WAVES ON A SLOPING BEACH

By W. E. WILLIAMS

Received 21 December 1959

ABSTRACT. A method employed by the author for a class of boundary-value problems arising in diffraction theory is applied to the problem of the oscillations of water waves on a sloping beach. The two cases of long and short waves are considered. The approach is valid for all values of the parameters and particular cases do not have to be treated separately. It is shown that the present approach enables certain solutions to be obtained which cannot arise in earlier solutions of the problem.

1. *Introduction.* In recent years a considerable amount of research has been carried out into the problem of the oscillations of water waves on a sloping beach. The first general solution for long waves was obtained independently, but by similar methods, by Peters (1) and Roseau (2). The case of short waves has also been recently investigated at great length by Roseau (3, 4). In the present paper a simpler and more general approach to the problem is given and it is shown that the solution for both long and short waves may be expressed, with very little analysis, in terms of one fundamental function. This function was first obtained by Peters and its properties have been investigated by Peters and also by the author (5, 6). The general method of solution of the probem is similar to that developed in connexion with similar boundary-value problems occurring in diffraction theory (5).

The form of the solution obtained here is similar to that of Roseau and Peters in that it is expressed as a contour integral. The contour employed is simpler than that of the earlier solutions and two solutions may be obtained by taking the real and imaginary parts of this contour integral. In the previous methods independent solutions were obtained by taking the real part of two completely different representations.

For short waves the contour used demonstrates the existence of certain solutions which cannot be obtained, in general, from Roseau's contours.

The present approach thus gives a unified treatment of the problem for both long and short waves and avoids the separate treatment of special cases. In particular, for short waves, the solution is obtained for all values of the parameters occurring and certain combinations of these parameters do not have to be treated separately as in Roseau's work.

For short waves the form of the solution is considerably simplified for certain values of the parameters and it is shown that these special cases may be deduced immediately from certain functional relationships obtained by the author.

2. *Representation of solutions.* The problem of waves on a sloping beach may be reduced (3) to that of obtaining a solution V of

$$\frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} - k^2 V = 0, \tag{1}$$

https://doi.org/10.1017/S0305004100034964 Published online by Cambridge University Press

in the region $0 \leqslant \phi \leqslant \alpha$, all $\rho \geqslant 0$, and such that

$$\frac{\partial V}{\partial \phi} + V = 0 \quad \text{on} \quad \phi = 0 \quad \text{and} \quad \frac{\partial V}{\partial \phi} = 0 \quad \text{on} \quad \phi = \alpha.$$

The function V is to be bounded for large ρ and at every point in the sector apart from possibly $\rho = 0$. The case of long waves corresponds to $k < 1$ and short waves to $k > 1$.

By a method similar to that of (5) it may be shown that a solution of 1 is given by

$$V = \int_C z^{-1} \exp\{\tfrac{1}{2}k\rho(z+z^{-1})\}[g_1(z\exp i\phi) + g_2\{z\exp(-i\phi)\}]\,dz. \tag{2}$$

The contour C is to be such that the singularities of the integrand in (2) never cross C and the exponential in (2) is required to vanish at both end-points of C. Thus one possible contour is the curve C' lying completely in the half plane $0 \leqslant \arg z \leqslant \pi$, enclosing all the singularities of the integrand lying in this half plane, with one end-point at $z = \infty \exp i\pi$ and entering the origin tangentially to the real axis and from the left. Another possible contour is that obtained by reflecting C' in the real axis; this will be denoted by C''. Any linear combination of C' and C'' is also a possible contour for (2). The appropriate forms of C' and C'' are given in Figure 2 of (3).

We consider first the case of short waves, i.e. $k > 1$. By analysis similar to that of (5) it may be shown that the term in square brackets in (2) has the form

$$f[z\exp i\phi] + f[z\exp\{i(2\alpha - \phi)\}],$$

where

$$f(z) = \frac{(z-a)(z-b)}{(z+a)(z+b)} f(z\exp 2i\alpha), \tag{3}$$

and $a = i\exp(-i\theta)$, $b = i\exp i\theta$, where $k = 1/\cos\theta$, $0 < \theta < \frac{1}{2}\pi$.

Clearly a solution of (3) may be obtained in terms of $h(z)$, a solution of

$$h(z) = \frac{z-i}{z+i} h(z\exp 2i\alpha).$$

The form and detailed properties of this function $h(z)$ have been given in (5, 6) and a summary of the essential properties follows.

$h(z)$ is regular apart from poles on the unit circle at $\arg z = -\frac{1}{2}\pi - 2n\pi - 2r\alpha$ and $\frac{1}{2}\pi + 2(r+1)\alpha + 2n\pi$. Throughout this work r and n will refer to any positive integer or zero. $h(z)$ has zeros at $\arg z = \frac{3}{2}\pi + 2(r+1)\alpha + 2n\pi$ and $-\frac{3}{2}\pi - 2n\pi - 2r\alpha$. $h(z) \to 1$ at $|z| \to \infty$ and $h(z) \sim -iz^{\frac{1}{2}\tau}$, where $\tau = \pi/\alpha$, for small z. The following functional equations also hold

$$h(z) = -iz^{\frac{1}{2}\tau} h(z^{-1}\exp 2i\alpha), \tag{4}$$

$$h(z)\,h\{z\exp(-i\pi)\} = z^\tau/(z^\tau - \exp\tfrac{1}{2}i\pi\tau). \tag{5}$$

We now consider the solution V obtained by choosing C' as the contour of integration in (2). Clearly the solution will be bounded for large ρ if there are no poles of the integrand in $0 \leqslant \arg z < \frac{1}{2}\pi$ and we thus seek solutions of (3) satisfying this condition. Clearly a possible solution for $f(z)$ is given by

$$f_1(z) = h[z\exp(-i\theta)]\,h^{-1}[z\exp\{i(\theta - \pi)\}]. \tag{6}$$

https://doi.org/10.1017/S0305004100034964 Published online by Cambridge University Press

If $f(z)$ is set equal to $f_1(z)$ then the integrand of 2 will have poles on the unit circle at

$$\arg z = \tfrac{1}{2}\pi + 2(n+1)\pi + 2r\alpha - \phi - \theta,$$

$$\tfrac{1}{2}\pi + 2(n+1)\pi + 2(r-1)\alpha + \phi - \theta, \quad -\tfrac{1}{2}\pi - 2n\pi - 2r\alpha - \phi \pm \theta,$$

$$-\tfrac{1}{2}\pi - 2n\pi - 2(r+1)\alpha + \phi \pm \theta, \quad \tfrac{1}{2}\pi + 2n\pi + 2(r+1)\alpha + \theta - \phi, \quad \tfrac{1}{2}\pi + 2r\pi + 2r\alpha + \theta + \phi.$$

The function $f_1(z)$ thus produces a solution V which satisfies all the conditions imposed on it and this solution will be denoted by $V_1^{(0)}$. Other solutions, which will be denoted by $V_1^{(p)}$, are obtained by multiplying $f_1(z)$ by $z^{p\tau}$, where $|p|$ is an integer.

It may be easily verified that a second solution $V_2^{(0)}$ is obtained by setting $C = C''$ and $f = f_2$ in (2), where

$$f_2(z) = h(z \exp i\theta)\, h^{-1}[z \exp\{-i(\theta - \pi)\}].$$

The solutions obtained by multiplying $f_2(z)$ by $z^{p\tau}$ will be denoted by $V_2^{(p)}$.

In Roseau's work the solution V is essentially expressed as a contour integral over the curves $C' \pm C''$. Thus, in order to establish the relationship between the present work and that of Roseau, it is necessary to obtain forms of the function $f(z)$ which are valid for both C' and C''. The function $f_1(z)$ may not be used as there will be poles of the integrand at $\arg z = -\tfrac{1}{2}\pi - 2r\alpha \pm \phi + \theta$. These poles may be removed by multiplying $f_1(z)$ by a suitable factor and thus a possible form of $f(z)$ which can be used for both C' and C'' is

$$g_1(z) = f_1(z)\,[z^\tau \exp(-i\tau\theta) - \exp(-\tfrac{1}{2}i\pi\tau)].$$

A similar solution constructed from $f_2(z)$ is

$$g_2(z) = f_2(z)\,[z^\tau \exp i\tau\theta - \exp \tfrac{1}{2}i\pi\tau].$$

It can be shown from equation (5) that the ratio g_1/g_2 is a constant. The type of solution obtained by Roseau may thus be obtained by a suitable combination of the functions $V_1^{(0)}$, $V_1^{(1)}$, $V_2^{(0)}$, $V_2^{(1)}$. Other more general solutions can of course be obtained from $V_1^{(p)}$, $V_2^{(p)}$. All the solutions obtained tend to zero exponentially as $\rho \to \infty$.

3. *Properties of the solutions.* Equation (1) and the boundary conditions on V are real and thus both the real and imaginary parts of V are possible solutions of the boundary-value problem. We now consider some of the properties of these functions. These solutions, by construction, vanish at infinity and the integrands for $V_1^{(p)}$ and $V_2^{(p)}$ have no poles in the regions $0 \leqslant \arg z \leqslant \tfrac{1}{2}\pi$, $-\tfrac{1}{2}\pi \leqslant \arg z \leqslant 0$, respectively, also all the other poles are on the unit circle. There is thus no loss of generality in taking the contours C' and C'' to be such that C' maps into C'' under the transformation $z' = 1/z$. If this transformation is made in the expression for $V_1^{(p)}$ then it may be shown that $V_1^{(p)} = \exp(-2i\pi\tau)\, V_2^{(1-p)}$. Thus it is sufficient to consider the case of $V_1^{(p)}$ for arbitrary p. It is of interest to examine the behaviour of $V_1^{(p)}$ for small ρ and for this purpose it is convenient to employ the following lemma which is easily established by conventional techniques.

LEMMA. *If, for large z, $f(z) \sim Az^\lambda$, $\lambda > 0$ and, for small z, $f(z) \sim Bz^{-\mu}$, $\mu > 0$, then*

$$I = \int_C z^{-1} f(z \exp i\phi) \exp\{\tfrac{1}{2}k\rho(z + z^{-1})\}\, dz$$

https://doi.org/10.1017/S0305004100034964 Published online by Cambridge University Press

behaves as the most singular of

$$A\exp i\lambda(\phi+\pi)\,(\lambda-1)!\,(2/k\rho)^{\lambda},\quad B\exp -i\mu(\phi+\pi)\,(\mu-1)!\,(2/k\rho)^{\mu}$$

for small ρ. *If* $\lambda=\mu=0$ *then* $I\sim(A+B)\log\rho$ *for small* ρ.

This lemma shows that, for $p>0$, $V_1^{(p)}\sim C\rho^{-p\tau}$ for small ρ and that, for $p<0$, $V_1^{(p)}\sim D\rho^{p\tau}$. Also, for small ρ,

$$V_1^{(0)}\sim 2[1+\exp\{\tfrac{1}{2}i\tau(\pi-2\theta)\}]\log\rho. \tag{7}$$

Thus $V_1^{(0)}$ will have a logarithmic singularity at $\rho=0$ unless $\pi-2\theta=2m\alpha$, where m is an odd integer. Both the real and imaginary parts of $V_1^{(0)}$ represent possible solutions of our problem and thus, if $V_1^{(0)}$ does not have the form $[1+\{\exp\frac{1}{2}i\tau(\pi-2\theta)\}]\times$ (real function), we may conclude that there exists a solution of the boundary-value problem which is less singular than $\log\rho$ for small ρ. Since the integrand for $V_1^{(0)}$ is an extremely complicated function it seems reasonable, at first, to conclude that $V_1^{(0)}$ does not have this special form. However, a detailed analysis of the form of $V_1^{(0)}$ shows that it may, in fact, be written as $\exp\frac{1}{4}i\tau(\pi-2\theta)\times$ (real function). The analysis leading to this result will now be briefly sketched. The contour C' in the integral defining $V_1^{(0)}$ may be defined into the negative real axis; in this deformation some residues of the integrand may also be captured. Thus $V_1^{(0)}$ may be written as the sum of an integral (J say) along the real axis and a finite number of residue terms. These terms will be examined separately. We have that

$$J=\int_0^{\infty e^{i\pi}}\left\{\frac{h[z\exp\{i(\phi-\theta)\}]}{\overline{h[z\exp\{i(\phi+\theta-\pi)\}]}}+\frac{h[z\exp\{i(2\alpha-\phi-\theta)\}]}{\overline{h[z\exp\{i(2\alpha+\theta-\phi-\pi)\}]}}\right\}\exp\{\tfrac{1}{2}k\rho(z+z^{-1})\frac{dz}{z}. \tag{8}$$

If, in (8), z in the second term in curly brackets is replaced by $e^{2\pi i}/z$ then from (4) and (5) it can be shown that

$$J=\{1+\exp[\tfrac{1}{2}i\tau(\pi-2\theta)]\}\int_0^{\infty e^{i\pi}}\frac{h[z\exp\{i(\phi-\theta)\}]\,[z^{\tau}-\exp\{i\tau(\pi-\phi)\}]}{\overline{h[z\exp\{i(\theta+\phi-\pi)\}]}\,\overline{[z^{\tau}-\exp\{i\tau(\frac{3}{2}\pi-\phi-\theta)\}]}}\times\exp\{\tfrac{1}{2}k\rho(z+z^{-1})\}\frac{dz}{z}. \tag{9}$$

It is now a relatively simple matter to show that the integral in (9) is purely real; to complete the proof it is necessary to employ the identity (1)

$$\overline{h(z)}=h(\bar{z}\,e^{2i\alpha}), \tag{10}$$

where the bar denotes the complex conjugate.

The analysis of the residue terms is somewhat simpler and it is easily shown from (4), (5) and (10) that these terms have the form $\exp\frac{1}{4}i\tau(\pi-2\theta)\times$ (real function). Unlike J, however, the residue terms do not vanish for $\pi-2\theta=2$ (odd integer) α.

It is thus seen that $V_1^{(0)}$ does not supply a solution, valid for all θ, α which is bounded for small ρ. Such solutions are only obtained for $\pi-2\theta=2$ (odd integer) α, and the above analysis shows that for θ, α satisfying this identity the solution has a closed form. This is in general agreement with the results of Roseau (3). For all other values

https://doi.org/10.1017/S0305004100034964 Published online by Cambridge University Press

of the parameters equation (7) shows that $V_1^{(0)}$ has a logarithmic singularity at the origin. Roseau has obtained this type of solution except when $\alpha = \frac{1}{2}\pi$ (integer) and $3\pi - 2\theta = 2$ (odd integer) α. Thus the present approach yields solutions which have not been obtained by Roseau. This is essentially due to the fact that Roseau employs the contours $C' \pm C''$. In this analysis Roseau treats the case of $\pi - 2\theta = 2$ (integer) α as a special case, whereas the present approach presents a treatment valid for all values of the parameters and does not require a knowledge of the particular form taken by $h(z)$ for certain values of the parameters.

It is of interest from the point of view of the present work to examine briefly the circumstances which enable Roseau to treat $\pi - 2\theta = 2$ (integer) α as a special case. In his representation Roseau employs the curves $C' \pm C''$ and thus has to employ terms of the form $g_1(z)$ as use of the function $f_1(z)$ would yield inadmissible poles at $\arg z = -\frac{1}{2}\pi - 2r\alpha + \theta$. For $\pi - 2\theta = 2m\alpha$ these poles cancel with zeros at

$$\arg z = \tfrac{1}{2}\pi - 2r\alpha - \theta.$$

Thus for this particular case the function $f_1(z)$ may be used for $C' \pm C''$. This is in fact what Roseau has done in treating $\pi - 2\theta = 2$ (integer) α as a special case and his solution is a combination of $V_1^{(0)}$, $V_2^{(0)}$. For other values of θ and α Roseau has employed $g_1(z)$.

In view of the complicated form of $h(z)$ the functions appearing in the integrand of (2) are themselves fairly complicated. For the particular case of $\alpha = p\pi/2q$ relatively prime integers with p odd, the function $h(z)$ reduces to a ratio of finite products (5). The function $f_1(z)$ is of the form $h(z)\, h^{-1}[z \exp\{i(2\theta - \pi)\}]$ and thus, from (5) and the functional equation defining $h(z)$, $f_1(z)$ will have a simple form for $(2r+1)\pi - 2\theta = 2n\alpha$, where n and r are integers. Some of these special cases have been considered by Roseau.

4. *Solution for long waves.* We now consider the case of long waves, i.e. $k < 1$. In this case equation (3) will again hold if a, b are set equal to ir_1, ir_2, where

$$r_1, r_2 = \{1 \pm (1 - k^2)^{\frac{1}{2}}\}/k.$$

The corresponding solution of (3) which is equal to $f_1(z)$ for $k = 1$ is

$$H_1(z) = h(z/r_1)\, h^{-1}\{z \exp(-i\pi/r_2)\}.$$

It may be verified that, if $f(z)$ is set equal to $H_1(z)$, the corresponding integrand of (2) will have no poles in the region $-\frac{1}{2}\pi < \arg z < \frac{1}{2}\pi$ and that all the other poles of the integrand are within the circle of radius r_1, with its centre at the origin. Thus, if C' and C'' are such that they contain this circle, a solution V of the boundary-value problem is obtained by setting $C = C' \pm C''$ and $f(z) = H_1(z)$. Our lemma now shows that one solution will exist which is regular at $\rho = 0$ and another which has a logarithmic singularity at the origin. In the case of long waves these solutions can be generated by representations over the curves $C' \pm C''$ whereas, for short waves, only one of the curves C', C'' may be employed. This is the essential difference between the problem for long and short waves and it arises from the fact that for r_1, r_2 complex ($k > 1$) it is not possible to find a solution of (3) bounded for large and small values of

https://doi.org/10.1017/S0305004100034964 Published online by Cambridge University Press

z and such that the integrand of (2) has no poles in $-\frac{1}{2}\pi < \arg z < \frac{1}{2}\pi$. For $k < 1$, however, several such functions can be obtained, for example, $H_1(z)$. For $\phi = 0$ there will be poles of (2) on the imaginary axis but this does not violate the condition imposed on V at infinity. Amongst other valid solutions of (3) is $h(z/r_1)\,h(z/r_2)$, which is the solution employed by Peters (1); from (5) the relationship between this function and $H_1(z)$ may be obtained. Further solutions may be obtained by multiplying $H_1(z)$ by $z^{p\tau}$ and relations between the solutions obtained by using $C' + C''$ and those obtained from $C' - C''$ may be obtained by making the transformation $z' = 1/z$ and employing (4). Some relationships of this type have been given by Roseau (3, 4).

5. *Conclusions.* A simple direct method of obtaining solutions of equation (1) for all values of k has been presented. The method of solution is such that it is valid for all values of the parameters of the problem and particular values of the parameters do not require to be treated separately. Solutions of (1), logarithmic at the origin, have been obtained for all values of the parameters other than $\pi - 2\theta = 2$ (odd integer) α. Similar solutions have been obtained by Roseau but he did not obtain these solutions for $\alpha = \frac{1}{2}\pi$ (integer) and $3\pi - 2\theta = 2$ (odd integer) α. The present approach also yields solutions which have algebraic singularities at the origin.

REFERENCES

(1) Peters, A. S. *Comm. Pure Applied Maths.* 5 (1952), 87–108.
(2) Roseau, M. *C.R. Acad. Sci., Paris*, 232 (1951), 479–81.
(3) Roseau, M. *Comm. Pure Applied Maths.* 11 (1958), 433–93.
(4) Roseau, M. *J. Math.* 38 (1959), 25–61.
(5) Williams, W. E. *Proc. Roy. Soc.* A, 252 (1959), 376–94.
(6) Williams, W. E. To appear in *Proc. Camb. Phil. Soc.*

Department of Applied Mathematics
University of Liverpool

https://doi.org/10.1017/S0305004100034964 Published online by Cambridge University Press