c_{\mathrm{qe}}(\mathbf{Q}_1)$ and thus $\mathrm{Th}^c_{\mathrm{qe}}(\mathbf{Q}_0) \subseteq \mathrm{Th}^c_{\mathrm{qe}}(\mathbf{Q}_1)$. By reasoning analogously we obtain that $\mathrm{Th}^c_{\mathrm{qe}}(\mathbf{Q}_1) \subseteq \mathrm{Th}^c_{\mathrm{qe}}(\mathbf{Q}_0)$, hence $\mathrm{Th}^c_{\mathrm{qe}}(\mathbf{Q}_0) = \mathrm{Th}^c_{\mathrm{qe}}(\mathbf{Q}_1)$. It then follows by Proposition 3.26 above that $(\mathbf{Q}_0)_{\mathrm{CG}} = (\mathbf{Q}_1)_{\mathrm{CG}}$ and since both $\mathbf{Q}_0$ and $\mathbf{Q}_1$ are core-generated we conclude that $\mathbf{Q}_0 = \mathbf{Q}_1$.

Clause (3): $\tau_0(\phi) \equiv^c_{\mathbf{Q}_i} \tau_1(\phi)$ with $i \in \{0, 1\}$, for all $\phi \in \mathsf{Fm}$. By Clause (1), it suffices to prove that $\tau_0(\phi) \equiv^0_{\mathbf{Q}_0} \tau_1(\phi)$. By 4.6(W3), we have $\Delta_0(\tau_0(\phi)) \dashv\Vdash \Delta_1(\tau_1(\phi))$ and by Clause (2) this is equivalent to $\Delta_0(\tau_0(\phi)) \dashv\Vdash \Delta_0(\tau_1(\phi))$. It then follows by 4.6(W2) that $\tau_0(\phi) \equiv^0_{\mathbf{Q}_0} \tau_1(\phi)$. ⊣

We have thus established that every algebraizable weak logic has a unique equivalent algebraic semantics, up to equivalence under the core consequence relation. We conclude this section by proving an analogue of Fact 4.4 for weak logics.

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PROPOSITION 4.8. *Let* $\Vdash$ *be a weak logic, then*:

(1) *if Conditions* 4.6($W1$) *and* 4.6($W4$), *or Conditions* 4.6($W2$) *and* 4.6($W3$) *hold, then* $\Vdash$ *is loosely algebraized by* $(\mathbf{Q}, \tau, \Delta)$;
(2) *if* $\Vdash$ *is loosely algebraizable, then there are two finitary transformers* τ, Δ *witnessing this fact*.

PROOF. We prove (1). Suppose $\Vdash$ is a weak logic and let $(\mathbf{Q}, \tau, \Delta)$ satisfy 4.6(W1) and 4.6(W4). We verify that 4.6(W2) and 4.6(W3) hold as well. By 4.6(W4) $\Theta \models^c_{\mathbf{Q}} \varepsilon \approx \delta$ is equivalent to $\tau[\Delta(\Theta)] \models^c_{\mathbf{Q}} \tau(\Delta(\varepsilon, \delta))$, which by 4.6(W1) is equivalent to $\Delta(\Theta) \Vdash \Delta(\varepsilon, \delta)$, proving 4.6(W2). Also, for any formula ϕ, we have that $\tau(\phi) \equiv^c_{\mathbf{Q}} \tau(\phi)$, hence by 4.6(W4) $\tau[\Delta(\tau(\phi))] \equiv^c_{\mathbf{Q}} \tau(\phi)$ and by 4.6(W1) $\Delta(\tau(\phi)) \dashv\Vdash \phi$, proving 4.6(W3). If $(\mathbf{Q}, \tau, \Delta)$ satisfies 4.6(W2) and 4.6(W3), then we proceed analogously.

We prove (2). For any $\phi \in \mathsf{Fm}$, we have by 4.6(W3) that $\phi \dashv\Vdash \Delta[\tau(\phi)]$, thus by $\Vdash$ being finitary there is some $\tau_0(\phi) \subseteq \tau(\phi)$ such that $|\tau_0(\phi)| < \aleph_0$ and $\phi \dashv\Vdash \Delta[\tau_0(\phi)]$. Moreover, for any equation $\varepsilon \approx \delta \in \mathsf{Eq}$, we have by 4.6(W4) that $\varepsilon \approx \delta \equiv^c_{\mathbf{Q}} \tau[\Delta(\varepsilon, \delta)]$, so we obtain by finitarity a finite subset $\Delta_0(\varepsilon, \delta) \subseteq \Delta(\varepsilon, \delta)$ such that $\varepsilon \approx \delta \equiv^c_{\mathbf{Q}} \tau[\Delta_0(\varepsilon, \delta)]$. Finally, it follows by the choice of τ_0, Δ_0 that $\Delta[\tau(\phi)] \dashv\Vdash \Delta(\tau_0(\phi))$, hence $\tau(\phi) \equiv^c_{\mathbf{Q}} \tau_0(\phi)$. Similarly, from $\tau[\Delta(\varepsilon, \delta)] \equiv^c_{\mathbf{Q}} \tau[\Delta_0(\varepsilon, \delta)]$ we obtain $\Delta_0(\varepsilon, \delta) \dashv\Vdash \Delta(\varepsilon, \delta)$. Thus τ_0, Δ_0, and $\mathbf{Q}$ witness the algebraizability of $\Vdash$. ⊣

REMARK 4.9. As witnessed by the previous results, loose algebraizability satisfies the same uniqueness property of standard algebraizability. However, we believe this is too weak of a notion, as it does not really meet the fundamental intuition behind algebraizability. In fact, in contrast to the matrix semantics of logics (which we shall explore later in Section 7), the fundamental aspect of algebraizability is that it allows us to translate logical systems into equational consequences defined by classes of algebras, and not by arbitrary first-order structures containing relations. We will achieve this by considering strict algebraizability in the next section.

4.3. Strict algebraizability of weak logics. We introduce strict algebraizability as a refined notion of algebraizability for weak logical systems. As we mentioned in Remark 4.9, the problem with loose algebraizability is that it relates weak logics to (universal Horn) classes of first-order structures, and not really to algebras. To overcome this issue, we need to look for ways in which one can, so to speak, *eliminate* the core predicate $\mathsf{core}(A)$. The key idea is to restrict attention to classes of expanded algebras in which the core is already definable in the functional part of the signature. Furthermore, since we are dealing with quasivarieties and we want the validity of formulas to be preserved by the operators $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$, and $\mathbb{P}_{\mathrm{U}}$, we exclusively consider definability by means of equations. We make this explicit in the following definitions.

NOTATION 4.10. *If* A *is an* $\mathcal{L}$*-algebra and* $\Sigma(x)$ *a set of equations in the variable* x, *we let* $\Sigma(A) = \{a \in A : A \models \varepsilon(a) \approx \delta(a)$ *for all* $\varepsilon \approx \delta \in \Sigma\}$.

DEFINITION 4.11. An expanded algebra A is said to have an *equationally definable core* if there is a finite set of equations Σ in the variable x such that $\mathsf{core}(A) = \Sigma(A)$. A class of expanded algebras $\mathbf{K}$ is said to have a *(uniformly) equationally definable core* if there is a finite set of equations Σ such that $\mathsf{core}(A) = \Sigma(A)$ for all $A \in \mathbf{K}$.

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The key reason that explains our interest in quasivarieties of expanded algebras with an equationally definable core is that, given an algebra A and a finite set of equations Σ, there is a *unique* way to expand A into an expanded algebra with core defined by Σ. As the following proposition shows, this provides us with a canonical expansion of $\mathbf{Q}$ into a quasivariety of expanded algebras with core defined by Σ.

PROPOSITION 4.12. *Let* $\mathbf{Q}$ *be a quasivariety of* $\mathcal{L}$*-algebras and* $\Sigma(x)$ *a finite set of equations, then the class of structures* $(A, \mathsf{core}(A))$ *with* $A \in \mathbf{Q}$ *and* $\mathsf{core}(A) = \Sigma(A)$ *is a quasivariety of expanded algebras.*

PROOF. Let $\mathbf{Q}$ be a quasivariety of $\mathcal{L}$-algebras and let $\mathbf{K}$ be the class of structures $(A, \mathsf{core}(A))$ with $A \in \mathbf{Q}$ and $\mathsf{core}(A) = \Sigma(A)$. Let $\{\alpha_i \approx \beta_i : i \leqslant n\}$ enumerate all equations in Σ. Then consider the formulas

$$\Phi_i \coloneqq \forall x(\mathsf{core}(x) \to \alpha_i(x) \approx \beta_i(x))$$
$$\Psi \coloneqq \forall x(\bigwedge_{i \leqslant n} \alpha_i(x) \approx \beta_i(x) \to \mathsf{core}(x)).$$

Clearly, both Ψ and each Φ_i are universal Horn sentences, thus by Fact 2.14 they are preserved by the quasivariety operators $\mathbb{I}, \mathbb{S}, \mathbb{P}$, and $\mathbb{P}_{\mathrm{U}}$. It follows that $\mathsf{core}(A) = \Sigma(A)$ for all $A \in \mathbb{Q}(\mathbf{K})$, and thus that $\mathbf{K}$ is already a quasivariety. ⊣

REMARK 4.13. Crucially, the previous proposition cannot be extended to varieties of expanded algebras. In particular, if $\mathbf{K}$ is a class of expanded algebras with $\mathsf{core}(A) = \Sigma(A)$ for all $A \in \mathbf{K}$, it is not necessarily the case that $\mathsf{core}(A) = \Sigma(A)$ for all $A \in \mathbb{H}(\mathbf{K})$. For example, consider the equation $x \approx \neg x$, let A be any Boolean algebra with size $|A| \geqslant 2$ and let B be the trivial one-element Boolean algebra. Clearly, there is a surjective homomorphism $h : A \twoheadrightarrow B$ and, at the same time, we also have that $\Sigma(B) = B$ and $\Sigma(A) = \emptyset$, showing that $\mathsf{core}(B) \neq h(\mathsf{core}(A))$. This shows that we cannot replace quasivarieties by varieties in the statement of Proposition 4.12. We shall see later that this is possible in the restricted setting of fixed-point algebraizability.

It is easy to find concrete examples of quasivarieties of expanded algebras whose core is equationally definable. We list here some examples which are determined by some very basic equations.

EXAMPLE 4.14. Let M be a monoid in $\mathcal{L} = (\cdot, e)$ and define $\mathsf{core}(M) = \{a \in M : a^n = e^M\}$, then M is an expanded algebra with core defined by $\Sigma = \{x^n \approx e\}$.

EXAMPLE 4.15. Let $\mathbf{HA}$ be the variety of Heyting algebras and for all $A \in \mathbf{HA}$ let $\mathsf{core}(A) = A_\neg = \{x \in A : A \models x \approx \neg\neg x\}$, i.e., the core of A is its subset of regular elements. Let $\mathbf{ML}$ be the variety of all Medvedev algebras, then $\mathbf{ML}$ is generated by its subclass of core-generated Heyting algebras A with core $A_\neg$ (cf. [2, 3] and Fact 8.16 below). As we shall see in Section 8, this class plays an important role in the semantics of classical inquisitive propositional logic.

We define the strict version of algebraizability for weak logics by requiring that the core-generated algebras corresponding to a weak logic $\Vdash$ have the core defined by a finite set of equations Σ. This definition is significantly stronger than the one we introduced in the previous section, and it essentially rephrases the weak logic $\Vdash$

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in terms of the relative consequence relation $\models_{\mathbf{Q}}$ of the corresponding quasivariety of algebras.

DEFINITION 4.16. A weak logic $\Vdash$ is *strictly algebraizable* if it is loosely algebraized (in the sense of Definition 4.6) by $(\mathbf{Q}, \tau, \Delta)$ and, additionally, there is a finite set of equations Σ defining the core of the expanded algebras in $\mathbf{Q}$. We then say that $\Vdash$ is strictly algebraized by $(\mathbf{Q}, \Sigma, \tau, \Delta)$ and call $(\mathbf{Q}, \Sigma, \tau, \Delta)$ its *(equivalent) strict algebraic semantics*.

We can then strengthen Theorem 4.7 to the present context of core-generated quasivarieties with a definable core.

THEOREM 4.17. *If both the tuples* $(\mathbf{Q}_0, \Sigma_0, \tau_0, \Delta_0)$ *and* $(\mathbf{Q}_1, \Sigma_1, \tau_1, \Delta_1)$ *witness the strict algebraizability of* $\Vdash$, *then*:

(1) $\mathbf{Q}_0 = \mathbf{Q}_1$;
(2) $\Sigma_0 \equiv_{\mathbf{Q}_i} \Sigma_1$ *with* $i \in \{0, 1\}$;
(3) $\Delta_0(\varepsilon, \delta) \dashv\Vdash \Delta_1(\varepsilon, \delta)$, *for all* $\varepsilon \approx \delta \in \mathsf{Eq}$;
(4) $\tau_0(\phi) \equiv^{c}_{\mathbf{Q}_i} \tau_1(\phi)$ *with* $i \in \{0, 1\}$, *for all* $\phi \in \mathsf{Fm}$.

PROOF. Clauses (1), (3), and (4) follow immediately from Theorem 4.7, so we prove (2). We write $\mathsf{core}_0(A)$ and $\mathsf{core}_1(A)$ for the core predicates induced by Σ_0 and Σ_1 on A, respectively. First, notice that by (1) we have $\mathbf{Q}_0 = \mathbf{Q}_1$, so let $\mathbf{Q} = \mathbf{Q}_i$, $i \in \{0, 1\}$. Let $\alpha_0 \approx \beta_0 \in \Sigma_0$, then $\models^0_{\mathbf{Q}} \alpha_0 \approx \beta_0$ hence by 4.6(W2) we obtain $\varnothing \Vdash \Delta_0(\alpha_0, \beta_0)$ and thus $\varnothing \Vdash \Delta_1(\alpha_0, \beta_0)$. It follows that $\models^1_{\mathbf{Q}} \alpha_0 \approx \beta_0$, meaning that $(A, \mathsf{core}_1(A)) \models \alpha_0 \approx \beta_0$. Since this holds for all equations $\alpha_0 \approx \beta_0 \in \Sigma_0$, it follows that $\mathsf{core}_1(A) \subseteq \mathsf{core}_0(A)$. The other direction is proven analogously. ⊣

The following corollary follows immediately from Proposition 4.8.

COROLLARY 4.18. *Let* $\Vdash$ *be a standard logic, then*:

(1) *if Conditions* 4.6(*W*1) *and* 4.6(*W*4), *or Conditions* 4.6(*W*2) *and* 4.6(*W*3) *hold, then* $\Vdash$ *is algebraized by* $(\mathbf{Q}, \Sigma, \tau, \Delta)$;
(2) *if* $\Vdash$ *is algebraizable, then there are two finitary transformers* τ *and* Δ *witnessing this fact*.

4.4. Fixed-point algebraizability of weak logics. We introduce a third alternative definition of algebraizability for weak logics, which can be seen as a refinement of strict algebraizability. We first define the following notion of selector term, essentially from [1].

NOTATION 4.19. *Let* $t(x)$ *be unary* $\mathcal{L}$*-term, then we define recursively* $t^1(x) = t(x)$ *and* $t^{n+1}(x) = t(t^n(x))$ *for all* $1 \leq n < \omega$.

DEFINITION 4.20. Let $\mathbf{Q}$ be a quasivariety of $\mathcal{L}$-algebras and $\delta(x)$ be a unary term in $\mathcal{L}$, then we say that $\delta(x)$ is a *selector term for* $\mathbf{Q}$ if $\mathbf{Q} \models \delta(x) \approx \delta^2(x)$.

REMARK 4.21. Selector terms essentially identify the fixed points of polynomials in $\mathbf{Q}$. If there is some $n < \omega$ such that for all $m < \omega$ we have $\delta^{n+m}(x) = \delta^n(x)$, then in particular we obtain

$$\delta^n(\delta^n(a)) = \delta^{n+n}(a) = \delta^n(a)$$

for all $A \in \mathbf{Q}$. Thus δ^n is a selector term for $\mathbf{Q}$.

https://doi.org/10.1017/jsl.2026.10180 Published online by Cambridge University Press

DEFINITION 4.22. A weak logic $\Vdash$ is *fixed-point algebraizable* if it is strictly algebraized by a tuple $(\mathbf{Q}, \Sigma, \tau, \Delta)$ (as in Definition 4.16) and, additionally, $\Sigma = \{x \approx \delta(x)\}$ for some selector term $\delta(x)$.

The key idea in the previous definition is that the core of the expanded algebras corresponding to the logic $\Vdash$ is not simply defined by a finite set of equations, but by one single equation characterising it as the set of fixed points of a certain polynomial. The main motivation lies in the fact that fixed-point algebraizability behaves very well in the context of varieties, as the following result shows (this is essentially [18, Theorem 19] and [1, Theorem 3.14]).

PROPOSITION 4.23. *Let* $\mathbf{V}$ *be a variety of* $\mathcal{L}$*-algebras and* $\Sigma(x) = \{\delta(x) \approx x\}$ *for some selector term* δ*, then the class of structures* $(A, \mathsf{core}(A))$ *with* $A \in \mathbf{V}$ *and* $\mathsf{core}(A) = \Sigma(A)$ *is a variety of expanded algebras.*

PROOF. Let $\mathbf{V}$ be a variety of $\mathcal{L}$-algebras and let $\mathbf{K}$ be the class of structures $(A, \mathsf{core}(A))$ with $A \in \mathbf{V}$ and $\mathsf{core}(A) = \Sigma(A)$. By Proposition 4.12 it follows that $\mathbf{K}$ is a quasivariety. We show that $\mathbf{K}$ is also closed under homomorphic images. Let $A \in \mathbf{V}$ and consider a surjective homomorphism $h : A \to B$, it suffices to show that $\mathsf{core}(B) = h[\mathsf{core}(A)]$. Since $\mathsf{core}(A) = \Sigma(A)$ and $\mathsf{core}(B) = \Sigma(B)$, it follows from the fact that homomorphic images preserve the validity of equations that $h[\mathsf{core}(A)] \subseteq \mathsf{core}(B)$. Consider now some $b \in \mathsf{core}(B)$ and notice that since h is surjective there is some $a \in A$ with $h(a) = b$. Then we have that $h(\delta(a)) = \delta(h(a)) = h(a)$, given that $h(a) = b \in \mathsf{core}(B) = \Sigma(B)$. Since δ is a selector term, we have $A \models \delta(a) \approx \delta^2(a)$ and so $\delta(a) \in \mathsf{core}(A)$ and $b \in h[\mathsf{core}(A)]$. ⊣

REMARK 4.24. We notice that most of our examples of algebraizable weak logics are actually fixed-point algebraizable. However, in the present work we shall not focus on the special features of fixed-point algebraizability, as our interest is rather in the more general properties of loose and strict algebraizability. We refer the reader to [1] for an in-depth study of logics arising from selector terms.

§5. Schematic fragment and standard companions.

We start in this section to investigate the relation between the loose and the strict version of algebraizability. In particular, we introduce the notion of schematic fragment of a weak logic and relate the algebraizability of a weak logic to the standard algebraizability of its schematic fragment.

5.1. The schematic fragment of a weak logic.

Given a weak logic $\Vdash$, it is natural to ask if we can associate to it to some specific standard logical system. The study of negative variants of intermediate logics led to the notion of schematic fragment, which was originally introduced in [22, p. 545] (under the name of *standardization*) and further investigated in [3, 12]. Here we generalise it to arbitrary weak logics.

DEFINITION 5.1 (Schematic fragment). Let $\Vdash$ be a weak logic, we define its *schematic fragment* $\mathrm{Schm}(\Vdash)$ as follows:

$$\mathrm{Schm}(\Vdash) := \{(\Gamma, \phi) \,:\, \forall \sigma \in \mathsf{Subst}(\mathcal{L}),\ \sigma[\Gamma] \Vdash \sigma(\phi)\}$$

and we then also write $\Gamma \Vdash_{\mathrm{s}} \phi$ if $(\Gamma, \phi) \in \mathrm{Schm}(\Vdash)$.

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It is clear from the definition that $\mathrm{Schm}(\vdash\!\!\!|)$ is the largest standard logic contained in $\vdash\!\!\!|$. We use the schematic fragment of a weak logic to relate loose and strict algebraizability to standard algebraizability. To this end, we first introduce the following notion of (finite) representability of a weak logic.

NOTATION 5.2. *Let* $\Gamma \subseteq \mathsf{Fm}$ *be a set of formulas, we let* $\mathsf{At}[\Gamma]$ *be the closure of* Γ *under all atomic substitutions* $\sigma \in \mathsf{At}(\mathcal{L})$. *If* Θ *is a set of equations we similarly denote by* $\mathsf{At}[\Theta]$ *the closure of* Θ *under atomic substitutions.*

DEFINITION 5.3. We say that a weak logic $\vdash\!\!\!|$ is *representable* if there is a set of formulas Λ in one variable such that for all $\Gamma \cup \{\phi\} \subseteq \mathsf{Fm}$,

$$\Gamma \vdash\!\!\!| \phi \iff \Gamma \cup \mathsf{At}[\Lambda] \vdash\!\!\!|_{\mathrm{s}} \phi.$$

We say that $\vdash\!\!\!|$ is *finitely representable* if the condition above holds for some finite set $\Lambda \subseteq \mathsf{Fm}$.

PROPOSITION 5.4. *If a weak logic* $\vdash\!\!\!|$ *is loosely algebraized by* $(\mathbf{Q}, \tau, \Delta)$, *then its schematic fragment* $\mathrm{Schm}(\vdash\!\!\!|)$ *is algebraized by* $(\mathbf{Q}, \tau, \Delta)$ *as well.*

PROOF. Let $(\mathbf{Q}, \tau, \Delta)$ witness the loose algebraizability of $\vdash\!\!\!|$. In particular $\mathbf{Q}$ is a core-generated quasivariety of expanded algebra, and thus the collection of its algebraic reducts $\mathbf{Q}{\restriction}\mathcal{L}$ is a quasivariety of algebras. We claim that $(\mathbf{Q}{\restriction}\mathcal{L}, \tau, \Delta)$ witnesses the algebraizability of $\mathrm{Schm}(\vdash\!\!\!|)$. Notice that by Fact 4.4 it suffices to verify that $(\mathbf{Q}{\restriction}\mathcal{L}, \tau, \Delta)$ satisfies 4.2(A1) and 4.2(A4). Firstly, we consider 4.2(A1):

$$\begin{aligned}
\Gamma \vdash\!\!\!|_{\mathrm{s}} \phi &\iff \forall \sigma \in \mathsf{Subst}(\mathcal{L}),\ \sigma[\Gamma] \vdash\!\!\!| \sigma(\phi) && \text{(by definition)}\\
&\iff \forall \sigma \in \mathsf{Subst}(\mathcal{L}),\ \tau(\sigma[\Gamma]) \models^{c}_{\mathbf{Q}{\restriction}\mathcal{L}} \tau(\sigma(\phi)) && \text{(by 4.6(W2))}\\
&\iff \forall \sigma \in \mathsf{Subst}(\mathcal{L}),\ \sigma(\tau[\Gamma]) \models^{c}_{\mathbf{Q}{\restriction}\mathcal{L}} \sigma(\tau(\phi)) && \text{(by structurality of } \tau)\\
&\iff \tau[\Gamma] \models_{\mathbf{Q}{\restriction}\mathcal{L}} \tau(\phi) && \text{(by Proposition 3.21).}
\end{aligned}$$

Next, consider 4.2(A4). To this end, suppose that $\tau(\Delta(x, y)) \not\equiv_{\mathbf{Q}{\restriction}\mathcal{L}} x \approx y$, then by Proposition 3.21 and the fact that $\mathbf{Q}$ is core-generated, we get that there exists a substitution σ such that $\sigma(\tau(\Delta(x, y))) \not\equiv^{c}_{\mathbf{Q}} \sigma(x \approx y)$. Then, we obtain by the structurality of τ and Δ that $\tau(\Delta(\sigma(x), \sigma(y))) \not\equiv^{c}_{\mathbf{Q}} \sigma(x) \approx \sigma(y)$, contradicting the algebraizability of $\vdash\!\!\!|$. This completes the proof. ⊣

The following theorem allows us to relate loose and strict algebraizability of weak logics to the standard algebraizability of their schematic fragment.

THEOREM 5.5. *For a weak logic* $\vdash\!\!\!|$, *the following are equivalent*:

(1) $\vdash\!\!\!|$ *is strictly algebraizable*;
(2) $\vdash\!\!\!|$ *is loosely algebraizable and* $\vdash\!\!\!|$ *is finitely representable*;
(3) $\mathrm{Schm}(\vdash\!\!\!|)$ *is algebraizable and* $\vdash\!\!\!|$ *is finitely representable.*

PROOF. We first show that (1) entails (2). By definition, if $\vdash\!\!\!|$ is strictly algebraizable, then it is also loosely algebraizable. We show that it is also finitely

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representable. Let $(\mathbf{Q}, \Sigma, \tau, \Delta)$ be the strict algebraic semantics of $\Vdash$. In particular we have that $\mathsf{core}(A) = \Sigma(A)$ for all $A \in \mathbf{Q}$. We then have the following equivalences:

$$\begin{aligned}
&\Gamma \Vdash \phi \iff \tau[\Gamma] \models^{c}_{\mathbf{Q}} \tau(\phi) && \text{(by 4.6(W1))}\\
&\iff \tau[\Gamma] \cup \mathsf{At}[\Sigma] \models_{\mathbf{Q}} \tau(\phi) && \text{(by Lemma 3.23)}\\
&\iff \forall \sigma \in \mathsf{Subst}(\mathcal{L}),\ \sigma(\tau(\Gamma)) \cup \sigma(\mathsf{At}[\Sigma]) \models^{c}_{\mathbf{Q}} \sigma(\tau(\phi)) && \text{(by Proposition 3.21)}\\
&\iff \forall \sigma \in \mathsf{Subst}(\mathcal{L}),\ \tau(\sigma(\Gamma)) \cup \sigma(\mathsf{At}(\Sigma)) \models^{c}_{\mathbf{Q}} \tau(\sigma(\phi)) && \text{(by structurality of } \tau)\\
&\iff \forall \sigma \in \mathsf{Subst}(\mathcal{L}),\ \sigma[\Gamma] \cup \Delta(\sigma(\mathsf{At}(\Sigma))) \Vdash \sigma(\phi) && \text{(by 4.6(W2) and 4.6(W3))}\\
&\iff \forall \sigma \in \mathsf{Subst}(\mathcal{L}),\ \sigma[\Gamma] \cup \sigma(\mathsf{At}(\Delta(\Sigma))) \Vdash \sigma(\phi) && \text{(by structurality of } \Delta)\\
&\iff \Gamma \cup \mathsf{At}(\Delta(\Sigma)) \Vdash_{\mathrm{s}} \phi,
\end{aligned}$$

and thus $\Lambda = \Delta(\Sigma)$ witnesses the fact that $\Vdash$ is finitely representable.

The direction from (2) to (3) follows immediately by Proposition 5.4.

Finally, we show that (3) entails (1). Suppose that $\mathrm{Schm}(\Vdash)$ is algebraized by $(\mathbf{Q}, \tau, \Delta)$ and that $\Vdash$ is finitely represented via a set of formulas Λ. Let $\mathsf{core}(A) = \tau[\Lambda](A)$ for all $A \in \mathbf{Q}$ and consider $\mathbf{Q}' := \mathbb{Q}(\{\langle \mathsf{core}(A)\rangle : A \in \mathbf{Q}\})$, i.e., $\mathbf{Q}'$ is the quasivariety of expanded algebras generated by the algebras of the form $\langle \mathsf{core}(A)\rangle$, for $A \in \mathbf{Q}$. We then derive the following equivalences:

$$\begin{aligned}
\Gamma \Vdash \phi &\iff \Gamma \cup \mathsf{At}[\Lambda] \Vdash_{\mathrm{s}} \phi && \text{(by assumption)}\\
&\iff \tau[\Gamma] \cup \tau[\mathsf{At}[\Lambda]] \models_{\mathbf{Q}} \tau(\phi) && \text{(by 4.2(A1))}\\
&\iff \tau[\Gamma] \models^{c}_{\mathbf{Q}} \tau(\phi). && \text{(by Lemma 3.23)}\\
&\iff \tau[\Gamma] \models^{c}_{\mathbf{Q}'} \tau(\phi), && \text{(by Proposition 3.24)}
\end{aligned}$$

which establish 4.6(W1). We next verify that 4.6(W4) holds as well. Suppose that $\tau(\Delta(x, y)) \not\models^{c}_{\mathbf{Q}'} x \approx y$, then since $\mathsf{core}(A) = \tau[\Lambda](A)$ we obtain that

$$\tau(\Delta(x, y)) \cup \tau[\Lambda] \not\models_{\mathbf{Q}'} \{x \approx y\} \cup \tau[\Lambda],$$

which contradicts the fact that $\mathrm{Schm}(\Vdash)$ is algebraized by $(\mathbf{Q}, \tau, \Delta)$. ⊣

The following corollary is an immediate consequence of the proofs of the previous theorem and lemma.

COROLLARY 5.6. *A weak logic* $\Vdash$ *is strictly algebraized by* $(\mathbf{Q}, \Sigma, \tau, \Delta)$ *if and only if it is represented by* $\Delta[\Sigma]$ *and* $\mathrm{Schm}(\Vdash)$ *is algebraized by* $(\mathbf{Q}, \tau, \Delta)$.

5.2. The lattice of standard companions. The previous results established that, if $\Vdash$ is strictly algebraizable, then there is a finite set Λ witnessing its finite representability. Essentially, this means that the consequences in $\Vdash$ can be encoded as logical consequences in its schematic fragment $\Vdash_{\mathrm{s}}$, modulo the set of formulas Λ. However, the schematic fragment of $\Vdash$ is not necessarily the only standard logic which bears this property. We study what are the standard logics that, up to Λ, share the same consequences.

DEFINITION 5.7. Let $\Vdash$ be a strictly algebraizable weak logic which is finitely representable by Λ and with strict algebraic semantics $(\mathbf{Q}, \Sigma, \tau, \Delta)$. A standard logic $\vdash$ is a *standard companion* of $\Vdash$ if the following conditions hold:

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(1) $\Gamma \Vdash \phi \iff \Gamma \cup \mathsf{At}[\Lambda] \vdash \phi$;
(2) $\vdash$ is algebraized by a quasivariety $\mathbf{K}$ together with the transformers τ and Δ.

We denote by $\mathrm{St}(\Vdash)$ the family of all standard companions of $\Vdash$.

REMARK 5.8. We provide some explanations of the previous definition. Condition 5.7(1) is essentially the same condition of representability from Definition 5.3, but with respect to an arbitrary standard logic. Thus Condition 5.7(1) identifies those standard logics that, up to Λ, deliver the same weak logic $\Vdash$. However, this condition by itself is quite weak, and thus we focus on logics that satisfy also condition 5.7(2), i.e., that can be algebraized *via* the same transformers τ and Δ as $\Vdash$. Notice that, in the specific cases of polyatomic logics and negative variants, these families had already been identified and studied in [1, 3], respectively.

Our underlying intuition is that the family of standard companions defined above must give rise to a corresponding notion on the side of quasivarieties of algebras.

DEFINITION 5.9. Let $\mathbf{Q}$ be a core-generated quasivariety of expanded algebras with core defined by a finite set of equations $\Sigma(x)$. Let $\mathbf{K}$ be an arbitrary quasivariety of algebras (in the same signature as $\mathbf{Q}$), then we say that $\mathbf{K}$ is a *standard companion* of $\mathbf{Q}$ if, for all $A \in \mathbf{K}$, we have $\langle \Sigma(A) \rangle \in \mathbf{Q}$.

LEMMA 5.10. *Let $\Vdash$ be a weak logic strictly algebraized by $(\mathbf{Q}, \Sigma, \tau, \Delta)$, and let $\vdash$ be a standard logic. Then $\vdash \in \mathrm{St}(\Vdash)$ if and only if $\vdash$ is algebraized by $(\mathbf{K}, \tau, \Delta)$ for some $\mathbf{K}$ standard companion of $\mathbf{Q}$.*

PROOF. We first prove the left-to-right direction. Since $\vdash \in \mathrm{St}(\Vdash)$ we have by definition that $\vdash$ is algebraized by some $(\mathbf{K}, \tau, \Delta)$, where τ and Δ also witness the strict algebraizability of $\Vdash$. In particular, notice that by the strict algebraizability of $\Vdash$ and Corollary 5.6 we have that $\Vdash$ is represented by $\Delta[\Sigma]$ over its schematic fragment, i.e., that $\Lambda = \Delta[\Sigma]$. Now, we consider $\mathbf{K}$ as a class of expanded algebras with core $\Sigma(A)$ for all $A \in \mathbf{K}$ and we let $\mathbf{C} = \mathbb{Q}(\{\langle \Sigma(A) \rangle : A \in \mathbf{K}\})$. Clearly, we have by Proposition 3.26 that $\Theta \models^c_{\mathbf{C}} \varepsilon \approx \delta$ holds if and only if $\Theta \models^c_{\mathbf{K}} \varepsilon \approx \delta$ does. Also, we have by strict algebraizability of $\Vdash$ and $\vdash$ that $\Theta \models^c_{\mathbf{K}} \varepsilon \approx \delta$ is equivalent to $\Theta \models^c_{\mathbf{Q}} \varepsilon \approx \delta$, since $\vdash \in \mathrm{St}(\Vdash)$ and $\Lambda = \Delta[\Sigma]$. Thus we obtain that $\mathrm{Th}^c_{\mathrm{qe}}(\mathbf{C}) = \mathrm{Th}^c_{\mathrm{qe}}(\mathbf{Q})$ and so by Proposition 3.26 $\mathbf{C} = \mathbf{Q}$, proving that $\mathbf{K}$ is a standard companion of $\mathbf{Q}$.

We consider the right-to-left direction. Suppose $\vdash$ is algebraized by $(\mathbf{K}, \tau, \Delta)$ and let $\mathbf{K}$ be a standard companion of $\mathbf{Q}$, we show that $\vdash$ is a standard companion of $\Vdash$. We view $\mathbf{K}$ also as a quasivariety of expanded algebras by letting, $\mathsf{core}(A) = \Sigma(A)$ for each $A \in \mathbf{K}$. We have the following equivalences:

$$\begin{aligned}
\Gamma \Vdash \phi &\iff \tau(\Gamma) \models^c_{\mathbf{Q}} \tau(\phi) && \text{(by 4.6(W1))}\\
&\iff \tau(\Gamma) \models^c_{\mathbf{K}} \tau(\phi) && \text{(by Definition 5.9 and Proposition 3.26)}\\
&\iff \tau(\Gamma) \cup \mathsf{At}[\Sigma] \models_{\mathbf{K}} \tau(\phi) && \text{(by Lemma 3.23)}\\
&\iff \Gamma \cup \Delta(\mathsf{At}[\Sigma]) \vdash \phi && \text{(by 4.2(A2) and 4.2(A3))}\\
&\iff \Gamma \cup \mathsf{At}(\Delta[\Sigma]) \vdash \phi && \text{(by structurality of } \Delta)
\end{aligned}$$

and thus, since $\Lambda = \Delta[\Sigma]$ (as we argued in the previous direction), it follows that $\vdash$ is a standard companion of $\Vdash$. This completes our proof. ⊣

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The correspondence provided by the previous lemma motivates the following definition. Intuitively, while the schematic fragment identifies the greatest standard logic contained in a weak logic, the following notion characterises the greatest quasivariety with core defined by Σ and which has the same core-generated substructures of a quasivariety $\mathbf{Q}$. The following definition essentially refines the notion of core superalgebra (Definition 3.22) in the setting with equationally definable core.

DEFINITION 5.11. We say that B is a Σ-*superalgebra* of A if $A \leqslant B$ and $\Sigma(A) = \Sigma(B)$. If $\mathbf{Q}$ is a class of algebras, then we write $\mathbb{C}_\Sigma(\mathbf{Q})$ for the class of all Σ-superalgebras of elements of $\mathbf{Q}$.

LEMMA 5.12. *Suppose* $\mathbf{K}$ *is a quasivariety of expanded algebras with core defined by* Σ, *then* $\mathbb{C}_\Sigma(\mathbf{K})$ *is also a quasivariety of expanded algebras.*

PROOF. We notice that $A \in \mathbb{C}_\Sigma(\mathbf{K})$ if and only if $A \models \forall x(\mathsf{core}(x) \leftrightarrow \bigwedge \Sigma(x))$, which can be rephrased by means of Horn formulas as in Proposition 4.12. Also, for all quasiequations $\bigwedge_{i \leqslant n} \varepsilon_i \approx \delta_i \to \varepsilon \approx \delta \in \mathrm{Th}^c_{\mathrm{qe}}(\mathbf{K})$ we have that this is equivalent to saying

$$A \models \forall x_1, \ldots \forall x_n \Big(\bigwedge_{1 \leqslant i \leqslant n} \mathsf{core}(x_i) \wedge \bigwedge_{i \leqslant n} \varepsilon_i \approx \delta_i \to \varepsilon \approx \delta \Big),$$

where these are also Horn formulas. It follows immediately by Fact 2.14 that $\mathbb{C}_\Sigma(\mathbf{K})$ is closed under $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$, and $\mathbb{P}_{\mathrm{U}}$. ⊣

We conclude this section with the following characterisation of the family of standard companions of a strictly algebraizable weak logic $\Vdash$. This generalises the previous results from [1, 3] to our abstract setting.

NOTATION 5.13. *Let* $\tau : \mathsf{Fm} \to \mathsf{Eq}$ *and let* $\mathbf{Q}$ *be a quasivariety, then we write* $\mathsf{Log}_\tau(\mathbf{Q})$ *for the set of all pairs* (Γ, ϕ) *such that* $\tau(\Gamma) \models_{\mathbf{Q}} \tau(\phi)$.

THEOREM 5.14. *Let* $\Vdash$ *be a weak logic algebraized by* $(\mathbf{Q}, \Sigma, \tau, \Delta)$, *then* $\mathrm{St}(\Vdash)$ *with the subset ordering* $\subseteq$ *forms a bounded lattice with maximum element* $\mathrm{Schm}(\Vdash)$ *and minimum element* $\mathsf{Log}_\tau(\mathbb{C}_\Sigma(\mathbf{Q}))$.

PROOF. Let $\vdash_0, \vdash_1 \in \mathrm{St}(\Vdash)$, then $\vdash_0 \cap \vdash_1$ is a standard logic and it is straightforward to verify that $\vdash_0 \cap \vdash_1 \in \mathrm{St}(\Vdash)$. Similarly, if $\vdash_0, \vdash_1 \in \mathrm{St}(\Vdash)$ and they are algebraized, respectively, by $(\mathbf{K}_0, \tau, \Delta)$ and $(\mathbf{K}_1, \tau, \Delta)$, then $\mathbf{K}_0 \cap \mathbf{K}_1$ is a quasivariety, and the logic $\vdash_2$ defined by letting

$$\Gamma \vdash_2 \phi \iff \tau(\Gamma) \models^c_{\mathbf{K}_0 \cap \mathbf{K}_1} \tau(\phi)$$

is the supremum of $\vdash_0$ and $\vdash_1$ in $\mathrm{St}(\Vdash)$. This shows that $(\mathrm{St}(\Vdash), \subseteq)$ is a lattice.

We show that $\mathrm{Schm}(\Vdash)$ is maximal in $\mathrm{St}(\Vdash)$. Recall that, by Corollary 5.6, we have that $\mathrm{Schm}(\Vdash) = \mathsf{Log}_\tau(\mathbf{Q})$. Suppose now that $\vdash \in \mathrm{St}(\Vdash)$ then by monotonicity $\Gamma \vdash \phi$ entails $\Gamma \cup \Lambda \vdash \phi$, and thus we have $\Gamma \Vdash \phi$. This shows that $\vdash \subseteq \Vdash$. Since $\vdash$ and $\mathrm{Schm}(\Vdash)$ are both closed under uniform substitution, it follows that $\vdash \subseteq \mathrm{Schm}(\Vdash)$.

We show that $\mathsf{Log}_\tau(\mathbb{C}_\Sigma(\mathbf{Q}))$ is minimal in $\mathrm{St}(\Vdash)$. If $\vdash \in \mathrm{St}(\Vdash)$, then by Lemma 5.10 $\vdash$ is algebraized by $(\mathbf{K}, \tau, \Delta)$, for some standard companion $\mathbf{K}$ of $\mathbf{Q}$. By Definition 5.11, it follows that $\mathbf{K} \subseteq \mathbb{C}_\Sigma(\mathbf{Q})$ and so $\mathsf{Log}_\tau(\mathbb{C}_\Sigma(\mathbf{Q})) \subseteq \mathsf{Log}_\tau(\mathbf{K}) = \vdash$. ⊣

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REMARK 5.15. We notice that, if $\Vdash$ is fixed-point algebraized by $(\mathbf{Q}, \Sigma, \tau, \Delta)$ with $\Sigma = \{x \approx \delta(x)\}$ as in Definition 4.22, then $\Theta \models^{c}_{\mathbf{Q}} \alpha \approx \beta$ if and only if $\sigma_\delta(\Theta) \models \sigma_\delta(\alpha) \approx \sigma_\delta(\beta)$, where σ_δ is the substitution $\sigma_\delta(x) = \delta(x)$ for all $x \in \mathsf{Var}$. Then, since $\Vdash_{\mathrm{s}}$ is the logic $\mathsf{Log}_\tau(\mathbf{Q})$, we obtain that $\Gamma \Vdash \phi$ if and only if $\sigma_\delta(\Gamma) \Vdash_{\mathrm{s}} \sigma_\delta(\phi)$. This correspondence showcases a special case where the equations identifying the core of the expanded algebra induce a translation between a logic and its schematic fragment. We refer the reader to [1] for an in-depth study of such translations and to [1, 23] for the relation between such translations and corresponding adjunctions between quasivarieties of algebras.

5.3. Example of bridge theorem: Deduction theorem. As stressed by Font in [16, p. 160], "Bridge theorems and transfer theorems are the ultimate justification of abstract algebraic logic." We show that this motivation remains applicable in our modified setting of weak logics. The key observation is that, in virtue of Theorem 5.5, we do not have to come up with novel algebraic descriptions of logical properties, but we can rely on the established characterisation of such properties for standard logics.

To showcase a concrete example, we look at one bridge theorem for strictly algebraizable weak logics, i.e., we consider the case of strictly algebraizable logics with a deduction–detachment theorem. We start by recalling the definition of this property and the standard bridge theorem relating logics with the deduction–detachment theorem and quasivarieties with the so-called EDPRC property.

DEFINITION 5.16. A weak (or standard) logic $\Vdash$ has the *deduction–detachment theorem (DDT)* if there is a finite set of formulas $I(x, y)$ such that, for every set of formulas Γ,

$$\Gamma \cup \{\phi\} \Vdash \psi \iff \Gamma \Vdash I(\phi, \psi).$$

DEFINITION 5.17. A quasivariety $\mathbf{K}$ has *equationally definable principal relative congruences* (EDPRC) if there is a finite set of equations $\Theta(x, y, z, v)$ such that, for every $A \in \mathbf{K}$ and $a, b, c, d \in A$,

$$\langle a, b\rangle \in \mathrm{Cn}_{\mathbf{K}}(c, d) \iff A \models \Theta(c, d, a, b),$$

where $\mathrm{Cn}_{\mathbf{K}}(c, d)$ is the $\mathbf{K}$-congruence of A induced by the equation $c \approx d$, i.e., the smallest congruence θ of A such that $(c, d) \in \theta$ and $A/\theta \in \mathbf{K}$ (cf. Notation 6.3).

The following result, due to Blok and Pigozzi, is one classical bridge theorem from abstract algebraic logic. We refer the reader to [16, Theorem 3.85] for a proof of this result.

FACT 5.18 (Blok–Pigozzi). *Let $\vdash$ be a standard algebraizable logic with equivalent algebraic semantics $\mathbf{Q}$, then $\vdash$ has a deduction–detachment theorem if and only if $\mathbf{Q}$ has EDPRC.*

In the case of a strictly algebraizable weak logic $\Vdash$, finite representability allows us to transfer the property of having the DDT from $\Vdash$ to $\Vdash_{\mathrm{s}}$. More precisely, we can prove the following proposition.

PROPOSITION 5.19. *Let $\Vdash$ be a strictly algebraizable weak logic, then $\Vdash$ has DDT if and only if $\Vdash_{\mathrm{s}}$ has DDT, witnessed by the same finite set of formulas $I(x, y)$.*

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PROOF. We first prove the right-to-left direction. Suppose $\Vdash_s$ has DDT, and notice that since $\Vdash$ is strictly algebraizable then it is finitely represented by a set of formulas Λ (by Theorem 5.5). We have the following equivalences:

$$\begin{aligned}
\Gamma \cup \{\phi\} \Vdash \psi &\iff \Gamma \cup \{\phi\} \cup \mathsf{At}[\Lambda] \Vdash_s \psi && \text{(by representability)}\\
&\iff \Gamma \cup \mathsf{At}[\Lambda] \Vdash_s I(\phi, \psi) && \text{(by assumption)}\\
&\iff \Gamma \Vdash I(\phi, \psi), && \text{(by representability)}
\end{aligned}$$

which establish that $\Vdash$ has DDT.

Consider now the left-to-right direction. We assume that $\Vdash$ has DDT and that this is witnessed by the finite set of formulas $I(x, y)$, then:

$$\begin{aligned}
\Gamma \cup \{\phi\} \Vdash_s \psi &\iff \forall \sigma \in \mathsf{Subst}(\mathcal{L}),\ \sigma(\Gamma \cup \{\phi\}) \Vdash \sigma(\psi) && \text{(by Definition 5.1)}\\
&\iff \forall \sigma \in \mathsf{Subst}(\mathcal{L}),\ \sigma(\Gamma) \Vdash I(\sigma(\phi), \sigma(\psi)) && \text{(by assumption)}\\
&\iff \Gamma \Vdash_s I(\phi, \psi), && \text{(by Definition 5.1)}
\end{aligned}$$

which proves our claim. ⊣

COROLLARY 5.20. *Let $\Vdash$ be a strictly algebraizable weak logic with equivalent algebraic semantics* $\mathbf{Q}$*, then $\Vdash$ has DDT if and only if* $\mathbf{Q}$ *has EDPRC.*

PROOF. Immediate from Fact 5.18 and Proposition 5.19. ⊣

REMARK 5.21. The previous result provides one example where one can obtain results about strictly algebraizable weak logics by investigating their schematic fragments. In particular, we did not need to come up with a novel algebraic characterisation for the presence of a deduction–detachment theorem in a strictly algebraizable weak logic $\Vdash$, and we could simply rely on Theorem 5.5 and the classical bridge theorem by Blok and Pigozzi. This shows that strict algebraizability provides an especially strong and well-behaved notion in the setting of weak logics. We point out that we ascribe the underlying reason of this phenomena to the validity of a strong version of the Isomorphism Theorem in the case of strict algebraizability (in contrast to loose algebraizability). We shall investigate this in Section 6.3.

§6. The isomorphism theorem. Blok and Pigozzi's isomorphism theorem [16, Section 3.5] is possibly the most important single result on algebraizability, as it relates the algebraizability of a logic to the existence, for every algebra, of an isomorphism between the lattice of deductive filters of the logic and the lattice of its relative congruences. We consider in this section analogues of this result for the setting of algebraizable weak logics. Firstly, in Section 6.1 we recall the isomorphism theorem for standard algebraizable logic. In Section 6.2 we provide a partial analogue of this result for loosely algebraizable weak logics, and then in Section 6.3 we prove a stronger result for the case of strictly algebraizable weak logics.

6.1. Standard algebraizable logics. We start by providing some preliminary definitions needed to state Blok and Pigozzi's result. We first recall the notion of deductive filter and fix some notation for algebraic congruences.

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DEFINITION 6.1. For any algebra A and standard logic $\vdash$, we say that $F \subseteq A$ is a *deductive filter* of $\vdash$ on A if:

$$\Gamma \vdash \phi \implies \forall h \in \mathsf{Hom}(\mathsf{Fm}, A),\ h[\Gamma] \subseteq F \text{ entails } h(\phi) \in F;$$

and we let $\mathsf{Fi}_{\vdash}(A)$ be the set of all deductive filters of $\vdash$ on A.

REMARK 6.2. We notice that $\mathsf{Fi}_{\vdash}(A)$ is closed under arbitrary intersections and thus that it forms a complete lattice under the subset ordering. Moreover, if F is a deductive filter and h is an endomorphism of A, then $h^{-1}(F)$ is also a deductive filter. We denote by $\mathsf{Fi}^{+}_{\vdash}(A)$ the lattice expansion $(\mathsf{Fi}_{\vdash}, \subseteq, \{h^{-1} : h \in \mathsf{End}(\mathcal{L})\})$. We refer the reader to [16, Section 2.3] for proofs of these facts.

NOTATION 6.3. *Given any algebra A we let $\mathsf{Con}(A)$ be the set of all congruences θ over A, and $\mathsf{Con}_{\mathbf{Q}}(A)$ be the set of all $\mathbf{Q}$-congruences of A, i.e., those congruences θ over A such that $A/\theta \in \mathbf{Q}$.*

REMARK 6.4. Similarly to the case of deductive filters, $\mathsf{Con}_{\mathbf{Q}}(A)$ is closed under arbitrary intersections and therefore it is also a lattice under the subset ordering. Additionally, it is possible to verify that $\mathsf{Con}_{\mathbf{Q}}(A)$ is closed under inverse endomorphisms of A. We then write $\mathsf{Con}^{+}_{\mathbf{Q}}(A)$ for this lattice expansion, i.e., $\mathsf{Con}^{+}_{\mathbf{Q}}(A) = (\mathsf{Con}_{\mathbf{Q}}(A), \subseteq, \{h^{-1} : h \in \mathsf{End}(A)\})$.

Finally, we introduce syntactical and semantical theories as follows.

DEFINITION 6.5. If $\vdash$ is a standard logic, then we denote by $\mathsf{Th}(\vdash)$ the set of all *(syntactic) theories* over $\vdash$, i.e., all sets $\Gamma \subseteq \mathsf{Fm}$ such that $\Gamma \vdash \phi$ entails $\phi \in \Gamma$. If $\mathbf{Q}$ is a quasivariety, then $\mathsf{Th}(\models_{\mathbf{Q}})$ denotes the set of *(semantical) theories* over $\mathbf{Q}$, i.e., the sets of equations $\Theta \subseteq \mathsf{Eq}$ such that $\Theta \models_{\mathbf{Q}} \alpha \approx \beta$ entails $\alpha \approx \beta \in \Theta$.

REMARK 6.6. We notice that $\mathsf{Th}(\vdash)$ forms a lattice under the subset relation, and that it is additionally closed under inverse substitutions. We refer by $\mathsf{Th}^{+}(\vdash)$ to the lattice expansion $(\mathsf{Th}(\vdash), \subseteq, \{\sigma^{-1} : \sigma \in \mathsf{Subst}(\mathcal{L})\})$. Similarly, also $\mathsf{Th}(\models_{\mathbf{Q}})$ forms a lattice under the subset ordering and is closed under inverse substitutions. We let $\mathsf{Th}^{+}(\models_{\mathbf{Q}}) = (\mathsf{Th}(\models_{\mathbf{Q}}), \subseteq, \{\sigma^{-1} : \sigma \in \mathsf{Subst}(\mathcal{L})\})$. These properties follow from the fact that the syntactic theories over $\vdash$ are exactly the deductive filters of $\vdash$ over Fm, and that, under the identification of equations and pairs of formulas, the semantic theories over $\models_{\mathbf{Q}}$ are the $\mathbf{Q}$-congruences of Fm.

Blok and Pigozzi's isomorphism theorem for standard logics provides a criterion to determine if a logic is algebraized by a quasivariety based on their associated lattices of filters and congruences. For a proof of the following theorem we refer the reader to [16, Section 3.5].

FACT 6.7 (Isomorphism theorem). *Let $\vdash$ be a standard logic and $\mathbf{Q}$ a quasivariety, then the following are equivalent*:

(1) $\vdash$ *is algebraizable with equivalent algebraic semantics* $\mathbf{Q}$;
(2) $\mathsf{Fi}^{+}_{\vdash}(A) \cong \mathsf{Con}^{+}_{\mathbf{Q}}(A)$, *for any algebra* A;
(3) $\mathsf{Th}^{+}(\vdash) \cong \mathsf{Th}^{+}(\models_{\mathbf{Q}})$.

REMARK 6.8. We clarify what are the underlying witnesses in the previous theorem (cf. [16, p. 150]). On the one hand, let $\vdash$ be an algebraizable logic with equivalent

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algebraic semantics $(\mathbf{Q}, \tau, \Delta)$. Then the associated isomorphism $\mathsf{Fi}^{+}_{\vdash}(A) \cong \mathsf{Con}^{+}_{\mathbf{Q}}(A)$ is given by the following map from filters to congruences:

$$\begin{aligned}\theta_{(-)} : \mathsf{Fi}_{\vdash}(A) &\longrightarrow \mathsf{Con}_{\mathbf{Q}}(A)\\ G &\longmapsto \theta_G := \{(a,b) \in A^2 : \Delta^A(a,b) \subseteq G\}\end{aligned}$$

and the following map from congruences to filters:

$$\begin{aligned}F_{(-)} : \mathsf{Con}_{\mathbf{Q}}(A) &\longrightarrow \mathsf{Fi}_{\vdash}(A)\\ \eta &\longmapsto F_\eta := \{a \in A : \tau^A(a) \subseteq \eta\},\end{aligned}$$

which can be shown to be inverse of each other. On the other hand, suppose $\Omega : \mathsf{Th}^{+}(\vdash) \cong \mathsf{Th}^{+}(\models_{\mathbf{Q}})$ is an isomorphism. Then the two transformers τ and Δ are defined as follows:

$$\begin{aligned}\tau(x) &= \sigma_x(\Omega(\mathrm{Cn}_{\vdash}(x)))\\ \Delta(x,y) &= \sigma_{x,y}(\Omega^{-1}(\mathrm{Cn}_{\mathbf{Q}}(x \approx y))),\end{aligned}$$

where $\mathrm{Cn}_{\vdash}$ and $\mathrm{Cn}_{\mathbf{Q}}$ denote, respectively, the closure consequence operators on the logic $\vdash$ and the quasivariety $\mathbf{Q}$, σ_x is the substitution sending every variable to x, and $\sigma_{x,y}$ is the substitution sending every variable but y to x and y to itself.

6.2. Loosely algebraizable weak logics. We prove in this section a (partial) version of the isomorphism theorem for loosely algebraizable weak logics. To this end, we start by introducing a version of deductive filters and congruences relative to core semantics.

DEFINITION 6.9. For any expanded algebra A and weak logic $\Vdash$, we say that $F \subseteq A$ is a *core filter* of $\Vdash$ over A if:

$$\Gamma \Vdash \phi \implies \forall h \in \mathsf{Hom}^{c}(\mathsf{Fm}, A),\ h[\Gamma] \subseteq F \text{ entails } h(\phi) \in F;$$

and we denote the set of core filters of A with respect to $\Vdash$ by $\mathsf{Fi}_{\Vdash}(A)$.

REMARK 6.10. If A is an expanded algebra and $\mathbf{Q}$ a quasivariety of algebras in the same algebraic signature of A, then we write $\mathsf{Con}(A)$ and $\mathsf{Con}_{\mathbf{Q}}(A)$ also for the congruences and the $\mathbf{Q}$-congruences of A, respectively. Notice that, if A is an expanded algebra and θ a congruence of its algebraic reduct, then the structure A/θ is simply the quotient of the algebraic reduct of A by θ with $\mathsf{core}(A/\theta) = \mathsf{core}(A)/\theta$. This corresponds to viewing A/θ as a strong homomorphic image of A (cf. Definition 2.8 and Remark 2.9).

DEFINITION 6.11. Given an expanded algebra A and a quasivariety $\mathbf{Q}$ of expanded algebras, a congruence $\theta \in \mathsf{Con}(A)$ is said to be a *core* $\mathbf{Q}$-congruence if $A/\theta \in \mathbf{Q}$. We write $\mathsf{Con}^{c}_{\mathbf{Q}}(A)$ for the set of all core $\mathbf{Q}$-congruences over A.

REMARK 6.12. Similarly to Remarks 6.2 and 6.4, it is possible to verify that both $\mathsf{Fi}_{\Vdash}(A)$ and $\mathsf{Con}^{c}_{\mathbf{Q}}(A)$ are closed under arbitrary intersections, and thus form complete lattices under the subset ordering.

LEMMA 6.13. *Let* $\mathbf{Q}$ *be a quasivariety of expanded algebras and let* A *be a core-generated expanded algebra, then*:

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(1) *if $G \subseteq A$ is a core filter of $\Vdash$, it is a deductive filter of its schematic fragment $\Vdash_s$;*
(2) *if θ is a core $\mathbf{Q}$-congruence of A, then it is also a $\mathbf{Q}\uparrow\mathcal{L}$-congruence.*

PROOF. Clause (2) follows immediately from the definition of core $\mathbf{Q}$-congruence. We thus consider clause (1). Let $G \subseteq A$ be a core filter of $\Vdash$, and let $\Gamma \Vdash_s \phi$, $h[\Gamma] \subseteq G$ for some $h \in \mathsf{Hom}(\mathsf{Fm}, A)$. Now, since A is core-generated, by Lemma 3.20 we can find a substitution σ and a core assignment g, such that $g(\sigma(x)) = h(x)$ for all $x \in \mathsf{Var}$. In particular, it follows that $g[\sigma[\Gamma]] \subseteq G$. Now, since $\Gamma \Vdash_s \phi$, it follows by uniform substitution that $\sigma[\Gamma] \Vdash_s \sigma(\phi)$ and therefore $\sigma[\Gamma] \Vdash \sigma(\phi)$. Now, since g is a core assignment and G a core filter with $g[\sigma[\Gamma]] \subseteq G$, it follows that $g(\sigma(\phi)) \in G$, whence $h(\phi) \in G$. This shows that G is a deductive filter with respect to $\Vdash_s$. ⊣

LEMMA 6.14. *Let $\mathbf{Q}$ be a quasivariety of expanded algebras, A be a core-generated expanded algebra, and let h be a strong endomorphism of A, then the following hold*:

(1) *if F is a core filter of $\Vdash$, then $h^{-1}(F)$ is also a core filter*;
(2) *if $\theta \in \mathsf{Con}^c_{\mathbf{Q}}(A)$ then $h^{-1}(\theta) \in \mathsf{Con}^c_{\mathbf{Q}}(A)$.*

Therefore, $\mathsf{Fi}_{\Vdash}(A)$ and $\mathsf{Con}^c_{\mathbf{Q}}(A)$ are two lattices commuting with all inverse strong endomorphisms of A.

PROOF. We already pointed out in Remark 6.12 that $\mathsf{Fi}_{\Vdash}(A)$ and $\mathsf{Con}^c_{\mathbf{Q}}(A)$ are (complete) lattices under the subset ordering, whence we simply prove (1) and (2).

We prove (1). Suppose $\Gamma \Vdash \phi$ and let $g \in \mathsf{Hom}^c(\mathsf{Fm}, A)$ be such that $g[\Gamma] \subseteq h^{-1}(F)$. Then $h(g[\Gamma]) \subseteq F$ and, since $h \circ g \in \mathsf{Hom}^c(\mathsf{Fm}, A)$, it follows that $h(g[\phi]) \subseteq F$ and thus $g(\phi) \in h^{-1}(F)$.

We prove (2). The fact that $h^{-1}(\theta) \in \mathsf{Con}(A)$ follows from Remark 6.4. We next show that $A/h^{-1}(\theta) \in \mathbf{Q}$. Consider a quasiequation $\bigwedge_{i\leqslant n} \varepsilon_i \approx \delta_i \to \alpha \approx \beta \in \mathrm{Th}^c_{\mathrm{qe}}(\mathbf{Q})$. Since $A/\theta \in \mathbf{Q}$ we have that $A/\theta \models^c \bigwedge_{i\leqslant n} \varepsilon_i \approx \delta_i \to \alpha \approx \beta$. Now, let $g \in \mathsf{Hom}^c(\mathsf{Fm}, A)$ be such that $g(\varepsilon_i)/h^{-1}(\theta) = g(\delta_i)/h^{-1}(\theta)$ for all $i\leqslant n$, then it follows that $h \circ g(\varepsilon_i)/\theta = h \circ g(\delta_i)/\theta$ for all $i\leqslant n$. Since h is a strong endomorphism of A we obtain that $h \circ g(\alpha)/\theta = h \circ g(\beta)/\theta$. It then follows that $g(\alpha)/h^{-1}(\theta) = g(\beta)/h^{-1}(\theta)$. This shows that $A/h^{-1}(\theta) \models^c \bigwedge_{i\leqslant n} \varepsilon_i \approx \delta_i \to \alpha \approx \beta$ and so $A/h^{-1}(\theta) \models^c \mathrm{Th}^c_{\mathrm{qe}}(\mathbf{Q})$. Finally, notice that since A is core generated, also $A/h^{-1}(\theta)$ is core generated, and thus we have from Proposition 3.26 that $A/h^{-1}(\theta) \in \mathbf{Q}$. We conclude that $h^{-1}(\theta) \in \mathsf{Con}^c_{\mathbf{Q}}(A)$. ⊣

As expanded algebras augment standard algebras with an additional core predicate, we now expand $\mathsf{Fi}^+_{\vdash}(A)$ and $\mathsf{Con}^+_{\mathbf{Q}}(A)$ in a similar way to capture core filters and congruences.

DEFINITION 6.15. Let $\Vdash$ be a weak logic and let $\mathbf{Q}$ be a quasivariety of expanded algebras, for any core-generated expanded algebra A, we write $\overline{\mathsf{Fi}}_{\Vdash}(A)$ for the structure

$$(\mathsf{Fi}_{\Vdash_s}(A),\ \mathsf{Fi}_{\Vdash}(A),\ \subseteq,\ \{\sigma^{-1} : \sigma \in \mathsf{sEnd}(A)\}$$

and we write $\overline{\mathsf{Con}}_{\mathbf{Q}}(A)$ for the structure

$$(\mathsf{Con}_{\mathbf{Q}\uparrow\mathcal{L}}(A),\ \mathsf{Con}^c_{\mathbf{Q}}(A),\ \subseteq,\ \{\sigma^{-1} : \sigma \in \mathsf{sEnd}(A)\}),$$

https://doi.org/10.1017/jsl.2026.10180 Published online by Cambridge University Press

where $\mathsf{sEnd}(A)$ is the set of all strong endomorphisms of the expanded algebra A (recall Definition 2.8).

REMARK 6.16. The key intuition in the context of weak logics is to consider *together* arbitrary filters and core filters, and to consider *together* arbitrary $\mathbf{Q}$-congruences and core $\mathbf{Q}$-congruences. This is possible because of Lemma 6.13, which makes sure that core filters are deductive filters, and that core congruences are congruences. Then, by Lemma 6.14, we additionally have that strict endomorphisms also preserve core filters and core congruences.

THEOREM 6.17. *Let $\Vdash$ be a weak logic and suppose it is loosely algebraized by $(\mathbf{Q}, \tau, \Delta)$, then for every core-generated expanded algebra A there is an isomorphism $\Omega : \overline{\mathsf{Fi}}_{\Vdash}(A) \cong \overline{\mathsf{Con}}_{\mathbf{Q}}(A)$.*

PROOF. Firstly, we define the following map from filters to congruences:

$$\begin{aligned} \theta_{(-)} : \mathsf{Fi}_{\Vdash_s}(A) &\longrightarrow \mathsf{Con}_{\mathbf{Q}\uparrow\mathcal{L}}(A) \\ G &\longmapsto \theta_G := \{(a,b) \in A^2 : \Delta^A(a,b) \subseteq G\} \end{aligned}$$

and the following map from congruences to filters:

$$\begin{aligned} F_{(-)} : \mathsf{Con}_{\mathbf{Q}\uparrow\mathcal{L}}(A) &\longrightarrow \mathsf{Fi}_{\Vdash_s}(A) \\ \eta &\longmapsto F_\eta := \{a \in A : \tau^A(a) \subseteq \eta\}. \end{aligned}$$

Now, notice that since $\Vdash$ is loosely algebraized by $(\mathbf{Q}, \tau, \Delta)$, then it follows from Proposition 5.4 that also $\Vdash_s$ is algebraized by $(\mathbf{Q}, \tau, \Delta)$. Since the two maps above are exactly the same that occur in the proof of the isomorphism theorem for standard logics (cf. Remark 6.8), it follows immediately that they describe an isomorphism $\mathsf{Fi}^+_{\Vdash_s}(A) \cong \mathsf{Con}^+_{\mathbf{Q}}(A)$. Thus, it suffices to show that $\theta_{(-)}$ sends core filters to core congruences, and that $F_{(-)}$ sends core congruences to core filters.

First, let G be a core filter over A, we show that θ_G is a core-$\mathbf{Q}$-congruence of A. Consider A/θ_G and notice that this is well-defined because θ_G is a congruence. Now, consider the expansion of A/θ_G defined by letting $\mathsf{core}(A/\theta_G) = \mathsf{core}(A)/\theta_G$, we claim that $(A/\theta_G, \mathsf{core}(A)/\theta_G) \in \mathbf{Q}$. Let $\bigwedge_{i\leqslant n} \varepsilon_i \approx \delta_i \models^c_{\mathbf{Q}} \alpha \approx \beta$ and suppose $h \in \mathsf{Hom}^c(\mathsf{Fm}, A/\theta_G)$ is such that $h(\varepsilon_i) = h(\delta_i)$ for all $i \leqslant n$. By algebraizability we have that $\bigcup_{i\leqslant n} \Delta(\varepsilon_i \approx \delta_i) \Vdash \Delta(\alpha, \beta)$ and so $h(\Delta(\alpha,\beta)) \subseteq G$. It follows that $(h(\alpha), h(\beta)) \in \theta_G$ and so $A/\theta_G \models^c \alpha \approx \beta$. This shows that $(A/\theta_G, \mathsf{core}(A)/\theta_G) \models^c \bigwedge_{i\leqslant n} \varepsilon_i \approx \delta_i \to \alpha \approx \beta$. Since $\mathbf{Q}$ is a core-generated quasivariety of expanded algebras, it follows from Proposition 3.26 that $(A/\theta_G, \mathsf{core}(A)/\theta_G) \in \mathbf{Q}$. We conclude that the map $\theta_{(-)}$ sends core filters to core-$\mathbf{Q}$-congruences.

Next, we show that for any core-$\mathbf{Q}$-congruence θ over A the filter F_θ is a core filter. Suppose $\Gamma \Vdash \phi$ and $h \in \mathsf{Hom}^c(\mathsf{Fm}, A)$ is such that $h[\Gamma] \subseteq F_\theta$. Then, we have by definition that $\tau^A(c) \subseteq \theta$ for every $c \in h[\Gamma]$. By algebraizability, notice that we have $\tau(\Gamma) \models^c_{\mathbf{Q}} \tau(\phi)$, thus we obtain $h(\tau(\phi)) = \tau^A(h(\phi)) \subseteq \theta$ and so $h(\phi) \in F_\theta$. This shows that F_θ is a core deductive filter and completes the proof. ⊣

Exactly as in the standard setting, it is possible to specialise the previous isomorphism to the specific case of filters and congruences of the (expanded) algebra of formulas Fm. We first define syntactic and semantic theories in the setting of weak logics.

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DEFINITION 6.18. If $\Vdash$ is a weak logic then we denote by $\mathsf{Th}(\Vdash)$ the set of all *(syntactic) theories* over $\Vdash$, i.e., all sets $\Gamma \subseteq \mathsf{Fm}$ such that if $\Gamma \Vdash \phi$ then $\phi \in \Gamma$. If $\mathbf{Q}$ is a core-generated quasivariety of expanded algebras, then $\mathsf{Th}(\models^c_{\mathbf{Q}})$ denotes the set of *(semantical) core theories* over $\mathbf{Q}$, i.e., those sets of equations $\Theta \subseteq \mathsf{Eq}$ such that $\Theta \models^c_{\mathbf{Q}} \alpha \approx \beta$ entails $\alpha \approx \beta \in \Theta$.

REMARK 6.19. It is straightforward to verify that, for $\Vdash$ a weak logic, the syntactic theories from Definition 6.18 are exactly the core filters of $\Vdash$ over $(\mathsf{Fm}, \mathsf{Var})$. Similarly, if $\mathbf{Q}$ is a quasivariety of expanded algebras, then the semantical core theories over $\mathbf{Q}$ are exactly the core congruences of $(\mathsf{Fm}, \mathsf{Var})$. Since $(\mathsf{Fm}, \mathsf{Var})$ is a core-generated algebra, we obtain as in Definition 6.15 an expanded lattice $\overline{\mathsf{Th}}(\Vdash)$ defined by

$$(\mathsf{Th}(\Vdash_{\mathrm{s}}),\ \mathsf{Th}(\Vdash),\ \subseteq,\ \{\sigma^{-1} : \sigma \in \mathsf{At}(\mathcal{L})\})$$

and an expanded lattice $\overline{\mathsf{Th}}(\models^c_{\mathbf{Q}})$ defined by

$$(\mathsf{Th}(\models_{\mathbf{Q}}),\ \mathsf{Th}(\models^c_{\mathbf{Q}}),\ \subseteq,\ \{\sigma^{-1} : \sigma \in \mathsf{At}(\mathcal{L})\}).$$

This clearly corresponds to the fact that theories over $\Vdash$ are also theories in $\Vdash_{\mathrm{s}}$, and theories over $\models^c_{\mathbf{Q}}$ are also theories over $\models_{\mathbf{Q}}$. Additionally, since the strong endomorphisms of $(\mathsf{Fm}, \mathsf{Var})$ are exactly the atomic substitutions, it follows from Lemma 6.14 that inverse atomic substitutions preserve elements in $\mathsf{Th}(\Vdash)$ and $\mathsf{Th}(\models^c_{\mathbf{Q}})$.

COROLLARY 6.20. *Let $\Vdash$ be a weak logic and suppose it is loosely algebraized by $(\mathbf{Q}, \tau, \Delta)$, then there is an isomorphism $\Omega : \overline{\mathsf{Th}}(\Vdash) \cong \overline{\mathsf{Th}}(\models^c_{\mathbf{Q}})$.*

PROOF. It follows immediately from Theorem 6.17 and Remark 6.19. ⊣

REMARK 6.21. A key aspect of Blok and Pigozzi's isomorphism theorem (Fact 6.7) is that one can recover the two transformers τ and Δ from the isomorphism between $\mathsf{Fi}^+_{\vdash}(A)$ and $\mathsf{Con}^+_{\mathbf{Q}}(A)$, thus showing algebraizability. We indicate however that this fact is not as clear in the setting of loose algebraizability. In fact, in the proof of the isomorphism theorem (cf. Remark 6.8 and [16, Section 3.5]) the two transformers are defined as follows:

$$\tau(x) = \sigma_x(\Omega(\mathrm{Cn}_{\vdash}(x)))$$
$$\Delta(x, y) = \sigma_{x,y}(\Omega^{-1}(\mathrm{Cn}_{\mathbf{Q}}(x \approx y))),$$

where $\mathrm{Cn}_{\vdash}$ and $\mathrm{Cn}_{\mathbf{Q}}$ denote, respectively, the closure consequence operators on the logic $\vdash$ and the quasivariety $\mathbf{Q}$, while σ_x is a substitution sending every variable to x, and $\sigma_{x,y}$ is one sending every variable but y to x and y to itself. If one simply mimics this proof in the setting of weak logics and lets, for example,

$$\tau(x) = \sigma_x(\Omega(\mathrm{Cn}_{\Vdash}(x))),$$

where $\mathrm{Cn}_{\Vdash}$ is the closure consequence operator of the weak logic $\Vdash$, then crucially it could be that structurality fails. In fact, we could have that $x \Vdash \delta(x)$ but also $\phi \nVdash \delta(\phi)$, witnessing the failure of uniform substitution. Let σ be the substitution sending every variable to ϕ, then we have

$$\delta(\phi) \in \mathrm{Cn}_{\Vdash}(\sigma(\mathrm{Cn}_{\Vdash}(x))) \text{ and } \delta(\phi) \notin \mathrm{Cn}_{\Vdash}(\sigma(x)),$$

https://doi.org/10.1017/jsl.2026.10180 Published online by Cambridge University Press

and therefore we obtain $\Omega(\mathrm{Cn}_{\Vdash\vdash}(\sigma(\mathrm{Cn}_{\Vdash\vdash}(x)))) \neq \Omega(\mathrm{Cn}_{\Vdash\vdash}(\sigma(x)))$. Clearly, we have that $\Omega(\mathrm{Cn}_{\Vdash\vdash}(\sigma(x))) = \Omega(\mathrm{Cn}_{\Vdash\vdash}(\phi))$, simply by the definition of σ. Assume additionally that Ω commutes with closure operators and substitution, i.e., that

$$\Omega(\mathrm{Cn}_{\Vdash\vdash}(\sigma(\Gamma))) = \mathrm{Cn}^c_{\mathbf{Q}}(\sigma(\Omega(\mathrm{Cn}_{\Vdash\vdash}(\Gamma))))$$

for all sets of formulas Γ and substitutions σ (and where $\mathrm{Cn}^c_{\mathbf{Q}}$ refers the closure operator associated with the core semantics of a quasivariety of core-generated expanded algebras $\mathbf{Q}$). Then we obtain

$$\begin{aligned}\Omega(\mathrm{Cn}_{\Vdash\vdash}(\sigma(\mathrm{Cn}_{\Vdash\vdash}(x)))) &= \mathrm{Cn}^c_{\mathbf{Q}}(\sigma(\Omega(\mathrm{Cn}_{\Vdash\vdash}(\mathrm{Cn}_{\Vdash\vdash}(x))))) \\ &= \mathrm{Cn}^c_{\mathbf{Q}}(\sigma(\Omega(\mathrm{Cn}_{\Vdash\vdash}(x)))) \\ &= \mathrm{Cn}^c_{\mathbf{Q}}(\sigma\sigma_x(\Omega(\mathrm{Cn}_{\Vdash\vdash}(x)))) \\ &= \mathrm{Cn}^c_{\mathbf{Q}}(\tau(\phi))\end{aligned}$$

and therefore $\mathrm{Cn}^c_{\mathbf{Q}}(\tau(\phi)) \neq \Omega(\mathrm{Cn}_{\Vdash\vdash}(\phi))$. This indicates that the usual proof of the isomorphism theorem does not work in the setting of loosely algebraizable weak logics. We leave it as a pointer for future works to establish if there is a version of the isomorphism theorem in the setting of loosely algebraizable weak logics.

6.3. Strictly algebraizable weak logics. In contrast to Remark 6.21, in the case of strictly algebraizable weak logics, we can use the fact that they are finitely representable to derive a full version of the isomorphism theorem. We first recall some preliminary definitions, which we employed informally already in Remark 6.21.

DEFINITION 6.22 (Closure operators). Let A be an expanded algebra, $\Vdash$ a weak logic, and $\mathbf{Q}$ a core-generated quasivariety of expanded algebras. We denote by $\mathrm{Cn}_{\Vdash\vdash}$ the closure operator on A defined by letting $\mathrm{Cn}_{\Vdash\vdash}(X)$ be the smallest core filter of A with respect to $\Vdash$ that contains $X \subseteq A$. Similarly, we denote by $\mathrm{Cn}^c_{\mathbf{Q}}$ the closure operator on A^2 defined by letting $\mathrm{Cn}^c_{\mathbf{Q}}(R)$ be the smallest congruence in $\mathsf{Con}^c_{\mathbf{Q}}(A)$ containing $R \subseteq A^2$.

REMARK 6.23. Recall that, in the expanded term algebra $(\mathsf{Fm}, \mathsf{Var})$, the core filters coincide exactly with the syntactical theories and the core congruences coincide with the semantical theories. In this case we recover the more intuitive definition of the closure operators $\mathrm{Cn}_{\Vdash\vdash}$ and $\mathrm{Cn}^c_{\mathbf{Q}}$, namely, for $\Theta \subseteq \mathsf{Eq}$ and $\Gamma \subseteq \mathsf{Fm}$,

$$\begin{aligned}\mathrm{Cn}_{\Vdash\vdash}(\Gamma) &= \{\phi \in \mathsf{Fm} : \Gamma \Vdash \phi\} \\ \mathrm{Cn}^c_{\mathbf{Q}}(\Theta) &= \{\alpha \approx \beta \in \mathsf{Eq} : \Theta \models^c_{\mathbf{Q}} \alpha \approx \beta\}.\end{aligned}$$

In the context of strictly algebraizable weak logics, it is also convenient to work with the following notions of Λ-filters and Σ-congruences.

DEFINITION 6.24. Let Λ be a set of formulas in Fm, then for any algebra A and standard logic $\vdash$, we say that $F \subseteq A$ is a Λ-*filter* over A with respect to $\vdash$ if

$$\Gamma \cup \mathsf{At}[\Lambda] \vdash \phi \implies \forall h \in \mathsf{Hom}(\mathsf{Fm}, A),\ h[\Gamma] \subseteq F \text{ entails } h(\phi) \in F.$$

We denote the set of Λ-filters of A with respect to $\vdash$ by $\Lambda\mathsf{Fi}_{\vdash}(A)$. Similarly, we let $\Lambda\mathsf{Th}(\vdash)$ be the set of all syntactical theories Γ over $\vdash$ such that $\mathsf{At}[\Lambda] \subseteq \Gamma$.

https://doi.org/10.1017/jsl.2026.10180 Published online by Cambridge University Press

DEFINITION 6.25. Given an expanded algebra A, a quasivariety of expanded algebras $\mathbf{Q}$, and a finite set of equations in one variable Σ, a congruence $\theta \in \mathsf{Con}_{\mathbf{Q}\restriction\mathcal{L}}(A)$ is said to be a Σ-*congruence* if $\Sigma(a) \subseteq \theta$ for all $a \in \mathsf{core}(A)$. We write $\Sigma\mathsf{Con}_{\mathbf{Q}\restriction\mathcal{L}}(A)$ for the set of all $\mathbf{Q}\restriction\mathcal{L}$-congruences of A which are also Σ-congruences. Similarly, we let $\Sigma\mathsf{Th}(\models_{\mathbf{Q}})$ denote the set of all semantical theories Θ such that $\mathsf{At}[\Sigma] \subseteq \Theta$.

REMARK 6.26. It is clear from the definition of Λ-filters and the monotonicity of $\vdash$ that the Λ-filters of $\vdash$ on A are a special kind of deductive filters of $\vdash$. Similarly, Σ-congruences are clearly a special kind of congruences. The set of Λ-filters as well as that of Σ-congruences are closed under arbitrary intersections and hence form complete lattices under the subset relation, but, crucially, they are not necessarily closed under arbitrary inverse endomorphisms.

LEMMA 6.27. *Let $\Vdash$ be a weak logic. If $\Vdash$ is represented by a set of formulas Λ then $\mathsf{Fi}_{\Vdash}(A) = \Lambda\mathsf{Fi}_{\Vdash_{\mathrm{s}}}(A)$ for any core-generated expanded algebra A. Moreover, if $\mathsf{Th}(\Vdash) = \Lambda\mathsf{Th}(\Vdash_{\mathrm{s}})$, then $\Vdash$ is represented by Λ.*

PROOF. We prove the first claim. Suppose $\Vdash$ is represented by Λ. Let $F \in \mathsf{Fi}_{\Vdash}(A)$, $\Gamma \cup \mathsf{At}[\Lambda] \Vdash_{\mathrm{s}} \phi$, and $h \in \mathsf{Hom}(\mathsf{Fm}, A)$ be such that $h[\Gamma] \subseteq F$. By the representability of $\Vdash$ we have that $\Gamma \Vdash \phi$. Since A is core-generated, by Lemma 3.20 there are a core-assignment $g \in \mathsf{Hom}^{c}(\mathsf{Fm}, A)$ and a substitution $\sigma \in \mathsf{Subst}(\mathcal{L})$ such that $h[\Gamma \cup \{\phi\}] = g[\sigma(\Gamma \cup \{\phi\})]$. It follows that $g[\sigma(\Gamma)] \subseteq F$ and so, since $\Gamma \Vdash \phi$, we obtain from the definition of core filters that $h(\phi) = g(\sigma(\phi)) \in F$. This shows that $\mathsf{Fi}_{\Vdash}(A) \subseteq \Lambda\mathsf{Fi}_{\Vdash_{\mathrm{s}}}(A)$.

Conversely, let $F \in \Lambda\mathsf{Fi}_{\Vdash_{\mathrm{s}}}(A)$. Suppose $\Gamma \Vdash \phi$ and let $h \in \mathsf{Hom}^{c}(\mathsf{Fm}, A)$ be such that $h[\Gamma] \subseteq F$. By representability we obtain that $\Gamma \cup \mathsf{At}[\Lambda] \Vdash_{\mathrm{s}} \phi$, thus it follows from $F \in \Lambda\mathsf{Fi}_{\Vdash_{\mathrm{s}}}(A)$ that $h(\phi) \in F$. This shows that $\Lambda\mathsf{Fi}_{\Vdash_{\mathrm{s}}}(A) \subseteq \mathsf{Fi}_{\Vdash}(A)$ and thus proves that $\mathsf{Fi}_{\Vdash}(A) = \Lambda\mathsf{Fi}_{\Vdash_{\mathrm{s}}}(A)$.

We now prove the second claim. Suppose we have that $\mathsf{Th}(\Vdash) = \Lambda\mathsf{Th}(\Vdash_{\mathrm{s}})$. Then, for any $\psi \in \mathsf{Fm}$ we have that

$$\begin{aligned} \mathrm{Cn}_{\Vdash}(\psi) &= \mathrm{Cn}_{\Vdash_{\mathrm{s}}}(\mathrm{Cn}_{\Vdash}(\psi) \cup \mathsf{At}[\Lambda]); \\ \mathrm{Cn}_{\Vdash_{\mathrm{s}}}(\psi \cup \mathsf{At}[\Lambda]) &= \mathrm{Cn}_{\Vdash}(\mathrm{Cn}_{\Vdash_{\mathrm{s}}}(\psi \cup \mathsf{At}[\Lambda])). \end{aligned}$$

Therefore, it follows from the first equality (and monotonicity of the consequence operators) that $\mathrm{Cn}_{\Vdash_{\mathrm{s}}}(\psi \cup \mathsf{At}[\Lambda]) \subseteq \mathrm{Cn}_{\Vdash}(\psi)$, and it follows from the second equality that $\mathrm{Cn}_{\Vdash}(\psi) \subseteq \mathrm{Cn}_{\Vdash_{\mathrm{s}}}(\psi \cup \mathsf{At}[\Lambda])$. This yields that $\mathrm{Cn}_{\Vdash}(\psi) = \mathrm{Cn}_{\Vdash_{\mathrm{s}}}(\psi \cup \mathsf{At}[\Lambda])$. Finally, we derive the following equivalences:

$$\begin{aligned} \Gamma \Vdash \phi &\iff \mathrm{Cn}_{\Vdash}(\phi) \subseteq \mathrm{Cn}_{\Vdash}(\Gamma) \\ &\iff \mathrm{Cn}_{\Vdash_{\mathrm{s}}}(\phi \cup \mathsf{At}[\Lambda]) \subseteq \mathrm{Cn}_{\Vdash_{\mathrm{s}}}(\Gamma \cup \mathsf{At}[\Lambda]) \\ &\iff \Gamma \cup \mathsf{At}[\Lambda] \Vdash_{\mathrm{s}} \phi, \end{aligned}$$

which prove the representability of $\Vdash$ via Λ. ⊣

LEMMA 6.28. *Let $\mathbf{Q}$ be a core-generated quasivariety with core defined by a set of equations Σ, then $\mathsf{core}(A) = \Sigma(A)$ entails $\mathsf{Con}^{c}_{\mathbf{Q}}(A) = \Sigma\mathsf{Con}_{\mathbf{Q}\restriction\mathcal{L}}(A)$ for any core-generated expanded algebra A.*

https://doi.org/10.1017/jsl.2026.10180 Published online by Cambridge University Press

PROOF. Suppose that $\mathsf{core}(A) = \Sigma(A)$ for all $A \in \mathbf{Q}$. First, let $\theta \in \mathsf{Con}^c_{\mathbf{Q}}(A)$, then it follows that $A/\theta \in \mathbf{Q}$ and so that $\mathsf{core}(A/\theta) = \Sigma[A/\theta]$. This yields $\Sigma(a) \subseteq \theta$ for all $a \in \mathsf{core}(A)$, showing that $\theta \in \Sigma\mathsf{Con}_{\mathbf{Q}\restriction\mathcal{L}}(A)$. Conversely, suppose $\theta \in \Sigma\mathsf{Con}_{\mathbf{Q}\restriction\mathcal{L}}(A)$, then in particular $A/\theta \in \mathbf{Q}\restriction\mathcal{L}$. Since $\mathbf{Q}$ has core defined by the set of equations Σ, the unique expansion of A/θ which belongs to $\mathbf{Q}$ is $(A/\theta, \Sigma(A/\theta)) \in \mathbf{Q}$. It follows that $\mathsf{core}(A/\theta) = \Sigma(A/\theta)$ and so that $\theta \in \mathsf{Con}^c_{\mathbf{Q}}(A)$. ⊣

We use the previous lemmas to obtain the following isomorphism theorem for strictly algebraizable weak logics. This result displays how the setting of strict algebraizability better reproduces the original approach from Blok and Pigozzi in the context of logics without uniform substitutions.

THEOREM 6.29. *Let* $\Vdash$ *be a weak logic and* $\mathbf{Q}$ *a core-generated quasivariety of expanded algebras with core defined by a finite set of equations* Σ. *The following statements are equivalent*:

(1) $\Vdash$ *is strictly algebraized by* $(\mathbf{Q}, \Sigma, \tau, \Delta)$.
(2) *There is a finite set of formulas* Λ *such that, for every core-generated expanded algebra* A, *it holds that* $\mathsf{Fi}_{\Vdash}(A) = \Lambda\mathsf{Fi}_{\Vdash_{\mathrm{s}}}(A)$ *and* $\mathsf{Con}^c_{\mathbf{Q}}(A) = \Sigma\mathsf{Con}_{\mathbf{Q}\restriction\mathcal{L}}(A)$. *Moreover, there is an isomorphism* $\Omega : \overline{\mathsf{Fi}}_{\Vdash}(A) \cong \overline{\mathsf{Con}}_{\mathbf{Q}}(A)$.
(3) *There is a finite set of formulas* Λ *such that* $\mathsf{Th}(\Vdash) = \Lambda\mathsf{Th}(\Vdash_{\mathrm{s}})$ *and* $\mathsf{Th}(\models^c_{\mathbf{Q}}) = \Sigma\mathsf{Th}(\models_{\mathbf{Q}})$. *Moreover, there is an isomorphism* $\Omega : \overline{\mathsf{Th}}(\Vdash) \cong \overline{\mathsf{Th}}(\models^c_{\mathbf{Q}})$.

PROOF. Direction from (1) to (2) follows immediately from Theorem 6.17 and Lemmas 6.27 and 6.28. Direction from (2) to (3) follows from Remark 6.19. It remains to show that (3) entails (1).

Firstly, notice that the isomorphism $\Omega : \overline{\mathsf{Th}}(\Vdash) \cong \overline{\mathsf{Th}}(\models^c_{\mathbf{Q}})$ induces an isomorphism $\mathsf{Th}^+(\Vdash_{\mathrm{s}}) \cong \mathsf{Th}^+(\models_{\mathbf{Q}})$, thus by the standard isomorphism theorem it immediately follows that $\Vdash_{\mathrm{s}}$ is algebraized by $(\mathbf{Q}, \tau, \Delta)$, where

$$\tau(x) = \sigma_x(\Omega(\mathrm{Cn}_{\Vdash_{\mathrm{s}}}(x)))$$
$$\Delta(x, y) = \sigma_{x,y}(\Omega^{-1}(\mathrm{Cn}_{\mathbf{Q}\restriction\mathcal{L}}(x \approx y))),$$

as in Remark 6.8. In particular, σ_x is the substitution sending every variable to x, and $\sigma_{x,y}$ is the substitution sending every variable but y to x and y to itself. Now, since Ω is an isomorphism, the least syntactic theory in $\Vdash$ must be mapped to the least theory in core semantics, i.e., $\Omega(\mathrm{Cn}_{\Vdash}(\emptyset)) = \mathrm{Cn}^c_{\mathbf{Q}}(\emptyset)$. By assumption, we have that $\mathrm{Cn}_{\Vdash}(\emptyset) = \mathrm{Cn}_{\Vdash_{\mathrm{s}}}(\mathsf{At}[\Lambda])$ and $\mathrm{Cn}^c_{\mathbf{Q}}(\emptyset) = \mathrm{Cn}_{\mathbf{Q}\restriction\mathcal{L}}(\mathsf{At}[\Sigma])$, thus we obtain that $\Omega(\mathrm{Cn}_{\Vdash_{\mathrm{s}}}(\mathsf{At}[\Lambda])) = \mathrm{Cn}_{\mathbf{Q}\restriction\mathcal{L}}(\mathsf{At}[\Sigma])$. Let σ_δ the substitution sending x to δ and $\sigma_{\alpha,\beta}$ the substitution sending x to α and y to β. Then we obtain that

$$\sigma_x\left(\Omega\left(\mathrm{Cn}_{\Vdash_{\mathrm{s}}}\left(\bigcup_{\delta\in\mathsf{At}[\Lambda]}\sigma_\delta(x)\right)\right)\right) = \sigma_x\left(\mathrm{Cn}_{\mathbf{Q}\restriction\mathcal{L}}\left(\bigcup_{\alpha\approx\beta\in\mathsf{At}[\Sigma]}\sigma_{\alpha,\beta}(x \approx y)\right)\right)$$
$$\subseteq \mathrm{Cn}_{\mathbf{Q}\restriction\mathcal{L}}\left(\sigma_x\left(\bigcup_{\alpha\approx\beta\in\mathsf{At}[\Sigma]}\sigma_{\alpha,\beta}(x \approx y)\right)\right)$$

https://doi.org/10.1017/jsl.2026.10180 Published online by Cambridge University Press

as σ_x is an atomic substitution. It follows that $\Sigma \models_{\mathbf{Q}\uparrow\mathcal{L}} \tau(\Lambda)$. Similarly, we have

$$\sigma_{x,y}\left(\Omega^{-1}\left(\mathrm{Cn}_{\mathbf{Q}\uparrow\mathcal{L}}\left(\bigcup_{\alpha\approx\beta\in\mathsf{At}[\Sigma]}\sigma_{\alpha,\beta}(x\approx y)\right)\right)\right) = \sigma_{x,y}\left(\mathrm{Cn}_{\Vdash_{\mathrm{s}}}\left(\bigcup_{\delta\in\mathsf{At}[\Lambda]}\sigma_\delta(x)\right)\right)$$
$$\subseteq \mathrm{Cn}_{\Vdash_{\mathrm{s}}}\left(\sigma_{x,y}\left(\bigcup_{\delta\in\mathsf{At}[\Lambda]}\sigma_\delta(x)\right)\right),$$

which shows that $\Lambda \Vdash_{\mathrm{s}} \Delta(\Sigma)$. By the algebraizability of $\Vdash_{\mathrm{s}}$ and $\Sigma \models_{\mathbf{Q}\uparrow\mathcal{L}} \tau(\Lambda)$ this entails that $\tau(\Lambda) \equiv_{\mathbf{Q}\uparrow\mathcal{L}} \Sigma$ and $\Delta(\Sigma) \Vdash_{\mathrm{s}} \Lambda$, $\Lambda \Vdash_{\mathrm{s}} \Delta(\Sigma)$. Also, from the assumption that $\mathsf{Th}(\Vdash) = \Lambda\mathsf{Th}(\Vdash_{\mathrm{s}})$ and Lemma 6.27 we also obtain that Λ witnesses the finite representability of $\Vdash$. We finally conclude from Corollary 5.6 that $\Vdash$ is strictly algebraized by $(\mathbf{Q}, \Sigma, \tau, \Delta)$. ⊣

§7. Aside: Matrix semantics. In the previous sections we have focused on loosely and strictly algebraizable weak logics, thus extending the notion of algebraizability to the setting of logics without uniform substitution. However, in the setting of abstract algebraic logic, algebraizability makes only for one of several properties, and it could be seen as the concept carving out the most well-behaved family of logical systems. On the converse direction, in this section we work towards an increased level of generality and we study the matrix semantics of arbitrary weak logics. In fact, while not every logic is algebraizable, it can be shown that every logic admits a matrix semantics (see, e.g., [16, Theorem 4.16]). Furthermore, Dellunde and Jansana [14] provided a characterisation of the class of matrices of a (possibly infinitary) logic in terms of some model-theoretic results for first-order logic without equality. We show in this section that similar results can be proved in the context of weak logics. We prove in Section 7.1 that every weak logic is complete with respect to a suitable class of so-called bimatrices, and we show in Section 7.2 that Dellunde and Jansana's results are still applicable in our setting.

7.1. Completeness of matrix semantics. We briefly recall the matrix semantics for standard logics. Intuitively, the idea is to work with first-order structures with a predicate $\mathsf{truth}(A)$ which encodes the "truth set" of the algebra A.

DEFINITION 7.1. A *(logical) matrix* of type $\mathcal{L}$ is a pair $(A, \mathsf{truth}(A))$, where A is an $\mathcal{L}$-algebra and $\mathsf{truth}(A) \subseteq \mathsf{dom}(A)$.

Matrices induce a consequence relation over propositional formulas analogously as classes of algebras do. However, notice that here we work directly with propositional formulas (i.e., terms in the language $\mathcal{L}$) and not with equations. This corresponds to the fact that these logics do not necessarily correspond to quasiequational theories over classes of algebras.

NOTATION 7.2. *For notational convenience, in this section we denote logics and weak logics by $L, L_0, L_1, \ldots$. We then write $\Gamma \vdash_L \phi$ if $(\Gamma, \phi) \in L$. If L is a weak logic, then we write $\Gamma \Vdash_L \phi$ if $(\Gamma, \phi) \in L$.*

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DEFINITION 7.3. Let $\mathbf{K}$ be a class of $\mathcal{L}$-matrices and let $\Gamma \cup \{\phi\}$ be a set of propositional formulas, then we let

$$\Gamma \models_{\mathbf{K}} \phi \iff \text{for all } A \in \mathbf{K},\ h \in \mathsf{Hom}(\mathsf{Fm}, A), \text{ if } h[\Gamma] \subseteq \mathsf{truth}(A), \text{ then } h(\phi) \in \mathsf{truth}(A).$$

Given a logic L, we say that $(A, \mathsf{truth}(A))$ is a *model* of L and write $(A, \mathsf{truth}(A)) \models L$ if, for every $\Gamma \cup \{\phi\} \subseteq \mathcal{L}$, $\Gamma \vdash_L \phi$ entails $\Gamma \models_{\{A\}} \phi$. We often abbreviate this latter expression by $\models_{\{A\}} \Gamma$.

We refer the reader to [16, Section 4] for a detailed study of matrix semantics in the context of standard propositional logics. In particular, [16, Theorem 4.16] states that *every logic is complete with respect to a class of matrices*. We show that one can obtain the same result in the setting of weak logics. First, we extend the matrix semantics to the setting of weak logics by introducing a further predicate, i.e., by viewing them as structures in an algebraic language $\mathcal{L}$ augmented by two unary predicates, one for the core set of A and one for the truth set of A.

DEFINITION 7.4. The tuple $(A, \mathsf{truth}(A), \mathsf{core}(A))$ is a *(logical) bimatrix* of type $\mathcal{L}$ if A is an $\mathcal{L}$-algebra, $\mathsf{truth}(A) \subseteq \mathsf{dom}(A)$ and $\mathsf{core}(A) \subseteq \mathsf{dom}(A)$.

NOTATION 7.5. *As in the case of expanded algebras, we write* $\mathsf{Hom}^c(\mathsf{Fm}, A)$ *for the set of all assignments* $h : \mathsf{Fm} \to A$ *such that* $h[\mathsf{Var}] \subseteq \mathsf{core}(A)$.

Bimatrices induce a consequence relation analogous to that of expanded algebras by restricting attention to assignments over core elements.

DEFINITION 7.6. Let $\mathbf{K}$ be a class of $\mathcal{L}$-bimatrices and let $\Gamma \cup \{\phi\}$ be a set of propositional formulas, then we let

$$\Gamma \models^c_{\mathbf{K}} \phi \iff \text{for all } A \in \mathbf{K},\ h \in \mathsf{Hom}^c(\mathsf{Fm}, A), \text{ if } h[\Gamma] \subseteq \mathsf{truth}(A), \text{ then } h(\phi) \in \mathsf{truth}(A).$$

Given a weak logic L, we say that $(A, \mathsf{truth}(A), \mathsf{core}(A))$ is a *model* of L, and write $(A, \mathsf{truth}(A), \mathsf{core}(A)) \models L$ if, for every $\Gamma \cup \{\phi\} \subseteq \mathcal{L}$, $\Gamma \vdash_L \phi$ entails $\Gamma \models^c_{\{A\}} \phi$. For set of formulas Γ and a bimatrix A, we write $A \models^c \Gamma$ if $\models^c_{\{A\}} \Gamma$.

Thus, the main intuition behind bimatrices is the same of expanded algebras: we add a new predicate specifying the core of the matrix, in order to consider only the assignments sending atomic formulas to elements of the core. As shown by the following proposition, bimatrices give rise to several weak logics. As we stressed already before (cf. Remark 2.3), the finitary requirement in the following definition is not necessary *per se*, but we need it as we are focusing on finitary logical systems.

DEFINITION 7.7. Let $\mathbf{K}$ be a class of $\mathcal{L}$-bimatrices, then $\mathsf{Log}(\mathbf{K})$ is the set of all pairs (Γ, ϕ) with $\Gamma \cup \{\phi\} \subseteq \mathsf{Fm}$ such that $\Gamma_0 \models^c_{\mathbf{K}} \phi$ for some finite $\Gamma_0 \subseteq \Gamma$. We say that a weak logic L is complete with respect to $\mathbf{K}$ if $L = \mathsf{Log}(\mathbf{K})$.

PROPOSITION 7.8. *Let* $\mathbf{K}$ *be a class of bimatrices, then* $\mathsf{Log}(\mathbf{K})$ *is a weak logic.*

PROOF. Let $L = \mathsf{Log}(\mathbf{K})$, then Conditions (1)–(3) from the definition of consequence relation (Definition 2.2) are immediately valid by Definition 7.6.

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Additionally, if $\Gamma \Vdash_L \phi$, then by definition there is some finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models^c_{\mathbf{K}} \phi$, and therefore $\Gamma_0 \Vdash_L \phi$, which shows that L is also finitary.

Finally, we show that L is closed under atomic substitutions. Suppose $\Gamma \models^c_{\mathbf{K}} \phi$ but $\sigma[\Gamma] \not\models^c_{\mathbf{K}} \sigma(\phi)$ for some atomic substitution σ. Then there are a bimatrix $M \in \mathbf{K}$ and a core assignment $h : \mathsf{Fm} \to M$ such that $h[\sigma[\Gamma]] \subseteq \mathsf{truth}(M)$ but $h(\sigma(\phi)) \notin \mathsf{truth}(M)$. Now, since h is a core assignment and σ an atomic substitution, it follows that $h \circ \sigma \in \mathsf{Hom}^c(\mathsf{Fm}, A)$, contradicting $\Gamma \models^c_M \phi$. ⊣

By the previous proposition we have that every class of bimatrices determines a weak logic. Additionally, we can also show that every weak logic is complete with respect to a class of bimatrices.

DEFINITION 7.9. For every weak logic $\Vdash$ and every set of propositional formulas $\Gamma \subseteq \mathsf{Fm}$, we let $M^\Gamma_\Vdash$ be the bimatrix with domain $\mathsf{dom}(M^\Gamma_\Vdash) = \mathsf{Fm}$ and predicates $\mathsf{truth}(M^\Gamma_\Vdash) = \mathrm{Cn}_\Vdash(\Gamma)$ and $\mathsf{core}(M^\Gamma_\Vdash) = \mathsf{Var}$. We let $\mathbf{M}_\Vdash$ be the class of all bimatrices $M^\Gamma_\Vdash$ for $\Gamma \subseteq \mathsf{Fm}$.

THEOREM 7.10. *Every weak logic $\Vdash$ is complete with respect to the class $\mathbf{M}_\Vdash$.*

PROOF. We need to show that $\Vdash = \mathsf{Log}(\mathbf{K})$. Firstly, suppose towards contradiction that $\Gamma \Vdash \phi$ but $\Gamma \not\models^c_{\mathbf{M}_\Vdash} \phi$. Then there is a bimatrix $M^\Delta_\Vdash$ and a core assignment $h : \mathsf{Fm} \to M^\Delta_\Vdash$ such that $h[\Gamma] \subseteq \mathsf{truth}(M^\Delta_\Vdash)$ and $h(\phi) \notin \mathsf{truth}(M^\Delta_\Vdash)$. Consider now the substitution σ defined letting $\sigma(x) = h(x)$ for all $x \in \mathsf{Var}$. Notice in particular that this is well-defined because the domain of $M^\Delta_\Vdash$ is Fm. Since h is a core assignment and $\mathsf{core}(M^\Gamma_\Vdash) = \mathsf{Var}$, it follows that σ is an atomic substitution, thus we obtain that $\sigma(\Gamma) \Vdash \sigma(\phi)$. Now, since $h[\Gamma] \subseteq \mathsf{truth}(M^\Delta_\Vdash)$ it follows in particular that $\Delta \Vdash \sigma(\Gamma)$ and so by transitivity $\Delta \Vdash \sigma(\phi)$. Since $\sigma(\phi) = h(\phi) \notin \mathsf{truth}(M^\Delta_\Vdash)$, this contradicts the definition of $\mathsf{truth}(M^\Delta_\Vdash)$. It follows that $\Gamma \models^c_{\mathbf{M}_\Vdash} \phi$.

Conversely, suppose $\Gamma \models^c_{\mathbf{M}_\Vdash} \phi$ and let $\mathsf{id}_{\mathsf{Fm}} : \mathsf{Fm} \to M^\Gamma_\Vdash$ be the identity map. Then clearly $\mathsf{id}_{\mathsf{Fm}}[\mathsf{Var}] \subseteq \mathsf{Var}$ and $\mathsf{id}_{\mathsf{Fm}}[\Gamma] \subseteq \mathsf{truth}(M^\Gamma_\Vdash)$. Since $\Gamma \models^c_{\mathbf{M}_\Vdash} \phi$ we then obtain $\phi = \mathsf{id}_{\mathsf{Fm}}(\phi) \in \mathsf{truth}(M^\Gamma_\Vdash)$, showing $\phi \in \mathrm{Cn}_\Vdash(\Gamma)$ and thus $\Gamma \Vdash \phi$. ⊣

7.2. Connections to model theory without equality. In the previous section we have established that every class of bimatrices defines a weak logic and, conversely, that every weak logic is complete with respect to a class of bimatrices. Here we next consider what is exactly the class of *all* bimatrices defined by a weak logic L, i.e., the class of all bimatrices M for which $M \models^c \Gamma$ entails $M \models^c \phi$ whenever $\Gamma \Vdash_L \phi$. In the standard context this issue was first considered by Czelakowski [13], who characterized the class of all matrices which are models of a logic. Here we follow however the later work of Dellunde and Jansana in [14], which provided a novel proof of Czelakowski's result by employing the fact that (finitary) propositional logics can be translated into Horn theories *without equality*. We start by reviewing this translation, which is essentially a generalisation of what we already considered in Remark 3.18.

PROPOSITION 7.11. *Let $\mathcal{L}$ be an algebraic language, $\Gamma \cup \phi \subseteq \mathsf{Fm}$ and $|\Gamma| < \aleph_0$, then we can translate the consequence relations from Definitions 7.3 and 7.6 as follows*:

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(a) *let* $\mathbf{K}$ *be a class of matrices then*:

$$\Gamma \models_{\mathbf{K}} \phi \iff \mathbf{K} \models \forall x_0, \dots, \forall x_n \Big(\bigwedge_{\gamma \in \Gamma} \mathsf{truth}(\gamma(\bar{x})) \to \mathsf{truth}(\phi(\bar{x})) \Big);$$

(b) *let* $\mathbf{K}$ *be a class of bimatrices, then*:

$$\Gamma \models^{c}_{\mathbf{K}} \phi \iff \mathbf{K} \models \forall x_0, \dots, \forall x_n \Big(\bigwedge_{\gamma \in \Gamma} \mathsf{truth}(\gamma(\bar{x})) \wedge \bigwedge_{i \leqslant n} \mathsf{core}(x_i) \to \mathsf{truth}(\phi(\bar{x})) \Big).$$

PROOF. This follows immediately from the definition of $\models_{\mathbf{K}}$ and $\models^{c}_{\mathbf{K}}$. ⊣

It follows from the previous proposition that standard logics in the language $\mathcal{L}$ can be encoded by Horn theories in $\mathcal{L} \cup \{\mathsf{truth}\}$, while weak logics can be encoded in $\mathcal{L} \cup \{\mathsf{truth}, \mathsf{core}\}$. This motivates the following definitions.

NOTATION 7.12. *Let* $\Gamma \cup \{\phi\} \subseteq \mathsf{Fm}$, *then we write* $\Phi(\Gamma, \phi)$ *for the first-order formula*

$$\forall x_0, \dots, \forall x_n \Big(\bigwedge_{\gamma \in \Gamma} \mathsf{truth}(\gamma(\bar{x})) \wedge \bigwedge_{i \leqslant n} \mathsf{core}(x_i) \to \mathsf{truth}(\phi(\bar{x})) \Big).$$

DEFINITION 7.13. Let $\Vdash$ be a weak logic, then we let $\mathsf{Horn}(\Vdash)$ be the Horn theory obtained by letting $\Phi(\Gamma, \phi) \in \mathsf{Horn}(\Vdash)$ whenever $\Gamma \Vdash \phi$. We write $\mathrm{Mod}(\Vdash)$ for the class of structures $\mathrm{Mod}(\mathsf{Horn}(\Vdash))$.

While Czelakowski's original approach in [13] was specifically tailored to logical matrices, Dellunde and Jansana considered arbitrary model classes axiomatized by Horn theories without equality, thus making it possible to apply their results to the setting of bimatrices and expanded algebras. We recall from Notation 2.10 that the operator $\mathbb{H}_{\mathrm{s}}$ refers to the closure under *strict* homomorphic images, and thus $\mathbb{H}_{\mathrm{s}}^{-1}(\mathbf{K})$ is the class of all structures A that are preimages of some strict homomorphism of some structure in $\mathbf{K}$. We stress that Dellunde and Jansana's results (from [14]) hold in the setting *without* the equality symbol. In particular, when in the rest of this section we consider Horn formulas, we always restrict attention to Horn formulas not containing the equality symbol. For clarity, we introduce the following notation.

NOTATION 7.14. *We write* $\mathcal{L}^{-}$ *to refer to a fixed first-order language without the equality symbol* $\approx$. *Also, we write* $\mathcal{L}^{-}$ *for the collections of all* (*first-order*) *formulas in this language* (*which clearly do not contain the equality symbol*).

FACT 7.15 (Dellunde and Jansana). *Let* $\mathbf{K}$ *be a class of* $\mathcal{L}^{-}$*-structures, then the following are equivalent*:

(1) $\mathbf{K}$ *is axiomatised by strict universal Horn formulas in* $\mathcal{L}^{-}$;
(2) $\mathbf{K}$ *is closed under* $\mathbb{H}_{\mathrm{s}}^{-1}, \mathbb{H}_{\mathrm{s}}, \mathbb{S}, \mathbb{P}, \mathbb{P}_{\mathrm{U}}$ *and contains a trivial structure*;
(3) $\mathbf{K} = \mathbb{H}_{\mathrm{s}}^{-1}\mathbb{H}_{\mathrm{s}}\mathbb{S}\mathbb{P}\mathbb{P}_{\mathrm{U}}(\mathbf{K}_0)$ *for some class* $\mathbf{K}_0$ *of* $\mathcal{L}$*-structures containing a trivial structure.*

As we mentioned in Fact 2.14, the validity of Horn formulas is always preserved under the operators $\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{P}_{\mathrm{U}}$. From the previous theorem it follows that strict Horn formulas in a language without equality are also preserved under the operators $\mathbb{H}_{\mathrm{s}}^{-1}$ and $\mathbb{H}_{\mathrm{s}}$. From this fact and the previous theorem we immediately obtain as a corollary

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the following characterization of the class of models $\mathrm{Mod}(\Vdash)$, where $\Vdash = \mathsf{Log}(\mathbf{K})$ for some class of bimatrices $\mathbf{K}$.

COROLLARY 7.16. *Let* $\mathbf{K}$ *be a class of bimatrices and let* $T = \mathsf{Horn}(\mathsf{Log}(\mathbf{K}))$, *then*:

$$\mathrm{Mod}(T) = \mathbb{H}_{\mathrm{s}}^{-1}\mathbb{H}_{\mathrm{s}}\mathbb{S}\mathbb{P}\mathbb{P}_{\mathrm{U}}(\mathbf{K}').$$

where $\mathbf{K}'$ *is* $\mathbf{K}$ *together with some trivial bimatrices.*

Notice that, by Theorem 7.10, every weak logic $\Vdash$ is complete with respect to a class of bimatrices $\mathbf{M}_{\Vdash}$ and so the previous theorem applies to all weak logics $\Vdash$ and provides us with a characterization of the class of all bimatrices defined by $\Vdash$.

Importantly, however, one can see that the class of all matrices $\mathrm{Mod}(T)$ above does not meet the intuition about the "right" semantics of a (weak) logic, and will contain several pathological examples. For example, in the case of the standard logic CPC, the class $\mathrm{Mod}(\mathtt{CPC})$ is *not* the class of Boolean algebras, but rather the class $\mathbb{H}_{\mathrm{s}}^{-1}(\mathbf{BA})$, where here we identify $\mathbf{BA}$ with the class of expansions $(B, \{1\})$ such that B is a Boolean algebra (for example, the so-called "benzene ring" is a matrix model of CPC, but it is not a Boolean algebra, cf. [16, Example 4.79]). In order to identify the non-pathological models of a propositional logic, Dellunde and Jansana focused in [14] on the so-called *reduced structures*. Since they work at the general level of model theory without equality, it is again straightforward to adapt their results to our current setting of bimatrices.

DEFINITION 7.17. Let $\mathcal{L}$ be any first-order language, M an $\mathcal{L}$-structure, and $X \subseteq M$, then we write $\mathcal{L}(X)$ for the set of all $\mathcal{L}$-formulas with parameters in X.

DEFINITION 7.18. Let M be a first-order structure, $D \subseteq M$, and $\bar{a} \in M^{<\omega}$, we let the *type without equality of* $\bar{a}$ *over* D *in* M be the following set of equality-free formulas:

$$\mathrm{tp}^{-}_{M}(\bar{a}/D) = \{\phi(x) \in \mathcal{L}^{-}(D) : M \models \phi(\bar{a})\}.$$

Then, the *Leibniz congruence* is the relation $\sim_*$ on A defined by letting, for $a, b \in M$,

$$a \sim_* b \iff \mathrm{tp}^{-}_{M}(a/M) = \mathrm{tp}^{-}_{M}(b/M).$$

A model M is *reduced* if the Leibniz congruence $\sim_*$ over M is the identity.

REMARK 7.19. As shown in [14], $\sim_*$ is the largest non-trivial congruence relation on M, meaning that it is the largest congruence θ over its algebraic reduct such that, for any relation symbol $R \in \mathcal{L}$, if $(a_i, b_i) \in \theta$ for all $1 \leqslant i \leqslant n$ then $A \models R(a_1, \ldots, a_n)$ if and only if $A \models R(b_1, \ldots, b_n)$. As we stressed already in Remark 2.9, the projections induced by congruences respecting this conditions are always strict homomorphisms. Additionally, since $\sim_*$ is the greatest such congruence, any strict homomorphism from $M/\sim_*$ onto some further structure N must be the identity.

NOTATION 7.20. *Let* M *be an* $\mathcal{L}^{-}$*-structure and let* $\sim_*$ *be the Leibniz congruence over* M, *then we write* M^* *for the quotient structure* $M/\sim_*$. *For any class operator* $\mathbb{O}$, *we let* $\mathbb{O}^*(C) := \{A^* : A \in \mathbb{O}(C)\}$, *and for any class* $\mathbf{K}$ *we let* $\mathbf{K}^* = \{A^* : A \in \mathbf{K}\}$. *Finally, we let* $\mathrm{Mod}^*(T) = \{A^* : A \in \mathrm{Mod}(T)\}$.

REMARK 7.21. By Fact 7.15, the validity of (strict) universal Horn formulas in a language without equality is closed by $\mathbb{H}_{\mathrm{s}}$, and thus $\mathrm{Mod}^*(T) \subseteq \mathrm{Mod}(T)$.

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From Fact 7.15, one immediately obtains the following theorem [14, Theorem 18] and corollary. Notice that our formulation differs from the original one as Dellunde and Jansana assume that closure operators are already closed under isomorphic copies.

FACT 7.22 (Dellunde and Jansana). *Let* $\mathbf{K}$ *be a class of reduced* $\mathcal{L}^{-}$*-structures, then the following are equivalent*:

(1) $\mathbf{K}$ *is the class of reduced models of an* $\mathcal{L}^{-}$*-universal Horn theory*;
(2) $\mathbf{K}$ *is closed under the operators* $\mathbb{I}^{*}, \mathbb{S}^{*}, \mathbb{P}^{*}, \mathbb{P}^{*}_{\mathrm{U}}$;
(3) $\mathbf{K} = \mathbb{I}^{*}\mathbb{S}^{*}\mathbb{P}^{*}\mathbb{P}^{*}_{\mathrm{U}}(\mathbf{K}_0)$ *for some class* $\mathbf{K}_0$ *of* $\mathcal{L}^{-}$*-structures.*

COROLLARY 7.23. *Let* $\mathbf{K}$ *be a class of reduced bimatrices,* $T = \mathsf{Horn}(\mathsf{Log}(\mathbf{K}))$*, then*:

$$\mathrm{Mod}^{*}(T) = \mathbb{I}^{*}\mathbb{S}^{*}\mathbb{P}^{*}\mathbb{P}^{*}_{\mathrm{U}}(\mathbf{K}).$$

Thus, since every weak logic is complete with respect to a class of bimatrices, the previous corollary provides us with a characterization of the class of reduced bimatrices defined by any weak logic $\Vdash$. We conclude our abstract study of matrix semantics by showing that, in the case of a loosely algebraizable weak logic $\Vdash$, its reduced core-generated bimatrices coincide with the core-generated expanded algebras of its equivalent algebraic semantics. We first define these notions and notice that Proposition 3.21 easily extends to this setting.

DEFINITION 7.24. Let M be a bimatrix, we say that M is core-generated if $M = \langle \mathsf{core}(M) \rangle$. We then write $\mathrm{Mod}_{\mathrm{CG}}(\Vdash)$ for the subclass of $\mathrm{Mod}(\Vdash)$ consisting only of core-generated structures. We then let $\mathrm{Mod}^{*}_{\mathrm{CG}}(\Vdash) = (\mathrm{Mod}(\Vdash))^{*}$.

PROPOSITION 7.25. *Let* A *be a core-generated bimatrix, then* $\sigma(\Theta) \models^{c}_{\{A\}} \sigma(\varepsilon \approx \delta)$ *for all* $\sigma \in \mathsf{Subst}(\mathcal{L})$ *holds if and only if* $\Theta \models_{\{A\}} \varepsilon \approx \delta$.

PROOF. This follows by reasoning as in Lemma 3.20 and Proposition 3.21. ⊣

We can then characterise the reduced, core-generated bimatrices of a loosely algebraizable weak logics $\Vdash$ by directly applying the standard version of this result. We recall the following classical result from [16, Theorem 4.60].

FACT 7.26. *Let* $\vdash$ *be an algebraizable standard logic with equivalent algebraic semantics* $(\mathbf{Q}, \tau, \Delta)$*. Then* $(A, \mathsf{truth}(A)) \in \mathrm{Mod}^{*}(\vdash)$ *if and only if* $A \in \mathbf{Q}$ *and* $\mathsf{truth}(A) = \{a \in A : A \models \tau^{A}(a)\}$.

COROLLARY 7.27. *Let* $\Vdash$ *be a loosely algebraizable weak logic with equivalent algebraic semantics* $(\mathbf{Q}, \tau, \Delta)$*. Then* $(A, \mathsf{truth}(A), \mathsf{core}(A)) \in \mathrm{Mod}^{*}_{\mathrm{CG}}(\Vdash)$ *if and only if* $(A, \mathsf{core}(A)) \in \mathbf{Q}_{\mathrm{CG}}$ *and* $\mathsf{truth}(A) = \{a \in A : A \models \tau^{A}(a)\}$.

PROOF. Firstly, notice that if $\Vdash$ is loosely algebraized by $(\mathbf{Q}, \tau, \Delta)$, then by Proposition 5.4 we have that $\Vdash_{\mathrm{s}}$ is also algebraized by $(\mathbf{Q}, \tau, \Delta)$.

Suppose now that $(A, \mathsf{truth}(A), \mathsf{core}(A)) \in \mathrm{Mod}^{*}_{\mathrm{CG}}(\Vdash)$, then it follows by Proposition 7.25 that $(A, \mathsf{truth}(A)) \in \mathrm{Mod}(\Vdash_{\mathrm{s}})$. We claim that $(A, \mathsf{truth}(A))$ is also reduced. If $a, b \in A$ are distinct elements, then since $\Vdash_{\mathrm{s}}$ is algebraized by $(\mathbf{Q}, \tau, \Delta)$ we have that $A \models a \not\approx b$ is equivalent to $A \not\models \Delta^{A}(a, b)$. But then we have that $\Delta(x, a) \in \mathrm{tp}^{-}_{M}(a/A)$ but $\Delta(x, a) \notin \mathrm{tp}^{-}_{M}(b/A)$, showing that $(A, \mathsf{truth}(A))$ is reduced. It thus follows that $(A, \mathsf{truth}(A)) \in \mathrm{Mod}^{*}(\Vdash_{\mathrm{s}})$. So, by Fact 7.26 and the fact that A is core-generated, we then conclude that $(A, \mathsf{core}(A)) \in \mathbf{Q}_{\mathrm{CG}}$ and $\mathsf{truth}(A) = \{a \in A : A \models \tau^{A}(a)\}$.

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Conversely, suppose that $(A, \mathsf{core}(A)) \in \mathbf{Q}_{\mathrm{CG}}$ and $\mathsf{truth}(A) = \{a \in A : A \models \tau^A(a)\}$. By Fact 7.26 and the fact that $\Vdash_{\mathrm{s}}$ is algebraized by $(\mathbf{Q}, \tau, \Delta)$, we have that $(A, \mathsf{truth}(A)) \in \mathrm{Mod}^*(\Vdash_{\mathrm{s}})$. Also, since $(A, \mathsf{core}(A))$ is core-generated, it follows from Proposition 7.25 that $(A, \mathsf{truth}(A), \mathsf{core}(A)) \in \mathrm{Mod}^*_{\mathrm{CG}}(\Vdash)$. This concludes our proof. ⊣

§8. Application: Inquisitive and dependence logics. We turn in this section to one application of the abstract machinery that we studied, i.e., the case of (propositional) inquisitive and dependence logics. These logical systems make for two interesting examples of logics where uniform substitution fails, and which have also been studied from the algebraic point of view. In particular, an algebraic semantics for the classical version of inquisitive logic InqB was introduced in [2] (although some preliminary inquiry into the subject was provided already in [28]) and further investigated in [3]. Such semantics was later generalised in [26] to the intuitionistic logic InqI, and to both the classical and intuitionistic version of dependence logic $\mathtt{InqB}^{\otimes}$ and $\mathtt{InqI}^{\otimes}$. Since these logical systems do not satisfy the rule of uniform substitution, it has so far been an open question whether they are in any sense unique—as a matter of fact, another (similar) semantics to InqI was suggested independently in [25]. The notion of algebraizability of weak logics that we have introduced in this article provides us with a framework to make sense of this question. In this section, we build on these previous works and relate them to the notion of algebraizability from the present article. Our main result is that the classical versions of inquisitive and dependence logic InqB and $\mathtt{InqB}^{\otimes}$ are strictly algebraizable, while their intuitionistic versions InqI and $\mathtt{InqI}^{\otimes}$ are only loosely so.

8.1. Inquisitive and dependence logic. We introduce in this section the intuitionistic and classical propositional variants of inquisitive and dependence logic. We refer the reader to [12, 22] for the original presentation of classical propositional inquisitive logic, to [30] for classical propositional dependence logic, and to [11] for their intuitionistic versions. We make explicit in the following remark the language in which we formulate these logics. Notice that while our presentation follows essentially [11], our notation is as in [26].

CONTEXT 8.1. We let $\mathcal{L}_{\mathtt{IPC}}$ be the propositional language $\mathcal{L}_{\mathtt{IPC}} = \{\wedge, \vee, \to, \bot, \top\}$, i.e., $\mathcal{L}_{\mathtt{IPC}}$ is simply the usual language of propositional intuitionistic logic. With some abuse of notation, we denote by $\mathcal{L}_{\mathtt{IPC}}$ also the set of all propositional formulas recursively built from Var in this syntax. Also, we let $\mathcal{L}^{\otimes}_{\mathtt{IPC}}$ be the propositional language $\mathcal{L}^{\otimes}_{\mathtt{IPC}} = \{\wedge, \vee, \otimes, \to, \bot, \top\}$, namely, the expansion of $\mathcal{L}_{\mathtt{IPC}}$ by a novel *tensor disjunction* operation $\otimes$. We denote by $\mathcal{L}^{\otimes}_{\mathtt{IPC}}$ also the set of all propositional formulas recursively built from Var in the syntax $\mathcal{L}^{\otimes}_{\mathtt{IPC}}$.

NOTATION 8.2. *We define the inquisitive operation* $?\phi := \phi \vee \neg\phi$. *We treat negation as a defined operation and we define it by* $\neg\phi := \phi \to \bot$. *Also, we write* $\phi \leftrightarrow \psi$ *as an abbreviation for* $\phi \to \psi \wedge \psi \to \phi$.

DEFINITION 8.3. A formula of $\mathcal{L}^{\otimes}_{\mathtt{IPC}}$ is *standard* if it does not contain the symbol $\vee$. We write $\mathcal{L}_{\mathtt{CL}}$ for the signature $\mathcal{L}_{\mathtt{CL}} = \{\wedge, \to, \bot, \top\}$ and for the set of formulas determined by it. Similarly, we write $\mathcal{L}^{\otimes}_{\mathtt{CL}}$ for the signature $\mathcal{L}^{\otimes}_{\mathtt{CL}} = \{\wedge, \to, \otimes, \bot, \top\}$ and for its corresponding set of formulas.

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We recall briefly the standard semantics of inquisitive and dependence logic, i.e., their *team* (or *state*) *semantics*. Intuitively, while in the standard semantics of classical propositional logic a formula is evaluated by a truth-table, i.e., by an assignment of atomic variables into $\{0, 1\}$, in the classical version of team semantics it is evaluated by a *set* of such assignments. Similarly, while in the standard semantics of intuitionistic propositional logic a formula is evaluated at a node of a poset, in the intuitionistic version of team semantics it is evaluated by a *set* of such nodes. We make this idea precise via the following definitions from [11].

DEFINITION 8.4. A *Kripke frame* is a partial order $\mathfrak{F} = (W, R)$. A *Kripke model* is a pair $\mathfrak{M} = (\mathfrak{F}, V)$, where $\mathfrak{F}$ is a Kripke frame and $V : W \to \wp(\mathsf{Var})$ is a valuation of atomic formulas such that, if $p \in V(w)$ and wRv, then $p \in V(v)$. We say that a Kripke frame (resp. model) $\mathfrak{F} = (W, R)$ is *classical* if R is the identity relation.

DEFINITION 8.5. Let $\mathfrak{M} = (W, R, V)$ be a Kripke model. A *team* is a subset $t \subseteq W$ of the set of possible worlds. A team s is an *extension* of a team t if $s \subseteq R[t]$.

REMARK 8.6. Crucially, an element in a Kripke model $(\mathfrak{F}, V)$ is essentially a classical assignment, and we write $w(p) = 1$ if and only if $p \in V(w)$ (when the underlying valuation V is clear from the context). This reflects the main underlying intuition that teams are essentially sets of assignments.

We can next define the team semantics of the formulas in $\mathcal{L}^{\otimes}_{\mathtt{IPC}}$, and thus of the classical and inquisitive variant of inquisitive and dependence logic.

DEFINITION 8.7. Let $\mathfrak{M} = (W, R, V)$ be a Kripke model. The notion of a formula $\phi \in \mathcal{L}^{\otimes}_{\mathtt{IPC}}$ being *true in a team* $t \subseteq W$ is defined as follows:

$$
\begin{array}{lcl}
\mathfrak{M}, t \models p & \iff & \forall w \in t\ (w(p) = 1), \\
\mathfrak{M}, t \models \bot & \iff & t = \varnothing, \\
\mathfrak{M}, t \models \psi \vee \chi & \iff & \mathfrak{M}, t \models \psi \text{ or } \mathfrak{M}, t \models \chi, \\
\mathfrak{M}, t \models \psi \wedge \chi & \iff & \mathfrak{M}, t \models \psi \text{ and } \mathfrak{M}, t \models \chi, \\
\mathfrak{M}, t \models \psi \otimes \chi & \iff & \exists s, r \subseteq t \text{ such that } s \cup r = t \text{ and } \mathfrak{M}, s \models \psi, \mathfrak{M}, r \models \chi, \\
\mathfrak{M}, t \models \psi \to \chi & \iff & \forall s\ (\text{if } s \subseteq R[t] \text{ and } \mathfrak{M}, s \models \psi \text{ then } \mathfrak{M}, s \models \chi).
\end{array}
$$

If Γ is a set of formulas, then we write $\mathfrak{M}, t \models \Gamma$ if $\mathfrak{M}, t \models \phi$ for all $\phi \in \Gamma$. Also, we write $\mathfrak{M} \models \phi$ if $\mathfrak{M}, t \models \phi$ for all $t \subseteq W$ and we write $\mathfrak{F} \models \phi$ if $(\mathfrak{F}, V) \models \phi$ for all valuations V over W.

DEFINITION 8.8. We define the following logical systems:

(a) the system $\Vdash_{\mathtt{InqI}}$ of intuitionistic inquisitive logic is the consequence relation in the language $\mathcal{L}_{\mathtt{IPC}}$ over the class of all Kripke models $\mathfrak{M}$, namely,

$$
\Gamma \Vdash_{\mathtt{InqI}} \phi \iff \mathfrak{M}, t \models \Gamma \text{ entails } \mathfrak{M}, t \models \phi \quad \text{for all Kripke models } \mathfrak{M} \text{ and teams } t \subseteq \mathfrak{M};
$$

(b) the system $\Vdash_{\mathtt{InqB}}$ of classical inquisitive logic is the consequence relation in the language $\mathcal{L}_{\mathtt{IPC}}$ over the class of all classical Kripke models $\mathfrak{M}$, namely,

$$
\Gamma \Vdash_{\mathtt{InqB}} \phi \iff \mathfrak{M}, t \models \Gamma \text{ entails } \mathfrak{M}, t \models \phi \quad \text{for all classical Kripke models } \mathfrak{M} \text{ and teams } t \subseteq \mathfrak{M};
$$

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(c) the system $\Vdash_{\mathtt{InqI}^{\otimes}}$ of intuitionistic dependence logic is the consequence relation in the language $\mathcal{L}^{\otimes}_{\mathtt{IPC}}$ over the class of all Kripke models $\mathfrak{M}$, namely,

$$\Gamma \Vdash_{\mathtt{InqI}^{\otimes}} \phi \iff \mathfrak{M}, t \models \Gamma \text{ entails } \mathfrak{M}, t \models \phi \quad \text{for all Kripke models } \mathfrak{M} \text{ and teams } t \subseteq \mathfrak{M};$$

(d) the system $\Vdash_{\mathtt{InqB}^{\otimes}}$ of classical dependence logic is the consequence relation in the language $\mathcal{L}^{\otimes}_{\mathtt{IPC}}$ over the class of all classical Kripke models $\mathfrak{M}$, namely,

$$\Gamma \Vdash_{\mathtt{InqB}^{\otimes}} \phi \iff \mathfrak{M}, t \models \Gamma \text{ entails } \mathfrak{M}, t \models \phi \quad \text{for all classical Kripke models } \mathfrak{M} \text{ and teams } t \subseteq \mathfrak{M}.$$

REMARK 8.9. Notice that in [11] the previous systems were introduced simply as the sets of validities of the consequence relations from Definition 8.8. However this does not really make a difference, since these logics are all finitary and satisfy the deduction theorem. In particular, the fact that each of the previous consequence relations is finitary follows from [11, Corollary 4.22], while the deduction theorem is essentially [11, Proposition 4.3]. It then follows that, for any $L \in \{\mathtt{InqB}, \mathtt{InqB}^{\otimes}, \mathtt{InqI}, \mathtt{InqI}^{\otimes}\}$,

$$\Gamma \Vdash_L \phi \iff \Vdash_L \bigwedge_{\psi \in \Gamma_0} \psi \to \phi \text{ for some finite } \Gamma_0 \subseteq \Gamma,$$

showing that each consequence relation $\Vdash_L$ is determined by its set of validities.

The next proposition makes it explicit that these logics are proper examples of weak logics. The failure of uniform substitution in these logics has been pointed out since their introduction in [11, 12, 30]. We provide details of the following proposition (which essentially develops Example 3.5) for completeness of exposition, and since the lack of uniform substitution provides the key motivation of our abstract work.

PROPOSITION 8.10. *Let* $L \in \{\mathtt{InqB}, \mathtt{InqB}^{\otimes}, \mathtt{InqI}, \mathtt{InqI}^{\otimes}\}$, *then* $\Vdash_L$ *is a weak logic, and it is not closed under uniform substitution*.

PROOF. Let $L \in \{\mathtt{InqB}, \mathtt{InqB}^{\otimes}, \mathtt{InqI}, \mathtt{InqI}^{\otimes}\}$, then the fact that $\Vdash_L$ satisfies Conditions (1)–(3) from Definition 2.2 follows from the definition of the semantic consequence relation $\models$ from Definition 8.7. Condition (4) follows from Remark 8.9 (thus in particular from [11, Corollary 4.22]). Finally, the fact that $\Vdash_L$ is closed under atomic substitution is immediate to verify using the team semantics from above.

It remains to show that $\Vdash_L$ is not closed under uniform substitution. We prove this rigorously only for $\mathtt{InqB}$ and $\mathtt{InqB}^{\otimes}$ and mention an example that applies also to $\mathtt{InqI}$ and $\mathtt{InqI}^{\otimes}$ below in Remark 8.11. Suppose thus $L \in \{\mathtt{InqB}, \mathtt{InqB}^{\otimes}\}$, we formalize the model that we sketched in Example 3.5. Let $\mathfrak{F} = (\{a, b, c, d\}, =)$ be a classical Kripke frame, let $V : \mathsf{Var} \to \wp(\mathfrak{F})$ be a valuation such that $V(p) = \{a, b\}$ and let $\mathfrak{M} = (\mathfrak{F}, V)$.

Firstly, we use the team semantics from Definition 8.7 to show that $\mathfrak{M}, \{b, d\} \models \neg\neg ?p$. Suppose on the contrary that there is a team $t \subseteq \{b, d\}$ such that $\mathfrak{M}, t \models \neg ?p$. Then, we have in particular that either $\mathfrak{M}, \{b\} \not\models ?p$ or $\mathfrak{M}, \{d\} \not\models ?p$, which are clearly false. We thus conclude that $\mathfrak{M}, \{b, d\} \models \neg\neg ?p$. Additionally, we notice that

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$\mathfrak{M}, \{b, d\} \not\models ?p$. In fact, $\mathfrak{M}, \{b, d\} \not\models p$ since $d(p) = 0$, and $\mathfrak{M}, \{b, d\} \not\models \neg p$ since $b(p) = 1$. This shows that $\neg\neg ?p \not\Vdash_{\mathtt{InqB}} ?p$ and $\neg\neg ?p \not\Vdash_{\mathtt{InqB}^{\otimes}} ?p$.

However, suppose that $(\mathfrak{M}, t) \models \neg\neg p$ for some classical Kripke model $\mathfrak{M} = (W, R, V)$ and $t \subseteq W$. Then for every $w \in t$ we have $(\mathfrak{M}, \{w\}) \not\models \neg p$, which entails that $w(p) = 1$ for all $w \in t$, and thus that $(\mathfrak{M}, t) \models p$. In other words, we proved that $\neg\neg p \Vdash_L p$ in any of the logics $L \in \{\mathtt{InqB}, \mathtt{InqB}^{\otimes}\}$. We conclude that $\mathtt{InqB}$ and $\mathtt{InqB}^{\otimes}$ are not closed under the substitution $p \mapsto ?p$. ⊣

REMARK 8.11. We mention a counterexample to uniform substitution that applies to all $L \in \{\mathtt{InqB}, \mathtt{InqB}^{\otimes}, \mathtt{InqI}, \mathtt{InqI}^{\otimes}\}$. One can verify that:

$$\Vdash_L (p \to (q \vee r)) \to ((p \to q) \vee (p \to r)),$$

but the result of the substitution $p \mapsto q \vee r$ is not a validity of these logics:

$$\not\Vdash_L ((q \vee r) \to (q \vee r)) \to (((q \vee r) \to q) \vee ((q \vee r) \to r)).$$

We refer the reader to [11, 4.5] for an explanation of this fact and a lengthier discussion of the failure of uniform substitution in propositional inquisitive and dependence logic.

8.2. Strict algebraizability of $\mathtt{InqB}$ and $\mathtt{InqB}^{\otimes}$. We prove in this section the strict algebraizability of the classical versions of inquisitive and dependence logic. We recall some important facts and definitions.

DEFINITION 8.12. A *Heyting algebra* H is a bounded distributive lattice with an operation $\to$ such that, for all $a, b, c \in H$,

$$a \wedge b \leqslant c \iff a \leqslant b \to c,$$

where $\leqslant$ is the lattice order. The negation of $a \in H$ is $\neg a := a \to \bot$. An element $a \in H$ is *regular* if $a = \neg\neg a$. We write $H_\neg$ for the subset of regular elements of H, and we say that a Heyting algebra H is *regular* if $H = \langle H_\neg \rangle$. We say that a variety $\mathbf{V}$ of Heyting algebras is *regularly generated* if it is core-generated with $\mathsf{core}(H) = H_\neg$ for all $H \in \mathbf{V}$.

REMARK 8.13. We recall that an *intermediate logic* is a standard logic $\mathtt{L}$ such that $\mathtt{IPC} \subseteq \mathtt{L} \subseteq \mathtt{CPC}$, where $\mathtt{IPC}$ denotes the intuitionistic propositional calculus, and $\mathtt{CPC}$ the classical propositional calculus. Every intermediate logic $\mathtt{L}$ is algebraized by the variety of Heyting algebras $\mathrm{Var}(\mathtt{L})$ defined by

$$\mathrm{Var}(\mathtt{L}) = \{H \in \mathbf{HA} : H \models \phi \approx 1 \text{ for all } \phi \in \mathcal{L}_{\mathtt{IPC}} \text{ such that } \vdash_{\mathtt{L}} \phi\}$$

and by the transformers (τ, Δ) given by $\tau(\phi) = (\phi \approx 1)$ and $\Delta(\alpha, \beta) = \alpha \leftrightarrow \beta$ (cf. [16, Example 3.34]).

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The following two definitions are less standard. First, we recall the DNA-logics from Example 3.5. Negative variants were originally considered in [22].

DEFINITION 8.14. Let L be an intermediate logic, then its negative variant $\mathtt{L}^{\neg}$ is the set of formulas $\mathtt{L}^{\neg} = \{\phi[\neg p_0, \dots, \neg p_n / p_0, \dots, p_n] : \phi \in \mathcal{L}_{\mathtt{IPC}} \text{ and } \vdash_{\mathtt{L}} \phi\}$. A DNA-logic is the negative variant of some intermediate logic.

Additionally, we recall Medvedev's logic of finite problems, which was originally introduced by Medvedev in [21]. This is an intermediate logic defined in terms of validity in a specific class of Kripke frames. We direct the reader to [8, Section 2.2] for details about the Kripke semantics of intermediate logics.

DEFINITION 8.15. We recall that the intermediate logic ML, known as Medvedev's logic of finite problems, is the logic of all Kripke frames of the form $(\wp^+(s), \supseteq)$, where s is a finite set and $\wp^+(s) = \wp(s)\backslash\{\emptyset\}$.

The next fact collects together some results by Ciardelli [10] on the schematic variant of InqB, and the characterization of regularly generated varieties from [3]. From these it is then immediate to prove the strict algebraizability of InqB.

FACT 8.16.
(1) $\mathtt{ML}^{\neg} = \mathtt{InqB}$ *and* $\mathrm{Schm}(\mathtt{InqB}) = \mathtt{ML}$.
(2) $\mathrm{Var}(\mathtt{ML})$ *is regularly generated.*
(3) *Let* $\tau(\phi) = (\phi \approx 1)$ *for all* $\phi \in \mathsf{Fm}$, *then for all* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\mathtt{IPC}}$, $\Gamma \Vdash_{\mathtt{InqB}} \phi$ *is equivalent to* $\tau(\Gamma) \models^{c}_{\mathrm{Var}(\mathtt{ML})} \tau(\phi)$, *where* $\mathsf{core}(H) = H_{\neg}$ *for all* $H \in \mathrm{Var}(\mathtt{ML})$.

PROOF. Considering Clause (1), both $\mathtt{ML}^{\neg} = \mathtt{InqB}$ and $\mathrm{Schm}(\mathtt{InqB}) = \mathtt{ML}$ were proved by Ciardelli in [10]. Clause (2) follows immediately from (1) together with Proposition 4.17 from [3]. Clause (3) is essentially the main algebraic completeness result for inquisitive logic shown in [2] (and later in [3]). However, in these articles completeness is formulated with respect to the intermediate logics KP and ND, thus we explain how to obtain completeness with respect to $\mathrm{Var}(\mathtt{ML})$ as stated in (3). Firstly, by Theorem 3.32 in [3] we have that $\Gamma \models_{\mathtt{InqB}} \phi$ is equivalent to $\Gamma \models^{c}_{\mathrm{Var}(\mathtt{ND})} \phi$, where ND is a specific intermediate logic and $\mathsf{core}(H) = H_{\neg}$ for all $H \in \mathrm{Var}(\mathtt{ND})$. Additionally, it follows from [3, Theorem 5.9] that ND and ML contain the same regularly generated Heyting algebras. It then follows from Proposition 3.26 that $\Gamma \Vdash_{\mathtt{InqB}} \phi$ is equivalent to $\Gamma \models^{c}_{\mathrm{Var}(\mathtt{ML})} \phi$. ⊣

THEOREM 8.17. InqB *is strictly algebraizable.*

PROOF. Let $\tau(\phi) = \phi \approx 1$, $\Delta(x, y) = x \leftrightarrow y$, and $\Sigma = \{x \approx \neg\neg x\}$. We prove that $(\mathrm{Var}(\mathtt{ML}), \Sigma, \tau, \Delta)$ strictly algebraizes InqB. Firstly, by Fact 8.16(2) above we have that $\mathrm{Var}(\mathtt{ML})$ is core-generated with core defined by Σ. Then, by Proposition 4.8 it suffices to show that $(\mathrm{Var}(\mathtt{ML}), x \approx \neg\neg x, \phi \approx 1, x \leftrightarrow y)$ satisfies 4.6(W1) and 4.6(W4). By Fact 8.16(3), Condition 4.6(W1) follows immediately. Moreover, for all $H \in \mathrm{Var}(\mathtt{ML})$ and $x, y \in H$, we have that $x = y$ if and only if $x \leqslant y$ and $y \leqslant x$. This is equivalent to $H \models x \to y \approx 1$ and $H \models y \to x \approx 1$. It follows that $x \approx y \equiv^{c}_{\mathrm{Var}(\mathtt{ML})} \{x \to y \approx 1, y \to x \approx 1\}$, showing that 4.6(W4) holds. It follows that InqB is strictly algebraizable. ⊣

To extend the previous theorem to dependence logic, we firstly need to introduce a suitable notion of dependence algebras. We refer the reader to [5, p. 57] for the definition of subdirectly irreducible algebra.

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DEFINITION 8.18. An $\mathtt{InqB}^{\otimes}$-algebra A is a structure in the signature $\mathcal{L}^{\otimes}_{\mathtt{IPC}}$ satisfying the following conditions:

(a) $A{\restriction}\{\vee,\wedge,\to,\bot\} \in \mathrm{Var}(\mathtt{ML})$;
(b) $A_{\neg}{\restriction}\{\otimes,\wedge,\to,\bot\} \in \mathbf{BA}$;
(c) $A \models \forall x\forall y\forall z\ (x \otimes (y \vee z) \approx (x \otimes y) \vee (x \otimes z))$;
(d) $A \models \forall x\forall y\forall z\forall k\ ((x \to z) \to (y \to k) \approx (x \otimes y) \to (z \otimes k))$;

we then let $\mathsf{InqBAlg}^{\otimes}$ be the variety of all $\mathtt{InqB}^{\otimes}$-algebras and $\mathsf{InqBAlg}^{\otimes}_{\mathsf{FRSI}}$ be its subclass of finite, regular, and subdirectly irreducible elements.

REMARK 8.19. Our definition is essentially from [26, Definition 2.2], with the difference that here we assume that the equations hold in the full algebra and not only in the subalgebra generated by the core. Since our results deal with core semantics and core-generated structure, this does not affect the validity of the results from [26].

The previous class of algebras was shown to provide a sound and complete semantics for $\mathtt{InqB}^{\otimes}$. We recall the following fact from [26, Theorem 2.15, Theorem 3.20] and use it to show that $\mathtt{InqB}^{\otimes}$ is strictly algebraizable. Notice that, since $A{\restriction}\{\vee,\wedge,\to,\bot\} \in \mathrm{Var}(\mathtt{ML})$, for $A \in \mathsf{InqBAlg}^{\otimes}$, it follows that the notion of regular elements and the subset $A_{\neg}$ are well defined in this context.

FACT 8.20. *Let* $\tau(\phi) = (\phi \approx 1)$ *for all* $\phi \in \mathsf{Fm}$, *then for all* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}^{\otimes}_{\mathtt{IPC}}$, *we have* $\Gamma \Vdash_{\mathtt{InqB}^{\otimes}} \phi$ *if and only if* $\tau(\Gamma) \models^{c}_{\mathsf{InqBAlg}^{\otimes}_{\mathsf{FRSI}}} \tau(\phi)$, *where* $\mathsf{core}(A) = A_{\neg}$ *for all* $A \in \mathsf{InqBAlg}^{\otimes}_{\mathsf{FRSI}}$.

THEOREM 8.21. $\mathtt{InqB}^{\otimes}$ *is strictly algebraizable.*

PROOF. This follows from Fact 8.20 analogously to Theorem 8.17, by choosing for equivalent algebraic semantics the variety $\mathbb{V}(\mathsf{InqBAlg}^{\otimes}_{\mathsf{FRSI}})$, the transformers $\tau(\phi) := \phi \approx 1$ and $\Delta(x,y) = x \to y \wedge y \to x$, and the finite set of formulas defining the core $\Sigma = \{x \approx \neg\neg x\}$. ⊣

8.3. Loose algebraizability of $\mathtt{InqI}$ and $\mathtt{InqI}^{\otimes}$. We show in this section that $\mathtt{InqI}$ and $\mathtt{InqI}^{\otimes}$ are both loosely algebraizable, but they are not strictly so. The loose algebraizability of these logics is essentially a corollary of the algebraic completeness result from [26]. We start by introducing the algebraic semantics for $\mathtt{InqI}$ and $\mathtt{InqI}^{\otimes}$ described in [26]. As in Definition 8.18 we slightly modify the definitions from [26] so that the equations defining $\mathtt{InqI}$-algebras and $\mathtt{InqI}^{\otimes}$-algebra are valid in the entire structure and not only in the subalgebra generated by the core. As stressed in Remark 8.19, this does not affect the validity of the completeness results from [26].

DEFINITION 8.22. A *Brouwerian semilattice* B is a bounded join-semilattice with an additional operation $\to$ such that, for all $a, b, c \in B$,

$$a \wedge b \leqslant c \iff a \leqslant b \to c,$$

and we write **BS** for the class of all Browerian semilattices.

DEFINITION 8.23. An $\mathtt{InqI}$-algebra A is an expanded algebra in the signature $\mathcal{L}_{\mathtt{IPC}}$ satisfying the following conditions:

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(a) $A\restriction\{\vee,\wedge,\to,\bot\}\in\mathbf{HA}$;
(b) $\mathsf{core}(A)\restriction\{\wedge,\to,\bot\}\in\mathbf{BS}$;
(c) $A\models\forall x\forall y\forall a\ (\mathsf{core}(a)\to(a\to(x\vee y)\approx(a\to x)\vee(a\to y)))$;

and we then write InqIAlg for the class of all InqI-algebras.

DEFINITION 8.24. An $\mathtt{InqI}^{\otimes}$-algebra A is an expanded algebra in the signature $\mathcal{L}^{\otimes}_{\mathtt{IPC}}$ satisfying the following conditions:

(a) $A\restriction\{\vee,\wedge,\to,\bot\}\in\mathbf{HA}$;
(b) $\mathsf{core}(A)\restriction\{\wedge,\otimes\to,\bot\}\in\mathbf{HA}$;
(c) $A\models\forall x\forall y\forall a\ (\mathsf{core}(a)\to(a\to(x\vee y)\approx(a\to x)\vee(a\to y)))$;
(d) $A\models\forall x\forall y\forall z\ (x\otimes(y\vee z)\approx(x\otimes y)\vee(x\otimes z))$;
(e) $A\models\forall x\forall y\forall z\forall k\ ((x\to z)\to(y\to k)\approx(x\otimes y)\to(z\otimes k))$;

and we then write $\mathsf{InqIAlg}^{\otimes}$ for the class of all $\mathtt{InqI}^{\otimes}$-algebras.

We next recall the algebraic completeness result for intuitionistic inquisitive and dependence logic. This follows immediately from [26, Proposition 2.15]. From this, it is immediately to show that both InqI and $\mathtt{InqI}^{\otimes}$ are loosely algebraizable.

FACT 8.25. *Let* $\tau:\mathsf{Fm}\to\mathsf{Eq}$ *be defined by* $\tau(x)=(x\approx 1)$, *then the following completeness results hold*:

(1) *For all* $\Gamma\cup\{\phi\}\subseteq\mathcal{L}_{\mathtt{IPC}}$, $\Gamma\Vdash_{\mathtt{InqI}}\phi$ *if and only if* $\tau(\Gamma)\models^{c}_{\mathsf{InqIAlg}}\tau(\phi)$.
(2) *For all* $\Gamma\cup\{\phi\}\subseteq\mathcal{L}^{\otimes}_{\mathtt{IPC}}$, $\Gamma\Vdash_{\mathtt{InqI}^{\otimes}}\phi$ *if and only if* $\tau(\Gamma)\models^{c}_{\mathsf{InqIAlg}^{\otimes}}\tau(\phi)$.

THEOREM 8.26. InqI *and* $\mathtt{InqI}^{\otimes}$ *are loosely algebraizable.*

PROOF. Consider the logic InqI. Let $\mathbf{Q}$ be the quasivariety of expanded algebras generated by $\mathsf{InqIAlg}_{\mathrm{CG}}$, i.e., $\mathbf{Q}=\mathbb{Q}(\mathsf{InqIAlg}_{\mathrm{CG}})$, $\tau(\phi)=(\phi\approx 1)$, and $\Delta(\alpha,\beta)=(\alpha\to\beta)\wedge(\beta\to\alpha)$. Notice that, since every $H\in\mathbf{Q}\restriction\{\vee,\wedge,\to,\bot\}$ is a Heyting algebra, it follows that $\tau(\Delta(\alpha,\beta))=\alpha\leftrightarrow\beta\approx 1$ is equivalent to $\alpha\approx\beta$. This establishes 4.6(W4). Moreover, we have by the algebraic completeness 8.25(1) and by Proposition 3.24 that

$$\Gamma\Vdash_{\mathtt{InqI}}\phi\iff\tau(\Gamma)\models^{c}_{\mathbf{Q}}\tau(\phi),$$

which establishes 4.6(W1). It follows from Proposition 4.8 that $(\mathbf{Q},\tau,\Delta)$ loosely algebraizes InqI. The loose algebraizability of $\mathtt{InqI}^{\otimes}$ follows analogously. ⊣

While the loose algebraizability of InqI and $\mathtt{InqI}^{\otimes}$ follows straightforwardly from [26], the fact that they are not strictly algebraizabile is substantially more subtle. In principle, to show that a logic is not strictly algebraizable one should test infinitely many finite sets of equations to see whether any of them defines the core of the algebras in the corresponding quasivariety of expanded algebras, which is clearly a no-go. However, the failure of strict algebraizability in InqI and $\mathtt{InqI}^{\otimes}$ is witnessed by a simpler pattern. Namely, we can find two inquisitive (dependence) algebras H, K which have the same algebraic reduct, but different cores. Clearly, if a logic is strictly algebraizable this cannot happen, as the core is uniquely determined by a finite set of equations Σ. We make this intuition precise in the following proof.

THEOREM 8.27. InqI *and* $\mathtt{InqI}^{\otimes}$ *are not strictly algebraizable.*

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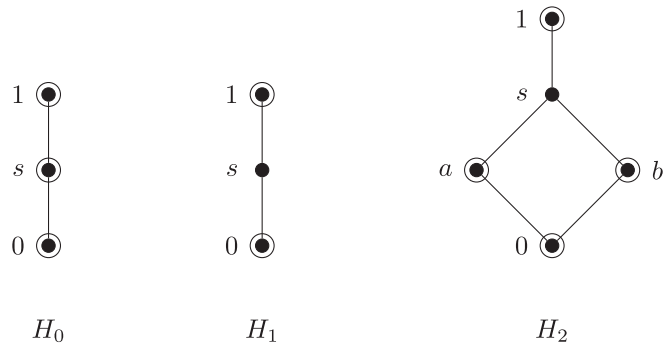


FIGURE 2. The inquisitive (dependence) algebras from Theorem 8.27.

PROOF. We prove this for $\mathtt{InqI}$, as the proof easily adapts to $\mathtt{InqI}^{\otimes}$. Firstly, notice that if $(\mathbf{Q}, \Sigma, \tau, \Delta)$ strictly algebraizes $\mathtt{InqI}$, then clearly $(\mathbf{Q}, \tau, \Delta)$ is also a witness of its loose algebraizability. It then follows from Theorems 8.26 and 4.7 that $\mathbf{Q} = \mathbb{Q}(\mathsf{InqIAlg}_{\mathrm{CG}})$. Now, by directly checking the definition of $\mathtt{InqI}$-algebras from Definition 8.23 (or, alternatively, by applying the categorical duality between finite Kripke frames and finite, well-connected, core-generated, $\mathtt{InqI}$-algebras from [26]), one can verify that the expanded algebras H_0 and H_2 from Figure 2 (with circles indicating which elements are in the core) belong to $\mathsf{InqIAlg}_{\mathrm{CG}}$. Furthermore, since $H_1 \leqslant H_2$, it also follows that $H_1 \in \mathbb{Q}(\mathsf{InqIAlg}_{\mathrm{CG}})$. Crucially, H_0 and H_1 have the same algebraic reduct but different subsets of core elements. In particular, assuming that $(\mathbf{Q}, \Sigma, \tau, \Delta)$ strictly algebraizes $\mathtt{InqI}$, we obtain that $\mathsf{core}(H_0) = \Sigma(H_0) = \Sigma(H_1) = \mathsf{core}(H_1)$, contradicting the fact that $\mathsf{core}(H_0) \neq \mathsf{core}(H_1)$. It follows that $\mathtt{InqI}$ is not strictly algebraizable. The same argument also works for $\mathtt{InqI}^{\otimes}$, as it suffices to notice that the core subsets of H_0, H_1, and H_2 can all be expanded to form Heyting algebras with an additional disjunction $\otimes$, which additionally satisfies the axioms from Definition 8.24 (this is done in full generality in [26, Section 4.3]). ⊣

COROLLARY 8.28. $\mathtt{InqI}$ *and* $\mathtt{InqI}^{\otimes}$ *are not finitely representable.*

PROOF. Immediate from Theorems 8.26, 8.27, and 5.5. ⊣

QUESTION 8.29. *We notice that in the proof of Theorem* 8.27 *we were in a sense lucky, i.e., we proved the impossibility of strict algebraizability by showcasing two expanded algebras with the same algebraic reduct and different cores. Must such situation always happen whenever a weak logic is loosely algebraizable, but not strictly so? More precisely, consider the following properties of a weak logic* $\vdash\!\!\!\models$:

(i) $\vdash\!\!\!\models$ *is loosely algebraized by a tuple* $(\mathbf{Q}, \tau, \Delta)$;
(ii) $\vdash\!\!\!\models$ *is not strictly algebraizable*;
(iii) *if* $A \restriction \mathcal{L} = B \restriction \mathcal{L}$ *then* $\mathsf{core}(A) = \mathsf{core}(B)$.

Is it possible to find a weak logic $\vdash\!\!\!\models$ *which satisfies the properties (i)–(iii) from above?*

QUESTION 8.30. *We notice that the argument for the loose algebraizability of* $\mathtt{InqI}$ *and* $\mathtt{InqI}^{\otimes}$ *easily generalises to the entire class of intermediate inquisitive* (*and*

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dependence) *logics from* [26]. *Now, as mentioned in Fact* 8.16, *the schematic fragment of* InqB *is* ML. *Moreover, as pointed out in* [15], *it is a corollary of the main result from* [17] *that the schematic fragment of* InqI *is* ML *as well. In other words, all the intermediate inquisitive logics* L *have the same schematic fragment* ML. *It thus follows from our Theorem* 5.5 *that the strictly algebraizable intermediate inquisitive logics are exactly those obtained by closure under modus ponens of sets of the form* $\mathtt{ML} \cup \mathsf{At}[\Lambda]$, *for some finite set of formulas* $\Lambda \subseteq \mathcal{L}_{\mathtt{IPC}}$ *in one variable. Is it possible to refine this characterisation? For example, can one provide a semantical characterisation of the intermediate inquisitive and dependence logics which are strictly algebraizable? We leave this and the previous question as pointers for future investigations.*

Acknowledgments. The authors would like to thank Tommaso Moraschini for useful discussions and pointers to the literature, Fan Yang and Fredrik Nordvall Forsberg for their valuable comments and remarks to an early draft of this article, and one anonymous reviewer for their careful reading of the article and their many very useful suggestions.

Funding. The second author was supported by grant 336283 of the Academy of Finland and Research Funds of the University of Helsinki, and by an InDam postdoc scholarship.

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https://doi.org/10.1017/jsl.2026.10180 Published online by Cambridge University Press