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Green's function for a general system of hyperbolic-parabolic balance laws with applications

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We consider a general system of hyperbolic-parabolic balance laws, which contains both second-order dissipation represented by a viscosity matrix and lower-order damping or relaxation. As dictated by physics, the viscosity matrix and the Jacobian matrix of the lower-order term are usually rank deficient. We study the Green's function of its linearization around a constant equilibrium state. The Green's function is constructed using heat kernels along equilibrium characteristic directions and delta functions along other predetermined directions. Detailed error estimates are given in the space-time pointwise sense. Our main result is then applied to two different models of polyatomic gas flows. The first one is a model of translational and vibrational non-equilibrium flows with a general setting in the fundamental thermodynamic relations. The second one is a two-temperature Navier–Stokes type model for polyatomic gas flows, proposed by Aoki et al., with specific equations of state. For each model, we give an explicit formulation of Green's function for the Cauchy problem.

Keywords: Cauchy problem; Green's function; hyperbolic-parabolic balance laws; polyatomic gas flows; space-time pointwise estimates; thermal non-equilibrium flows

2020 *Mathematics Subject Classification:* 35Q35; 35E05; 35B40; 76N15

1. Introduction

We consider a general system of hyperbolic-parabolic balance laws,

$$w_t + f(w)_x = [B(w)w_x]_x + r(w), \quad x \in \mathbb{R},\ t \in \mathbb{R}^+, \tag{1.1}$$

where $w, f, r \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n\times n}$. Here, $w = w(x,t)$ is the unknown density function with components representing mass density, momentum density, etc.; f is the flux function; and r represents external forces, relaxation, chemical reaction,

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and so forth. The matrix B is known as a viscosity matrix. It is for viscosity, heat conduction, species diffusion, etc. We are interested in the Cauchy problem of (1.1) with the initial condition

$$w(x,0) = w_0(x), \qquad x \in \mathbb{R}. \tag{1.2}$$

Many interesting physical models take the form (1.1). Usually, the flux function satisfies an entropy condition so that the corresponding inviscid form

$$w_t + f(w)_x = 0 \tag{1.3}$$

is hyperbolic [5]. On the other hand, the viscosity matrix is rank deficient as dictated by physics. Examples are those from continuum mechanics with a continuity equation. Thus, (1.1) is hyperbolic-parabolic rather than uniformly parabolic. The appearance of the lower-order term makes (1.1) a system of balance laws rather than conservation laws. In many physical applications, the Jacobian matrix of the lower-order term is also rank deficient. An appropriate set of structural conditions needs to be imposed on (1.1) so that there is sufficient coupling among its equations to have dissipation of solutions.

An important example of (1.1) is polyatomic gas flows in both translational and vibrational non-equilibrium [4, 21],

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = (\mu u_x)_x, \\ (\rho E)_t + (\rho E u + p u)_x = (\mu u u_x + \kappa T_{1x} + \nu \rho e_{2x})_x, \\ (\rho e_2)_t + (\rho e_2 u)_x = (\nu \rho e_{2x})_x + \rho \frac{e_2^* - e_2}{\tau}. \end{cases} \tag{1.4}$$

Here, ρ, u and p are the density, velocity and pressure, respectively. The total specific energy is

$$E = e + \frac{1}{2}u^2, \qquad e = e_1 + e_2, \tag{1.5}$$

where the internal energy e consists of two parts: e_2 is the non-equilibrium vibrational energy, and e_1 is the rest of the internal energy.

The dissipation parameters are the viscosity coefficient μ, the thermal conductivity κ, and the self-diffusion coefficient ν. They are introduced to compensate for the non-equilibrium in the translational mode through the Chapman–Enskog expansion. On the other hand, the vibrational mode is singled out as the non-equilibrium mode. Such a setup is based on the fact that the relaxations from translational non-equilibrium and from vibrational non-equilibrium to their respective local equilibrium states are on very different time scales.

Therefore, we need two sets of thermodynamic variables. One is for the translational mode and for all other internal modes that are at the same pace with the translational mode, and the other is for the vibrational mode exclusively. We use subscript '1' for the former and '2' for the latter, like e_1 and e_2 introduced above. Thus, T_1 in (1.4) stands for the translational temperature. In (1.4), we use e_2^* for the local equilibrium value of e_2 and τ for the vibrational relaxation time scale. Both e_2^* and τ are known functions of the thermodynamic variables of set one.

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The two sets of thermodynamic variables obey two different thermodynamic equations:

$$T_1 ds_1 = de_1 + pdv, \qquad v = 1/\rho, \tag{1.6}$$

$$T_2 ds_2 = de_2. \tag{1.7}$$

Here, v is the specific volume, T_2 is the vibrational temperature, and s_1 and s_2 are the translational and vibrational entropies, respectively. The total entropy is

$$s = s_1 + s_2. \tag{1.8}$$

From (1.6) and (1.7), two of the thermodynamic variables in set one are independent, and one in set two is independent. Therefore, (1.4) is a closed system of those variables and u.

Recently, Aoki et al. proposed a two-temperature Navier–Stokes type model for polyatomic gas flows [22]. The system is derived from the polyatomic version of the ellipsoidal-statistical model rather than the original Boltzmann equation. In one space dimension, the model reads

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + \rho R T_1)_x = (\mu_{\rm tr} u_x)_x, \\ [\rho(\frac{3+D}{2} RT + \frac{1}{2}u^2)]_t + [\rho u(\frac{3+D}{2} RT + RT_1 + \frac{1}{2}u^2)]_x = [\frac{2}{5}\lambda_{\rm tr}(\frac{3+D}{2}T_x + T_{1x}) + \mu_{\rm tr} u u_x]_x, \\ (\rho T_2)_t + (\rho u T_2)_x = \frac{2}{DR}(\lambda_{\rm int} T_{2x})_x + \theta A_c(T)\rho^2 (T - T_2). \end{cases} \tag{1.9}$$

Here, ρ and u are the density and velocity, respectively, as in (1.4). Meanwhile, T_1 and T_2 are the temperatures of the translational mode and internal modes, respectively. R is the gas constant per unit mass, while $D \geq 2$ is a constant for the number of internal degrees of freedom of gas molecules. The temperature T is

$$T = \frac{3T_1 + DT_2}{3 + D}. \tag{1.10}$$

The viscosity coefficient and the thermal conductivities for the translational temperature and for the internal temperature are

$$\mu_{\rm tr} = \mu_{\rm tr}(T, T_1), \quad \lambda_{\rm tr} = \lambda_{\rm tr}(T, T_1), \quad \lambda_{\rm int} = \lambda_{\rm int}(T, T_1) = \frac{D}{5}\lambda_{\rm tr}, \tag{1.11}$$

respectively. Also, $\theta \in (0, 1]$ is a constant parameter, and $A_c(T)$ is a function of T such that $A_c(T)\rho$ is the collision frequency of the gas molecules.

The setup of (1.9) is different from that of (1.4). Here, the translational mode is separated from D (at least two) relaxing internal modes. The pressure and the internal energy are

$$p = \rho R T_1, \quad e = \frac{3 + D}{2} RT = \frac{3}{2} RT_1 + \frac{D}{2} RT_2 \equiv e_1 + e_2, \tag{1.12}$$

respectively. Equation (1.9) is a system of four equations for four unknowns ρ, u, T_1 and T_2, while (1.12) provides related thermodynamic variables.

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Both models, (1.4) and (1.9), are examples of the general system (1.1). In fact, as discussed in Section 5, (1.4) and (1.9) are formally very similar. Our main results in this paper apply to both of them.

There are two special cases of (1.1). The first one is $r = 0$ but $B \neq 0$. It is a system of hyperbolic-parabolic conservation laws,

$$w_t + f(w)_x = [B(w)w_x]_x. \tag{1.13}$$

Structural conditions for (1.13) are proposed in [9, 11, 19]. Under those conditions, global well-posedness of (1.13), (1.2) is established for small data solutions [9]. L^2 time asymptotic behaviour of solutions is studied in [10]. Pointwise space-time asymptotic behaviour is studied in [14]. As a direct consequence, results in L^p spaces, $1 \leq p \leq \infty$, are obtained. The innovative approach in [14] is a combination of Green's function estimates for the corresponding linear system and estimates on wave interaction. The approach is extended to other types of problems afterwards [7, 16, 17, 24, 26].

The second special case of (1.1) is $B = 0$ but $r \neq 0$. This is a system of hyperbolic balance laws,

$$w_t + f(w)_x = r(w). \tag{1.14}$$

There is an extensive literature about (1.14) on structural conditions and global well-posedness, see [3, 6, 12, 23] and references therein. In particular, a set of structural conditions is proposed in [29], which simplifies the set proposed in [12] and is parallel to those conditions developed for (1.13) in [9, 11, 19]. Green's function estimates are obtained in [1, 26]. Pointwise space-time asymptotic behaviour is studied in [24], also see [28].

For the more general system (1.1), rigorous mathematical treatments are much more recent. Although its counterparts in multi-space dimensions, with space dimensions at least three, have been studied in [9], the technique used there heavily depends on the fast decay of solutions in higher space dimensions and does not apply to one or two space dimensions. A set of structural conditions for (1.1) is proposed in [31]. It is a natural extension of the classical works in [9, 11, 19] for (1.13). It reduces to those conditions in the special case (1.13), and to the set proposed in [29] in the case of (1.14) as well. The set is proposed for all space dimensions and applicable to the physical model of polyatomic gas flows in both translational and vibrational non-equilibrium.

Under the structural conditions proposed in [31], a general L^2 theory for small, classical solutions to (1.1), (1.2) and to its multi-dimensional counterparts has been developed in recent years. This includes global existence via the energy method in [31] for all space dimensions, and L^p ($p \geq 2$) time decay in [32] for multi-space dimensions and in [34] for one space dimension. Time asymptotic behaviour of solutions in multi-space dimensions is studied in [33].

The result of time asymptotic behaviour for multi-space dimensions obtained in [33] does not apply to one space dimension. This is because plane wave solutions have less space to diffuse and hence the time decay is slower. The asymptotic behaviour of solutions has an intrinsically different picture in one space dimension than in multi-space dimensions, where the phenomenon is basically linear. In one

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space dimension, by contrast, an asymptotic solution of (1.1), (1.2) necessarily includes Burgers waves if the reduced characteristic fields are genuinely nonlinear.

A roadmap for studying the solution behaviour of (1.1), (1.2) in depth is as follows. One first considers the linearized system of (1.1) around a constant equilibrium state $\bar{w}$, with $r(\bar{w}) = 0$. It reads

$$w_t + Aw_x = Bw_{xx} + Lw, \tag{1.15}$$

where

$$A = f'(\bar{w}), \quad B = B(\bar{w}), \quad L = r'(\bar{w}). \tag{1.16}$$

The goal of this part is to obtain detailed estimates on the Green's function. It is done under the linear version of the structural conditions proposed in [31]. One first finds the Green's function in the Fourier space as the solution matrix of an ODE system. Then estimates on the inverse transform are carried out to obtain a formulation of the Green's function in the physical space. The formulation is explicit in leading terms, using coefficients of (1.15), and precise and detailed in higher-order terms.

The second part of studying the solution behaviour is to write (1.1) as a linear system with nonlinear source terms. Based on properties of the flux function f and information contained in the Green's function estimates, one constructs an asymptotic component for each characteristic field. After extracting the explicit, predetermined asymptotic solution, one presents the remainder using Duhamel's principle. Estimation on wave interaction then follows. The expected results are space-time estimates on the remainder, which further imply L^p, $1 \leq p \leq \infty$, convergence rates of the remainder.

While the above roadmap is along the lines of the special cases (1.13) and (1.14), see [14, 24], the analysis is much more complicated here due to the intricate interwinding of the matrices A, B and L in the Green's function of (1.15). The involvement of f in the construction of an asymptotic solution in the second part of the roadmap adds additional difficulties. Due to the great complexity, we focus on obtaining Green's function estimates for (1.15) in the current paper. Using the results obtained here, we are able to construct a time asymptotic solution to the Cauchy problem of the polyatomic gas flow model (1.4) and study solution behaviour at large time in a recent paper [35]. The extension of the study of asymptotic behaviour to the general system (1.1), (1.2), however, is left to future work.

The study of Green's function estimates in the pointwise sense has its own significance in the PDE theory. The original approach developed in [14, 15, 25] for (1.13) has been extended to apply to other problems [7, 17, 18, 20, 26, 27]. In particular, using rotational invariance, the approach can be extended to study problems in three space dimensions, see [13, 30] and references therein.

In the current paper, we study Green's function estimates in the space-time pointwise sense for (1.15) under reasonable and physical assumptions, with applications to the polyatomic gas flow models (1.4) and (1.9). In the special cases when one of the matrices L and B becomes zero, our results reduce to those obtained in [14] for hyperbolic-parabolic conservation laws and in [26] for hyperbolic balance laws.

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Our focus in the current study is on the new challenge of the intertwining of the three coefficient matrices A, B and L. These matrices together with their coupling determine various parameters in the leading terms of the Green's function. The parameters are to be found employing ideas from Kato's perturbation theory [8] and from Chapman–Enskog expansions.

The plan of the paper is as follows. In Section 2, we set up our problem, impose hypotheses, and state main results. In Section 3, we consider the Green's function in the Fourier space and carry out the corresponding spectral analysis. Parameters in the leading terms are obtained in this section. In Section 4, we study the inverse Fourier transform and prove the main results. In Section 5, we apply our results to the polyatomic gas flow models (1.4) and (1.9). Finally, in the Appendix we list some alternative steps in finding parameters in the leading terms of the Green's function.

2. Main results

In this section, we consider the general linear system (1.15),

$$w_t + Aw_x = Bw_{xx} + Lw, \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \tag{2.1}$$

where $w = w(x,t)$ is an n-vector, and $A, B, L \in \mathbb{R}^{n\times n}$ are constant matrices. As dictated by physics, B and L usually are rank deficient. We are interested in the Green's function for the Cauchy problem of (2.1).

For this, we impose the following structural conditions.

Assumption 2.1. (i) *There exists a symmetric, positive definite matrix A_0, such that A_0A is symmetric, A_0B is symmetric, positive semi-definite, and A_0L is symmetric, negative semi-definite.*

(ii) *There are n_1 conservation laws in (2.1). That is, there is a partition $n = n_1 + n_2$, $n_1, n_2 \geq 0$, such that*

$$L = \begin{pmatrix} 0_{n_1\times n_1} & 0_{n_1\times n_2} \\ L_{21} & L_{22} \end{pmatrix}, \tag{2.2}$$

where $L_{21} \in \mathbb{R}^{n_2\times n_1}$ and $L_{22} \in \mathbb{R}^{n_2\times n_2}$ is nonsingular (if $n_2 > 0$).

(iii) *Let $\mathbb{N}_1$ be the null space of B and $\mathbb{N}_2$ be the null space of L. Then $\mathbb{N}_1 \cap \mathbb{N}_2$ contains no eigenvectors of A.*

Remark 2.2. In Assumption 2.1, the first hypothesis is for the symmetrizability of (2.1). This is consistent with the entropy condition for a system of nonlinear hyperbolic conservation laws [5], which is extended to nonlinear hyperbolic-parabolic conservation laws [9] and hyperbolic-parabolic balance laws [31]. The second hypothesis is to set the number of conservation laws in (2.1) to n_1 definitely. It is satisfied by many physical models including the linearizations of (1.4) and (1.9). The third hypothesis is known as the Kawashima–Shizuta condition [19]. It implies the full coupling of A with B and L in (2.1), which in turn gives rise to the full dissipation of solutions as seen in the Green's function estimates.

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We now define the Green's function for the Cauchy problem of (2.1). It is the $n \times n$ matrix $G(x,t)$ that satisfies

$$\begin{cases} G_t + AG_x = BG_{xx} + LG, \\ G(x,0) = \delta(x)I, \end{cases} \tag{2.3}$$

where δ is the Dirac δ-function, and I is the $n \times n$ identity.

For convenience we introduce the following notation for a heat kernel,

$$H(x,t;\lambda,\mu) = \frac{1}{\sqrt{4\pi\mu t}} e^{-\frac{(x-\lambda t)^2}{4\mu t}}, \tag{2.4}$$

where $\mu > 0$ and λ are constants.

THEOREM 2.3. *Let Assumption 2.1 be satisfied and $l \geq 0$ be an integer. For $x \in \mathbb{R}$ and $t > 0$, the Green's function G of the Cauchy problem of (2.1) satisfies*

$$\begin{aligned} \frac{\partial^l}{\partial x^l} G(x,t) = {} & \frac{\partial^l}{\partial x^l} \sum_{j=1}^{m} H(x,t;\lambda_j^{(r)},\mu_j) P_j + O(1)(t+1)^{-\frac{1}{2}} t^{-\frac{l}{2}} \sum_{j=1}^{m} H(x,t;\lambda_j^{(r)},C) \\ & + \sum_{j=1}^{m'} e^{-\beta_j t} \sum_{k=0}^{l} \delta^{(l-k)}(x-\alpha_j t) Q_{j,k}(t). \end{aligned} \tag{2.5}$$

Here, m and m' are nonnegative integers; $\lambda_j^{(r)}$, $1 \leq j \leq m$, and α_j, $1 \leq j \leq m'$, are constants; μ_j, $1 \leq j \leq m$, and β_j, $1 \leq j \leq m'$, are positive constants; P_j, $1 \leq j \leq m$, are constant projections; $Q_{j,k}(t)$, $1 \leq j \leq m'$, $0 \leq k \leq l$, are $n \times n$ matrix-valued polynomials in t with degrees not larger than k; $C > 0$ is a constant; and δ is the Dirac δ-function. The parameters $m, m', \lambda_j^{(r)}, \alpha_j, \mu_j, \beta_j, P_j$ and $Q_{j,0}$ in (2.5) are determined from the coefficient matrices A, B and L through Procedure 2.9 below and have explicit formulation.

REMARK 2.4. In (2.5), there is a singularity $t^{-\frac{l+1}{2}}$ at $t = 0$. Such a singularity is intrinsic if $B \neq 0$. (One can think of the heat equation.) On the other hand, if $B = 0$, i.e., (2.1) is hyperbolic, the singularity is artificial. In this case, (2.5) has an alternative that explicitly excludes the singularity as follows.

Let $\chi(\mathcal{A})$ be the characteristic function of a set $\mathcal{A}$ and sinc be the unnormalized sinc function,

$$\operatorname{sinc} x = \frac{\sin x}{x}.$$

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For a pair of large but fixed constants $K, N > 0$, if $B = 0$ in (2.1) then (2.5) can be replaced by

$$\frac{\partial^l}{\partial x^l} G(x,t) = \chi(|x| \le Kt) \frac{N}{\pi} \operatorname{sinc}(Nx) * \left[\frac{\partial^l}{\partial x^l} \sum_{j=1}^{m} H(x,t;\lambda_j^{(r)}, \mu_j) P_j \right] + O(1)(t+1)^{-\frac{l+2}{2}} \sum_{j=1}^{m} e^{-\frac{(x-\lambda_j^{(r)} t)^2}{Ct}} + \sum_{j=1}^{m'} e^{-\beta_j t} \sum_{k=0}^{l} \delta^{(l-k)}(x - \alpha_j t) Q_{j,k}(t) \tag{2.6}$$

for $x \in \mathbb{R}$ and $t > 0$. Here, the notation '$*$' means convolution.

We notice that in the first term of (2.6), the singularity $t^{-\frac{l+1}{2}}$ from derivatives of heat kernels is eliminated through the convolution with the sinc function. This can be seen by integration by parts and properties of the sinc function and heat kernels.

In the case $B = 0$, while (2.6) is more accurate, (2.5) is much easier to apply since every term on the right-hand side is locally defined. At the same time, in the application to nonlinear systems through Duhamel's principle, the singularity $t^{-\frac{l+1}{2}}$ in (2.5) can be bypassed via integration by parts [24, 26, 27]. More specifically, the l-th derivative of G is used to estimate $\partial^l w / \partial x^l$. From PDE theory, the initial data w_0 is necessary to have at least the same regularity. Thus, the contribution from initial data can be expressed as convolutions $G * \frac{\partial^l}{\partial x^l} w_0$ and $\frac{\partial^l}{\partial x^l} G * w_0$ through integration by parts. We then divide the temporal domain $t > 0$ as two parts, $(0,1] \cup [1,\infty)$. For $t \le 1$ we use $G * \frac{\partial^l}{\partial x^l} w_0$ since time-decay rates are not a concern, $O(1) = O(1)(t+1)^{-l/2}$. On the other hand, for $t \ge 1$ we use $\frac{\partial^l}{\partial x^l} G * w_0$ to obtain the wanted time decay, noting $t^{-l/2} = O(1)(t+1)^{-l/2}$. A similar technique is used to handle the contribution from nonlinear source terms. When linearizing (1.1), nonlinear source terms inherit the same regularity from w.

Remark 2.5. The first term on the right-hand side of (2.5) is the leading term and represents the contribution from the equilibrium system. This can be seen by considering the system

$$w_t + A w_x = \varepsilon B w_{xx} + \frac{1}{\varepsilon} L w \tag{2.7}$$

and its Green's function $G(x,t;\varepsilon)$. By a rescaling one has $G(x,t;\varepsilon) = \frac{1}{\varepsilon} G(x/\varepsilon, t/\varepsilon; 1)$. Now let $\varepsilon \to 0^+$. The second and third terms on the right-hand side of (2.5) (with $l = 0$) disappear while the first term survives and its heat kernels become δ-functions. In this case, (2.7) is reduced to the inviscid, equilibrium system of (2.1) and $G(x,t;\varepsilon)$ approaches to its Green's function.

The formulation of parameters in (2.5) is an application of a reduction process in Kato's perturbation theory [8]. It has an intricate intertwining of the three matrices A, B and L. To make our formulation more manageable, we first introduce new variables to reduce L in (2.2) to a block diagonal structure.

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For our convenience, we introduce notations as follows for a partition of a matrix $Z \in \mathbb{R}^{n\times n}$ according to (ii) of Assumption 2.1. We write

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}, \tag{2.8}$$

where $Z_{11} \in \mathbb{R}^{n_1\times n_1}$, $Z_{12} \in \mathbb{R}^{n_1\times n_2}$, $Z_{21} \in \mathbb{R}^{n_2\times n_1}$ and $Z_{22} \in \mathbb{R}^{n_2\times n_2}$.

We introduce new variables also according to (ii) of Assumption 2.1. Let

$$\begin{aligned} w &= \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \qquad w_1 \in \mathbb{R}^{n_1},\ w_2 \in \mathbb{R}^{n_2}; \\ \tilde{w} &= \begin{pmatrix} w_1 \\ L_{21}w_1 + L_{22}w_2 \end{pmatrix} \equiv \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix}. \end{aligned} \tag{2.9}$$

Here, we have used the partition (2.8) for L.

From (ii) of Assumption 2.1, L_{22} is invertible. We thus have

$$\tilde{w}_w = \begin{pmatrix} I_{n_1\times n_1} & 0_{n_1\times n_2} \\ L_{21} & L_{22} \end{pmatrix}, \qquad w_{\tilde{w}} = \tilde{w}_w^{-1} = \begin{pmatrix} I_{n_1\times n_1} & 0_{n_1\times n_2} \\ -L_{22}^{-1}L_{21} & L_{22}^{-1} \end{pmatrix}. \tag{2.10}$$

Noting $\tilde{w} = \tilde{w}_w w$, from (2.1), we have the equation for $\tilde{w}$ as follows,

$$\tilde{w}_t + \tilde{A}\tilde{w}_x = \tilde{B}\tilde{w}_{xx} + \tilde{L}\tilde{w}, \tag{2.11}$$

where

$$\tilde{A} = \tilde{w}_w A w_{\tilde{w}}, \quad \tilde{B} = \tilde{w}_w B w_{\tilde{w}}, \quad \tilde{L} = \tilde{w}_w L w_{\tilde{w}}. \tag{2.12}$$

Let

$$\tilde{A}_0 = w_{\tilde{w}}^t A_0 w_{\tilde{w}}. \tag{2.13}$$

Then

$$\tilde{A}_0\tilde{A} = w_{\tilde{w}}^t A_0 A w_{\tilde{w}}, \qquad \tilde{A}_0\tilde{B} = w_{\tilde{w}}^t A_0 B w_{\tilde{w}}, \qquad \tilde{A}_0\tilde{L} = w_{\tilde{w}}^t A_0 L w_{\tilde{w}}. \tag{2.14}$$

Thus, under Assumption 2.1, $\tilde{A}_0$ is symmetric, positive definite; $\tilde{A}_0\tilde{A}$ is symmetric; $\tilde{A}_0\tilde{B}$ is symmetric, positive semi-definite; and $\tilde{A}_0\tilde{L}$ is symmetric, negative semi-definite. One can show that the intersection of null spaces of $\tilde{B}$ and $\tilde{L}$ contains no eigenvectors of $\tilde{A}$. On the other hand, (ii) in Assumption 2.1 implies

$$\tilde{L} = \tilde{w}_w L w_{\tilde{w}} = \begin{pmatrix} 0_{n_1\times n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & L_{22} \end{pmatrix}$$

by (2.10) and (2.12). Therefore, in the new system (2.11) for the new variable $\tilde{w}$, all coefficient matrices satisfy Assumption 2.1, with a simplified, block-diagonal form of $\tilde{L}$.

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The block-diagonal form of $\tilde{L}$ greatly simplifies the formulation of parameters in our main estimate (2.5). At the same time, the Green's function $\tilde{G}$ for (2.11) is related to G for (2.1) by

$$\tilde{G}(x,t) = \tilde{w}_w G(x,t) w_{\tilde{w}}, \tag{2.15}$$

and hence G can be easily recovered from $\tilde{G}$. Therefore, we drop the tilde accent, and consider (2.1) with a simplified L.

Assumption 2.1'. (i) *There exists a symmetric, positive definite matrix A_0, such that A_0A is symmetric, A_0B is symmetric, positive semi-definite, and A_0L is symmetric, negative semi-definite.*

(ii) *The matrix L has a block-diagonal form*

$$L = \mathrm{diag}(0_{n_1\times n_1}, L_{22}), \tag{2.16}$$

where $L_{22} \in \mathbb{R}^{n_2\times n_2}$ is invertible.

(iii) *Let $\mathbb{N}_1$ be the null space of B and $\mathbb{N}_2$ be the null space of L. Then $\mathbb{N}_1 \cap \mathbb{N}_2$ contains no eigenvectors of A.*

Next, we cite some known results under Assumption 2.1'.

Lemma 2.6. ([24]) *Under* (i) *and* (ii) *of Assumption 2.1', A_0 is block-diagonal in the partition of* (2.8)*,*

$$A_0 = \mathrm{diag}(A_{11}^0, A_{22}^0). \tag{2.17}$$

Lemma 2.7. *Under* (i) *of Assumption 2.1', the matrices A, B and L are diagonalizable with real eigenvalues. The eigenvalues of B are nonnegative, and the eigenvalues of L are nonpositive.*

Lemma 2.6 is easily verified by comparing A_0L and L^tA_0, using (2.16). For Lemma 2.7, we observe that A is similar to

$$A_0^{\frac{1}{2}} A (A_0^{\frac{1}{2}})^{-1} = (A_0^{\frac{1}{2}})^{-1}(A_0A)(A_0^{\frac{1}{2}})^{-1}, \tag{2.18}$$

where $A_0^{\frac{1}{2}}$ is a square root of A_0, chosen as symmetric, positive definite. The right-hand side of (2.18) is symmetric and hence diagonalizable with real eigenvalues under (i) of Assumption 2.1'. We can discuss B and L similarly.

Let

$$A^{(r)} = A_{11} \in \mathbb{R}^{n_1\times n_1}, \tag{2.19}$$

which is the part of A related to the reduced system (the equilibrium system). From (2.17), we have

$$A_0A = \begin{pmatrix} A_{11}^0 A^{(r)} & A_{11}^0 A_{12} \\ A_{22}^0 A_{21} & A_{22}^0 A_{22} \end{pmatrix}.$$

Under (i) of Assumption 2.1', A_0A is symmetric, $A_{11}^0A^{(r)}$ is also symmetric. Noting A_0 is symmetric, positive definite, so is A_{11}^0. Thus, following the discussion above for Lemma 2.7, we have

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LEMMA 2.8. *Under* (i) *of Assumption 2.1', $A^{(r)}$ is diagonalizable with real eigenvalues.*

Now we formulate the parameters in our main estimate (2.5) under Assumption 2.1'. For simplicity, we assume that all eigenvalues of $A^{(r)}$ are simple in the following procedure, and postpone the formulation related to multiple eigenvalues to Appendix A.

PROCEDURE 2.9. The parameters in *(2.5)* are evaluated as follows.

Step 1. Assume that all eigenvalues of $A^{(r)}$ are simple,

$$\lambda_1^{(r)} < \lambda_2^{(r)} < \cdots < \lambda_{n_1}^{(r)}.$$

These are $\lambda_j^{(r)}$ in *(2.5)*, and $m = n_1$ therein. Let the normalized left eigenvector (row vector) and right eigenvector of $A^{(r)}$ corresponding to $\lambda_j^{(r)}$ be $l_j^{(r)}$ and $r_j^{(r)}$, respectively,

$$A^{(r)} r_j^{(r)} = \lambda_j^{(r)} r_j^{(r)}, \qquad l_j^{(r)} A^{(r)} = \lambda_j^{(r)} l_j^{(r)}, \qquad 1 \le j \le n_1,$$
$$\begin{pmatrix} l_1^{(r)} \\ \vdots \\ l_{n_1}^{(r)} \end{pmatrix} \begin{pmatrix} r_1^{(r)} & \cdots & {}_{n_1}^{(r)} \end{pmatrix} = I_{n_1 \times n_1}.$$

Then corresponding to each $\lambda_j^{(r)}$, P_j in *(2.5)* is given by

$$P_j = \operatorname{diag}\left(r_j^{(r)} l_j^{(r)}, 0_{n_2 \times n_2}\right), \qquad 1 \le j \le n_1. \tag{2.20}$$

Step 2. From Lemma 2.7, L has spectral decomposition

$$L = \sum_{j=1}^{s} -\sigma_j Q_j, \tag{2.21}$$

with $\sigma_j > 0$, $1 \le j \le s$. Here, each $-\sigma_j$ is a negative eigenvalue of L, with the corresponding eigenprojection Q_j. Let Q_0 be the eigenprojection corresponding to the eigenvalue zero,

$$Q_0 = I - \sum_{j=1}^{s} Q_j = \operatorname{diag}(I_{n_1 \times n_1}, 0_{n_2 \times n_2}). \tag{2.22}$$

Define

$$S = \sum_{j=1}^{s} \frac{1}{\sigma_j} Q_j. \tag{2.23}$$

Then $-S$ is the value at zero of the reduced resolvent of L with respect to the eigenvalue zero. Using *(2.16)* one can show that

$$S = \operatorname{diag}(0_{n_1 \times n_1}, S_{22}) \tag{2.24}$$

under the partition *(2.8)*. It is clear that S_{22} is symmetric, positive definite.

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Now for each heat kernel in the first term of *(2.5)*, corresponding to $\lambda_j^{(r)}$ we have

$$\begin{aligned}\mu_j &= (l_j^{(r)}, 0_{1\times n_2})(ASA+B)\begin{pmatrix} r_j^{(r)} \\ 0_{n_2\times 1}\end{pmatrix} \\ &= l_j^{(r)}(A_{12}S_{22}A_{21}+B_{11})r_j^{(r)}, \qquad 1\le j\le n_1.\end{aligned} \tag{2.25}$$

Here, we have applied *(2.24)*. It is shown in Section 3 that $\mu_j > 0$, $1 \le j \le n_1$.

Step 3. From Lemma 2.7, B has distinct eigenvalues

$$0 = \sigma_0' < \sigma_1' < \cdots < \sigma_{s'}',$$

with corresponding eigenprojections $Q_0', Q_1', \ldots, Q_{s'}'$, and hence B has spectral decomposition

$$B = \sum_{j=1}^{s'} \sigma_j' Q_j'. \tag{2.26}$$

Similar to *(2.18)*, $A_0^{\frac{1}{2}} B (A_0^{\frac{1}{2}})^{-1}$ is symmetric. Thus, for $0 \le j \le s'$, $A_0^{\frac{1}{2}} Q_j' (A_0^{\frac{1}{2}})^{-1}$ is symmetric.

Now we form a matrix $Q_0' A Q_0'$, which is similar to

$$A_0^{\frac{1}{2}}(Q_0'AQ_0')(A_0^{\frac{1}{2}})^{-1} = \big[A_0^{\frac{1}{2}}Q_0'(A_0^{\frac{1}{2}})^{-1}\big]\big[A_0^{\frac{1}{2}}A(A_0^{\frac{1}{2}})^{-1}\big]\big[A_0^{\frac{1}{2}}Q_0'(A_0^{\frac{1}{2}})^{-1}\big].$$

Since the right-hand side is symmetric, $Q_0' A Q_0'$ is diagonalizable. Next, we have its spectral decomposition in the range of Q_0',

$$Q_0'AQ_0' = \sum_{k=1}^{s''} \alpha_k \mathcal{Q}_k, \qquad \sum_{k=1}^{s''} \mathcal{Q}_k = Q_0'. \tag{2.27}$$

The distinct eigenvalues α_k are those appearing in the third term of *(2.5)*.

Step 4. Related to *(2.26)*, we define a matrix

$$S' = \sum_{j=1}^{s'} \frac{1}{\sigma_j'} Q_j', \tag{2.28}$$

which is the value at zero of the reduced resolvent of B with respect to the eigenvalue zero. Then for each $\mathcal{Q}_k$ associated with a distinct α_k in *(2.27)*, we form a matrix $\mathcal{Q}_k(AS'A-L)\mathcal{Q}_k$ and find its spectral decomposition in the range of $\mathcal{Q}_k$,

$$\mathcal{Q}_k(AS'A-L)\mathcal{Q}_k = \sum_{k'=1}^{s_k} \beta_{kk'}\mathcal{Q}_{kk'}, \qquad \sum_{k'=1}^{s_k}\mathcal{Q}_{kk'} = \mathcal{Q}_k. \tag{2.29}$$

This is possible since $\mathcal{Q}_k(AS'A-L)\mathcal{Q}_k$ is diagonalizable via an argument similar to the one that gives *(2.27)*. It is shown in Section 3 that all $\beta_{kk'}$ are positive. Now we collect all distinct pairs $(\alpha_k, \beta_{kk'})$, $1 \le k \le s''$, $1 \le k' \le s_k$. The total number

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of those pairs is m' in the third term of (2.5), with $(\alpha_k, \beta_{kk'})$ being (α_j, β_j) and $\mathcal{Q}_{kk'}$ being $Q_{j,0}$ therein.

In the special case that all α_k in (2.27) are simple, i.e., all $\mathcal{Q}_k$ are rank one, we can simplify Step 4 as Step 4′ in Appendix A.

Procedure 2.9 is justified in Section 3. In particular, we will show the positivity of μ_j and β_j as a consequence of the crucial assumption of coupling, which is stated in (iii) of Assumption 2.1'.

For an arbitrary vector $\eta_2 \in \mathbb{R}^{n_2}$ we form a vector $\eta = \begin{pmatrix} 0_{n_1\times 1} \\ \eta_2 \end{pmatrix} \in \mathbb{R}^n$. From (2.20), we have

$$\eta^t P_j = 0, \qquad P_j \eta = 0, \qquad 1 \le j \le n_1.$$

This implies that when G is multiplied by η^t from the left, or by η from the right, the first term (the leading term) on the right-hand side of (2.5) disappears. In this case, we can further refine the higher order term (the second term) in (2.5).

We notice that each P_j is a subprojection of Q_0, see (2.20) and (2.22). Thus, η can be characterized as those satisfying $\eta^t Q_0 = 0$ or $Q_0 \eta = 0$. Recall that the formulation of P_j and Q_0 is after the change of variables (2.9), and hence under Assumption 2.1'. Applying (2.15) and (2.10), we recover P_j in (2.5) under Assumption 2.1 for the original variables,

$$P_j = \tilde{w}_w^{-1} \begin{pmatrix} r_j^{(r)} l_j^{(r)} & \\ & 0_{n_2\times n_2} \end{pmatrix} w_{\tilde{w}}^{-1} = \begin{pmatrix} r_j^{(r)} l_j^{(r)} & 0_{n_1\times n_2} \\ -L_{22}^{-1} L_{21} r_j^{(r)} l_j^{(r)} & 0_{n_2\times n_2} \end{pmatrix}, \qquad 1 \le j \le n_1.$$

At the same time, we find the eigenprojection Q_0 of L in (2.2) corresponding to the eigenvalue zero as

$$Q_0 = \begin{pmatrix} I_{n_1\times n_1} & 0_{n_1\times n_2} \\ -L_{22}^{-1} L_{21} & 0_{n_2\times n_2} \end{pmatrix}.$$

It is straightforward to verify that each P_j is a subprojection of Q_0. Thus, for η satisfying $\eta^t Q_0 = 0$ or $Q_0 \eta = 0$, multiplying G by η^t from the left or by η from the right will eliminate the first term on the right-hand side of (2.5). We now formulate a refinement under such a situation.

THEOREM 2.10 *Let $\eta \in \mathbb{R}^n$ be a constant vector such that $\eta^t Q_0 = 0$, where Q_0 is the eigenprojection of L corresponding to the eigenvalue zero. Then under the same assumptions and same notations as in Theorem 2.3, we have*

$$\begin{aligned} \frac{\partial^l}{\partial x^l}(\eta^t G)(x,t) = \frac{\partial^{l+1}}{\partial x^{l+1}} \sum_{j=1}^{m} H(x,t;\lambda_j^{(r)},\mu_j)\eta_j^t + O(1)(t+1)^{-\frac{1}{2}} t^{-\frac{l+1}{2}} \sum_{j=1}^{m} H(x,t;\lambda_j^{(r)},C) \\ + \sum_{j=1}^{m'} e^{-\beta_j t} \sum_{k=0}^{l} \delta^{(l-k)}(x-\alpha_j t)\eta^t Q_{j,k}(t), \end{aligned} \tag{2.30}$$

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where $\eta_j \in \mathbb{R}^n$, $1 \le j \le m$, are constant vectors. Similarly, if $Q_0\eta = 0$ we have

$$\frac{\partial^l}{\partial x^l}(G\eta)(x,t) = \frac{\partial^{l+1}}{\partial x^{l+1}} \sum_{j=1}^{m} H(x,t;\lambda_j^{(r)},\mu_j)\eta_j + O(1)(t+1)^{-\frac{1}{2}} t^{-\frac{l+1}{2}} \sum_{j=1}^{m} H(x,t;\lambda_j^{(r)},C) + \sum_{j=1}^{m'} e^{-\beta_j t} \sum_{k=0}^{l} \delta^{(l-k)}(x-\alpha_j t) Q_{j,k}(t)\eta. \tag{2.31}$$

3. Spectral analysis

The proofs of Theorems 2.3 and 2.10 consist of two steps. The first step is to perform estimation on the Green's function G and its derivatives in the Fourier space via spectral analysis. This is to be done in the current section. At the same time, we also justify Procedure 2.9. In particular, we prove that μ_j and β_j in (2.5) are positive. The second step is to perform related estimation in the physical space to obtain (2.5), (2.30), and (2.31). This is to be done in Section 4, where we also give an outline of obtaining (2.6)

From the discussion in Section 2, to prove the main estimate (2.5), we only need to prove it under Assumption 2.1'. After that, (2.15) and (2.10) give the intended result under Assumption 2.1 as stated in Theorem 2.3. The simplification makes the formulation of parameters in (2.5) more manageable. To recover the result under Assumption 2.1, we only need to modify the projections P_j and $Q_{j,k}$ via (2.15). Such a simplification also applies to (2.30) and (2.31) in Theorem 2.10 (also to (2.6) in Remark 2.4).

In fact, we can further simplify Assumption 2.1' in the proof of our main results. For this, again we take $A_0^{\frac{1}{2}}$, a square root of A_0, as a symmetric, positive definite matrix, and write (2.1) in its equivalent form,

$$\bar{w}_t + \bar{A}\bar{w}_x = \bar{B}\bar{w}_{xx} + \bar{L}\bar{w}, \tag{3.1}$$

where

$$\bar{w} = A_0^{\frac{1}{2}} w, \qquad \bar{A} = A_0^{\frac{1}{2}} A (A_0^{\frac{1}{2}})^{-1},$$
$$\bar{B} = A_0^{\frac{1}{2}} B (A_0^{\frac{1}{2}})^{-1}, \qquad \bar{L} = A_0^{\frac{1}{2}} L (A_0^{\frac{1}{2}})^{-1}.$$

It is straightforward to verify that under Assumption 2.1', all matrices $\bar{A}, \bar{B}$ and $\bar{L}$ are symmetric, see (2.18). In fact, Assumption 2.1' is satisfied by $\bar{A}, \bar{B}$ and $\bar{L}$ (with $A_0 = I$) as long as it is satisfied by A, B and L.

Note that the Green's function $\bar{G}$ for (3.1) is related to G for (2.1) by

$$\bar{G}(x,t) = A_0^{\frac{1}{2}} G(x,t) (A_0^{\frac{1}{2}})^{-1}. \tag{3.2}$$

Similar to the above discussion on Assumption 2.1 vs. Assumption 2.1', we can drop the bar accent in (3.1), and study (2.1) under the further simplified version of assumptions.

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ASSUMPTION 3.1. (i) *Let A be symmetric, B be symmetric, positive semi-definite, and L be symmetric, negative semi-definite.*

(ii) *The matrix L is block-diagonal,*

$$L = \operatorname{diag}(0_{n_1\times n_1}, L_{22}), \tag{3.3}$$

where $L_{22} \in \mathbb{R}^{n_2\times n_2}$ is symmetric, negative definite.

(iii) *Let $\mathbb{N}_1$ be the null space of B and $\mathbb{N}_2$ be the null space of L. Then $\mathbb{N}_1 \cap \mathbb{N}_2$ contains no eigenvectors of A.*

In this section, we carry out Green's function estimation for (2.1) in the Fourier space under Assumption 3.1. In Section 4, we perform Green's function estimation in the physical space, also under Assumption 3.1.

Let the Fourier transform of a function $g(x)$ be $\hat{g}(\xi)$,

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} g(x)e^{-ix\xi}\,dx, \qquad g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(\xi)e^{ix\xi}\,d\xi. \tag{3.4}$$

Taking the Fourier transform of (2.3) with respect to x gives us

$$\begin{cases} \hat{G}_t + i\xi A\hat{G} = (i\xi)^2 B\hat{G} + L\hat{G}, \\ \hat{G}(\xi, 0) = I. \end{cases} \tag{3.5}$$

Solving (3.5), we have

$$\begin{cases} \hat{G}(\xi, t) = e^{tE(i\xi)}, \\ E(i\xi) = L - i\xi A + (i\xi)^2 B. \end{cases} \tag{3.6}$$

Consider

$$E(z) = L - zA + z^2 B \tag{3.7}$$

as a matrix-valued function of a complex variable z. We state some basic facts about its spectral decomposition. They are from complex analysis and perturbation theory, see [8].

First, the number n' of distinct eigenvalues of $E(z)$ is constant if z is not one of exceptional points, of which there are only a finite number in the whole complex plane. In each simply connected domain containing no exceptional points, the eigenvalues of $E(z)$ can be expressed as n' holomorphic functions,

$$\tilde{\lambda}_1(z), \tilde{\lambda}_2(z), \ldots, \tilde{\lambda}_{n'}(z),$$

which have constant multiplicities $\tilde{m}_1, \tilde{m}_2, \ldots, \tilde{m}_{n'}$, respectively, $\sum_{j=1}^{n'} \tilde{m}_j = n$.

The $\tilde{\lambda}_j(z)$, $1 \le j \le n'$, are branches of one or several algebraic functions, and are everywhere continuous. These algebraic functions will also be denoted by $\tilde{\lambda}_j(z)$. An exceptional point z_0 is either a branch point of some of the $\tilde{\lambda}_j(z)$ or a regular point for all of them; in the latter case the values of some of the different $\tilde{\lambda}_j(z)$

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coincide at $z = z_0$. Thus, there is always a splitting of eigenvalue at (and only at) an exceptional point.

In general (i.e., without appropriate assumptions), $E(z)$ in the form of (3.7) is not diagonalizable, and has the spectral representation

$$E(z) = \sum_{j=1}^{n'} \tilde{\lambda}_j(z)\tilde{P}_j(z) + \sum_{j=1}^{n'} \tilde{D}_j(z) \tag{3.8}$$

for any z that is not an exceptional point. Here, $\tilde{P}_j(z)$ and $\tilde{D}_j(z)$ are the eigenprojection and the eigennilpotent of $\tilde{\lambda}_j(z)$, respectively,

$$\begin{aligned} &\tilde{\lambda}_j(z) \neq \tilde{\lambda}_k(z) \quad \text{for } j \neq k; \\ &\tilde{P}_j(z)\tilde{P}_k(z) = \delta_{jk}\tilde{P}_j(z), \qquad \sum_{j=1}^{n'} \tilde{P}_j(z) = I; \\ &\tilde{D}_j^{\tilde{m}_j}(z) = 0, \qquad \tilde{P}_j(z)\tilde{D}_j(z) = \tilde{D}_j(z)\tilde{P}_j(z) = \tilde{D}_j(z), \qquad 1 \leq j, k \leq n'. \end{aligned} \tag{3.9}$$

The eigenprojections $\tilde{P}_j(z)$ and the eigennilpotents $\tilde{D}_j(z)$ are also holomorphic in each simply connected domain containing no exceptional points, being branches of one or several analytic functions (also denoted by $\tilde{P}_j(z)$ and $\tilde{D}_j(z)$) and with only algebraic singularities.

The analytic functions $\tilde{P}_j(z)$ and $\tilde{\lambda}_j(z)$ have common branch points of the same order. Let z_0 be an exceptional point. When z describes a small circle around z_0, $\{\tilde{\lambda}_1(z), \tilde{\lambda}_2(z), \ldots, \tilde{\lambda}_{n'}(z)\}$ and $\{\tilde{P}_1(z), \tilde{P}_2(z), \cdots, \tilde{P}_{n'}(z)\}$ will undergo the same permutation among themselves after analytic continuation. We emphasize that z_0 may or may not be a branch point, and $\tilde{\lambda}_j(z)$ is always continuous there, while $\tilde{P}_j(z)$ in general has a pole at z_0 even when z_0 is not a branch point. If z_0 is indeed a branch point, we have the following lemma due to Butler [2], see Theorem 1.9 of Chapter 2 in [8].

LEMMA 3.2. *If $z = z_0$ is a branch point of $\tilde{\lambda}_j(z)$ (and therefore also of $\tilde{P}_j(z)$) of order $p - 1 \geq 1$, then $\tilde{P}_j(z)$ has a pole there. That is, the Laurent expansion of $\tilde{P}_j(z)$ in powers of $(z - z_0)^{1/p}$ necessarily contains negative powers.*

From (3.7), $E(z)$ is real, symmetric for real z. This implies that $E(z)$ is diagonalizable on the real axis of the complex plane. Therefore, in (3.8) $\tilde{D}_j(z) = 0$, $1 \leq j \leq n'$, on the real axis. By analytic continuation, they are zero on the whole complex plane, and (3.8) is reduced to

$$E(z) = \sum_{j=1}^{n'} \tilde{\lambda}_j(z)\tilde{P}_j(z) \tag{3.10}$$

for every z that is not an exceptional point.

Since $E(z)$ is real, symmetric for real z, $\tilde{P}_j(z)$, $1 \leq j \leq n'$, are symmetric with $|\tilde{P}_j(z)| = 1$ on the real axis. By Lemma 3.2, every real z is a regular point. In

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particular, $z = 0$ is a regular point. Due to the fact that $\tilde{\lambda}_j(z)$ is continuous at $z = 0$ and $\tilde{P}_j(z)$ does not have a pole there, they are holomorphic at the origin. We thus have their Taylor expansions: for $1 \le j \le n'$,

$$\begin{aligned} \tilde{\lambda}_j(z) &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \tilde{\lambda}_j^{(k)}(0), \\ \tilde{P}_j(z) &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \tilde{P}_j^{(k)}(0). \end{aligned} \tag{3.11}$$

We comment that $z = 0$ is in general an exceptional point. If L is rank deficient, as been seen below, the eigenvalue zero of L can split into some $\tilde{\lambda}_j(z)$ at the origin. Assumption 3.1 (i), which is a linear version of entropy condition, guarantees that the point is neither a branch point for those $\tilde{\lambda}_j(z)$ nor a pole for the corresponding $\tilde{P}_j(z)$.

Next, we combine (3.7) and (3.10) to have

$$E(z) = L - zA + z^2 B = \sum_{j=1}^{n'} \tilde{\lambda}_j(z) \tilde{P}_j(z). \tag{3.12}$$

Setting $z = 0$ gives us

$$L = \sum_{j=1}^{n'} \tilde{\lambda}_j(0) \tilde{P}_j(0).$$

The spectral decomposition of L is obtained by combining those terms with the same $\tilde{\lambda}_j(0)$-value. Using the notation (2.21) for the decomposition we write

$$L = \sum_{j=1}^{n'} \tilde{\lambda}_j(0) \tilde{P}_j(0) = \sum_{j=1}^{s} -\sigma_j Q_j, \qquad \sigma_j > 0. \tag{3.13}$$

Thus, we have the following lemma.

LEMMA 3.3. *The eigenvalues $\tilde{\lambda}_j(z)$ and the eigenprojections $\tilde{P}_j(z)$, $1 \le j \le n'$, of $E(z)$ are divided into two groups. In the first group, $\tilde{\lambda}_j(0) = 0$ while the corresponding $\tilde{P}_j(0)$ is a sub-projection of*

$$Q_0 = I - \sum_{j=1}^{s} Q_j = \operatorname{diag}(I_{n_1 \times n_1}, 0_{n_2 \times n_2}), \tag{3.14}$$

which is the eigenprojection of L corresponding to the eigenvalue zero, see (2.22). In the second group, each $\tilde{\lambda}_j(0)$ is some $-\sigma_k < 0$, $1 \le k \le s$, while $\tilde{P}_j(0)$ is a sub-projection of the corresponding Q_k.

From Lemma 3.3 and (3.3), it is straightforward to verify the block-diagonal structure of $\tilde{P}_j(0)$ as follows.

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LEMMA 3.4. *For the two groups defined in Lemma 3.3, we have the following. In group one,*

$$\tilde{P}_j(0) = \mathrm{diag}((\tilde{P}_j(0))_{11}, 0_{n_2\times n_2}), \qquad \sum_j \tilde{P}_j(0) = Q_0. \tag{3.15}$$

Here, we have used the partition defined in (2.8). Similarly, in group two,

$$\tilde{P}_j(0) = \mathrm{diag}(0_{n_1\times n_1}, (\tilde{P}_j(0))_{22}). \tag{3.16}$$

Now we denote the number of $\tilde{\lambda}_j(z)$ (and hence the number of $\tilde{P}_j(z)$) in group one as n_1'. By appropriately arranging the order, we let the first n_1' pairs of $\tilde{\lambda}_j(z)$ and $\tilde{P}_j(z)$ be in group one.

LEMMA 3.5. *Under the notations in (2.19) and (3.14), we have*

$$-Q_0AQ_0 = \sum_{j=1}^{n_1'} \tilde{\lambda}_j'(0)\tilde{P}_j(0), \qquad \sum_{j=1}^{n_1'} \tilde{P}_j(0) = Q_0, \tag{3.17}$$

i.e.

$$-A^{(r)} = \sum_{j=1}^{n_1'} \tilde{\lambda}_j'(0)(\tilde{P}_j(0))_{11}, \qquad \sum_{j=1}^{n_1'} (\tilde{P}_j(0))_{11} = I_{n_1\times n_1}. \tag{3.18}$$

Proof. From (3.12) and (3.9), we have

$$M(z) \equiv \left(\sum_{j=1}^{n_1'} \tilde{P}_j(z)\right) E(z) \left(\sum_{j=1}^{n_1'} \tilde{P}_j(z)\right) = \sum_{j=1}^{n_1'} \tilde{\lambda}_j(z)\tilde{P}_j(z). \tag{3.19}$$

Comparing the coefficients of z on both sides and noting (3.15) give us

$$M'(0) = Q_0L\sum_{j=1}^{n_1'} \tilde{P}_j'(0) - Q_0AQ_0 + \sum_{j=1}^{n_1'} \tilde{P}_j'(0)LQ_0 = \sum_{j=1}^{n_1'} \tilde{\lambda}_j'(0)\tilde{P}_j(0).$$

By the definition of Q_0, the equation is simplified to

$$M'(0) = -Q_0AQ_0 = \sum_{j=1}^{n_1'} \tilde{\lambda}_j'(0)\tilde{P}_j(0). \tag{3.20}$$

□

By combining terms with the same value of $\tilde{\lambda}_j'(0)$, the right-hand side of (3.17) gives us the spectral decomposition of $-Q_0AQ_0$ in the range of Q_0. Similarly, after such combination, the right-hand side of (3.18) gives us the spectral decomposition of $-A^{(r)}$. In particular, $-\tilde{\lambda}_j'(0)$ is an eigenvalue of $A^{(r)}$.

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Let the distinct eigenvalues of $A^{(r)}$ be $\lambda_j^{(r)}$, with multiplicities m_j, $1 \le j \le n_1''$, $m_1 + \cdots + m_{n''} = n_1$. We then divide terms on the right-hand side of (3.17) into groups such that each group has the same value of $\tilde{\lambda}_j'(0)$ and combine each group. This gives us

$$Q_0 A Q_0 = \sum_{j=1}^{n_1''} \lambda_j^{(r)} \mathcal{P}_j, \qquad \sum_{j=1}^{n_1''} \mathcal{P}_j = Q_0. \tag{3.21}$$

Here, each $\mathcal{P}_j$ is an eigenprojection of $Q_0 A Q_0$ in the range of Q_0, corresponding to $\lambda_j^{(r)}$ and with rank m_j. By appropriately arranging the order of $\tilde{\lambda}_j'(0)$ in (3.17), we set

$$\mathcal{P}_j = \sum_{k=m_{j-1}'+1}^{m_j'} \tilde{P}_k(0), \qquad 1 \le j \le n_1'', \tag{3.22}$$

where $m_0' = 0$ and $m_{n_1''}' = n_1'$.

LEMMA 3.6. *Under the notations of* (2.23) *and* (3.22)*, for each* $1 \le j \le n_1''$ *we have*

$$\mathcal{P}_j (B + ASA) \mathcal{P}_j = \frac{1}{2} \sum_{k=m_{j-1}'+1}^{m_j'} \tilde{\lambda}_k''(0) \tilde{P}_k(0). \tag{3.23}$$

Proof. For S defined in (2.23) it is straightforward to verify that

$$LS = SL = -\sum_{j=1}^{s} Q_j = Q_0 - I, \qquad Q_0 S = S Q_0 = 0. \tag{3.24}$$

We now find $\sum_{j=1}^{n_1'} \tilde{P}_j'(0)$. For this, we consider

$$E(z) \sum_{j=1}^{n_1'} \tilde{P}_j(z) = \sum_{j=1}^{n_1'} \tilde{\lambda}_j(z) \tilde{P}_j(z), \tag{3.25}$$

where we have applied (3.12) and (3.9). Comparing the z-terms on both sides of (3.25) gives us

$$L \sum_{j=1}^{n_1'} \tilde{P}_j'(0) - A Q_0 = \sum_{j=1}^{n_1'} \tilde{\lambda}_j'(0) \tilde{P}_j(0).$$

Multiplying the resulted equation by S from the left, we have

$$SL \sum_{j=1}^{n_1'} \tilde{P}_j'(0) - SAQ_0 = 0, \tag{3.26}$$

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where we have used (3.24) and the fact that for $1 \le j \le n_1'$, $\tilde{P}_j(0)$ is a sub-projection of Q_0. Similarly, we also have

$$\sum_{j=1}^{n_1'} \tilde{P}_j'(0) LS - Q_0 AS = 0. \tag{3.27}$$

We sum up (3.26) and (3.27) to have

$$SL \sum_{j=1}^{n_1'} \tilde{P}_j'(0) + \sum_{j=1}^{n_1'} \tilde{P}_j'(0) LS = SAQ_0 + Q_0 AS. \tag{3.28}$$

Then we substitute SL and LS from (3.24) into (3.28) and simplify. We arrive at

$$Q_0 \sum_{j=1}^{n_1'} \tilde{P}_j'(0) + \sum_{j=1}^{n_1'} \tilde{P}_j'(0) Q_0 - 2 \sum_{j=1}^{n_1'} \tilde{P}_j'(0) = SAQ_0 + Q_0 AS. \tag{3.29}$$

Next, we consider the following identity, justified by the definition of eigenprojections as given in (3.9),

$$\left[\sum_{j=1}^{n_1'} \tilde{P}_j(z) \right] \left[\sum_{j=1}^{n_1'} \tilde{P}_j(z) \right] = \sum_{j=1}^{n_1'} \tilde{P}_j(z).$$

Comparing the z-terms on both sides gives us

$$\left[\sum_{j=1}^{n_1'} \tilde{P}_j(0) \right] \left[\sum_{j=1}^{n_1'} \tilde{P}_j'(0) \right] + \left[\sum_{j=1}^{n_1'} \tilde{P}_j'(0) \right] \left[\sum_{j=1}^{n_1'} \tilde{P}_j(0) \right] = \sum_{j=1}^{n_1'} \tilde{P}_j'(0).$$

Noting $Q_0 = \sum_{j=1}^{n_1'} \tilde{P}_j(0)$, see (3.17), we write

$$Q_0 \sum_{j=1}^{n_1'} \tilde{P}_j'(0) + \sum_{j=1}^{n_1'} \tilde{P}_j'(0) Q_0 = \sum_{j=1}^{n_1'} \tilde{P}_j'(0). \tag{3.30}$$

Substituting (3.30) into (3.29) and simplifying, we arrive at

$$-\sum_{j=1}^{n_1'} \tilde{P}_j'(0) = SAQ_0 + Q_0 AS. \tag{3.31}$$

Now for $M(z)$ defined in (3.19), we consider $\frac{1}{2}M''(0)$. Noting (3.17) and Q_0 being the eigenprojection of L corresponding to the eigenvalue zero, from (3.19), we have

$$\frac{1}{2}M''(0) = Q_0 B Q_0 - Q_0 A \sum_{j=1}^{n_1'} \tilde{P}_j'(0) - \sum_{j=1}^{n_1'} \tilde{P}_j'(0) A Q_0 + \left[\sum_{j=1}^{n_1'} \tilde{P}_j'(0) \right] L \left[\sum_{j=1}^{n_1'} \tilde{P}_j'(0) \right]. \tag{3.32}$$

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Substituting (3.31) into (3.32) and simplifying, we arrive at

$$\frac{1}{2}M''(0) = Q_0BQ_0 + Q_0ASAQ_0 + Q_0AQ_0AS + SAQ_0AQ_0 + Q_0ASAQ_0 + Q_0ASLSAQ_0.$$

Applying (3.24) to the last term on the right, we further have

$$\frac{1}{2}M''(0) = Q_0BQ_0 + Q_0AQ_0AS + SAQ_0AQ_0 + Q_0ASAQ_0. \tag{3.33}$$

With the indexing in (3.22), from (3.19), we have

$$\left[\sum_{k=m'_{j-1}+1}^{m'_j}\tilde{P}_k(z)\right]M(z)\left[\sum_{k=m'_{j-1}+1}^{m'_j}\tilde{P}_k(z)\right] = \left[\sum_{k=m'_{j-1}+1}^{m'_j}\tilde{\lambda}_k(z)\tilde{P}_k(z)\right]\left[\sum_{k=m'_{j-1}+1}^{m'_j}\tilde{P}_k(z)\right]$$

for $1 \le j \le n''_1$. Comparing the z^2-terms on both sides, noting $M(0) = 0$ and using the notation in (3.22), we have

$$\begin{aligned}&\frac{1}{2}\mathcal{P}_jM''(0)\mathcal{P}_j + \sum_{k=m'_{j-1}+1}^{m'_j}\tilde{P}'_k(0)M'(0)\mathcal{P}_j + \mathcal{P}_jM'(0)\sum_{k=m'_{j-1}+1}^{m'_j}\tilde{P}'_k(0)\\ =&\frac{1}{2}\sum_{k=m'_{j-1}+1}^{m'_j}\tilde{\lambda}''_k(0)\tilde{P}_k(0)\mathcal{P}_j + \sum_{k=m'_{j-1}+1}^{m'_j}\tilde{\lambda}'_k(0)\tilde{P}'_k(0)\mathcal{P}_j\\ &+\left[\sum_{k=m'_{j-1}+1}^{m'_j}\tilde{\lambda}'_k(0)\tilde{P}_k(0)\right]\left[\sum_{k=m'_{j-1}+1}^{m'_j}\tilde{P}'_k(0)\right].\end{aligned}$$

Substituting (3.20) into the equation and noting that all $\tilde{\lambda}'_k(0)$ on the right-hand side have the same value by our indexing, we simplify the equation to

$$\frac{1}{2}\mathcal{P}_jM''(0)\mathcal{P}_j = \frac{1}{2}\sum_{k=m'_{j-1}+1}^{m'_j}\tilde{\lambda}''_k(0)\tilde{P}_k(0)\mathcal{P}_j = \frac{1}{2}\sum_{k=m'_{j-1}+1}^{m'_j}\tilde{\lambda}''_k(0)\tilde{P}_k(0). \tag{3.34}$$

Finally, we substitute (3.33) into (3.34), noting (3.24) and that each $\mathcal{P}_j$ is a sub-projection of Q_0. This gives us (3.23). □

From (3.22) and (3.23), after combining terms of the same value of $\tilde{\lambda}''_k(0)$, the right-hand side of (3.23) gives us the spectral decomposition of the matrix on the left-hand side in the range of $\mathcal{P}_j$, $1 \le j \le n''_1$. Thus, each $\frac{1}{2}\tilde{\lambda}''_k(0)$, $m'_{j-1}+1 \le k \le m'_j$, is an eigenvalue. Our next lemma shows that these eigenvalues are positive.

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LEMMA 3.7. *For $1 \le j \le n_1''$, $m_{j-1}' + 1 \le k \le m_j'$, we have $\tilde{\lambda}_k''(0) > 0$.*

Proof. For $1 \le j \le n_1''$ let

$$M_j = \mathcal{P}_j(B + ASA)\mathcal{P}_j.$$

From (3.23), each $\frac{1}{2}\tilde{\lambda}_k''(0)$, $m_{j-1}' + 1 \le k \le m_j'$, is an eigenvalue of M_j in the range of $\mathcal{P}_j$. From (2.23), S is symmetric, positive semi-definite. Thus, M_j is symmetric, positive semi-definite. This implies $\tilde{\lambda}_k''(0) \ge 0$.

Now we show $\tilde{\lambda}_k''(0) \ne 0$ by way of contradiction. Suppose $\tilde{\lambda}_k''(0) = 0$. Let v be a corresponding eigenvector in the range of $\mathcal{P}_j$,

$$M_j v = 0, \qquad \mathcal{P}_j v = v \ne 0. \tag{3.35}$$

Then

$$v^t M_j v = (\mathcal{P}_j v)^t B(\mathcal{P}_j v) + (A\mathcal{P}_j v)^t S(A\mathcal{P}_j v) = v^t B v + (Av)^t S(Av) = 0.$$

Since B and S are both positive semi-definite, we have

$$v^t B v = 0, \qquad (Av)^t S(Av) = 0,$$

which further implies

$$Bv = 0, \qquad SAv = 0. \tag{3.36}$$

Note that $\mathcal{P}_j$ is a subprojection of Q_0, see (3.21). Thus, v is in the range of Q_0 and hence in the null space of L. Meanwhile, (3.36) shows that v is also in the null space of B. We therefore only need to prove that v is an eigenvector of A to reach a contradiction to (iii) of Assumption 3.1.

For this, we observe that Q_0 is also the eigenprojection of S corresponding to the eigenvalue zero, see (2.23). From (3.36), Av is in the null space of S. Thus,

$$Av = Q_0 Av.$$

With (3.35) and (3.21), we have

$$Av = Q_0 A\mathcal{P}_j v = (Q_0 A Q_0)\mathcal{P}_j v = \lambda_j^{(r)}\mathcal{P}_j v = \lambda_j^{(r)} v,$$

which shows that v is an eigenvector of A. □

We combine Lemmas 3.3-3.7 to have a summary on the leading terms as follows. The eigenvalues $\tilde{\lambda}_j(z)$, and hence the eigenprojections $\tilde{P}_j(z)$, of $E(z)$ are divided into two groups. In the first group, for small z eigenvalues split away from the

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eigenvalue zero of L,

$$\tilde{\lambda}_j(z) = \tilde{\lambda}_j'(0)z + \frac{1}{2}\tilde{\lambda}_j''(0)z^2 + O(z^3), \qquad 1 \le j \le n_1'. \tag{3.37}$$

Each $\tilde{P}_j(0)$ is a sub-projection of Q_0, the eigenprojection of L corresponding to the eigenvalue zero, and takes the form

$$\begin{aligned} &\tilde{P}_j(0) = \operatorname{diag}\left((\tilde{P}_j(0))_{11}, 0_{n_2\times n_2}\right), \quad 1 \le j \le n_1', \\ &\sum_{j=1}^{n_1'} (\tilde{P}_j(0))_{11} = I_{n_1\times n_1}. \end{aligned} \tag{3.38}$$

The $\tilde{\lambda}_j'(0)$, $1 \le j \le n_1'$, in (3.37) are eigenvalues of $-A^{(r)}$. In fact,

$$-A^{(r)} = \sum_{j=1}^{n_1'} \tilde{\lambda}_j'(0)(\tilde{P}_j(0))_{11}, \tag{3.39}$$

which is equivalent to

$$-Q_0AQ_0 = \sum_{j=1}^{n_1'} \tilde{\lambda}_j'(0)\tilde{P}_j(0). \tag{3.40}$$

By combining the terms on the right-hand sides of (3.39) and (3.40) for the same values of $\tilde{\lambda}_j'(0)$, we obtain spectral decompositions of $-A^{(r)}$ and $-Q_0AQ_0$, respectively. Here, the latter is in the range of Q_0, which gives us

$$Q_0AQ_0 = \sum_{j=1}^{n_1''} \lambda_j^{(r)}\mathcal{P}_j, \qquad \mathcal{P}_j = \sum_{k=m_{j-1}'+1}^{m_j'} \tilde{P}_k(0), \qquad 1 \le j \le n_1'', \tag{3.41}$$

with $m_0' = 0$ and $m_{n_1''}' = n_1'$. Here, $\lambda_j^{(r)}$, $1 \le j \le n_1''$, are distinct eigenvalues of Q_0AQ_0 in the range of Q_0 and hence distinct eigenvalues of $A^{(r)}$.

The $\frac{1}{2}\tilde{\lambda}_j''(0)$, $1 \le j \le n_1'$, in (3.37) are positive. In fact, they are nonzero eigenvalues of a symmetric, positive semi-definite matrix. With the notations in (3.41), we have, for $1 \le j \le n_1''$,

$$\mathcal{P}_j(B + ASA)\mathcal{P}_j = \sum_{k=m_{j-1}'+1}^{m_j'} \frac{1}{2}\tilde{\lambda}_k''(0)\tilde{P}_k(0). \tag{3.42}$$

By combining terms with the same value of $\tilde{\lambda}_k''(0)$ on the right, we obtain the spectral decomposition for the matrix on the left in the range of $\mathcal{P}_j$. This gives us

$$\mathcal{P}_j(B + ASA)\mathcal{P}_j = \sum_{l=1}^{m_j''} \mu_{jl}\mathcal{P}_{jl}, \qquad 1 \le j \le n_1'', \tag{3.43}$$

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where each μ_{jl} is a distinct $\frac{1}{2}\tilde{\lambda}_k''(0)$, $m_{j-1}'+1 \le k \le m_j'$, $\mathcal{P}_{jl}$ is the sum of the corresponding $\tilde{P}_k(0)$'s that share the same value of $\frac{1}{2}\tilde{\lambda}_k''(0)$, and $\sum_{l=1}^{m_j''} \mathcal{P}_{jl} = \mathcal{P}_j$.

Now we collect all distinct pairs $(\lambda_j^{(r)}, \mu_{jl})$, $1 \le j \le n_1''$, $1 \le l \le m_j''$. The total number of them is

$$m = \sum_{j=1}^{n_1''} m_j''. \tag{3.44}$$

We rename all μ_{jl} and $\mathcal{P}_{jl}$, $1 \le j \le n_1''$, $1 \le l \le m_j''$, as μ_j and P_j, $1 \le j \le m$, respectively. Meanwhile, we let each distinct $\lambda_j^{(r)}$ repeat m_j'' times and rename all of them as $\lambda_j^{(r)}$, $1 \le j \le m$. In this way we obtain triples $(\lambda_j^{(r)}, \mu_j, P_j)$, $1 \le j \le m$.

Using those triples we construct a matrix-valued function

$$E^*(z) = \sum_{j=1}^{m} (-\lambda_j^{(r)} z + \mu_j z^2) P_j. \tag{3.45}$$

From the construction of the triples it is clear that

$$E^*(z) = -zQ_0AQ_0 + z^2 \sum_{j=1}^{n_1''} \mathcal{P}_j(B + ASA)\mathcal{P}_j = \sum_{j=1}^{n_1'} [\tilde{\lambda}_j'(0)z + \frac{1}{2}\tilde{\lambda}_j''(0)z^2]\tilde{P}_j(0), \tag{3.46}$$

see (3.40)-(3.43).

Now we write (3.12) as

$$E(z) = \sum_{j=1}^{n_1'} \tilde{\lambda}_j(z)\tilde{P}_j(z) + \sum_{j=n_1'+1}^{n'} \tilde{\lambda}_j(z)\tilde{P}_j(z), \tag{3.47}$$

where the second summation is for eigen-pairs in the second group. In particular, for small z each $\tilde{\lambda}_j(z)$ in the second summation is perturbed away from a negative eigenvalue of L, see Lemma 3.3 and (3.13). Therefore, the second summation in (3.47) has a spectral gap and represents exponential decay in G in large time.

We compare (3.46) with the first summation in (3.47) to observe that $E^*(z)$ captures the leading term of the summation. Therefore, $E^*(z)$ generates the leading term of G in large time. We thus define $G^*(x,t)$ through its Fourier transform,

$$\hat{G}^*(\xi,t) = e^{tE^*(i\xi)} = \sum_{j=1}^{n_1'} e^{[\tilde{\lambda}_j'(0)i\xi - \frac{1}{2}\tilde{\lambda}_j''(0)\xi^2]t}\tilde{P}_j(0) \equiv \sum_{j=1}^{m} e^{(-\lambda_j^{(r)} i\xi - \mu_j\xi^2)t} P_j \tag{3.48}$$

in parallel to (3.6). Here, through a consolidation of like terms in the first summation, the last summation of (3.48) defines $m, \lambda_j^{(r)}$, μ_j and P_j in Theorem 2.3.

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Taking the inverse Fourier transform gives us

$$G^*(x,t) = \sum_{j=1}^{m} H(x,t;\lambda_j^{(r)},\mu_j)P_j, \tag{3.49}$$

which is the leading term in (2.5) for $l = 0$.

In the special case that all eigenvalues of $A^{(r)}$ are simple, i.e., $m_j = 1, 1 \le j \le n_1''$, we simplify the formulation of parameters in (3.49) as follows.

From (3.39)-(3.41), we immediately have

$$n_1' = n_1'' = n_1, \qquad m_j' = j, \quad 1 \le j \le n_1. \tag{3.50}$$

Since each $\mathcal{P}_j$ in (3.41) is rank-one, (3.42)-(3.44) give us $m_j'' = 1$, $1 \le j \le n_1$, and

$$m = n_1'' = n_1.$$

Now for each eigenvalue $\lambda_j^{(r)}$ of $A^{(r)}$, let the normalized left eigenvector and right eigenvector of $A^{(r)}$ corresponding to $\lambda_j^{(r)}$ be $l_j^{(r)}$ and $r_j^{(r)}$, respectively, see Step 1 in Procedure 2.9. Then (3.38), (3.39) and (3.50) imply that $P_j = \tilde{P}_j(0)$ is given by (2.20).

From (3.41)-(3.43), we have $\mathcal{P}_{j1} = \mathcal{P}_j = \tilde{P}_j(0)$ for $1 \le j \le n_1$ and

$$\tilde{P}_j(0)(B + ASA)\tilde{P}_j(0) = \mu_j \tilde{P}_j(0). \tag{3.51}$$

Noting $P_j = \tilde{P}_j(0)$ is given by (2.20), we multiply (3.51) by $(l_j^{(r)}, 0_{1\times n_2})$ from the left and by $\begin{pmatrix} r_j^{(r)} \\ 0_{n_2\times 1} \end{pmatrix}$ from the right. This gives us (2.25). Thus, we have justified Steps 1 and 2 in Procedure 2.9.

We have given a detailed spectral analysis of $E(z)$ for small z, which induces the leading term $G^*(x,t)$ of $G(x,t)$ that governs the long-time behaviour. The short-term behaviour, on the other hand, is induced by the spectral analysis of $E(z)$ for large z. It gives rise to the part related to δ-functions in (2.5).

We write

$$E(z) = L - zA + z^2B = z^2(B - \frac{1}{z}A + \frac{1}{z^2}L) \equiv z^2\check{E}(\frac{1}{z}), \tag{3.52}$$

i.e.,

$$\check{E}(\check{z}) = B - \check{z}A + \check{z}^2L. \tag{3.53}$$

Here, to consider $z \to \infty$ is equivalent to consider $\check{z} = \frac{1}{z} \to 0$. Noting $-\check{E}$ takes the same form as E, with the triple $(-A,-B,-L)$ corresponds to (A,L,B). Under Assumption 3.1, the triple $(-A,-B,-L)$ has the same properties as (A,L,B) except that $-B$ does not have a block-diagonal structure as L, see (3.3). Thus, we have similar results for $-\check{E}(\check{z})$ as for $E(z)$, except for minor modifications related to the block-diagonal structure of the leading terms in the eigenprojections. We state those results as follows, omitting the proofs as they are parallel to those for $E(z)$.

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From (3.10) and (3.52), we have

$$\check{E}(\check{z}) = \check{z}^2 E(\frac{1}{\check{z}}) = \check{z}^2 \sum_{j=1}^{n'} \tilde{\lambda}_j(\frac{1}{\check{z}})\tilde{P}_j(\frac{1}{\check{z}}).$$

Thus, we have the spectral decomposition for $\check{E}(\check{z})$ as

$$\check{E}(\check{z}) = \sum_{j=1}^{n'} \check{\lambda}_j(\check{z})\check{P}_j(\check{z}), \tag{3.54}$$

where

$$\check{\lambda}_j(\check{z}) = \check{z}^2\tilde{\lambda}_j(\frac{1}{\check{z}}), \qquad \check{P}_j(\check{z}) = \tilde{P}_j(\frac{1}{\check{z}}), \qquad 1 \le j \le n'.$$

Next, from (3.53) and (3.54), we have

$$B = \sum_{j=1}^{n'} \check{\lambda}_j(0)\check{P}_j(0) = \sum_{j=1}^{s'} \sigma'_j Q'_j, \qquad \sigma'_j > 0, \tag{3.55}$$

where the right hand side is the spectral decomposition of B, obtained by combining terms with the same value of $\check{\lambda}_j(0)$, see (2.26). Since B is symmetric, positive semi-definite, we rearrange the order of $\check{\lambda}_j(\check{z})$ to have the following lemma.

LEMMA 3.8. *The eigenvalues $\check{\lambda}_j(\check{z})$ and the corresponding eigenprojections $\check{P}_j(\check{z})$, $1 \le j \le n'$, of $\check{E}(\check{z})$ are divided into two groups. For $1 \le j \le \check{n}'_1$, each eigenvalue splits away from the eigenvalue zero of B for small $\check{z}$, $\check{\lambda}_j(0) = 0$, and the corresponding $\check{P}_j(0)$ is a sub-projection of*

$$Q'_0 = I - \sum_{j=1}^{s'} Q'_j = \sum_{j=1}^{\check{n}'_1} \check{P}_j(0), \tag{3.56}$$

which is the eigenprojection of B corresponding to the eigenvalue zero. For $\check{n}'_1+1 \le j \le n'$, on the other hand, each $\check{\lambda}_j(0)$ is some $\sigma'_k > 0$, while $\check{P}_j(0)$ is a sub-projection of the corresponding Q'_k.

LEMMA 3.9. *With Q'_0 defined in (3.56), we have*

$$-Q'_0 A Q'_0 = \sum_{j=1}^{\check{n}'_1} \check{\lambda}'_j(0)\check{P}_j(0). \tag{3.57}$$

Lemma 3.9 indicates that $\check{\lambda}'_j(0)$, $1 \le j \le \check{n}'_1$, are eigenvalues of $Q'_0(-A)Q'_0$ when restricted to the range of Q'_0. By combining terms with the same value of $\check{\lambda}'_j(0)$ on the right , (3.57) gives us the spectral decomposition of $Q'_0(-A)Q'_0$ in the range of

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Q_0'. Thus, letting the distinct $-\check{\lambda}_j'(0)$, $1 \le j \le \check{n}_1'$, be $\alpha_1, \dots, \alpha_{s''}$, we have

$$Q_0' A Q_0' = \sum_{j=1}^{s''} \alpha_j \mathcal{Q}_j, \qquad \sum_{j=1}^{s''} \mathcal{Q}_j = Q_0'. \tag{3.58}$$

By appropriately rearranging the order of $\check{\lambda}_j'(0)$ in (3.57), we have

$$\mathcal{Q}_j = \sum_{k=\check{m}_{j-1}'+1}^{\check{m}_j'} \check{P}_k(0), \qquad 1 \le j \le s'', \tag{3.59}$$

with $\check{m}_0' = 0$ and $\check{m}_{s''}' = \check{n}_1'$.

Next, using the notations in (3.55), we define

$$S' = \sum_{j=1}^{s'} \frac{1}{\sigma_j'} Q_j'. \tag{3.60}$$

LEMMA 3.10. *For each* $1 \le j \le s''$,

$$\mathcal{Q}_j (L - AS'A)\mathcal{Q}_j = \frac{1}{2} \sum_{k=\check{m}_{j-1}'+1}^{\check{m}_j'} \check{\lambda}_k''(0) \check{P}_k(0), \tag{3.61}$$

with $\check{\lambda}_k''(0) < 0$, $\check{m}_{j-1}' + 1 \le k \le \check{m}_j'$.

In (3.61), by combining terms on the right-hand side with the same value of $\check{\lambda}_k''(0)$, we obtain the spectral decomposition of the matrix on the left-hand side in the range of $\mathcal{Q}_j$. We thus write

$$\mathcal{Q}_j (AS'A - L)\mathcal{Q}_j = \sum_{l=1}^{s_j} \beta_{jl} \mathcal{Q}_{jl}, \qquad 1 \le j \le s'', \tag{3.62}$$

where each β_{jl} is a distinct $-\frac{1}{2}\check{\lambda}_k''(0)$, $\check{m}_{j-1}' + 1 \le k \le \check{m}_j'$, $\mathcal{Q}_{jl}$ is the sum of the corresponding $\check{P}_k(0)$'s that share the same value of $-\frac{1}{2}\check{\lambda}_k''(0)$, and $\sum_{l=1}^{s_j} \mathcal{Q}_{jl} = \sum_{k=\check{m}_{j-1}'+1}^{\check{m}_j'} \check{P}_k(0) = \mathcal{Q}_j$. We note that all $\beta_{jl} > 0$.

We then collect all distinct pairs (α_j, β_{jl}), $1 \le j \le s''$, $1 \le l \le s_j$. The total number of them is

$$m' = \sum_{j=1}^{s''} s_j. \tag{3.63}$$

We renames the triples $(\alpha_j, \beta_{jl}, \mathcal{Q}_{jl})$, $1 \le j \le s''$, $1 \le l \le s_j$, as $(\alpha_j, \beta_j, Q_{j,0})$, $1 \le j \le m'$. Together with m' defined in (3.63), these are parameters appearing in the third term on the right-hand side of (2.5). Clearly, from the construction described above, they are obtained through Steps 3 and 4 in Procedure 2.9. In

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particular, if all α_j's in (3.58) are simple, the process is reduced to Step $4'$ in Appendix A.

We combine (3.52) and (3.54) to write

$$E(z)=\sum_{j=1}^{\check{n}_1'} z^2\check{\lambda}_j(\frac{1}{z})\check{P}_j(\frac{1}{z})+\sum_{j=\check{n}_1'+1}^{n'} z^2\check{\lambda}_j(\frac{1}{z})\check{P}_j(\frac{1}{z}). \tag{3.64}$$

Applying Lemma 3.8, we have

$$z^2\check{\lambda}_j(\frac{1}{z})=\begin{cases}\check{\lambda}_j'(0)z+\frac{1}{2}\check{\lambda}_j''(0)+\frac{1}{6}\check{\lambda}_j'''(0)\frac{1}{z}+\cdots, & 1\le j\le \check{n}_1',\\ \check{\lambda}_j(0)z^2+\check{\lambda}_j'(0)z+\cdots, & \check{n}_1'+1\le j\le n',\end{cases} \tag{3.65}$$

and for $\check{n}_1'+1\le j\le n'$, each $\check{\lambda}_j(0)$ is a positive eigenvalue of B.

Using (3.64) and (3.65), we extract leading terms of $E(z)$ as $z\to\infty$ as follows. Let

$$E_0^{\#}(z)=\sum_{j=1}^{\check{n}_1'}\Big[\check{\lambda}_j'(0)z+\frac{1}{2}\check{\lambda}_j''(0)\Big]\check{P}_j(0). \tag{3.66}$$

From (3.57)-(3.59), (3.61)-(3.63) and the definition of $(\alpha_j,\beta_j,Q_{j,0})$, $1\le j\le m'$, we have

$$\begin{aligned}E_0^{\#}(z)&=z\sum_{j=1}^{\check{n}_1'}\check{\lambda}_j'(0)\check{P}_j(0)+\sum_{j=1}^{\check{n}_1'}\frac{1}{2}\check{\lambda}_j''(0)\check{P}_j(0)\\&=-z\sum_{j=1}^{s''}\alpha_j\mathcal{Q}_j+\sum_{j=1}^{s''}\sum_{k=\check{m}_{j-1}'+1}^{\check{m}_j'}\frac{1}{2}\check{\lambda}_k''(0)\check{P}_k(0)\\&=-z\sum_{j=1}^{s''}\alpha_j\sum_{l=1}^{s_j}\mathcal{Q}_{jl}-\sum_{j=1}^{s''}\sum_{l=1}^{s_j}\beta_{jl}\mathcal{Q}_{jl}\\&=-\sum_{j=1}^{s''}\sum_{l=1}^{s_j}(\alpha_j z+\beta_{jl})\mathcal{Q}_{jl}\\&=-\sum_{j=1}^{m'}(\alpha_j z+\beta_j)Q_{j,0}.\end{aligned} \tag{3.67}$$

Consider the semigroup generated by $E_0^{\#}(i\xi)$,

$$\hat{G}_0^{\#}(\xi,t)=e^{tE_0^{\#}(i\xi)}=\sum_{j=1}^{m'}e^{(-\alpha_j i\xi-\beta_j)t}Q_{j,0}. \tag{3.68}$$

Its inverse Fourier transform is

$$G_0^{\#}(x,t)=\sum_{j=1}^{m'}e^{-\beta_j t}\delta(x-\alpha_j t)Q_{j,0},$$

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which is the third term on the right-hand side of (2.5) when $l = 0$. For $l \geq 1$, the inverse transform of $(i\xi)^l \hat{G}_0^{\#}(\xi, t)$ gives the lth derivatives of the δ-functions. To completely extract the singular part, i.e., the third term on the right-hand side of (2.5), we need an expansion of $(i\xi)^l \hat{G}(\xi, t)$ into higher-order terms as follows.

From (3.6), (3.64) and (3.65), for $\xi \to \infty$ we have

$$\begin{aligned}
(i\xi)^l \hat{G}(\xi,t) =& \sum_{j=1}^{\check{n}_1'} (i\xi)^l e^{(i\xi)^2 \check{\lambda}_j(1/i\xi)t} \check{P}_j(\frac{1}{i\xi}) \\
&+ \sum_{j=\check{n}_1'+1}^{n'} (i\xi)^l e^{(i\xi)^2 \check{\lambda}_j(1/i\xi)t} \check{P}_j(\frac{1}{i\xi}) \\
=& \sum_{j=1}^{\check{n}_1'} e^{[i\xi\check{\lambda}_j'(0)+\frac{1}{2}\check{\lambda}_j''(0)]t} (i\xi)^l \check{P}_j(\frac{1}{i\xi}) \\
&+ \sum_{j=1}^{\check{n}_1'} e^{[i\xi\check{\lambda}_j'(0)+\frac{1}{2}\check{\lambda}_j''(0)]t} \Big\{ e^{[(i\xi)^2\check{\lambda}_j(1/i\xi) - i\xi\check{\lambda}_j'(0) - \frac{1}{2}\check{\lambda}_j''(0)]t} - 1 \Big\} (i\xi)^l \check{P}_j(\frac{1}{i\xi}) \\
&+ \sum_{j=\check{n}_1'+1}^{n'} O(1)(i\xi)^l e^{[-\xi^2\check{\lambda}_j(0)+i\xi\check{\lambda}_j'(0)]t + O(1)t}.
\end{aligned}$$

We take Taylor expansion on the right-hand side to arrive at

$$\begin{aligned}
(i\xi)^l \hat{G}(\xi,t) =& \sum_{j=1}^{\check{n}_1'} e^{[i\xi\check{\lambda}_j'(0)+\frac{1}{2}\check{\lambda}_j''(0)]t} \Big[(i\xi)^l \check{P}_j(0) + \sum_{k=1}^{l} (i\xi)^{l-k} Q_{j,k}(t) \Big] \\
&+ \sum_{j=1}^{\check{n}_1'} e^{[i\xi\check{\lambda}_j'(0)+\frac{1}{2}\check{\lambda}_j''(0)]t} (i\xi)^{-1} Q_{j,l+1}(t) \\
&+ \sum_{j=1}^{\check{n}_1'} e^{[i\xi\check{\lambda}_j'(0)+\frac{1}{2}\check{\lambda}_j''(0)]t} \Big[O(1)(i\xi)^{-2}(1 + t + \cdots + t^{l+1} + t^{l+2} e^{O(t/i\xi)}) \Big] \\
&+ \sum_{j=\check{n}_1'+1}^{n'} O(1)(i\xi)^l e^{[-\xi^2\check{\lambda}_j(0)+i\xi\check{\lambda}_j'(0)]t + O(1)t},
\end{aligned} \tag{3.69}$$

where we have used $Q_{j,k}(t)$ to denote an $n \times n$ matrix whose entries are polynomials in t with degrees not larger than k. Similar to the derivation of $\hat{G}_0^{\#}$, see (3.66)-(3.68),

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we further simplify (3.69) to

$$\begin{aligned}(i\xi)^l\hat{G}(\xi,t) =& \sum_{j=1}^{m'} e^{(-\alpha_j i\xi-\beta_j)t}\sum_{k=0}^{l}(i\xi)^{l-k}Q_{j,k}(t) + \sum_{j=1}^{m'} e^{(-\alpha_j i\xi-\beta_j)t}(i\xi)^{-1}Q_{j,l+1}(t)\\ &+ O(1)e^{-\beta t}\sum_{j=1}^{m'} e^{-\alpha_j i\xi t}(i\xi)^{-2}\\ &+ \sum_{j=\tilde{n}_1'+1}^{n'} O(1)(i\xi)^l e^{[-\xi^2\tilde{\lambda}_j(0)+i\xi\tilde{\lambda}_j'(0)]t+O(1)t}\\ \equiv& \hat{G}_l^{\#}(\xi,t)+\hat{R}_{l1}^{\#}(\xi,t)+\hat{R}_{l2}^{\#}(\xi,t)+\hat{R}_{l3}^{\#}(\xi,t),\end{aligned} \tag{3.70}$$

where $\beta = \frac{1}{2}\min_{1\le j\le m'}\beta_j > 0$.

The leading term on the right-hand side of (3.70) is $\hat{G}_l^{\#}(\xi,t)$. Taking the inverse Fourier transform we have

$$G_l^{\#}(x,t) = \sum_{j=1}^{m'} e^{-\beta_j t}\sum_{k=0}^{l}\delta^{(l-k)}(x-\alpha_j t)Q_{j,k}(t). \tag{3.71}$$

This is the third term on the right-hand side of (2.5).

In summary, in this section, we have constructed the leading terms of $(i\xi)^l\hat{G}^(\xi,t)$ for both small ξ and large ξ, respectively. The former is $(i\xi)^l\hat{G}^*(\xi,t)$, and the latter is $\hat{G}_l^{\#}(\xi,t)$, see (3.48) and (3.70).

4. Estimation on inverse Fourier transforms

Based on the spectral analysis carried out in the Fourier space in Section 3, we now consider inverse transforms to obtain estimates for the Green's function and its derivatives in the physical space. For convenience, we use C for a generic, positive constant in this section.

4.1. Proof of Theorem 2.3

With the key estimates (3.48) and (3.70), most of this part is parallel to the case of hyperbolic-parabolic conservation laws, discussed in [14]. We include this part for completeness, with some updates or simplifications. Readers are referred to [14] for more details.

With $G^*(x,t)$ and $G_l^{\#}(x,t)$ defined in (3.49) and (3.71), respectively, we set

$$R^{(l)}(x,t) = \frac{\partial^l}{\partial x^l}G(x,t) - \frac{\partial^l}{\partial x^l}G^*(x,t) - G_l^{\#}(x,t). \tag{4.1}$$

We only need to show that $R^{(l)}(x,t)$ is the second term on the right-hand side of (2.5) to prove Theorem 2.3.

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For this, we use inverse Fourier transforms as follows,

$$\begin{aligned} R^{(l)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}\hat{R}^{(l)}(\xi,t)\,d\xi \\ =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}[(i\xi)^l\hat{G}(\xi,t) - (i\xi)^l\hat{G}^*(\xi,t) - \hat{G}_l^{\#}(\xi,t)]\,d\xi \\ =& R_1^{(l)}(x,t) + R_2^{(l)}(x,t) + R_3^{(l)}(x,t), \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} R_1^{(l)}(x,t) =& \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} e^{ix\xi}(i\xi)^l[\hat{G}(\xi,t) - \hat{G}^*(\xi,t)]\,d\xi, \\ R_2^{(l)}(x,t) =& \frac{1}{2\pi}\left(\int_{-\infty}^{-N} + \int_{N}^{\infty}\right) e^{ix\xi}[(i\xi)^l\hat{G}(\xi,t) - \hat{G}_l^{\#}(\xi,t)]\,d\xi, \\ R_3^{(l)}(x,t) =& \frac{1}{2\pi}\left(\int_{-N}^{-\epsilon} + \int_{\epsilon}^{N}\right) e^{ix\xi}(i\xi)^l\hat{G}(\xi,t)\,d\xi \\ & -\frac{1}{2\pi}\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right) e^{ix\xi}(i\xi)^l\hat{G}^*(\xi,t)\,d\xi - \frac{1}{2\pi}\int_{-N}^{N} e^{ix\xi}\hat{G}_l^{\#}(\xi,t)\,d\xi, \end{aligned} \tag{4.3}$$

and $\epsilon > 0$ small and $N > 0$ large are to be chosen.

To estimate $R_1^{(l)}$ we recall (3.6),(3.47) and (3.48) to have

$$\begin{aligned} R_1^{(l)}(x,t) =& \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} e^{ix\xi}(i\xi)^l[e^{tE(i\xi)} - e^{tE^*(i\xi)}]\,d\xi \\ =& \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} e^{ix\xi}(i\xi)^l\left\{\sum_{j=1}^{n_1'} e^{\tilde{\lambda}_j(i\xi)t}\tilde{P}_j(i\xi) + \sum_{j=n_1'+1}^{n'} e^{\tilde{\lambda}_j(i\xi)t}\tilde{P}_j(i\xi)\right. \\ & \left. - \sum_{j=1}^{n_1'} e^{[\tilde{\lambda}_j'(0)i\xi - \frac{1}{2}\tilde{\lambda}_j''(0)\xi^2]t}\tilde{P}_j(0)\right\} d\xi \\ \equiv& \sum_{j=1}^{n_1'} R_{1,j}^{(l)}(x,t) + \sum_{j=n_1'+1}^{n'} R_{1,j}^{(l)}(x,t), \end{aligned} \tag{4.4}$$

where for $1 \le j \le n_1'$,

$$R_{1,j}^{(l)}(x,t) = \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} e^{ix\xi}(i\xi)^l\{e^{\tilde{\lambda}_j(i\xi)t}\tilde{P}_j(i\xi) - e^{[\tilde{\lambda}_j'(0)i\xi - \frac{1}{2}\tilde{\lambda}_j''(0)\xi^2]t}\tilde{P}_j(0)\}\,d\xi, \tag{4.5}$$

and for $n_1' + 1 \le j \le n'$,

$$R_{1,j}^{(l)}(x,t) = \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} e^{ix\xi}(i\xi)^l e^{\tilde{\lambda}_j(i\xi)t}\tilde{P}_j(i\xi)\,d\xi. \tag{4.6}$$

We note that $\tilde{\lambda}_j(z)$ and $\tilde{P}_j(z)$ are holomorphic at the origin. By taking ϵ small, $\tilde{\lambda}_j(z)$ and $\tilde{P}_j(z)$ are holomorphic in a 2ϵ-neighbourhood of the origin, which allows

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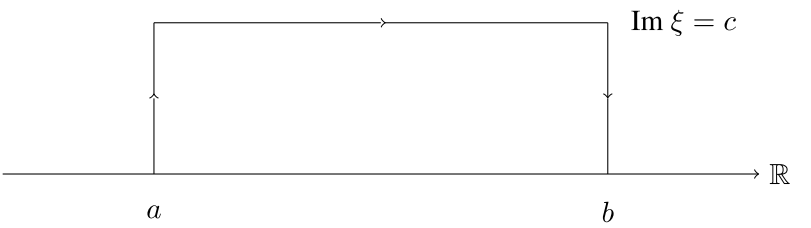


Figure 1. Integration path for inverse Fourier transforms.

us to apply the Cauchy integral theorem there. For this, we regard ξ as a complex variable. For our convenience, we also introduce the path

$$\begin{aligned}\sigma(a,b,c) =&\{\xi \,|\, \mathrm{Re}\,\xi = a, \mathrm{Im}\,\xi \text{ is from } 0 \text{ to } c\} \cup \{\xi \,|\, \mathrm{Im}\,\xi = c, \mathrm{Re}\,\xi \text{ is from } a \text{ to } b\}\\ &\cup \{\xi \,|\, \mathrm{Re}\,\xi = b, \mathrm{Im}\,\xi \text{ is from } c \text{ to } 0\}\end{aligned} \tag{4.7}$$

for $a, b, c \in \mathbb{R}$, see figure 1.

We first estimate $R_{1,j}^{(l)}$ for $1 \le j \le n_1'$, see (4.5). If ξ is in the 2ϵ-neighbourhood of the origin, we use Taylor expansion to write

$$\begin{aligned}&e^{\tilde{\lambda}_j(i\xi)t}\tilde{P}_j(i\xi) - e^{[\tilde{\lambda}_j'(0)i\xi - \frac{1}{2}\tilde{\lambda}_j''(0)\xi^2]t}\tilde{P}_j(0)\\ =&i\xi e^{[\tilde{\lambda}_j'(0)i\xi - \frac{1}{2}\tilde{\lambda}_j''(0)\xi^2]t}[O(1) + O(1)\xi^2 t e^{O(1)(i\xi)^3 t}]\\ =&i\xi e^{[-\lambda_k^{(r)} i\xi - \mu_k\xi^2]t}[O(1) + O(1)\xi^2 t e^{O(1)(i\xi)^3 t}],\end{aligned} \tag{4.8}$$

where we have applied (3.48), and k is an integer determined by j, $1 \le k \le m$. Now we consider two cases.

Case 1 is for $|x - \lambda_k^{(r)} t|/(2\mu_k t) \le \frac{1}{2}\epsilon$. In this case, we replace $[-\epsilon, \epsilon]$ by $\sigma(-\epsilon, \epsilon, c)$ in (4.5), with $c = (x - \lambda_k^{(r)} t)/(2\mu_k t)$. Note that $\sigma(-\epsilon, \epsilon, c)$ is in the 2ϵ-neighbourhood of the origin. Applying (4.8), we reduce (4.5) to

$$R_{1,j}^{(l)}(x,t) = \frac{1}{2\pi}\int_{\sigma(-\epsilon,\epsilon,c)} (i\xi)^{l+1} e^{i\xi(x-\lambda_k^{(r)}t) - \mu_k\xi^2 t}[O(1) + O(1)\xi^2 t e^{O(1)(i\xi)^3 t}]\, d\xi. \tag{4.9}$$

Along the middle part of $\sigma(-\epsilon, \epsilon, c)$, with the specific choice of c, we have $\xi = \zeta + i(x - \lambda_k^{(r)} t)/(2\mu_k t)$, and hence

$$i\xi(x - \lambda_k^{(r)} t) = i\zeta(x - \lambda_k^{(r)} t) - \frac{(x - \lambda_k^{(r)} t)^2}{2\mu_k t}, \qquad -\epsilon \le \zeta \le \epsilon. \tag{4.10}$$

Similarly, along the first and the third parts of $\sigma(-\epsilon, \epsilon, c)$, $\xi = \mp\epsilon + i\eta$, and

$$i\xi(x - \lambda_k^{(r)} t) = \mp i\epsilon(x - \lambda_k^{(r)} t) - (x - \lambda_k^{(r)} t)\eta, \qquad \eta \text{ between } 0 \text{ and } c. \tag{4.11}$$

We note that in (4.11), $(x - \lambda_k^{(r)} t)\eta \ge 0$.

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Substituting (4.10) and (4.11) into (4.9) and taking $\epsilon > 0$ small, we have

$$|R_{1,j}^{(l)}(x,t)| \leq C\int_{-\epsilon}^{\epsilon}\left[|\zeta|^{l+1}+\left(\frac{|x-\lambda_k^{(r)}t|}{2\mu_k t}\right)^{l+1}\right]e^{-\frac{(x-\lambda_k^{(r)}t)^2}{4\mu_k' t}-\frac{\mu_k}{2}\zeta^2 t}$$
$$\times\left[1+\zeta^2 t+\frac{(x-\lambda_k^{(r)}t)^2}{t}\right]d\zeta + C\epsilon^{l+2}e^{-\frac{3}{4}\epsilon^2\mu_k t}[1+\epsilon^2 t e^{C\epsilon^3 t}],$$

where $\mu_k' > \mu_k$ is a generic constant. Thus, by adjusting μ_k' slightly larger, we further have

$$|R_{1,j}^{(l)}(x,t)| \leq C\int_{-\epsilon}^{\epsilon}\left[|\zeta|^{l+1}+\left(\frac{|x-\lambda_k^{(r)}t|}{2\mu_k t}\right)^{l+1}\right]e^{-\frac{(x-\lambda_k^{(r)}t)^2}{4\mu_k' t}-\frac{\mu_k}{4}\zeta^2 t}d\zeta + C\epsilon^{l+2}e^{-\frac{1}{2}\epsilon^2\mu_k t}. \tag{4.12}$$

Now we consider $t \gtrsim 1$. If $t \geq 1$, we use $e^{-\frac{\mu_k}{4}\zeta^2 t}$ to absorb $\zeta^2 t$ and $e^{-\frac{(x-\lambda_k^{(r)}t)^2}{4\mu_k' t}}$ to absorb $(x-\lambda_k^{(r)}t)^2/(4\mu_k^2 t)$. Then we evaluate the integral in (4.12) to arrive at

$$|R_{1,j}^{(l)}(x,t)| \leq C\int_{-\epsilon}^{\epsilon} t^{-\frac{l+1}{2}}e^{-\frac{(x-\lambda_k^{(r)}t)^2}{4\mu_k' t}-\frac{\mu_k}{8}\zeta^2 t}d\zeta + Ce^{-t/C}$$
$$\leq Ct^{-\frac{l+2}{2}}e^{-\frac{(x-\lambda_k^{(r)}t)^2}{4\mu_k' t}} + Ce^{-t/C} \leq C(t+1)^{-\frac{l+2}{2}}e^{-\frac{(x-\lambda_k^{(r)}t)^2}{4\mu_k' t}} + Ce^{-t/C}.$$

If $t \leq 1$, (4.12) gives us

$$|R_{1,j}^{(l)}(x,t)| \leq C\epsilon^{l+2}e^{-\frac{(x-\lambda_k^{(r)}t)^2}{4\mu_k' t}} + C\epsilon^{l+2}e^{-\frac{1}{2}\epsilon^2\mu_k t}$$
$$\leq C(t+1)^{-\frac{l+2}{2}}e^{-\frac{(x-\lambda_k^{(r)}t)^2}{4\mu_k' t}} + Ce^{-t/C}.$$

Therefore, we obtain a key estimate in Case 1: for $t > 0$, $x \in \mathbb{R}$,

$$|R_{1,j}^{(l)}(x,t)| \leq C(t+1)^{-\frac{l+2}{2}}e^{-\frac{(x-\lambda_k^{(r)}t)^2}{4\mu_k' t}} + Ce^{-t/C}, \qquad 1 \leq j \leq n_1'. \tag{4.13}$$

Case 2 is for $|x-\lambda_k^{(r)}t|/(2\mu_k t) \geq \frac{1}{2}\epsilon$. In this case, we replace $[-\epsilon,\epsilon]$ by $\sigma(-\epsilon,\epsilon,c)$ in (4.5), with $c = \frac{\epsilon}{2}\mathrm{sign}(x-\lambda_k^{(r)}t)$. Since $\sigma(-\epsilon,\epsilon,c)$ is in the 2ϵ-neighbourhood of

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the origin, we still have (4.9) for $1 \le j \le n_1'$. Similar to the derivation of (4.12),

$$\begin{aligned}|R_{1,j}^{(l)}(x,t)| \le & \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon}\left[|\zeta| + \left(\frac{\epsilon}{2}\right)\right]^{l+1} e^{-\frac{\epsilon}{2}|x-\lambda_k^{(r)}t| - \mu_k\zeta^2 t + \frac{\epsilon^2}{4}\mu_k t} \\ & \left[O(1) + O(1)\epsilon^2 t e^{C\epsilon^3 t}\right] d\zeta + \frac{1}{\pi}\int_0^{\frac{\epsilon}{2}} (\eta+\epsilon)^{l+1} e^{-\eta|x-\lambda_k^{(r)}t| - \epsilon^2\mu_k t + \eta^2\mu_k t} \\ & \left[O(1) + O(1)\epsilon^2 t e^{C\epsilon^3 t}\right] d\eta \\ \le & C\int_{-\epsilon}^{\epsilon} \epsilon^{l+1} e^{-\frac{1}{4}\epsilon^2\mu_k t - \mu_k\zeta^2 t}(1+\epsilon^2 t e^{C\epsilon^3 t})\, d\zeta + Ce^{-t/C} \le Ce^{-t/C}.\end{aligned} \tag{4.14}$$

Combining (4.13) and (4.14), for $1 \le j \le n_1'$ we have

$$|R_{1,j}^{(l)}(x,t)| \le C(t+1)^{-\frac{l+2}{2}} e^{-\frac{(x-\lambda_k^{(r)}t)^2}{4\mu_k' t}} + Ce^{-t/C}, \qquad x \in \mathbb{R}, \quad t > 0. \tag{4.15}$$

Now we estimate $R_{1,j}^{(l)}$ for $n_1' + 1 \le j \le n'$. From (4.6), we have

$$R_{1,j}^{(l)}(x,t) = \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} (i\xi)^l e^{ix\xi + \tilde{\lambda}_j(0)t + O(i\xi)t} O(1)\, d\xi. \tag{4.16}$$

Noting that for $n_1' + 1 \le j \le n'$, $\tilde{\lambda}_j(z)$ is in group two, see the definition of n_1' beneath Lemma 3.4. Then by Lemma 3.3, each $\tilde{\lambda}_j(0)$ is some $-\sigma_k < 0$, a negative eigenvalue of L. Therefore, (4.16) gives us

$$|R_{1,j}^{(l)}(x,t)| \le Ce^{-t/C}, \qquad t \ge 0, \tag{4.17}$$

for $n_1' + 1 \le j \le n'$.

Substituting (4.15) and (4.17) into (4.4), we arrive at

$$|R_1^{(l)}(x,t)| \le C\sum_{k=1}^{m}(t+1)^{-\frac{l+2}{2}} e^{-\frac{(x-\lambda_k^{(r)}t)^2}{4\mu_k' t}} + Ce^{-t/C}, \qquad x \in \mathbb{R}, \quad t > 0. \tag{4.18}$$

Next, we consider $R_2^{(l)}$ in (4.3). Substituting (3.70) into (4.3) gives us

$$\begin{aligned}R_2^{(l)}(x,t) = & \frac{1}{2\pi}\left(\int_{-\infty}^{-N} + \int_N^{\infty}\right) e^{ix\xi}[\hat{R}_{l1}^{\#}(\xi,t) + \hat{R}_{l2}^{\#}(\xi,t) + \hat{R}_{l3}^{\#}(\xi,t)]\, d\xi \\ \equiv & R_{21}^{(l)}(x,t) + R_{22}^{(l)}(x,t) + R_{23}^{(l)}(x,t),\end{aligned} \tag{4.19}$$

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where

$$
\begin{aligned}
R_{21}^{(l)}(x,t) &= \frac{1}{2\pi}\left(\int_{-\infty}^{-N}+\int_{N}^{\infty}\right) e^{ix\xi}\hat{R}_{l1}^{\#}(\xi,t)\,d\xi \\
&= \frac{1}{2\pi}\left(\int_{-\infty}^{-N}+\int_{N}^{\infty}\right) \sum_{j=1}^{m'} e^{i(x-\alpha_j t)\xi-\beta_j t}(i\xi)^{-1}\,d\xi Q_{j,l+1}(t), \\
R_{22}^{(l)}(x,t) &= \frac{1}{2\pi}\left(\int_{-\infty}^{-N}+\int_{N}^{\infty}\right) e^{ix\xi}\hat{R}_{l2}^{\#}(\xi,t)\,d\xi \\
&= e^{-\beta t}\left(\int_{-\infty}^{-N}+\int_{N}^{\infty}\right) O(1)\sum_{j=1}^{m'} e^{i(x-\alpha_j t)\xi}(i\xi)^{-2}\,d\xi, \\
R_{23}^{(l)}(x,t) &= \frac{1}{2\pi}\left(\int_{-\infty}^{-N}+\int_{N}^{\infty}\right) e^{ix\xi}\hat{R}_{l3}^{\#}(\xi,t)\,d\xi \\
&= \left(\int_{-\infty}^{-N}+\int_{N}^{\infty}\right) \sum_{j=\check{n}_1'+1}^{n'} O(1)(i\xi)^l e^{-\xi^2\check{\lambda}_j(0)t+i(x+\check{\lambda}_j'(0)t)\xi+O(1)t}\,d\xi,
\end{aligned}
$$

where $Q_{j,l+1}(t)$ are $n\times n$ matrices whose entries are polynomials in t with degrees not larger than $l+1$; $\beta>0$ is a constant; and $\check{\lambda}_j(0)$ are positive eigenvalues of B for $\check{n}_1'+1\le j\le n'$.

Noting the integral in $R_{21}^{(l)}$ is for the principal value and by properties of the sinc function, we have

$$
|R_{21}^{(l)}(x,t)| \le \frac{1}{2\pi}\sum_{j=1}^{m'} e^{-\beta_j t}\left|\left(\int_{-\infty}^{-N}+\int_{N}^{\infty}\right)\frac{\sin((x-\alpha_j t)\xi)}{\xi}\,d\xi\right| |Q_{j,l+1}(t)| \le Ce^{-t/C}.
$$

It is also straightforward to have

$$
|R_{22}^{(l)}(x,t)| \le Ce^{-t/C}, \qquad |R_{23}^{(l)}(x,t)| \le Ct^{-\frac{l+1}{2}}e^{-t/C}.
$$

Thus,

$$
|R_{2}^{(l)}(x,t)| \le Ct^{-\frac{l+1}{2}}e^{-t/C} + Ce^{-t/C}. \tag{4.20}
$$

To consider $R_3^{(l)}$ in (4.3), we write

$$
R_3^{(l)}(x,t) = R_{31}^{(l)}(x,t) + R_{32}^{(l)}(x,t) + R_{33}^{(l)}(x,t), \tag{4.21}
$$

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where

$$\begin{aligned} R_{31}^{(l)}(x,t) &= \frac{1}{2\pi}\left(\int_{-N}^{-\epsilon}+\int_{\epsilon}^{N}\right) e^{ix\xi}(i\xi)^l \hat{G}(\xi,t)\,d\xi, \\ R_{32}^{(l)}(x,t) &= -\frac{1}{2\pi}\left(\int_{-\infty}^{-\epsilon}+\int_{\epsilon}^{\infty}\right) e^{ix\xi}(i\xi)^l \hat{G}^*(\xi,t)\,d\xi, \\ R_{33}^{(l)}(x,t) &= -\frac{1}{2\pi}\int_{-N}^{N} e^{ix\xi}\hat{G}_l^{\#}(\xi,t)\,d\xi. \end{aligned} \tag{4.22}$$

We also cite Lemma 3.2 in [34], which modifies a result from [9].

LEMMA 4.1. *There exist positive constants C and c such that*

$$|e^{tE(i\xi)}| \le Ce^{-\frac{c\xi^2 t}{1+\xi^2}}, \qquad \xi\in\mathbb{R}, \quad t\ge 0. \tag{4.23}$$

Applying (3.6) and (4.23) to $(4.22)_1$, we have

$$\begin{aligned} |R_{31}^{(l)}(x,t)| &= \frac{1}{2\pi}\left|\left(\int_{-N}^{-\epsilon}+\int_{\epsilon}^{N}\right) e^{ix\xi}(i\xi)^l e^{tE(i\xi)}\,d\xi\right| \\ &\le C\left(\int_{-N}^{-\epsilon}+\int_{\epsilon}^{N}\right)|\xi|^l e^{-\frac{c\xi^2 t}{1+\xi^2}}\,d\xi \le Ce^{-t/C}. \end{aligned} \tag{4.24}$$

Applying (3.48) and (3.70) to $(4.22)_2$ and $(4.22)_3$, respectively, we have

$$\begin{aligned} |R_{32}^{(l)}(x,t)| &\le C\left(\int_{-\infty}^{-\epsilon}+\int_{\epsilon}^{\infty}\right)|\xi|^l\sum_{j=1}^{m} e^{-\mu_j\xi^2 t}\,d\xi \le Ct^{-\frac{l+1}{2}}e^{-t/C}, \\ |R_{33}^{(l)}(x,t)| &\le C\int_{-N}^{N}\sum_{j=1}^{m'} e^{-\beta_j t}\sum_{k=0}^{l}|\xi|^{l-k}|Q_{j,k}(t)|\,d\xi \le Ce^{-t/C}. \end{aligned} \tag{4.25}$$

Thus, (4.21) gives us

$$|R_3^{(l)}(x,t)| \le Ct^{-\frac{l+1}{2}}e^{-t/C} + Ce^{-t/C}. \tag{4.26}$$

We now substitute (4.18), (4.20) and (4.26) into (4.2) to arrive at

$$|R^{(l)}(x,t)| \le C\sum_{k=1}^{m}(t+1)^{-\frac{l+2}{2}}e^{-\frac{(x-\lambda_k^{(r)}t)^2}{4\mu_k' t}} + Ce^{-t/C} + Ct^{-\frac{l+1}{2}}e^{-t/C}. \tag{4.27}$$

Here, (4.27) holds for $x\in\mathbb{R}$ and $t>0$.

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Next, we consider large x/t. Suppose $|x/t| \geq K$ for a large positive constant K. We consider

$$\tilde{R}^{(l)}(x,t) \equiv \frac{\partial^l}{\partial x^l} G(x,t) - G_l^{\#}(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi} [(i\xi)^l \hat{G}(\xi,t) - \hat{G}_l^{\#}(\xi,t)] \, d\xi. \quad (4.28)$$

By (3.6) and (3.70),

$$\tilde{R}^{(l)}(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi} [(i\xi)^l e^{tE(i\xi)} - \sum_{j=1}^{m'} e^{(-\alpha_j i\xi - \beta_j)t} \sum_{k=0}^{l} (i\xi)^{l-k} Q_{j,k}(t)] \, d\xi.$$

Noting that the integrand on the right-hand side is an entire function, we apply the Cauchy integral theorem and move the integration path to $\sigma = \sigma(-N, N, \nu x/t)$, where $\nu > 0$ is a constant to be determined and $N > 0$ is large. For large N and K we apply (3.70) on σ. That is

$$\tilde{R}^{(l)}(x,t) = \frac{1}{2\pi} \int_{\sigma} e^{ix\xi} [\hat{R}_{l1}^{\#}(\xi,t) + \hat{R}_{l2}^{\#}(\xi,t) + \hat{R}_{l3}^{\#}(\xi,t)] \, d\xi, \quad (4.29)$$

with $\hat{R}_{l1}^{\#}, \hat{R}_{l2}^{\#}$ and $\hat{R}_{l3}^{\#}$ defined in (3.70). By the formulation of $\hat{R}_{l1}^{\#}, \hat{R}_{l2}^{\#}$ and $\hat{R}_{l3}^{\#}$, it is clear that as $N \to \infty$, the integrals on the line segments $\{\pm N + i\eta \,|\, \eta \text{ between } 0 \text{ and } \nu x/t\}$ approach to zero for fixed x and $t > 0$. Therefore, we further reduce (4.29) to

$$\begin{aligned} \tilde{R}^{(l)}(x,t) &= \frac{1}{2\pi} \int_{\mathrm{Im}\xi = \frac{\nu x}{t}} e^{ix\xi} [\hat{R}_{l1}^{\#}(\xi,t) + \hat{R}_{l2}^{\#}(\xi,t) + \hat{R}_{l3}^{\#}(\xi,t)] \, d\xi \\ &= \tilde{R}_1^{(l)}(x,t) + \tilde{R}_2^{(l)}(x,t) + \tilde{R}_3^{(l)}(x,t), \end{aligned} \quad (4.30)$$

where

$$\tilde{R}_j^{(l)}(x,t) = \frac{1}{2\pi} \int_{\mathrm{Im}\xi = \frac{\nu x}{t}} e^{ix\xi} \hat{R}_{lj}^{\#}(\xi,t) \, d\xi, \qquad 1 \leq j \leq 3. \quad (4.31)$$

From (3.70), we have

$$|\tilde{R}_1^{(l)}(x,t)| \leq \frac{1}{2\pi} \sum_{j=1}^{m'} e^{-\frac{\nu x}{t}(x - \alpha_j t) - \beta_j t} \left| \int_{-\infty}^{\infty} \frac{e^{i\zeta(x-\alpha_j t)}}{i\zeta - \frac{\nu x}{t}} \, d\zeta \right| |Q_{j,l+1}(t)|. \quad (4.32)$$

By algebraic manipulation, we have

$$\frac{1}{i\zeta - \frac{\nu x}{t}} = \frac{1}{i\zeta} - \frac{\frac{\nu x}{t}}{\zeta^2 + (\frac{\nu x}{t})^2} - \frac{1}{i\zeta} \frac{(\frac{\nu x}{t})^2}{\zeta^2 + (\frac{\nu x}{t})^2}.$$

Noting that the integral is for the principal value, we have

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{i\zeta(x-\alpha_j t)}}{i\zeta - \frac{\nu x}{t}} \, d\zeta &= \int_{-\infty}^{\infty} \frac{\sin(\zeta(x - \alpha_j t))}{\zeta} \left[1 - \frac{(\frac{\nu x}{t})^2}{\zeta^2 + (\frac{\nu x}{t})^2} \right] d\zeta \\ &\quad - \int_{-\infty}^{\infty} \frac{\cos(\zeta(x - \alpha_j t))}{\zeta^2 + (\frac{\nu x}{t})^2} \frac{\nu x}{t} \, d\zeta. \end{aligned}$$

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Thus,

$$\begin{aligned}\left|\int_{-\infty}^{\infty}\frac{e^{i\zeta(x-\alpha_j t)}}{i\zeta-\frac{\nu x}{t}}\,d\zeta\right| \le& \pi+\int_{|\zeta|<|x-\alpha_j t|^{-1}}\left|\frac{\sin(\zeta(x-\alpha_j t))}{\zeta}\right|d\zeta\\ &+\int_{|\zeta|>|x-\alpha_j t|^{-1}}\frac{(\frac{\nu x}{t})^2}{|\zeta|[\zeta^2+(\frac{\nu x}{t})^2]}\,d\zeta+\int_{-\infty}^{\infty}\frac{|\frac{\nu x}{t}|}{\zeta^2+(\frac{\nu x}{t})^2}\,d\zeta\\ \le& C+\ln\left[1+\left(\frac{\nu x}{t}\right)^2|x-\alpha_j t|^2\right].\end{aligned}\tag{4.33}$$

Substituting (4.33) into (4.32) and taking $K\ge 3\max_{1\le j\le m'}|\alpha_j|$ give us

$$|\tilde{R}_1^{(l)}(x,t)|\le Ce^{-\frac{\nu x^2}{2t}-t/C}.\tag{4.34}$$

From (3.70) and (4.31), and by taking $K>0$ large and $\nu>0$ such that $\nu\max_{\check{n}_1'+1\le j\le n'}\check{\lambda}_j(0)\le\frac{1}{2}$, we also have

$$\begin{aligned}|\tilde{R}_2^{(l)}(x,t)|\le& Ce^{-\frac{\nu x^2}{2t}-t/C}\int_{-\infty}^{\infty}\frac{d\zeta}{\zeta^2+(\frac{\nu x}{t})^2}\le Ce^{-\frac{\nu x^2}{2t}-t/C},\\ |\tilde{R}_3^{(l)}(x,t)|\le& Ce^{-\frac{\nu x^2}{2t}+C|x|}\sum_{j=\check{n}_1'+1}^{n'}\int_{-\infty}^{\infty}\left(|\zeta|^l+\left|\frac{\nu x}{t}\right|^l\right)e^{-\check{\lambda}_j(0)\zeta^2 t}\,d\zeta\\ \le& Ce^{-\frac{\nu x^2}{2t}+C|x|}\left(t^{-\frac{l+1}{2}}+t^{-\frac{1}{2}}\left|\frac{\nu x}{t}\right|^l\right)\\ \le& Ce^{-\frac{\nu x^2}{2t}+C\frac{x^2}{Kt}}\left(t^{-\frac{l+1}{2}}+t^{-\frac{1}{2}}\left|\frac{\nu x}{t}\right|^l\right)\\ \le& Ce^{-\frac{\nu x^2}{4t}}t^{-\frac{l+1}{2}}.\end{aligned}\tag{4.35}$$

Combining (4.30), (4.34), and (4.35), we arrive at

$$|\tilde{R}^{(l)}(x,t)|\le Ce^{-\frac{\nu x^2}{4t}}(e^{-t/C}+t^{-\frac{l+1}{2}})\tag{4.36}$$

for $|x|/t\ge K$, with $K>0$ large.

We now finish the proof of Theorem 2.3 by showing that $R^{(l)}(x,t)$ is the second term on the right-hand side of (2.5). Fix a large $K>2\max_{1\le j\le m}|\lambda_j^{(r)}|$ such that (4.36) holds. Then for $|x|/t\ge K$,

$$e^{-\frac{\nu x^2}{8t}}\le e^{-\frac{(x-\lambda_1^{(r)}t)^2}{Ct}}e^{\frac{(1+|\lambda_1^{(r)}|/K)^2x^2}{Ct}-\frac{\nu x^2}{8t}}\le e^{-\frac{(x-\lambda_1^{(r)}t)^2}{Ct}}\tag{4.37}$$

by choosing a large $C>0$. Thus, (4.36) is reduced to

$$|\tilde{R}^{(l)}(x,t)|\le C(t+1)^{-\frac{1}{2}}t^{-\frac{l+1}{2}}e^{-\frac{(x-\lambda_1^{(r)}t)^2}{Ct}}.\tag{4.38}$$

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From (4.1), (4.28), (4.38), (3.49), and (2.4), we have

$$\begin{aligned}|R^{(l)}(x,t)| \le & |\tilde{R}^{(l)}(x,t)| + |\frac{\partial^l}{\partial x^l} G^*(x,t)| \\ \le & C(t+1)^{-\frac{1}{2}} t^{-\frac{l+1}{2}} e^{-\frac{(x-\lambda_1^{(r)}t)^2}{Ct}} + \sum_{j=1}^{m} |\frac{\partial^l}{\partial x^l} H(x,t;\lambda_j^{(r)},\mu_j) P_j| \\ \le & C(t+1)^{-\frac{1}{2}} t^{-\frac{l+1}{2}} \sum_{j=1}^{m} e^{-\frac{(x-\lambda_j^{(r)}t)^2}{Ct}}.\end{aligned}$$

This gives us the second term on the right-hand side of (2.5).

For $|x|/t \le K$, on the other hand, we recall (4.27). Note that for a fixed $C_1 > 0$,

$$\begin{aligned}e^{-t/C_1} \le & e^{-\frac{(x-\lambda_1^{(r)}t)^2}{Ct}} e^{\frac{(|x|/t+|\lambda_1^{(r)}|)^2 t}{C} - \frac{t}{C_1}} \\ \le & e^{-\frac{(x-\lambda_1^{(r)}t)^2}{Ct}} e^{\frac{(K+|\lambda_1^{(r)}|)^2 t}{C} - \frac{t}{C_1}} \le e^{-\frac{(x-\lambda_1^{(r)}t)^2}{Ct}}\end{aligned} \tag{4.39}$$

by choosing $C > 0$ large. We thus reduce (4.27) to

$$|R^{(l)}(x,t)| \le C \sum_{k=1}^{m} (t+1)^{-\frac{1}{2}} t^{-\frac{l+1}{2}} e^{-\frac{(x-\lambda_k^{(r)}t)^2}{Ct}}$$

for $|x|/t \le K$. This also gives us the second term on the right-hand side of (2.5). Now we have finished the proof of Theorem 2.3.

4.2. Justification of Remark 2.4

Now we justify Remark 2.4 to remove the singularity $t^{-\frac{l+1}{2}}$ in the special case $B = 0$. For this, we modify $R^{(l)}$ in (4.1) and set

$$\bar{R}^{(l)}(x,t) = \frac{\partial^l}{\partial x^l} G(x,t) - \frac{N}{\pi} \operatorname{sinc}(Nx) * \frac{\partial^l}{\partial x^l} G^*(x,t) - G_l^{\#}(x,t). \tag{4.40}$$

Noting that the Fourier transform of $\frac{N}{\pi} \operatorname{sinc}(Nx)$ is the characteristic function $\chi(|\xi| < N)$, we let

$$\bar{R}^{(l)}(x,t) = R_1^{(l)}(x,t) + R_2^{(l)}(x,t) + \bar{R}_3^{(l)}(x,t), \tag{4.41}$$

where $R_1^{(l)}$ and $R_2^{(l)}$ are defined in (4.3), and

$$\begin{aligned}\bar{R}_3^{(l)}(x,t) = & \frac{1}{2\pi} \left(\int_{-N}^{-\epsilon} + \int_{\epsilon}^{N} \right) e^{ix\xi} (i\xi)^l \hat{G}(\xi,t)\, d\xi \\ & - \frac{1}{2\pi} \left(\int_{-N}^{-\epsilon} + \int_{\epsilon}^{N} \right) e^{ix\xi} (i\xi)^l \hat{G}^*(\xi,t)\, d\xi - \frac{1}{2\pi} \int_{-N}^{N} e^{ix\xi} \hat{G}_l^{\#}(\xi,t)]\, d\xi.\end{aligned}$$

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Comparing with $R_3^{(l)}$ in (4.3), we have

$$\bar{R}_3^{(l)}(x,t) = R_{31}^{(l)}(x,t) + \bar{R}_{32}^{(l)}(x,t) + R_{33}^{(l)}(x,t), \tag{4.42}$$

where $R_{31}^{(l)}$ and $R_{33}^{(l)}$ are defined in (4.22), and

$$\bar{R}_{32}^{(l)}(x,t) = -\frac{1}{2\pi}\left(\int_{-N}^{-\epsilon} + \int_{\epsilon}^{N}\right) e^{ix\xi}(i\xi)^l \hat{G}^*(\xi,t)\, d\xi. \tag{4.43}$$

Next, we note that when $B = 0$, $\hat{R}_{l3}^{\#} = 0$ in (3.70), see Lemma 3.8. This implies $R_{23}^{(l)} = 0$ in (4.19), and hence we modify (4.20) to have

$$|R_2^{(l)}(x,t)| \le Ce^{-t/C}. \tag{4.44}$$

From (4.43) and (3.48) it is straightforward to have

$$|\bar{R}_{32}^{(l)}(x,t)| \le C\left(\int_{-N}^{-\epsilon} + \int_{\epsilon}^{N}\right) |\xi|^l \sum_{j=1}^{m} e^{-\mu_j \xi^2 t}\, d\xi \le Ce^{-t/C}.$$

Thus, comparing $\bar{R}_3^{(l)}$ in (4.42) with $R_3^{(l)}$ in (4.21), we modify (4.26) to have

$$|\bar{R}_3^{(l)}(x,t)| \le Ce^{-t/C}. \tag{4.45}$$

Combining (4.41), (4.18), (4.44), and (4.45) gives us

$$|\bar{R}^{(l)}(x,t)| \le C\sum_{k=1}^{m}(t+1)^{-\frac{l+2}{2}} e^{-\frac{(x-\lambda_k^{(r)}t)^2}{4\mu_k' t}} + Ce^{-t/C}.$$

For $|x|/t \le K$ we apply (4.39) to absorb the last term on the right -hand side and have

$$|\bar{R}^{(l)}(x,t)| \le C\sum_{k=1}^{m}(t+1)^{-\frac{l+2}{2}} e^{-\frac{(x-\lambda_k^{(r)}t)^2}{Ct}}. \tag{4.46}$$

Noting the definition of G^* and $G_l^{\#}$ in (3.49) and (3.71), respectively, (2.6) is justified for the case $|x|/t \le K$ by combining (4.40) and (4.46).

For the case $|x|/t \ge K$ we recall $\tilde{R}^{(l)}(x,t)$ defined in (4.28). Since $\hat{R}_{l3}^{\#} = 0$ for $B = 0$, $\tilde{R}_3^{(l)}$ in (4.30) is zero. Thus, (4.36) becomes

$$|\tilde{R}^{(l)}(x,t)| \le Ce^{-\frac{\nu x^2}{2t} - t/C}.$$

Applying (4.37), we write

$$|\tilde{R}^{(l)}(x,t)| \le C(t+1)^{-\frac{l+2}{2}} e^{-\frac{(x-\lambda_1^{(r)}t)^2}{Ct}}. \tag{4.47}$$

Combining (4.28) and (4.47), we justify (2.6) for the case $|x|/t \ge K$.

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4.3. Proof of Theorem 2.10

Next, we go back to the general case of (2.1) and prove Theorem 2.10 to obtain (2.30) and (2.31). Let $\eta \in \mathbb{R}^n$ be a constant vector such that $\eta^t Q_0 = 0$, where Q_0 is the eigenprojection of L corresponding to the eigenvalue zero. Recall that we only need to prove Theorem 2.10 under Assumption 3.1. Thus, from (3.14) η takes the form

$$\eta^t = (0_{1\times n_1}, \eta_*^t), \tag{4.48}$$

where $\eta_* \in \mathbb{R}^{n_2}$.

From (3.38), (3.48) and (4.48) one has $\eta^t \hat{G}^*(\xi,t) = 0$. Thus, we need to expand $\hat{G}(\xi,t)$ for small ξ into higher orders to capture the leading term of $\eta^t \hat{G}(\xi,t)$. For this, we recall (3.6) and (3.47),

$$\hat{G}(\xi,t) = e^{tE(i\xi)} = \sum_{j=1}^{n_1'} e^{\tilde{\lambda}_j(i\xi)t}\tilde{P}_j(i\xi) + \sum_{j=n_1'+1}^{n'} e^{\tilde{\lambda}_j(i\xi)t}\tilde{P}_j(i\xi). \tag{4.49}$$

For $1 \le j \le n_1'$ we consider

$$\eta^t \tilde{P}_j(i\xi) = \eta^t[\tilde{P}_j(0) + i\xi\tilde{P}_j'(0) + O(1)(i\xi)^2] = i\xi[\eta^t\tilde{P}_j'(0)] + O(1)\xi^2 \equiv i\xi\eta_j^t + O(1)\xi^2. \tag{4.50}$$

Applying (4.49) and (4.50), we have

$$(i\xi)^l \eta^t \hat{G}(\xi,t) = \sum_{j=1}^{n_1'} (i\xi)^l e^{\tilde{\lambda}_j(i\xi)t}[i\xi\eta_j^t + O(1)\xi^2] + \sum_{j=n_1'+1}^{n'} (i\xi)^l e^{\tilde{\lambda}_j(i\xi)t}\eta^t\tilde{P}_j(i\xi). \tag{4.51}$$

Similar to (3.48), the leading term on the right-hand side is

$$(i\xi)^{l+1}\hat{g}^*(\xi,t) \equiv (i\xi)^{l+1}\sum_{j=1}^{n_1'} e^{[\tilde{\lambda}_j'(0)i\xi - \frac{1}{2}\tilde{\lambda}_j''(0)\xi^2]t}\eta_j^t = (i\xi)^{l+1}\sum_{j=1}^{m} e^{(-\lambda_j^{(r)}i\xi - \mu_j\xi^2)t}\eta_j^t, \tag{4.52}$$

where in the last step we sum up those $\eta_j^t = \eta^t\tilde{P}_j'(0)$, $1 \le j \le n_1'$, that share the same pair of $(\lambda_j^{(r)}, \mu_j)$ and still name those corresponding to distinct pairs η_j^t. The new $\eta_j \in \mathbb{R}^n$, $1 \le j \le m$, are constant vectors, appearing in the first term on the right-hand side of (2.30).

From (4.52), we see that its inverse Fourier transform represents the leading term of $\frac{\partial^l}{\partial x^l}(\eta^t G)(x,t)$. This is

$$\frac{\partial^{l+1}}{\partial x^{l+1}} g^*(x,t) = \frac{\partial^{l+1}}{\partial x^{l+1}} \sum_{j=1}^{m} H(x,t;\lambda_j^{(r)},\mu_j)\eta_j^t. \tag{4.53}$$

Thus, parallel to (4.1), we set

$$R_*^{(l)}(x,t) = \frac{\partial^l}{\partial x^l}(\eta^t G)(x,t) - \frac{\partial^{l+1}}{\partial x^{l+1}} g^*(x,t) - \eta^t G_l^{\#}(x,t), \tag{4.54}$$

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and we only need to show that $R_*^{(l)}(x,t)$ is the second term on the right-hand side of (2.30), see (3.71).

For this, we modify (4.2) and write

$$R_*^{(l)}(x,t) = R_{*1}^{(l)}(x,t) + \eta^t R_2^{(l)}(x,t) + R_{*3}^{(l)}(x,t), \tag{4.55}$$

where $R_2^{(l)}$ is defined in (4.3), and

$$\begin{aligned} R_{*1}^{(l)}(x,t) =& \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} e^{ix\xi}[(i\xi)^l \eta^t \hat{G}(\xi,t) - (i\xi)^{l+1}\hat{g}^*(\xi,t)]\,d\xi, \\ R_{*3}^{(l)}(x,t) =& \eta^t R_{31}^{(l)}(x,t) + R_{*32}^{(l)}(x,t) + \eta^t R_{33}^{(l)}(x,t), \end{aligned} \tag{4.56}$$

with $R_{31}^{(l)}(x,t)$ and $R_{33}^{(l)}(x,t)$ defined in (4.22), and

$$R_{*32}^{(l)}(x,t) = -\frac{1}{2\pi}\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right) e^{ix\xi}(i\xi)^{l+1}\hat{g}^*(\xi,t)\,d\xi.$$

Noting the similarity between $(i\xi)^{l+1}\hat{g}^*$ to $(i\xi)^l\eta^t\hat{G}$ and $(i\xi)^l\hat{G}^*$ to $(i\xi)^l\hat{G}$, we follow the argument that leads (4.3) to (4.18). First, by substituting (4.51) and (4.52) into (4.56), we have

$$\begin{aligned} R_{*1}^{(l)}(x,t) =& \sum_{j=1}^{n_1'} \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} e^{ix\xi}(i\xi)^l \{e^{\tilde{\lambda}_j(i\xi)t}[i\xi\eta_j^t + O(1)\xi^2] - e^{[\tilde{\lambda}_j'(0)i\xi - \frac{1}{2}\tilde{\lambda}_j''(0)\xi^2]t} i\xi\eta_j^t\}\,d\xi \\ &+ \sum_{j=n_1'+1}^{n'} \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} e^{ix\xi}(i\xi)^l e^{\tilde{\lambda}_j(i\xi)t}\eta^t\tilde{P}_j(i\xi)\,d\xi \\ \equiv& \sum_{j=1}^{n_1'} R_{*1,j}^{(l)}(x,t) + \sum_{j=n_1'+1}^{n'} R_{*1,j}^{(l)}(x,t). \end{aligned} \tag{4.57}$$

By Taylor expansion we simplify $R_{*1,j}^{(l)}$ for $1 \le j \le n_1'$ as

$$R_{*1,j}^{(l)}(x,t) = \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} e^{ix\xi}(i\xi)^{l+2} e^{(-\lambda_k^{(r)} i\xi - \mu_k\xi^2)t}[O(1) + O(1)\xi^2 t e^{O(1)\xi^3 t}]\,d\xi,$$

where $1 \le k \le m$ is some integer. After moving the path of integration, we note that $R_{*1,j}^{(l)}$, $1 \le j \le n_1'$, are parallel to $R_{1,j}^{(l)}$ in (4.9), with an extra factor $i\xi$ in the integrand. On the other hand, for $n_1'+1 \le j \le n'$, $R_{*1,j}^{(l)}$ are identical to $\eta^t R_{1,j}^{(l)}$, see (4.6). Thus, following the estimation for $R_1^{(l)}$ that leads to (4.18), we have

$$|R_{*1}^{(l)}(x,t)| \le C\sum_{k=1}^{m}(t+1)^{-\frac{l+3}{2}} e^{-\frac{(x-\lambda_k^{(r)}t)^2}{4\mu_k' t}} + Ce^{-t/C} \tag{4.58}$$

for $x \in \mathbb{R}$ and $t > 0$.

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For the second term on the right-hand side of (4.55), we apply (4.20) to have

$$|\eta^t R_2^{(l)}(x,t)| \le Ct^{-\frac{l+1}{2}} e^{-t/C} + Ce^{-t/C}. \tag{4.59}$$

For the third term, we follow the estimation of $R_{32}^{(l)}$ (see (4.25)) to have

$$|R_{*32}^{(l)}(x,t)| \le Ct^{-\frac{l+2}{2}} e^{-t/C}. \tag{4.60}$$

Substituting (4.24), (4.25) and (4.60) into (4.56) gives us

$$|R_{*3}^{(l)}(x,t)| \le Ce^{-t/C} + Ct^{-\frac{l+2}{2}} e^{-t/C}.$$

We thus combine (4.55), (4.58)-(4.60) to have, for $x \in \mathbb{R}$ and $t > 0$,

$$|R_*^{(l)}(x,t)| \le C\sum_{k=1}^{m}(t+1)^{-\frac{l+3}{2}} e^{-\frac{(x-\lambda_k^{(r)}t)^2}{4\mu_k^r t}} + Ce^{-t/C} + Ct^{-\frac{l+2}{2}} e^{-t/C}. \tag{4.61}$$

For $|x|/t \ge K$ with $K > 0$ large, we write (4.54) as

$$R_*^{(l)}(x,t) = \eta^t \tilde{R}^{(l)}(x,t) - \frac{\partial^{l+1}}{\partial x^{l+1}} g^*(x,t),$$

where $\tilde{R}^{(l)}$ is defined in (4.28). Applying (4.36) and (4.53) gives us

$$|R_*^{(l)}(x,t)| \le Ce^{-\frac{\nu x^2}{4t}}(e^{-t/C} + t^{-\frac{l+1}{2}}) + C\sum_{j=1}^{m} t^{-\frac{l+2}{2}} e^{-\frac{(x-\lambda_j^{(r)}t)^2}{Ct}} \tag{4.62}$$

for $|x|/t \ge K$.

Following the proof of Theorem 2.3 (starting right after (4.36)), we show that (4.61) and (4.62) imply

$$|R_*^{(l)}(x,t)| \le C(t+1)^{-\frac{1}{2}} t^{-\frac{l+2}{2}} \sum_{j=1}^{m} e^{-\frac{(x-\lambda_j^{(r)}t)^2}{Ct}},$$

which justifies (2.30). We also prove (2.31) in a similar way.

5. Applications

In this section, we apply our main results to polyatomic gas flows. For both models, (1.4) and (1.9), we construct their Green's function explicitly by applying Theorem 2.3 and by following Procedure 2.9.

5.1. Thermal non-equilibrium flows

We first consider the models (1.4) for gas flows in both translational and vibrational non-equilibrium, where the translational non-equilibrium is compensated by

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dissipation mechanisms while the vibrational non-equilibrium is singled out as the sole relaxation process.

Recall that two of the thermodynamic variables are independent in set one (for the translational mode and for all internal modes other than the vibrational mode), and one is independent in set two (for the non-equilibrium vibrational mode). We introduce the following notations,

$$p = p(v, e_1) = \tilde{p}(v, T_1), \qquad T_1 = T_1(v, e_1), \qquad e_2 = \omega(T_2). \tag{5.1}$$

Under those notations we impose the following physical assumptions,

$$\begin{aligned} &\tilde{p}_v = \frac{\partial}{\partial v}\tilde{p}(v, T_1) < 0, \qquad T_{1e_1} = \frac{\partial}{\partial e_1} T_1(v, e_1) > 0, \\ &p_{e_1} = \frac{\partial}{\partial e_1} p(v, e_1) \neq 0, \qquad \omega'(T) > 0. \end{aligned} \tag{5.2}$$

We note that a state is an equilibrium state if and only if $T_2 = T_1$. Let $\bar{\rho}, \bar{e}_1 > 0$ be constants, $\bar{T}_1 = T_1(1/\bar{\rho}, \bar{e}_1)$, $\bar{e}_2 = \omega(\bar{T}_1)$ and $\bar{e} = \bar{e}_1 + \bar{e}_2$. For $w = (\rho, \rho u, \rho E, \rho e_2)^t$ in (1.4), we set the corresponding constant equilibrium state as

$$\bar{w} = (\bar{\rho}, 0, \bar{\rho}\bar{e}, \bar{\rho}\bar{e}_2)^t. \tag{5.3}$$

Here, without loss of generality, we have taken $\bar{u} = 0$.

To linearize (1.4) around $\bar{w}$, we note that (1.4) is in the form of (1.1),

$$w_t + f(w)_x = [B(w)w_x]_x + r(w), \tag{5.4}$$

where

$$\begin{aligned} &f(w) = uw + p(0, 1, u, 0)^t, \qquad r(w) = \Big(0, 0, 0, \rho[\omega(T_1) - \omega(T_2)]/\tau\Big)^t, \\ &B(w) = \mu v B_1 + \kappa v B_2 + \nu B_3, \\ &B_1 = \begin{pmatrix} 0 \\ 1 \\ u \\ 0 \end{pmatrix} (-u, 1, 0, 0), \qquad B_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} (-e_2, 0, 0, 1), \\ &B_2 = -\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} (vT_{1v} + T_{1e_1}(e_1 - \frac{u^2}{2}), T_{1e_1}u, -T_{1e_1}, T_{1e_1}). \end{aligned} \tag{5.5}$$

The linearization of (1.4) around $\bar{w}$ is

$$w_t + f'(\bar{w})w_x = B(\bar{w})w_{xx} + r'(\bar{w})(w - \bar{w}).$$

We reset the perturbation $w - \bar{w}$ as w to arrive at (2.1),

$$w_t + Aw_x = Bw_{xx} + Lw, \tag{5.6}$$

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where

$$A = f'(\bar{w}), \qquad B = B(\bar{w}), \qquad L = r'(\bar{w}). \tag{5.7}$$

The model for gas flows in both translational and vibrational nonequilibrium is considered in [31] for space dimensions $m \geq 1$, which includes (5.4), (5.5) as a special case of $m = 1$. The formulation of $f'(w)$ and $r'(w)$ is given in Section 4 therein. We cite those results and set w to the equilibrium state $\bar{w}$ to have explicit expressions for A, B and L in (5.7) as follows,

$$\begin{aligned}
A &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -v(vp_v + p_{e_1}e_1) & 0 & vp_{e_1} & -vp_{e_1} \\ 0 & e + vp & 0 & 0 \\ 0 & e_2 & 0 & 0 \end{pmatrix}_{\bar{w}}, \\
B &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mu v & 0 & 0 \\ -\kappa v(vT_{1v} + T_{1e_1}e_1) - \nu e_2 & 0 & \kappa v T_{1e_1} & -\kappa v T_{1e_1} + \nu \\ -\nu e_2 & 0 & 0 & \nu \end{pmatrix}_{\bar{w}}, \\
L &= -\frac{\omega'(T_1)}{\tau}\Big|_{\bar{w}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ vT_{1v} + e_1 T_{1e_1} - \dfrac{e_2}{\omega'(T_1)} & 0 & -T_{1e_1} & T_{1e_1} + \dfrac{1}{\omega'(T_1)} \end{pmatrix}_{\bar{w}}.
\end{aligned} \tag{5.8}$$

A set of structural conditions is proposed in [31] for a general system of hyperbolic-parabolic balance laws, see Assumption 1.1 therein. Our Assumption 2.1 here is a simplified, linear version of it in the one-dimensional case. It has been verified in [31] that under the physical assumptions (5.2), the nonlinear system (5.4), (5.5) and its multi-dimensional counterpart satisfy the structural conditions for a variety of choices in dissipation parameters, see Propositions 4.1 and 4.2 therein. Therefore, our linear system (5.6), (5.8) automatically satisfies Assumption 2.1. Precisely, we have the following lemma. Here, for simplicity, we use bar accent to denote a quantity in the equilibrium state $\bar{w}$.

LEMMA 5.1. *Under the assumptions in* (5.2)*, the linear system* (5.6), (5.8) *satisfies Assumption 2.1 for the following choices of dissipation parameters: (a)* $\bar{\mu}, \bar{\kappa}, \bar{\nu} > 0$*; (b)* $\bar{\mu}, \bar{\kappa} > 0$ *and* $\bar{\nu} = 0$*; (c)* $\bar{\kappa}, \bar{\nu} > 0$ *and* $\bar{\mu} = 0$*; and (d)* $\bar{\kappa} > 0$ *and* $\bar{\mu} = \bar{\nu} = 0$*.*

In Proposition 4.1 of [31], it is shown that (i) and (ii) of Assumption 2.1 are satisfied under (5.2) and for $\bar{\kappa}, \bar{\mu}, \bar{\nu} \geq 0$. In particular, the matrix A_0 in (i) is the Hessian of $(-\rho s)$, taken at the constant equilibrium state $\bar{w}$. Here, s is the total entropy, see (1.8). In Proposition 4.2 therein, it is further shown that (iii) of Assumption 2.1 is satisfied for the choices of parameters (a)-(d) in Lemma 5.1. Here, we note that $\bar{\kappa}$ is positive in all four cases.

In fact, we can consider $\bar{\kappa} = 0$ as well. Following a similar argument, one can show that (iii) of Assumption 2.1 is satisfied in case (e) $\bar{\mu}, \bar{\nu} > 0$ and $\bar{\kappa} = 0$, and

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in case (f) $\bar{\nu} > 0$ and $\bar{\mu} = \bar{\kappa} = 0$. On the other hand, (iii) of Assumption 2.1 fails when $\bar{\kappa} = \bar{\nu} = 0$ and $\bar{\mu} \geq 0$. For conciseness, we omit details of verification.

We now apply Theorem 2.3 to (5.6), (5.8). We use Procedure 2.9 to explicitly construct the Green's function G.

For this, we use the new variable $\tilde{w}$ as defined in (2.9). Noting $n_1 = 3$ and $n_2 = 1$, we use four components to write

$$w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} \rho - \bar{\rho} \\ \rho u \\ \rho E - \bar{\rho}\bar{e} \\ \rho e_2 - \bar{\rho}\bar{e}_2 \end{pmatrix}, \qquad \tilde{w} = \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \tilde{w}_3 \\ \tilde{w}_4 \end{pmatrix}, \tag{5.9}$$

where

$$\begin{aligned} &\tilde{w}_1 = w_1, \quad \tilde{w}_2 = w_2, \quad \tilde{w}_3 = w_3, \\ &\tilde{w}_4 = -\frac{\omega'(T_1)}{\tau}\bigg|_{\bar{w}} \left\{ \left[vT_{1v} + e_1 T_{1e_1} - \frac{e_2}{\omega'(T_1)} \right]_{\bar{w}} w_1 - [T_{1e_1}]_{\bar{w}} w_3 \right. \\ &\qquad \left. + \left[T_{1e_1} + \frac{1}{\omega'(T_1)} \right]_{\bar{w}} w_4 \right\} \equiv k_1 w_1 + k_3 w_3 + k_4 w_4. \end{aligned} \tag{5.10}$$

Here, we note that

$$k_4 = -\left. \frac{\omega'(T_1) T_{1e_1} + 1}{\tau} \right|_{\bar{w}} < 0. \tag{5.11}$$

Applying (2.10) and (2.12), we have

$$\begin{aligned} &\tilde{w}_w = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ k_1 & 0 & k_3 & k_4 \end{pmatrix}, \qquad w_{\tilde{w}} = \tilde{w}_w^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -k_1/k_4 & 0 & -k_3/k_4 & 1/k_4 \end{pmatrix}, \\ &\tilde{A} = \tilde{w}_w A w_{\tilde{w}}, \qquad \tilde{B} = \tilde{w}_w B w_{\tilde{w}}, \qquad \tilde{L} = \tilde{w}_w L w_{\tilde{w}} = \operatorname{diag}(0, 0, 0, k_4). \end{aligned} \tag{5.12}$$

Note that $(\tilde{A}, \tilde{B}, \tilde{L})$ is the triple that satisfies Assumption 2.1', which Procedure 2.9 is based upon. Thus, the following formulation is constructed with $(\tilde{A}, \tilde{B}, \tilde{L})$.

In Step 1, we solve the eigen-problem of the third principal minor of $\tilde{A}$. First, by direct calculation we have

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ a_{21} & 0 & a_{23} & a_{24} \\ 0 & a_{32} & 0 & 0 \\ 0 & a_{42} & 0 & 0 \end{pmatrix}, \tag{5.13}$$

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where

$$\begin{aligned}
a_{21} &= -\left[v(vp_v + p_{e_1}e_1) - \frac{k_1}{k_4}(vp_{e_1})\right]_{\bar{w}} \\
&= \left.\frac{-v(vp_v + p_{e_1}e) + \omega'(T_1)v^2(-p_vT_{1e_1} + T_{1v}p_{e_1})}{\omega'(T_1)T_{1e_1} + 1}\right|_{\bar{w}}, \\
a_{23} &= \left[vp_{e_1} + \frac{k_3}{k_4}vp_{e_1}\right]_{\bar{w}} = \left.\frac{vp_{e_1}}{\omega'(T_1)T_{1e_1} + 1}\right|_{\bar{w}}, \\
a_{24} &= -\left.\frac{vp_{e_1}}{k_4}\right|_{\bar{w}} = \left.\frac{vp_{e_1}\tau}{\omega'(T_1)T_{1e_1} + 1}\right|_{\bar{w}}, \\
a_{32} &= (e + vp)|_{\bar{w}}, \\
a_{42} &= \Big[k_1 + k_3(e + vp) + k_4e_2\Big]_{\bar{w}} = -\left[\frac{\omega'(T_1)}{\tau}(vT_{1v} - T_{1e_1}vp)\right]_{\bar{w}} \\
&= \left[\frac{\omega'(T_1)vT_1p_{e_1}}{\tau}\right]_{\bar{w}}.
\end{aligned} \tag{5.14}$$

Here, we have applied the identity

$$p_{e_1} = (pT_{1e_1} - T_{1v})/T_1 \tag{5.15}$$

for a_{42}. Thus, the third principal minor is

$$\tilde{A}^{(r)} = \begin{pmatrix} 0 & 1 & 0 \\ a_{21} & 0 & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix},$$

which has three distinct eigenvalues

$$\lambda_0^{(r)} = 0, \qquad \lambda_\pm^{(r)} = \pm\sqrt{a_{21} + a_{23}a_{32}}.$$

Using (5.14), we simplify the expression under the square root,

$$\begin{aligned}
a_{21} + a_{23}a_{32} &= \left[v^2(-p_v + \frac{pp_{e_1} + \omega'(T_1)T_{1v}p_{e_1}}{\omega'(T_1)T_{1e_1} + 1})\right]_{\bar{w}} \\
&= \left[v^2(-p_v + \frac{pp_{e_1} + e_{2v}^*p_{e_1}}{e_{2e_1}^* + 1})\right]_{\bar{w}} \equiv \bar{c}^2,
\end{aligned} \tag{5.16}$$

where we have used (5.1) to have $e_2^* = \omega(T_1) = \omega(T_1(v, e_1))$.

It has been shown in [30] that $\bar{c}^2$ defined in (5.16) is a positive number. In fact, $\bar{c} > 0$ is the equilibrium speed of sound at $\bar{w}$, see (2.24) and (1.33) in [30], also see (1.25) in [26] for the formulation under the Lagrangian coordinates. Thus, the

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three distinct eigenvalues of $\tilde{A}^{(r)}$ are

$$\lambda_0^{(r)} = 0, \qquad \lambda_\pm^{(r)} = \pm\bar{c}. \tag{5.17}$$

It is straightforward to find normalized left eigenvectors and right eigenvectors as

$$\begin{aligned} l_0^{(r)} &= \frac{1}{\bar{c}^2}(a_{32}, 0, -1) \qquad l_\pm^{(r)} = \frac{1}{2\bar{c}^2}(a_{21}, \pm\bar{c}, a_{23}), \\ r_0^{(r)} &= \begin{pmatrix} a_{23} \\ 0 \\ -a_{21} \end{pmatrix}, \qquad r_\pm^{(r)} = \begin{pmatrix} 1 \\ \pm\bar{c} \\ a_{32} \end{pmatrix}. \end{aligned} \tag{5.18}$$

The corresponding projections are

$$\tilde{P}_0 = \frac{1}{\bar{c}^2}\begin{pmatrix} a_{23}a_{32} & 0 & -a_{23} & 0 \\ 0 & 0 & 0 & 0 \\ -a_{21}a_{32} & 0 & a_{21} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{P}_\pm = \frac{1}{2\bar{c}^2}\begin{pmatrix} a_{21} & \pm\bar{c} & a_{23} & 0 \\ \pm a_{21}\bar{c} & \bar{c}^2 & \pm a_{23}\bar{c} & 0 \\ a_{21}a_{32} & \pm a_{32}\bar{c} & a_{23}a_{32} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.19}$$

We comment that the matrix $\tilde{A}^{(r)}$ is the Jacobian matrix of the flux function of the corresponding equilibrium equations. In that case, $e_2 = e_2^* = \omega(T_1)$, and hence the equation $e = e_1 + e_2^*(v, e_1)$ implicitly defines e_1 as a function of e and v, which we write $e_1 = e_1^{(r)}(v, e)$. This allows us to regard p as a function of v and e as well,

$$p = p(v, e_1^{(r)}(v, e)) \equiv p^{(r)}(v, e).$$

Using the notations in (5.1), we can verify that

$$a_{21} = \left[-v(vp_v^{(r)} + p_e^{(r)}e)\right]_{\bar{w}}, \qquad a_{23} = \left[vp_e^{(r)}\right]_{\bar{w}}.$$

Together with a_{32} in (5.14), these are compared with corresponding entries in A, see (5.8). Therefore, the eigenvalues $\{0, \pm\bar{c}\}$ and the corresponding eigenprojections $\{r_0^{(r)}l_0^{(r)}, r_\pm^{(r)}l_\pm^{(r)}\}$, which give rise to $\{\tilde{P}_0, \tilde{P}_\pm\}$, are from the spectral decomposition of the Jacobian matrix of the flux function related to the equilibrium flow.

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To go back to the original variable w we set

$$P_0 = w_{\tilde{w}}\tilde{P}_0\tilde{w}_w = \frac{1}{\bar{c}^2}\begin{pmatrix} a_{23}a_{32} & 0 & -a_{23} & 0 \\ 0 & 0 & 0 & 0 \\ -a_{21}a_{32} & 0 & a_{21} & 0 \\ -\frac{1}{k_4}(k_1a_{23}-k_3a_{21})a_{32} & 0 & \frac{1}{k_4}(k_1a_{23}-k_3a_{21}) & 0 \end{pmatrix},$$

$$\begin{aligned} P_\pm &= w_{\tilde{w}}\tilde{P}_\pm\tilde{w}_w \\ &= \frac{1}{2\bar{c}^2}\begin{pmatrix} a_{21} & \pm\bar{c} & a_{23} & 0 \\ \pm a_{21}\bar{c} & \bar{c}^2 & \pm a_{23}\bar{c} & 0 \\ a_{21}a_{32} & \pm a_{32}\bar{c} & a_{23}a_{32} & 0 \\ -\frac{1}{k_4}(k_1+k_3a_{32})a_{21} & \mp\frac{1}{k_4}(k_1+k_3a_{32})\bar{c} & -\frac{1}{k_4}(k_1+k_3a_{32})a_{23} & 0 \end{pmatrix}. \end{aligned} \tag{5.20}$$

Next, we continue into Step 2 of Procedure 2.9 to find diffusion coefficients. From $\tilde{L}$ in (5.12), we get

$$\tilde{S} = -\frac{1}{k_4}\mathrm{diag}(0_{3\times3}, 1).$$

This gives us

$$\begin{aligned} \tilde{A}\tilde{S}\tilde{A} &= -\frac{1}{k_4}[(0,0,0,1)\tilde{A}^t]^t(0,0,0,1)\tilde{A} \\ &= -\frac{1}{k_4}(0,,a_{24},0,0)^t(0,a_{42},0,0) = -\frac{1}{k_4}a_{24}a_{42}(0,,1,0,0)^t(0,1,0,0) \\ &= \left.\frac{\tau\omega'(T_1)v^2T_1p_{e_1}^2}{[\omega'(T_1)T_{1e_1}+1]^2}\right|_{\bar{w}}(0,,1,0,0)^t(0,1,0,0), \end{aligned} \tag{5.21}$$

where we have applied (5.13), (5.14) and (5.10). Using (5.21) and (5.18), we compute the contribution from the relaxation as follows,

$$\begin{aligned} &(l_0^{(r)},0)\tilde{A}\tilde{S}\tilde{A}\begin{pmatrix} r_0^{(r)} \\ 0 \end{pmatrix} = 0, \\ &(l_\pm^{(r)},0)\tilde{A}\tilde{S}\tilde{A}\begin{pmatrix} r_\pm^{(r)} \\ 0 \end{pmatrix} = \frac{1}{2}\left.\frac{\tau\omega'(T_1)v^2T_1p_{e_1}^2}{[\omega'(T_1)T_{1e_1}+1]^2}\right|_{\bar{w}} > 0. \end{aligned} \tag{5.22}$$

Similarly, the contribution from dissipation mechanisms is

$$(l_j^{(r)},0)\tilde{B}\begin{pmatrix} r_j^{(r)} \\ 0 \end{pmatrix} = (l_j^{(r)},0)\tilde{w}_wBw_{\tilde{w}}\begin{pmatrix} r_j^{(r)} \\ 0 \end{pmatrix}, \qquad j=0,\pm. \tag{5.23}$$

By (5.18) and (5.12),

$$\begin{aligned} &(l_0^{(r)},0)\tilde{w}_w = \frac{1}{\bar{c}^2}(a_{32},0,-1,0), \\ &([r_0^{(r)}]^t,0)w_{\tilde{w}}^t = (a_{23},0,-a_{21},-\frac{k_1}{k_4}a_{23}+\frac{k_3}{k_4}a_{21}). \end{aligned}$$

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Thus,

$$(l_0^{(r)},0)\tilde{B}\begin{pmatrix} r_0^{(r)} \\ 0 \end{pmatrix} = \left\{ \frac{1}{\bar{c}^2}\kappa v \left[(vT_{1v} + T_{1e_1}e_1)a_{23} + T_{1e_1}(a_{21} - \frac{k_1}{k_4}a_{23} + \frac{k_3}{k_4}a_{21}) \right] \right.$$
$$\left. + \frac{\nu}{\bar{c}^2}(e_2 a_{23} + \frac{k_1}{k_4}a_{23} - \frac{k_3}{k_4}a_{21}) \right\}_{\bar{w}} \equiv \frac{1}{\bar{c}^2}kv\mu_{01} + \frac{\nu}{\bar{c}^2}\mu_{02}. \tag{5.24}$$

Next, we apply (5.10) and (5.14) to have

$$\begin{aligned} -\frac{k_1}{k_4}a_{23} + \frac{k_3}{k_4}a_{21} =& -\frac{1}{[\omega'(T_1)T_{1e_1}+1]^2_{\bar{w}}} \{-vp_{e_1}e_2 + \omega'(T_1)vp_{e_1}(T_{1v}v + T_{1e_1}e_1) \\ & -\omega'(T_1)T_{1e_1}v(vp_v + ep_{e_1}) + [\omega'(T_1)]^2 T_{1e_1}v^2(p_{e_1}T_{1v} - p_vT_{1e_1})\}_{\bar{w}} \\ =& -\frac{1}{[\omega'(T_1)T_{1e_1}+1]^2_{\bar{w}}} \{-vp_{e_1}e_2[1+\omega'(T_1)T_{1e_1}] \\ & +\omega'(T_1)v^2(p_{e_1}T_{1v} - p_vT_{1e_1})[1+\omega'(T_1)T_{1e_1}]\}_{\bar{w}} \\ =& \frac{1}{[\omega'(T_1)T_{1e_1}+1]_{\bar{w}}} \left[vp_{e_1}e_2 - \omega'(T_1)v^2(p_{e_1}T_{1v} - p_vT_{1e_1}) \right]_{\bar{w}}. \end{aligned}$$

This gives us

$$a_{21} - \frac{k_1}{k_4}a_{23} + \frac{k_3}{k_4}a_{21} = \left. \frac{-v(vp_v + p_{e_1}e_1)}{\omega'(T_1)T_{1e_1}+1} \right|_{\bar{w}}.$$

Applying the identity

$$\tilde{p}_v = p_v - T_{1v}p_{e_1}/T_{1e_1}, \tag{5.25}$$

we have

$$\begin{aligned} \mu_{01} &= \left[(vT_{1v} + T_{1e_1}e_1)a_{23} + T_{1e_1}(a_{21} - \frac{k_1}{k_4}a_{23} + \frac{k_3}{k_4}a_{21}) \right]_{\bar{w}} \\ &= \left\{ \frac{v}{\omega'(T_1)T_{1e_1}+1} [p_{e_1}(vT_{1v} + T_{1e_1}e_1) - T_{1e_1}(vp_v + p_{e_1}e_1)] \right\}_{\bar{w}} \\ &= \left[\frac{v}{\omega'(T_1)T_{1e_1}+1}(p_{e_1}vT_{1v} - T_{1e_1}vp_v) \right]_{\bar{w}} = \left. \frac{v^2T_{1e_1}(-\tilde{p}_v)}{\omega'(T_1)T_{1e_1}+1} \right|_{\bar{w}} > 0. \end{aligned} \tag{5.26}$$

Similarly,

$$\mu_{02} = \bar{e}_2 a_{23} + \frac{k_1}{k_4}a_{23} - \frac{k_3}{k_4}a_{21} = \left. \frac{\omega'(T_1)v^2T_{1e_1}(-\tilde{p}_v)}{\omega'(T_1)T_{1e_1}+1} \right|_{\bar{w}} > 0. \tag{5.27}$$

Substituting (5.26) and (5.27) into (5.24), we arrive at

$$(l_0^{(r)},0)\tilde{B}\begin{pmatrix} r_0^{(r)} \\ 0 \end{pmatrix} = \left. \frac{v^2T_{1e_1}(-\tilde{p}_v)[\kappa v + \nu\omega'(T_1)]}{\bar{c}^2[\omega'(T_1)T_{1e_1}+1]} \right|_{\bar{w}} \geq 0 \tag{5.28}$$

for $\bar{\kappa}, \bar{\nu} \geq 0$.

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Finally, combining (5.22) and (5.28), we obtain μ_0 as defined in (2.25),

$$\mu_0 = \left. \frac{v^2 T_{1e_1}(-\tilde{p}_v)[\kappa v + \nu\omega'(T_1)]}{\bar{c}^2[\omega'(T_1)T_{1e_1}+1]} \right|_{\bar{w}} > 0 \tag{5.29}$$

for cases (a)-(f). (On the other hand, $\mu_0 = 0$ if $\bar{\kappa} = \bar{\nu} = 0$ while $\bar{\mu} \geq 0$.)

With similar calculation we find

$$(l_\pm^{(r)}, 0)\tilde{B}\begin{pmatrix} r_\pm^{(r)} \\ 0 \end{pmatrix} = \left\{ \frac{v^2 p_{e_1}^2 T_1[\kappa v + \nu\omega'(T_1)]}{2\bar{c}^2[\omega'(T_1)T_{1e_1}+1]^2} + \frac{\mu v}{2} \right\}_{\bar{w}} \geq 0 \tag{5.30}$$

for $\bar{\kappa}, \bar{\nu}, \bar{\mu} \geq 0$. Therefore, applying (5.22) and (5.30) to (2.25) gives us

$$\mu_\pm = \left. \frac{v^2 p_{e_1}^2 T_1}{2[\omega'(T_1)T_{1e_1}+1]^2}\{\tau\omega'(T_1) + [\kappa v + \nu\omega'(T_1)]/\bar{c}^2\} + \frac{\mu}{2}v \right|_{\bar{w}} > 0 \tag{5.31}$$

for $\bar{\kappa}, \bar{\nu}, \bar{\mu} \geq 0$.

We comment that $\mu_\pm$ are always positive while μ_0 can be zero if $\bar{\kappa} = \bar{\nu} = 0$. In that case, the Kawashima–Shizuta condition fails. Intuitively, we expect that the corresponding heat kernel degenerates to a non-decaying δ-function along the particle path. Indeed, it is verified for the inviscid case $\bar{\kappa} = \bar{\nu} = \bar{\mu} = 0$ in [26].

We now move on to Step 3 in Procedure 2.9. While the heat kernels in (2.5) are formulated uniformly above for cases (a)-(f), the singular part needs to be constructed case by case. Here, we consider the physically relevant one, $\bar{\kappa}, \bar{\nu}, \bar{\mu} > 0$. In this case, $\tilde{B}$ has four eigenvalues,

$$\sigma_0' = 0, \quad \sigma_1' = \mu v\Big|_{\bar{w}} \quad \sigma_2' = \kappa v T_{1e_1}\Big|_{\bar{w}} \quad \sigma_3' = \nu\Big|_{\bar{w}}, \tag{5.32}$$

see (5.12) and (5.8). The eigenprojection corresponding to zero is

$$\begin{aligned} \tilde{Q}_0' &= \tilde{w}_w(1, 0, e + vT_{1v}/T_{1e_1}, e_2)^t\Big|_{\bar{w}}(1,0,0,0)w_{\tilde{w}} \\ &= (1, 0, e + vT_{1v}/T_{1e_1}, 0)^t\Big|_{\bar{w}}(1,0,0,0) \equiv \eta_1(1,0,0,0). \end{aligned} \tag{5.33}$$

This implies

$$\tilde{Q}_0'\tilde{A}\tilde{Q}_0' = (1, 0, e + vT_{1v}/T_{1e_1}, 0)_{\bar{w}}^t(0,1,0,0)(1, 0, e + vT_{1v}/T_{1e_1}, 0)_{\bar{w}}^t(1,0,0,0) = 0. \tag{5.34}$$

The right-hand side of (5.34) is the spectral decomposition of $\tilde{Q}_0'\tilde{A}\tilde{Q}_0'$ in the range of $\tilde{Q}_0'$, i.e., $\alpha = 0$, $\tilde{\mathcal{Q}} = \tilde{Q}_0'$.

We continue into Step 4 of Procedure 2.9. For $\tilde{B}$ we find eigenprojections corresponding to σ_k', $1 \leq k \leq 3$, as

$$\begin{aligned} \tilde{Q}_1' &= \tilde{w}_w(0,1,0,0)^t(0,1,0,0)w_{\tilde{w}}, \\ \tilde{Q}_2' &= \tilde{w}_w(0,0,1,0)^t(-vT_{1v}/T_{1e_1} - e_1, 0, 1, -1)\Big|_{\bar{w}} w_{\tilde{w}}, \\ \tilde{Q}_3' &= \tilde{w}_w(0,0,1,1)^t(-\bar{e}_2, 0, 0, 1)w_{\tilde{w}}. \end{aligned} \tag{5.35}$$

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By (2.28),

$$\tilde{S}' = \left[\frac{1}{\mu v}\tilde{Q}_1' + \frac{1}{\kappa v T_{1e_1}}\tilde{Q}_2' + \frac{1}{\nu}\tilde{Q}_3'\right]_{\bar{w}}. \tag{5.36}$$

Thus, applying (5.33), (5.12)-(5.14), (5.36), and (5.25), we have

$$\begin{aligned}\tilde{\mathcal{Q}}\tilde{A}\tilde{S}'\tilde{A}\tilde{\mathcal{Q}} &= \tilde{Q}_0'\tilde{A}\tilde{S}'\tilde{A}\tilde{Q}_0' = \eta_1(0,1,0,0)\tilde{S}'(0,-v^2(p_v - p_{e_1}T_{1v}/T_{1e_1}),0,0)^t(1,0,0,0)\\ &= \eta_1\left[\frac{1}{\mu v}v^2(-p_v + p_{e_1}T_{1v}/T_{1e_1})\right]_{\bar{w}}(1,0,0,0) = \left[-\frac{v}{\mu}\tilde{p}_v\right]_{\bar{w}}\tilde{Q}_0'.\end{aligned} \tag{5.37}$$

Similarly, from (5.12), we have

$$\tilde{\mathcal{Q}}\tilde{L}\tilde{\mathcal{Q}} = \tilde{Q}_0'\,\mathrm{diag}(0,0,0,k_4)\,\tilde{Q}_0' = 0. \tag{5.38}$$

Combining (5.37) and (5.38) gives us

$$\tilde{\mathcal{Q}}(\tilde{A}\tilde{S}'\tilde{A} - \tilde{L})\tilde{\mathcal{Q}} = \left[-\frac{v}{\mu}\tilde{p}_v\right]_{\bar{w}}\tilde{Q}_0', \tag{5.39}$$

where the right-hand side is the spectral decomposition of the matrix on the left-hand side in the range of $\tilde{\mathcal{Q}} = \tilde{Q}_0'$. Therefore, $m' = 1$,

$$\beta = \left[-\frac{v}{\mu}\tilde{p}_v\right]_{\bar{w}} > 0, \tag{5.40}$$

and $\tilde{Q}_{1,0} = \tilde{Q}_0'$. To go back to the original variable w we set

$$Q_{1,0} = w_{\tilde{w}}\tilde{Q}_0'\tilde{w}_w = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ e + \frac{vT_{1v}}{T_{1e_1}} & 0 & 0 & 0\\ e_2 & 0 & 0 & 0\end{pmatrix}_{\bar{w}}. \tag{5.41}$$

Using the information obtained in all four steps, we write the Green's function of the Cauchy problem of (5.6), (5.8), given by Theorem 2.3, as

$$\begin{aligned}G(x,t) =& H(x,t;-\bar{c},\mu_-)P_- + H(x,t;0,\mu_0)P_0 + H(x,t;\bar{c},\mu_+)P_+\\ &+ O(1)(t+1)^{-\frac{1}{2}}[H(x,t;-\bar{c},C) + H(x,t;0,C) + H(x,t;\bar{c},C)]\\ &+ e^{[\frac{v}{\mu}\tilde{p}_v]_{\bar{w}}t}\delta(x)Q_{1,0},\end{aligned} \tag{5.42}$$

where $\bar{c}, \mu_0, \mu_\pm, P_0, P_\pm$ and $Q_{1,0}$ are given in (5.16), (5.29), (5.31), (5.20) and (5.41), and $C > 0$ is some constant,

We comment that the heat kernels in (5.42) are common to all choices (a)–(f) of dissipation parameters, where μ_0 and $\mu_\pm$ have explicit dependence on those parameters in (5.29) and (5.31). On the other hand, the δ-function in (5.42) is for the physically relevant case (a). From the point of view of PDE theory, we list corresponding results for other cases to illustrate how the matrix B impacts the

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singular part in G. Since the construction for other cases is parallel to that of case (a), we omit details of the calculation.

We write G for all cases as

$$\begin{aligned} G(x,t) =& H(x,t;-\bar{c},\mu_-)P_- + H(x,t;0,\mu_0)P_0 + H(x,t;\bar{c},\mu_+)P_+ \\ &+ O(1)(t+1)^{-\frac{1}{2}}[H(x,t;-\bar{c},C) + H(x,t;0,C) + H(x,t;\bar{c},C)] + D. \end{aligned} \tag{5.43}$$

Then D is given below.

(b) For $\bar{\mu},\bar{\kappa}>0$ and $\bar{\nu}=0$,

$$D = e^{[\frac{v}{\mu}\tilde{p}_v]_{\bar{w}}t}\delta(x)Q_{1,0} + e^{-t/\bar{\tau}}\delta(x)Q_{2,0},$$

where $Q_{1,0}$ is given in (5.41), and

$$Q_{2,0} = (0,0,1,1)^t(-\bar{e}_2,0,0,1). \tag{5.44}$$

(c) For $\bar{\kappa},\bar{\nu}>0$ and $\bar{\mu}=0$,

$$D = e^{-\beta t}\delta(x+\alpha t)Q_{-,0} + e^{-\beta t}\delta(x-\alpha t)Q_{+,0},$$

where

$$\begin{aligned} &\alpha = \left[v\sqrt{-\tilde{p}_v}\right]_{\bar{w}} > 0, \qquad \beta = \left[\frac{vp_{e_1}^2T_1}{2\kappa T_{1e_1}^2}\right]_{\bar{w}} > 0, \\ &Q_{\pm,0} = \pm\frac{1}{2\alpha}(1,\pm\alpha,e+vT_{1v}/T_{1e_1},e_2)_{\bar{w}}^t(\pm\alpha,1,0,0). \end{aligned} \tag{5.45}$$

(d) For $\bar{\kappa}>0$ and $\bar{\mu}=\bar{\nu}=0$,

$$D = e^{-\beta t}\delta(x+\alpha t)Q_{-,0} + e^{-t/\bar{\tau}}\delta(x)Q_{2,0} + e^{-\beta t}\delta(x-\alpha t)Q_{+,0},$$

where α,β and $Q_{\pm,0}$ are those defined in (5.45), and $Q_{2,0}$ is defined in (5.44).

(e) We note that

$$\begin{aligned} c_f^2 \equiv (pp_{e_1}-p_v)v^2 &= -v^2\frac{\partial}{\partial v}p(v,e_1(v,s_1)) = -v^2\frac{\partial}{\partial v}\tilde{p}(v,T_1(v,s_1)) \\ &= -v^2\tilde{p}_v + v^2T_1\left[\frac{\partial}{\partial T_1}\tilde{p}(v,T_1)\right]^2T_{1e_1} > 0. \end{aligned} \tag{5.46}$$

In fact, $c_f>0$ is the frozen speed of sound. We set

$$\bar{c}_f = c_f\Big|_{\bar{w}}.$$

Then for $\bar{\mu},\bar{\nu}>0$, and $\bar{\kappa}=0$,

$$D = e^{-\beta_- t}\delta(x)Q_{-,0} + e^{-\beta_+ t}\delta(x)Q_{+,0},$$

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where

$$\begin{aligned}
\beta_\pm &= \frac{1}{2}(c_1 \pm \sqrt{c_2}),\\
c_1 &= \left[\frac{-vp_v + p_{e_1}vp}{\mu} + \frac{\omega'(T_1)T_{1e_1}}{\tau}\right]_{\bar w} = \left[\frac{c_f^2}{\mu v} + \frac{\omega'(T_1)T_{1e_1}}{\tau}\right]_{\bar w} > 0,\\
c_2 &= c_1^2 - 4\,\frac{\omega'(T_1)(-vp_vT_{1e_1} + p_{e_1}T_{1v}v)}{\mu\tau}\bigg|_{\bar w}\\
&= \left[\frac{-vp_v + p_{e_1}vp}{\mu} - \frac{\omega'(T_1)T_{1e_1}}{\tau}\right]_{\bar w}^2 + 4\,\frac{\omega'(T_1)p_{e_1}^2vT_1}{\mu\tau}\bigg|_{\bar w} > 0,\\
Q_{\pm,0} &= \pm\frac{1}{\beta_+ - \beta_-}\begin{pmatrix} 1\\ 0\\ e + \frac{vp_v+\mu\beta_\pm}{p_{e_1}}\\ e_2\end{pmatrix}_{\bar w}\Big(\frac{-vp_v - p_{e_1}e_1}{\mu} - \beta_\mp, 0, \frac{p_{e_1}}{\mu}, -\frac{p_{e_1}}{\mu}\Big)_{\bar w}.
\end{aligned} \tag{5.47}$$

Here, we have used (5.15) for c_2. We can also use (5.25) to write

$$c_2 = c_1^2 + 4\frac{\omega'(T_1)T_{1e_1}v\tilde p_v}{\mu\tau} < c_1^2.$$

Thus, $\beta_+ > \beta_- > 0$.

(f) For $\bar\nu > 0$ and $\bar\mu = \bar\kappa = 0$,

$$D = e^{-\beta t}\delta(x + \bar c_f t)Q_{-,0} + e^{-\beta_0 t}\delta(x)Q_{0,0} + e^{-\beta t}\delta(x - \bar c_f t)Q_{+,0},$$

where

$$\begin{aligned}
\beta &= \frac{v^2p_{e_1}^2T_1\omega'(T_1)}{2\tau c_f^2}\bigg|_{\bar w} > 0, \qquad \beta_0 = \frac{v^2T_{1e_1}(-\tilde p_v)\omega'(T_1)}{\tau c_f^2}\bigg|_{\bar w} > 0,\\
Q_{\pm,0} &= \frac{1}{2\bar c_f^2}\begin{pmatrix} 1\\ \pm c_f\\ e + vp\\ e_2\end{pmatrix}_{\bar w}\Big(-v(vp_v + p_{e_1}e_1), \pm c_f, vp_{e_1}, -vp_{e_1}\Big)_{\bar w},\\
Q_{0,0} &= \frac{1}{\bar c_f^2}\begin{pmatrix} vp_{e_1}\\ 0\\ v(vp_v + p_{e_1}e)\\ vp_{e_1}e_2\end{pmatrix}_{\bar w}\Big(e_1 + vp, 0, -1, 1\Big)_{\bar w}.
\end{aligned}$$

5.2. Two-temperature Navier–Stokes type model

Next, we consider the model (1.9), which is also in the form of (1.1),

$$w_t + f(w)_x = [B(w)w_x]_x + r(w), \tag{5.48}$$

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where $w = (\rho, \rho u, \rho E, \rho T_2)^t$ with $E = e + \frac{1}{2}u^2$,

$$
\begin{aligned}
&f(w) = uw + p(0,1,u,0)^t, \qquad r(w) = \Big(0,0,0,\frac{3}{3+D}\theta A_c(T)\rho^2(T_1 - T_2)\Big)^t,\\
&B(w) = \frac{1}{\rho}\big(\mu_{\rm tr}B_1 + \lambda_{\rm tr}B_{21} + \frac{2}{DR}\lambda_{\rm int}B_{22}\big),\\
&B_1 = \begin{pmatrix}0\\1\\u\\0\end{pmatrix}(-u,1,0,0), \qquad B_{22} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}(-T_2,0,0,1),\\
&B_{21} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\Big(\frac{u^2}{3R} - T_1 - \frac{D}{5}T_2, -\frac{2u}{3R}, \frac{2}{3R}, -\frac{2D}{15}\Big).
\end{aligned}
\tag{5.49}
$$

Here, we have applied (1.10) and (1.12). Noting $\lambda_{\rm int}$ and $\lambda_{\rm tr}$ are not independent, see (1.11), we further have

$$
B(w) = \frac{1}{\rho}\big(\mu_{\rm tr}B_1 + \lambda_{\rm tr}B_2\big), \tag{5.50}
$$

where

$$
\begin{aligned}
B_2 &= B_{21} + \frac{2}{5R}B_{22}\\
&= \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\Big(\frac{u^2}{3R} - T_1 - \frac{D}{5}T_2, -\frac{2u}{3R}, \frac{2}{3R}, -\frac{2D}{15}\Big) + \begin{pmatrix}0\\0\\0\\1\end{pmatrix}(-\frac{2}{5R}T_2, 0, 0, \frac{2}{5R}).
\end{aligned}
\tag{5.51}
$$

We recalled that a state is an equilibrium state if and only if $T = T_1 = T_2$. Thus, letting $\bar\rho, \bar T_1 > 0$ be constants, we take a constant equilibrium state as

$$
\bar w = (\bar\rho, 0, \bar\rho\bar e, \bar\rho\bar T_1)^t, \qquad \bar e = \frac{3+D}{2}R\bar T_1. \tag{5.52}
$$

We linearize (5.48) around $\bar w$ and reset $w - \bar w$ as w to have

$$
w_t + Aw_x = Bw_{xx} + Lw, \tag{5.53}
$$

where

$$
A = f'(\bar w), \qquad B = B(\bar w), \qquad L = r'(\bar w).
$$

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From (5.49)-(5.51) and by direct calculation, we further have

$$
\begin{aligned}
A &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & -\frac{D}{3}R \\ 0 & \frac{5+D}{2}R\bar{T}_1 & 0 & 0 \\ 0 & \bar{T}_1 & 0 & 0 \end{pmatrix}, \\
B &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\bar{\mu}_{\mathrm{tr}}}{\bar{\rho}} & 0 & 0 \\ -\bar{\lambda}_{\mathrm{tr}}\frac{5+D}{5}\frac{\bar{T}_1}{\bar{\rho}} & 0 & \bar{\lambda}_{\mathrm{tr}}\frac{2}{3R\bar{\rho}} & -\bar{\lambda}_{\mathrm{tr}}\frac{2D}{15\bar{\rho}} \\ -\bar{\lambda}_{\mathrm{tr}}\frac{2}{5R}\frac{\bar{T}_1}{\bar{\rho}} & 0 & 0 & \bar{\lambda}_{\mathrm{tr}}\frac{2}{5R\bar{\rho}} \end{pmatrix}, \\
L &= -\frac{3}{3+D}\theta A_c(\bar{T}_1)\bar{\rho}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\Big(0,0,-\frac{2}{3R},\frac{3+D}{3}\Big),
\end{aligned}
\tag{5.54}
$$

where $\bar{\mu}_{\mathrm{tr}} = \mu_{\mathrm{tr}}(\bar{T}_1, \bar{T}_1)$ and $\bar{\lambda}_{\mathrm{tr}} = \lambda_{\mathrm{tr}}(\bar{T}_1, \bar{T}_1)$.

LEMMA 5.2. *The linearized two-temperature Navier–Stokes type model* (5.53)*, with* A, B *and* L *defined in* (5.54)*, satisfies Assumption 2.1.*

Proof. To verify condition (i) in Assumption 2.1, we take a symmetrizer

$$
A_0 = \begin{pmatrix} \frac{5+D}{2}R\bar{T}_1^2 & 0 & -\bar{T}_1 & 0 \\ 0 & \bar{T}_1 & 0 & 0 \\ -\bar{T}_1 & 0 & \frac{2}{3R} & -\frac{D}{3} \\ 0 & 0 & -\frac{D}{3} & \frac{(3+D)D}{6}R \end{pmatrix},
$$

which is symmetric, positive definite, with

$$
\begin{aligned}
A_0A &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3}\bar{T}_1 & -\frac{D}{3}R\bar{T}_1 \\ 0 & \frac{2}{3}\bar{T}_1 & 0 & 0 \\ 0 & -\frac{D}{3}R\bar{T}_1 & 0 & 0 \end{pmatrix}, \\
A_0B &= \begin{pmatrix} \bar{\lambda}_{\mathrm{tr}}\frac{5+D}{5}\frac{\bar{T}_1^2}{\bar{\rho}} & 0 & -\bar{\lambda}_{\mathrm{tr}}\frac{2}{3R}\frac{\bar{T}_1}{\bar{\rho}} & \bar{\lambda}_{\mathrm{tr}}\frac{2D}{15}\frac{\bar{T}_1}{\bar{\rho}} \\ 0 & \bar{\mu}_{\mathrm{tr}}\frac{\bar{T}_1}{\bar{\rho}} & 0 & 0 \\ -\bar{\lambda}_{\mathrm{tr}}\frac{2}{3R}\frac{\bar{T}_1}{\bar{\rho}} & 0 & \bar{\lambda}_{\mathrm{tr}}\frac{4}{9R^2\bar{\rho}} & -\bar{\lambda}_{\mathrm{tr}}\frac{2D}{9R\bar{\rho}} \\ \bar{\lambda}_{\mathrm{tr}}\frac{2D}{15}\frac{\bar{T}_1}{\bar{\rho}} & 0 & -\bar{\lambda}_{\mathrm{tr}}\frac{2D}{9R\bar{\rho}} & \bar{\lambda}_{\mathrm{tr}}(\frac{D}{9}+\frac{1}{5})\frac{D}{\bar{\rho}} \end{pmatrix}, \\
A_0L &= -\frac{3D}{2(3+D)}\theta A_c(\bar{T}_1)R\bar{\rho}\begin{pmatrix} 0 \\ 0 \\ -\frac{2}{3R} \\ \frac{3+D}{3} \end{pmatrix}\Big(0,0,-\frac{2}{3R},\frac{3+D}{3}\Big).
\end{aligned}
$$

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It is clear that A_0A and A_0B are symmetric, and A_0L is symmetric, negative semi-definite. Note that $A_0^{1/2}BA_0^{-1/2} = A_0^{-1/2}A_0BA_0^{-1/2}$ is symmetric and similar to B. From (5.54), all eigenvalues of B are nonnegative. This implies that $A_0^{1/2}BA_0^{-1/2}$, and hence A_0B, is symmetric, positive semi-definite.

For condition *(ii)*, from (5.54) it is clear that (5.53) has $n_1 = 3$ conservation laws, with $L_{22} = -\theta A_c(\bar{T}_1)\bar{\rho} < 0$.

To verify condition *(iii)*, we note that the null space $\mathbb{N}_1$ of B is one-dimensional and the null space $\mathbb{N}_2$ of L is three-dimensional, as zero is a simple eigenvalue of B and a triple eigenvalue of L. By direct calculation one can find

$$\mathbb{N}_1 = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ \frac{3+D}{2}R\bar{T}_1 \\ \bar{T}_1 \end{pmatrix}\right\}, \qquad \mathbb{N}_2 = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{3+D}{2}R \\ 1 \end{pmatrix}\right\}.$$

Thus, $\mathbb{N}_1 \subset \mathbb{N}_2$ and hence $\mathbb{N}_1 \cap \mathbb{N}_2 = \mathbb{N}_1$. Again, by direct calculation one can verify that $\left(1, 0, \frac{3+D}{2}R\bar{T}_1, \bar{T}_1\right)^t$ is not an eigenvector of A. That is, $\mathbb{N}_1 \cap \mathbb{N}_2$ contains no eigenvectors of A. □

REMARK 5.3. In a recent paper [36], it is shown that the two-temperature Navier–Stokes type model (1.9) and its multi-dimensional full system satisfy a set of structural conditions. The set is an extended, nonlinear version of Assumption 2.1. This allows the L^2 theory for a general nonlinear system built in [31]-[34] to apply to small, classical solutions of (1.9) and its full system.

Next, we define the new variable $\tilde{w}$ as in (2.9). Noting $n_1 = 3$ we write

$$w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} \rho - \bar{\rho} \\ \rho u \\ \rho E - \bar{\rho}\bar{e} \\ \rho T_2 - \bar{\rho}\bar{T}_1 \end{pmatrix}, \qquad \tilde{w} = \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \tilde{w}_3 \\ \tilde{w}_4 \end{pmatrix}, \tag{5.55}$$

where

$$\begin{aligned} &\tilde{w}_1 = w_1, \quad \tilde{w}_2 = w_2, \quad \tilde{w}_3 = w_3, \\ &\tilde{w}_4 = -\frac{3}{3+D}\theta A_c(\bar{T}_1)\bar{\rho}\Big(-\frac{2}{3R}w_3 + \frac{3+D}{3}w_4\Big) \equiv k_3 w_3 + k_4 w_4. \end{aligned} \tag{5.56}$$

Then

$$\tilde{w}_w = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & k_3 & k_4 \end{pmatrix}, \qquad w_{\tilde{w}} = \tilde{w}_w^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{k_3}{k_4} & \frac{1}{k_4} \end{pmatrix}. \tag{5.57}$$

Correspondingly, we define

$$\tilde{A} = \tilde{w}_w A w_{\tilde{w}}, \quad \tilde{B} = \tilde{w}_w B w_{\tilde{w}}, \quad \tilde{L} = \tilde{w}_w L w_{\tilde{w}} = \text{diag}(0, 0, 0, k_4), \tag{5.58}$$

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which form the triple that satisfies Assumption 2.1'.

Now we follow Procedure 2.9 to construct the Green's function for (5.53), (5.54). From (5.54) and (5.56)-(5.58) and by direct calculation,

$$\begin{aligned}
&\tilde{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & 0 & 0 \\ 0 & a_{42} & 0 & 0 \end{pmatrix}, \\
&a_{23} = \frac{2}{3} + \frac{D}{3} R \frac{k_3}{k_4} = \frac{2}{3+D}, \qquad a_{24} = -\frac{D}{3} R \frac{1}{k_4} = \frac{DR}{3\bar{\rho}} [\theta A_c(\bar{T}_1)]^{-1}, \\
&a_{32} = \frac{5+D}{2} R\bar{T}_1, \qquad a_{42} = k_3 \frac{5+D}{2} R\bar{T}_1 + k_4 \bar{T}_1 = \frac{2}{3+D} \theta A_c(\bar{T}_1) \bar{\rho} \bar{T}_1.
\end{aligned} \tag{5.59}$$

Thus, the third principal minor of $\tilde{A}$ is

$$\tilde{A}^{(r)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{2}{3+D} \\ 0 & \frac{5+D}{2} R\bar{T}_1 & 0 \end{pmatrix},$$

which has three distinct eigenvalues,

$$\lambda_0^{(r)} = 0, \qquad \lambda_\pm^{(r)} = \pm \bar{c}, \quad \bar{c} = \sqrt{\frac{5+D}{3+D} R T_1}. \tag{5.60}$$

The normalized left eigenvectors and right eigenvectors are

$$\begin{aligned}
&l_0^{(r)} = \frac{1}{\bar{c}^2} \Big(\frac{5+D}{2} R\bar{T}_1, 0, -1 \Big), \qquad l_\pm^{(r)} = \frac{1}{2\bar{c}^2} \Big(0, \pm\bar{c}, \frac{2}{3+D} \Big), \\
&r_0^{(r)} = \begin{pmatrix} \frac{2}{3+D} \\ 0 \\ 0 \end{pmatrix}, \qquad r_\pm^{(r)} = \begin{pmatrix} 1 \\ \pm\bar{c} \\ \frac{5+D}{2} R\bar{T}_1 \end{pmatrix}.
\end{aligned} \tag{5.61}$$

The corresponding projections are

$$\tilde{P}_0 = \frac{1}{\bar{c}^2} \begin{pmatrix} \bar{c}^2 & 0 & -\frac{2}{3+D} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{P}_\pm = \frac{1}{2\bar{c}^2} \begin{pmatrix} 0 & \pm\bar{c} & \frac{2}{3+D} & 0 \\ 0 & \bar{c}^2 & \pm\frac{2}{3+D}\bar{c} & 0 \\ 0 & \pm\frac{5+D}{2} R\bar{T}_1 \bar{c} & \bar{c}^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.62}$$

To go back to the original variable w we set

$$P_0 = w_{\tilde{w}} \tilde{P}_0 \tilde{w}_w = \tilde{P}_0, \qquad P_\pm = w_{\tilde{w}} \tilde{P}_\pm \tilde{w}_w = \frac{1}{2\bar{c}^2} \begin{pmatrix} 0 & \pm\bar{c} & \frac{2}{3+D} & 0 \\ 0 & \bar{c}^2 & \pm\frac{2}{3+D}\bar{c} & 0 \\ 0 & \pm\frac{5+D}{2} R\bar{T}_1 \bar{c} & \bar{c}^2 & 0 \\ 0 & \pm\frac{\bar{c}^3}{R} & \frac{2\bar{c}^2}{(3+D)R} & 0 \end{pmatrix}. \tag{5.63}$$

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Next, we find diffusion coefficients. From (5.58), we have

$$\tilde{S} = -\frac{1}{k_4}\mathrm{diag}(0_{3\times 3}, 1).$$

With (5.59) and (5.56), we further have

$$\tilde{A}\tilde{S}\tilde{A} = -\frac{1}{k_4}a_{42}a_{24}(0,1,0,0)^t(0,1,0,0) = \frac{2D}{3(3+D)}\frac{R\bar{T}_1}{\theta A_c(\bar{T}_1)\bar{\rho}}(0,1,0,0)^t(0,1,0,0).$$

Applying (5.61) gives us the contribution from the relaxation,

$$\begin{aligned}(l_0^{(r)},0)\tilde{A}\tilde{S}\tilde{A}\begin{pmatrix} r_0^{(r)} \\ 0\end{pmatrix} &= 0,\\ (l_\pm^{(r)},0)\tilde{A}\tilde{S}\tilde{A}\begin{pmatrix} r_\pm^{(r)} \\ 0\end{pmatrix} &= \frac{DR\bar{T}_1}{3(3+D)\theta A_c(\bar{T}_1)\bar{\rho}}.\end{aligned} \tag{5.64}$$

Similarly, from (5.54), (5.56)–(5.58), (5.60), and (5.61), we have

$$\begin{aligned}(l_0^{(r)},0)\tilde{B}\begin{pmatrix} r_0^{(r)} \\ 0\end{pmatrix} &= \frac{1}{\bar{c}^2}\Big(\frac{5+D}{2}R\bar{T}_1, 0, -1, 0\Big)B\begin{pmatrix} \frac{2}{3+D} \\ 0 \\ 0 \\ 0\end{pmatrix} = \frac{2\bar{\lambda}_{\mathrm{tr}}}{5R\bar{\rho}},\\ (l_\pm^{(r)},0)\tilde{B}\begin{pmatrix} r_\pm^{(r)} \\ 0\end{pmatrix} &= \frac{1}{2\bar{c}^2}\Big(0, \pm\bar{c}, \frac{2}{3+D}, 0\Big)B\begin{pmatrix} 1 \\ \pm\bar{c} \\ \frac{5+D}{2}R\bar{T}_1 \\ \frac{1}{R}\bar{c}^2\end{pmatrix} = \frac{\bar{\mu}_{\mathrm{tr}}}{2\bar{\rho}} + \frac{2\bar{\lambda}_{\mathrm{tr}}}{5(3+D)R\bar{\rho}}.\end{aligned} \tag{5.65}$$

Combining (5.64) and (5.65) gives us

$$\begin{aligned}\mu_0 &= \frac{2\bar{\lambda}_{\mathrm{tr}}}{5R\bar{\rho}} > 0,\\ \mu_\pm &= \frac{DR\bar{T}_1}{3(3+D)\theta A_c(\bar{T}_1)\bar{\rho}} + \frac{\bar{\mu}_{\mathrm{tr}}}{2\bar{\rho}} + \frac{2\bar{\lambda}_{\mathrm{tr}}}{5(3+D)R\bar{\rho}} > 0.\end{aligned} \tag{5.66}$$

Next, we find the singular part in the Green's function. Note that B (and hence $\tilde{B}$) has four eigenvalues,

$$\sigma_0' = 0, \qquad \sigma_1' = \frac{\bar{\mu}_{\mathrm{tr}}}{\bar{\rho}}, \qquad \sigma_2' = \frac{2\bar{\lambda}_{\mathrm{tr}}}{3R\bar{\rho}}, \qquad \sigma_3' = \frac{2\bar{\lambda}_{\mathrm{tr}}}{5R\bar{\rho}}.$$

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By (5.54) and (5.56)-(5.58), the eigenprojection of $\tilde{B}$ corresponding to σ_0' is

$$\begin{aligned}\tilde{Q}_0' &= \tilde{w}_w \begin{pmatrix} 1 \\ 0 \\ \frac{3+D}{2}R\bar{T}_1 \\ \bar{T}_1 \end{pmatrix} (1,0,0,0) w_{\tilde{w}} = \begin{pmatrix} 1 \\ 0 \\ \frac{3+D}{2}R\bar{T}_1 \\ 0 \end{pmatrix} (1,0,0,0) \\ &\equiv \eta_1 (1,0,0,0).\end{aligned} \tag{5.67}$$

This implies

$$\tilde{Q}_0' \tilde{A} \tilde{Q}_0' = 0,$$

where the right-hand side is the spectral decomposition of $\tilde{Q}_0' \tilde{A} \tilde{Q}_0'$ in the range of $\tilde{Q}_0'$. That is, $\alpha = 0$ and $\tilde{\mathcal{Q}} = \tilde{Q}_0'$.

The eigenprojection of $\tilde{B}$ corresponding to σ_i', $1 \le i \le 3$, are

$$\begin{aligned}\tilde{Q}_1' &= \tilde{w}_w (0,1,0,0)^t (0,1,0,0) w_{\tilde{w}}, \\ \tilde{Q}_2' &= \tilde{w}_w (0,0,1,0)^t (-\frac{3}{2}R\bar{T}_1, 0, 1, -\frac{D}{2}R) w_{\tilde{w}}, \\ \tilde{Q}_3' &= \tilde{w}_w (0,0,\frac{DR}{2},1)^t (-\bar{T}_1,0,0,1) w_{\tilde{w}},\end{aligned}$$

and

$$\tilde{S}' = \frac{\bar{\rho}}{\bar{\mu}_{\rm tr}} \tilde{Q}_1' + \frac{3R\bar{\rho}}{2\bar{\lambda}_{\rm tr}} \tilde{Q}_2' + \frac{5R\bar{\rho}}{2\bar{\lambda}_{\rm tr}} \tilde{Q}_3'. \tag{5.68}$$

Thus, applying (5.67), (5.57)-(5.59) and (5.68), and by direct calculation,

$$\begin{aligned}&\tilde{\mathcal{Q}} \tilde{A} \tilde{S}' \tilde{A} \tilde{\mathcal{Q}} = \tilde{Q}_0' \tilde{A} \tilde{S}' \tilde{A} \tilde{Q}_0' = \frac{\bar{\rho}}{\bar{\mu}_{\rm tr}} a_{23} \frac{3+D}{2} R\bar{T}_1 \tilde{Q}_0' = \frac{\bar{\rho}}{\bar{\mu}_{\rm tr}} R\bar{T}_1 \tilde{Q}_0', \\ &\tilde{\mathcal{Q}} \tilde{L} \tilde{\mathcal{Q}} = \tilde{Q}_0' {\rm diag}(0,0,0,k_4) \tilde{Q}_0' = 0.\end{aligned}$$

These give us

$$\tilde{\mathcal{Q}} (\tilde{A} \tilde{S}' \tilde{A} - \tilde{L}) \tilde{\mathcal{Q}} = \frac{\bar{\rho} R \bar{T}_1}{\bar{\mu}_{\rm tr}} \tilde{Q}_0',$$

where the right-hand side is the spectral decomposition of the matrix on the left-hand side in the range of $\tilde{\mathcal{Q}} = \tilde{Q}_0'$. Therefore, $m' = 1$,

$$\beta = \frac{\bar{\rho} R \bar{T}_1}{\bar{\mu}_{\rm tr}} > 0,$$

and $\tilde{Q}_{1,0} = \tilde{Q}_0'$. To go back to the original variable w we set

$$Q_{1,0} = w_{\tilde{w}} \tilde{Q}_0' \tilde{w}_w = \begin{pmatrix} 1 \\ 0 \\ \frac{3+D}{2}R\bar{T}_1 \\ \bar{T}_1 \end{pmatrix} (1,0,0,0). \tag{5.69}$$

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Gathering the information obtained above, we formulate the Green's function for the Cauchy problem of (5.53), (5.54) as follows,

$$\begin{aligned} G(x,t) =& H(x,t;-\bar{c},\mu_-)P_- + H(x,t;0,\mu_0)P_0 + H(x,t;\bar{c},\mu_+)P_+ \\ &+ O(1)(t+1)^{-\frac{1}{2}}[H(x,t;-\bar{c},C) + H(x,t;0,C) + H(x,t;\bar{c},C)] \\ &+ e^{-\beta t}\delta(x)Q_{1,0}, \end{aligned} \tag{5.70}$$

where

$$\begin{aligned} &\bar{c} = \sqrt{\frac{5+D}{3+D}R\bar{T}_1}, \qquad \mu_0 = \frac{2\bar{\lambda}_{\rm tr}}{5R\bar{\rho}}, \\ &\mu_\pm = \frac{DR\bar{T}_1}{3(3+D)\theta A_c(\bar{T}_1)\bar{\rho}} + \frac{\bar{\mu}_{\rm tr}}{2\bar{\rho}} + \frac{2\bar{\lambda}_{\rm tr}}{5(3+D)R\bar{\rho}}, \qquad \beta = \frac{R\bar{T}_1\bar{\rho}}{\bar{\mu}_{\rm tr}}, \end{aligned} \tag{5.71}$$

$C > 0$ is a constant, and the projections $P_0, P_\pm$ and $Q_{1,0}$ are given in (5.62), (5.63) and (5.69).

We comment that while the setup of the two-temperature Navier–Stokes type model (1.9) is different from that of the thermal nonequilibrium flow model (1.4), the two systems have similar structure. In particular, the linearized system (5.53), (5.54) is parallel to (5.6), (5.8). Indeed, if we symbolically make the following correspondence between the two systems,

$$\bar{\mu} \longleftrightarrow \bar{\mu}_{\rm tr}, \qquad \bar{\kappa} \longleftrightarrow \bar{\lambda}_{\rm tr}, \qquad \bar{\nu} \longleftrightarrow \frac{2\bar{\lambda}_{\rm tr}}{5R\bar{\rho}}, \qquad \bar{\tau} \longleftrightarrow \frac{3+D}{3\theta A_c(\bar{T}_1)\bar{\rho}},$$

together with (1.10) and (1.12), we can derive the same formulation in (5.71) from (5.16), (5.29),(5.31) and (5.40).

Data availability statement
My manuscript has no associated data.

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Appendix A. Alternative steps for procedure 2.9

If $A^{(r)}$ defined in (2.19) has multiple eigenvalues, we modify Steps 1 and 2 of Procedure 2.9 to Steps $1'$ and $2'$ as follows.

Step 1′. Let $\lambda_1^{(r)} < \lambda_2^{(r)} < \cdots < \lambda_{\tilde{s}}^{(r)}$ be distinct eigenvalues of $A^{(r)}$. Find the spectral decomposition

$$A^{(r)} = \sum_{k=1}^{\tilde{s}} \lambda_k^{(r)} P_k^{(r)}, \tag{A.1}$$

where $P_k^{(r)}$, $1 \le k \le \tilde{s}$, is the eigenprojection of $A^{(r)}$ corresponding to $\lambda_k^{(r)}$.

Step 2′. Find the spectral decomposition of L,

$$L = \sum_{j=1}^{s} -\sigma_j Q_j, \tag{A.2}$$

with $\sigma_j > 0$, $1 \le j \le s$. Here each $-\sigma_j$ is a negative eigenvalue of L, with the corresponding eigenprojection Q_j. Let Q_0 be the eigenprojection of L corresponding to the eigenvalue zero,

$$Q_0 = \operatorname{diag}(I_{n_1 \times n_1}, 0_{n_2 \times n_2}). \tag{A.3}$$

Define

$$S = \sum_{j=1}^{s} \frac{1}{\sigma_j} Q_j, \tag{A.4}$$

which is the value at zero of the reduced resolvent of $-L$ with respect to the eigenvalue zero.

Now for each $P_k^{(r)}$ associated with a distinct $\lambda_k^{(r)}$ in (A.1), we form a matrix $\operatorname{diag}(P_k^{(r)}, 0_{n_2 \times n_2})(ASA+B)\operatorname{diag}(P_k^{(r)}, 0_{n_2 \times n_2})$ and find its spectral decomposition in the range of $\operatorname{diag}(P_k^{(r)}, 0_{n_2 \times n_2})$,

$$\begin{aligned} &\operatorname{diag}(P_k^{(r)}, 0_{n_2 \times n_2})(ASA + B)\operatorname{diag}(P_k^{(r)}, 0_{n_2 \times n_2}) = \sum_{k'=1}^{\tilde{s}_k} \mu_{kk'} P_{kk'}, \\ &\sum_{k'=1}^{\tilde{s}_k} P_{kk'} = \operatorname{diag}(P_k^{(r)}, 0_{n_2 \times n_2}). \end{aligned} \tag{A.5}$$

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Next, we collect all distinct pairs $(\lambda_k^{(r)}, \mu_{kk'})$, $1 \le k \le \tilde{s}$, $1 \le k' \le \tilde{s}_k$. The total number of those pairs is m in th first term of (2.5), with $(\lambda_k^{(r)}, \mu_{kk'})$ being $(\lambda_j^{(r)}, \mu_j)$ and $P_{kk'}$ being P_j therein.

After Steps $1'$ and $2'$, one continues with Steps 3 and 4 in Procedure 2.9.

In case α_k, $1 \le k \le s''$, in (2.27) are simple, we simplify Step 4 in Procedure 2.9 to Step $4'$ as follows. This is regardless of using Steps 1 and 2 or Steps $1'$ and $2'$.

Step $4'$. If all α_k is (2.27) are simple, i.e., all $\mathcal{Q}_k$ are rank one, then $m' = s''$ is the rank of Q_0'. Let the normalized left eigenvector (row vector) and right eigenvector of $Q_0' A Q_0'$ associated with α_j be l_j' and r_j', respectively,

$$Q_0' A Q_0' r_j' = \alpha_j r_j', \qquad l_j' Q_0' A Q_0' = \alpha_j l_j', \qquad 1 \le j \le m',$$
$$l_i' r_j' = \delta_{ij}, \qquad 1 \le i, j \le m'.$$

For each α_j in the third term of (2.5), the corresponding β_j and $Q_{j,0}$ are

$$\beta_j = l_j'(AS'A - L)r_j', \qquad Q_{j,0} = r_j' l_j' = \mathcal{Q}_j, \tag{A.6}$$

where S' is defined by (2.28).

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