Odd run sums and fractions

RON KNOTT*

A series of consecutive odd numbers has interesting properties useful for classroom investigation [1]. This Article was inspired by the series of fractions for $\frac{1}{2}$ using only sums of consecutive odd numbers

$$\frac{1}{3} = \frac{1+3}{5+7} = \frac{1+3+5}{7+9+11} = \ldots = \frac{1+3+\ldots+(2n-3)+(2n-1)}{(2n+1)+(2n+3)+\ldots+\big(2n+(2n-1)\big)}.$$

The nth term of this odd run pattern for $\frac{1}{3}$ is $\dfrac{n^2}{(2n)^2 - n^2} = \dfrac{1}{3}$.

This pattern of two infinite series of sums of runs of odd numbers is not unique for $\frac{1}{3}$ and we will show that all fractions have such an infinite form. We conclude by giving an easy method of generating such forms eminently suitable for the secondary school classroom.

We will call a series of consecutive odd numbers an *oddrun*. Its arithmetic expression as well as the numerical total will be called an *oddrunsum*. We allow a run of a single number unless otherwise stated. The number of odd numbers in the run is its *length* L and the average of the run its **centre** C.

The oddrun $17 + 19$ with sum 36 is of length 2 around a central number or average of 18 so that $36 = 2 \times 18$. Since the average C is the sum S divided by the length L, an oddrunsum S will be always be its length L times its centre C, all of which are integers. This is a useful feature by which to denote an oddrunsum S using the notation

$$S = L \otimes C = L \times C$$

$$36 = 2 \otimes 18 = 2 \times 18$$

for the oddrun $17 + 19$.

The $\otimes$ symbol is a combination of "o" for the odds and multiplication for the oddrun of length L with centre C whose value is the number $L \times C$.

Thus $6 \otimes 12$ is the oddrunsum of $L = 6$ consecutive odd numbers centred on the number $C = 12$, or $7 + 9 + 11 + 13 + 15 + 17$ with numerical value $6 \times 12 = 72$.

If the length is odd then the average will be the unique central number contained in the oddrun and if the length is even, the centre will be the even number between the two central odds. So for our oddruns *both L and C have the same parity*.

Another useful notation will be an oddrun beginning at the odd number B and ending at the odd number E, denoted by $B \oplus E$ since we are adding (+) odds (o) between these limits. The oddrun $6 \otimes 12$ above is the oddrun beginning at 7 and ending at 17, so we write it as $7 \oplus 17$.

* Visiting Teaching Fellow, Mathematics Department, University of Surrey

https://doi.org/10.1017/mag.2025.10068 Published online by Cambridge University Press

Conversions between these are $B \oplus E = \left(\frac{E-B}{2}+1\right) \otimes \left(\frac{E+B}{2}\right)$, B and E both odd and $L \otimes C = (C-L+1) \oplus (C+L-1)$.

For an oddrun, L and C must be both odd or both even.

If $L \leqslant C$ then all the odds in the run are positive. If $L > C$ then the run starts with a negative odd number. For example:

$$16 = 8 \otimes 2 = -5 + (-3) + (-1) + 1 + 3 + 5 + 7 + 9 = 2 \otimes 8.$$

$f \otimes g$ has the same arithmetical value as $g \otimes f$, namely fg. The oddrun with $f > g$ will begin with a negative odd number and all of which are cancelled out by positive odds in the run. This will reduce the expression to the other form with $f \leqslant g$, as in the example for 16 above.

Prime numbers have only one factor-pair so odd primes have a single oddrun of length 1.

Oddruns as the difference of two squares

A sum of consecutive odd numbers beginning at 1 is a square number:

$$1 \oplus (2n-1) = n^2.$$

This is a useful result for students to practise proof by induction [1].

So for any oddrun starting at a larger odd number, if we subtract the oddrunsum

$$1 \oplus (2m-1) = 1 + 3 + \ldots + (2m-1) = m^2$$

from the start of the longer one

$$1 \oplus (2n-1) = 1 + 3 + \ldots + (2m-1) + (2\boldsymbol{m}+1) + \ldots + (2\boldsymbol{n}-1) = n^2$$

we have a difference of two squares

$$(2m+1) \oplus (2n-1) = (2\boldsymbol{m}+1) + \ldots + (2\boldsymbol{n}-1) = n^2 - m^2.$$

Generating oddruns for a given sum

We now have a way of generating oddrunsums. A given sum N is a product of two factors. Each difference of two squares corresponds to an oddrunsum and each oddrunsum is a difference of two squares, where the second square can be zero.

$$m^2 - n^2 = (m-n)(m+n).$$

To find oddruns for sum N, we factorise into two ordered factors $L \times C$ where $L \leqslant C$ (to give the shorter oddrun without negative terms) and then select those pairs where both factors have the same parity. These are the $L \otimes C$ oddruns for N.

For example 36 has five ordered factor pairs

$$1 \times 36 = 2 \times 18 = 3 \times 12 = 4 \times 9 = 6 \times 6.$$

https://doi.org/10.1017/mag.2025.10068 Published online by Cambridge University Press

Only two pairs are of the same parity: 2×18 and 6×6.

$$2 \otimes 18 = 17 + 19 = 10^2 - 8^2$$

and

$$6 \otimes 6 = 1 + 3 + 5 + 7 + 9 + 11 = 6^2 - 0^2$$

so 36 has 2 oddrunsums.

Fermat, around 1643, knew that the number of factorisations of a number n as $n = a \times b$ is the same as the number of ways of expressing n as the difference of two squares [2].

Do all numbers have an oddrunsum?

An oddrunsum for $n = ab$ implies that $n = \left(\frac{b+a}{2}\right)^2 - \left(\frac{b-a}{2}\right)^2$. If n is an integer and we require that a and b both be integers then a and b must be both even or both odd.

If a and b are both even then $n = ab$ is a multiple of 4.

If both a and b are odd then so is n.

So if n is an oddrunsum then it cannot be twice an odd number. Twice an odd number will have a remainder 2 modulo 4 and such numbers are the only positive integers with no oddrunsums (we are allowing runs of length 1). This series of natural numbers that are oddrunsums are also the natural numbers that are squares or the difference of two squares and are all the natural numbers except those with a remainder of 2 modulo 4: 1, 3, 4, 5, 7, 8, 9, 11, 12 … which is Sloane's A042965 [3].

A formula for the number of odd runs

A formula for the number of oddruns for a given natural number sum n can be given using the prime factorisation of n.

If $n = 2^e m$ where m is odd then the number of oddrunsums for n is

$$e = 0 : \lceil \#Divs(m)/2 \rceil$$

$$e = 1 : 0$$

$$e \geqslant 2 : \lceil \#Divs(2^{e-2}m)/2 \rceil.$$

$\#Divs(m)$ means the number of divisors of m including 1 and m and $\lceil x \rceil$, the *ceiling* function of x, means x rounded up to the nearest larger integer or x itself if it is an integer.

These three cases correspond to:

If n is odd, $e = 0$ and all pairs of factors (an even number of pairs) are of the same odd parity. We ignore the pairs L, C with $L > C$, which is half of them. If n is a perfect square then the number of factors is odd. We need to include the factor pair with $L = C$ which is accounted for by rounding up

https://doi.org/10.1017/mag.2025.10068 Published online by Cambridge University Press

using the ceiling function.

If n is 1 then we have a prohibited value of n which is twice an odd number, with no oddrunsums.

If n is even then we must have both numbers even in a factor pair. We use two 2s one in each of the pair of factors, accounting for two of the 2s in the prime factorisation. The pairs in the factorisation of $n/4$ can then be used with these two 2s to make an pair of even factors for $L \times C$. We use half by excluding the pairs with $L > C$ the ceiling function to account for n being a square number.

This formula is given in an equivalent form and proved in [4].

Examples of each case are

(a) n odd:

For $n = 9$, a square number, factors are 1, 3, 9 giving two pairs $L \times C$ with $L \leqslant C$: 1×9 and 3×3. This is the first number with 2 oddrunsums: $9 = 1 \otimes 9$ and $9 = 3 \otimes 3 = 1 + 3 + 5$.

The first number with three oddruns is 45 with six factors 1, 3, 6, 9, 15, 45. Again with $L \leqslant C$ we have: $1 \otimes 45$, $3 \otimes 15$, $5 \otimes 9$ corresponding to 45, $13 + 15 + 17$ and $5 + 7 + 9 + 11 + 13$ respectively.

(b) n even and a multiple of 4

$n = 144 = 2^4 3^2$ has $5 \times 3 = 15$ factors 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72, 144 of which three pairs are of opposite parity: 1×144, 3×48 and 9×16. We can take off two 2s and use each to double each number in each pair of factors of $2^2 \times 3^2 = 36$, which has factor pairs 1×36, 2×18, 3×12, 4×9, 6×6. By doubling each number in these pairs we have 5 $L \otimes C$ oddruns with sum 144:

Factors	$L \otimes C$	$B \oplus E$	The odd run sum
1×36	$2 \otimes 72$	$71 \oplus 73$	$71 + 73$
2×18	$4 \otimes 36$	$33 \oplus 39$	$33 + 35 + 37 + 39$
3×12	$6 \otimes 24$	$19 \oplus 29$	$19 + 21 + \ldots + 27 + 29$
4×9	$8 \otimes 18$	$11 \oplus 25$	$11 + 13 + \ldots + 23 + 25$
6×6	$12 \otimes 12$	$1 \oplus 23$	$1 + 3 + \ldots + 21 + 23$

144 is the smallest number to have five oddruns. The list of record-breakers, that is the smallest number to have r runsums, for each $r = 1, 2, \ldots$ is 9, 45, 96, 144, 240, 768, 480, … A068314 [3] which, we note, is not monotonically increasing.

https://doi.org/10.1017/mag.2025.10068 Published online by Cambridge University Press

Oddruns for all fractions

We now turn our attention to fractions and show that all fractions N/D have an infinite number of expressions $\frac{Nk}{Dk}$ expressible as $\frac{L_N \otimes C_N}{L_D \otimes C_D}$, the ratio of two oddruns.

If N and D have the same parity then all $k \in \mathbb{N}^+$ can be used to express N and D as $L \otimes C$ where L and C have the same parity.

If N and D are of opposite parity then k must be even and both L and C can then be even in the numerator and in the denominator.

To avoid negative odd numbers, ensure that in $L \otimes C$ we have $L \leqslant C$.

Here are some examples for $\frac{3}{4}$:

$$\frac{3}{4} = \frac{3 \otimes (2k+1)}{2 \otimes (4k+2)} = \frac{3 \otimes 3}{2 \otimes 6} = \frac{3 \otimes 5}{2 \otimes 10} = \ldots = \frac{1+3+5}{5+7} = \frac{3+5+7}{9+11} = \frac{5+7+9}{13+15} = \ldots$$

$$\frac{3}{4} = \frac{(2k) \otimes 6}{(2k) \otimes 8} = \frac{2 \otimes 6}{2 \otimes 8} = \frac{4 \otimes 6}{4 \otimes 8} = \frac{6 \otimes 6}{6 \otimes 8} = \frac{5+7}{7+9} = \frac{3+5+7+9}{5+7+9+11} = \frac{1 \otimes 11}{3 \otimes 13}$$

but after this L exceeds C in the numerator. Moving the k to be a factor of each C gives an infinite pattern again:

$$\frac{3}{4} = \frac{2 \otimes (6k)}{2 \otimes (8k)} = \frac{2 \otimes 6}{2 \otimes 8} = \frac{2 \otimes 12}{2 \otimes 16} = \ldots = \frac{5+7}{7+9} = \frac{11+13}{15+17} = \frac{17+19}{23+25} = \ldots.$$

Other conditions may be put on the values of k such as:

$$\frac{3}{4} = \frac{k \otimes (3k)}{(2k) \otimes (2k)} = \frac{2 \otimes 6}{4 \otimes 4} = \frac{3 \otimes 9}{6 \otimes 6} = \ldots = \frac{5+7}{1+3+5+7} = \frac{7+9+11}{1 \otimes 11} = \frac{9+11+13+15}{1 \otimes 13} = \ldots$$

Even runsums?

Although this Article is about oddrunsums, if one of L and C is even and the other odd then $L \otimes C$ can represent a run of *consecutive even* numbers with Length L and Centre C.

The centre C will still be between two numbers of the run if the length is even and will be equal to a member of the run if the length is odd both for odd runs and even runs. For example:

$L = 4, C = 9 \quad 4 \otimes 9 = 6 + 8 + 10 + 12 \quad = 4 \times 9 = 36$
$L = 5, C = 10 \quad 5 \otimes 10 = 6 + 8 + 10 + 12 + 14 \quad = 5 \times 10 = 50.$

Here are some more examples of infinite series of fractions for $\frac{3}{4}$ with both odd and even runsums:

$$\frac{3}{4} = \frac{2+4+6}{1+3+5+7} = \frac{4+6+8}{3+5+7+9} = \frac{6+8+10}{5+7+9+11} = \ldots = \frac{3 \otimes 2k}{4 \otimes 2k}$$

$$\frac{3}{4} = \frac{3+5+7}{2+4+6+8} = \frac{5+7+9}{4+6+8+10} = \frac{7+9+11}{6+8+10+12} = \ldots = \frac{3 \otimes (2k+1)}{4 \otimes (2k+1)}$$

https://doi.org/10.1017/mag.2025.10068 Published online by Cambridge University Press

$$\frac{3}{4}=\frac{1+3+5}{2+4+6}=\frac{4+6+8}{6+8+10}=\frac{7+9+11}{10+12+14}=\frac{10+12+14}{14+16+18}=\ \ldots\ =\frac{3\otimes 3k}{3\otimes 4k}$$

There are more details and interactive online calculators on the author's webpage [5].

Acknowledgement

I thank the referee for suggesting improvements.

References

1. C. Sangwin, Sums of the first n odd integers, *Math. Gaz.* **107** (March 2023) pp. 10-24.
2. L. E. Dickson, *History of the theory of numbers: Vol 1 Divisibility and primality* Dover (2005), ch XIV, p. 357.
3. N. T. Sloane, *On-line encyclopedia of integer sequences* at https://oeis.org
4. E. T. H. Wang, Problem 1717, *Crux Math* (1993) p. 30.
5. R. Knott, Odd runs and their sums, accessed November 2024 at https://r-knott.surrey.ac.uk/runsums/oddrunsums.html

10.1017/mag.2025.10068 © The Authors, 2025
Published by Cambridge University Press
on behalf of The Mathematical Association

RON KNOTT
92 Pennine Road,
Horwich,
Bolton,
BL6 7HW
e-mail: *ronknott@mac.com*

The answers to the *Nemo* page from March 2025 on fractions were:

1. Charlotte Bronte	Villette	Chapter 7
2. Vladimir Nabokov	"That in Aleppo Once"	
3. Lawrence Durrell	Mountolive	Chapter V
4. D. H. Lawrence	Tortoises	
5. Charles Dickens	The Signalman	
6. Antony Trollope	The Prime Minister	Chapter 27

Congratulations to Martin Lukarevski on identifying all of these apart from the Durrell, which is from the third book in the Alexandria Quartet. This month we move from fractions to zero. Quotations are to be identified by reference to author and work. One of the quotations is a translation from Russian. Solutions are invited to the ωditor by 23rd September 2025.

1. Dead Feet tied himself in knots trying to explain why x to the power of zero always equals one.

Continued on page 246

https://doi.org/10.1017/mag.2025.10068 Published online by Cambridge University Press