ON THE DECREASE OF VELOCITY WITH DEPTH IN AN IRROTATIONAL WATER WAVE

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ABSTRACT. The following theorems are proved for irrotational surface waves of finite amplitude in a uniform, incompressible fluid:

(*a*) In any space-periodic motion (progressive or otherwise) in uniform depth, the mean square of the velocity is a decreasing function of the mean depth z below the surface. Hence the fluctuations in the mean pressure $\overline{p}$ increase with z.

(*b*) In any space-periodic motion in infinite depth, the particle motion tends to zero exponentially as z tends to infinity. The pressure fluctuations at great depths are therefore simultaneous, but they do not in general tend to zero.

(*c*) In a progressive periodic wave in uniform depth the mass-transport velocity is a decreasing function of the mean depth of a particle below the free surface, and the tangent to the velocity profile is vertical at the bottom. This result conflicts with observations in wave tanks, and shows that the waves cannot be wholly irrotational.

(*d*) Analogous results are proved for the solitary wave.

The present paper discusses the vertical attenuation of certain physical quantities associated with irrotational wave motion. The interest of these quantities arises in different connexions, but because of a similarity in the analytical methods employed it is convenient to consider them together.

In the well-known approximate theory of surface waves (see Lamb (2) chap. 9), in which squares or products of the displacements are neglected, both the pressure fluctuations and the particle motion decrease rapidly downward from the surface; in deep water this decrease is exponential. It was therefore surprising when Miche (6) discovered that in a standing wave there are second-order fluctuations in the pressure even at infinite depths. Similar pressure variations have been shown to occur in a more general case, when the motion consists, in the first approximation, of two waves of the same wave-length, though not necessarily the same amplitude, travelling in opposite directions (Longuet-Higgins (3)). It is of some interest, therefore, to consider whether, in higher approximations than the first, the particle motion possesses a property similar to the pressure fluctuations or whether it decays rapidly with depth in all cases.

The results proved in §§ 1 and 2 of the present paper are for any irrotational motion which is periodic in space—for example, a progressive or a standing wave, or a motion which in the first approximation is a combination of two progressive waves of equal length. Periodicity in time, however, is not required. In § 1 it is shown by a simple argument that the *mean* square of the velocity at a given depth z is a decreasing function of z, and hence also that the *mean* value of the dynamical part of the pressure increases with z. In § 2 upper bounds are found for the velocity; when the depth is infinite the velocity diminishes exponentially. These results are independent of the

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pressure condition at the free surface, and follow from purely kinematical considerations, assuming the fluid to be incompressible and the motion irrotational. No approximations depending on the smallness of the wave amplitude are made. In deep water, therefore, the decay of the particle motion must be exponential to all orders of approximation.

One consequence of this result is that the pressure fluctuations at great depths, though depending in general on the time t, must be uniform in space.

In § 3 the mass-transport velocity of the waves is considered. It was Stokes (8) who first showed that in a progressive irrotational wave the particle orbits are not closed, but that each particle has in general an average, non-zero forward motion (called the mass-transport velocity). Stokes's method was approximate only. An exact and very elegant geometrical demonstration of this property was given by Rayleigh (7). The existence of a forward mass-transport velocity relative to the fluid at great depths was shown by him to be a consequence of the irrotational character of the motion. However, Rayleigh's proof is valid only for water of infinite depth; his argument does not apply when the depth is uniform and finite. Recently Ursell (10) presented an analytical proof, found by him some years previously, which extends Rayleigh's result to water of finite depth; he showed that the mass-transport velocity must be an increasing, or at any rate a non-decreasing, function of the depth, and that the tangent to the (mass-transport) velocity profile is vertical at the bottom itself. These results are of considerable interest, since wave-tank experiments have shown that for real fluids there is, on the contrary, a strong forward velocity near the bottom, in water of moderate depth (see, for example, Bagnold (1)); hence there must be a strong vorticity near the bottom, and the assumption of irrotational motion is invalid for real waves. Bagnold's observations have been accounted for quantitatively by the present writer (4) by introducing the viscous terms into the equations of motion. The original discrepancy between theory and observation has in fact led, in the paper just quoted, to a fundamental re-examination of the theory of mass transport and of the diffusion of vorticity in an oscillatory motion.

The method of proof used by Ursell (10) was to translate the first part of Rayleigh's geometrical proof into analytical terms; that is, he supposed that the progressive wave was reduced to a steady state by superposing a constant velocity $-c$ on the system (c being the wave velocity) and then found the time T taken by a particle to travel along a stream-line over a complete wave-length in the form

$$T(\psi) = \int_0^{\phi(\lambda)} \left[\left(\frac{\partial x}{\partial \phi} \right)^2 + \left(\frac{\partial z}{\partial \phi} \right)^2 \right] d\phi,$$

where λ is the wave-length, x and z are horizontal and vertical coordinates, and ϕ and ψ are the potential and stream function respectively. Then, by expanding x and z in Fourier series he showed that T was an increasing function of ψ, and hence that the mass-transport velocity increased with height above the bottom. In § 3 of the present paper we shall give a slightly simpler proof which avoids the use of a Fourier series expansion; $T(\psi)$ is shown to be an increasing function of ψ by direct differentiation; the analysis is precisely similar to that used in § 1.

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Analogous results for an irrotational solitary wave are proved in § 4. Here the advantage of the present method compared with that of Ursell (10) is also apparent; for the present method avoids altogether the use of Fourier integrals.

The usual notation will be used: x and z are rectangular coordinates, z being measured vertically downwards from the mean surface level. u and w denote the components of the velocity, which is two-dimensional and takes place in the (x, z)-plane. λ is the wave-length, h the mean depth of the fluid, p the pressure and ρ the density. The velocity potential and the stream function are denoted by ϕ and ψ. The fluid is assumed to be uniform and incompressible. A bar will be used to denote mean values with respect to x over a complete wave-length.

1. *In any space-periodic motion the mean square velocity at a given depth is a decreasing function of z.*

Since the motion is irrotational the components of velocity are given by

$$u = \frac{\partial \phi}{\partial x}, \quad w = \frac{\partial \phi}{\partial z}, \tag{1}$$

where, by the equation of continuity,

$$\nabla^2 \phi = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{2}$$

The mean square of the velocity at a fixed depth is given by

$$\overline{q^2} = \frac{1}{\lambda}\int_0^\lambda (u^2 + w^2)\, dx = \frac{1}{\lambda}\int_0^\lambda \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] dx. \tag{3}$$

Differentiating with respect to z, we have

$$\begin{aligned} \frac{\partial}{\partial z}(\overline{q^2}) &= \frac{2}{\lambda}\int_0^\lambda \left[\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x \partial z} + \frac{\partial \phi}{\partial z}\frac{\partial^2 \phi}{\partial z^2}\right] dx \\ &= \frac{2}{\lambda}\int_0^\lambda \left[\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial \phi}{\partial z}\frac{\partial^2 \phi}{\partial x^2}\right] dx, \end{aligned} \tag{4}$$

by equation (2). But

$$\int_0^\lambda \left[\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x \partial z} + \frac{\partial \phi}{\partial z}\frac{\partial^2 \phi}{\partial x^2}\right] dx = \left[\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial z}\right]_0^\lambda = 0, \tag{5}$$

by the periodicity of the motion. Thus

$$\frac{\partial}{\partial z}(\overline{q^2}) = \frac{4}{\lambda}\int_0^\lambda \frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x \partial z}\, dx. \tag{6}$$

On differentiating a second time we have similarly

$$\begin{aligned} \frac{\partial^2}{\partial z^2}(\overline{q^2}) &= \frac{4}{\lambda}\int_0^\lambda \left[\left(\frac{\partial^2 \phi}{\partial x \partial z}\right)^2 + \frac{\partial \phi}{\partial x}\frac{\partial^3 \phi}{\partial x \partial z^2}\right] dx \\ &= \frac{4}{\lambda}\int_0^\lambda \left[\left(\frac{\partial^2 \phi}{\partial x \partial z}\right)^2 - \frac{\partial \phi}{\partial x}\frac{\partial^3 \phi}{\partial x^3}\right] dx \\ &= \frac{4}{\lambda}\int_0^\lambda \left[\left(\frac{\partial^2 \phi}{\partial x \partial z}\right)^2 + \left(\frac{\partial^2 \phi}{\partial x^2}\right)^2\right] dx. \end{aligned} \tag{7}$$

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The right-hand side of (7) can never be negative; so that $\partial(\overline{q^2})/\partial z$ is an increasing function of z. But, on the bottom $z = h$, $\partial\phi/\partial z$ vanishes, and hence also do $\partial^2\phi/\partial x\,\partial z$ and $\partial(\overline{q^2})/\partial z$. Therefore $\partial(\overline{q^2})/\partial z$ is never positive when $z \leqslant h$ and so $\overline{q^2}$ is a decreasing function of z. This proves the result.

(We may note that, since $\partial\phi/\partial z$ also satisfies Laplace's equation and is periodic in x,

$$\frac{\partial^3}{\partial z^3}(\overline{q^2}) = \frac{16}{\lambda}\int_0^\lambda \frac{\partial^2\phi}{\partial x\,\partial z}\frac{\partial^3\phi}{\partial x\,\partial z^2}\,dx \tag{8}$$

and

$$\frac{\partial^4}{\partial z^4}(\overline{q^2}) = \frac{16}{\lambda}\int_0^\lambda \left[\left(\frac{\partial^3\phi}{\partial x\,\partial z^2}\right)^2 + \left(\frac{\partial^3\phi}{\partial x^2\,\partial z}\right)^2\right] dx. \tag{9}$$

Hence, by exactly similar arguments,

$$\frac{\partial^3(\overline{q^2})}{\partial z^3} \leqslant 0, \quad \frac{\partial^4(\overline{q^2})}{\partial z^4} \geqslant 0, \tag{10}$$

and in general

$$\frac{\partial^{(2n-1)}(\overline{q^2})}{\partial z^{(2n-1)}} \leqslant 0, \quad \frac{\partial^{2n}(\overline{q^2})}{\partial z^{2n}} \geqslant 0, \tag{11}$$

for every positive integer n.)

Since the density of the kinetic energy is proportional to q^2 the result may be stated in the form: *the mean density of kinetic energy is a decreasing function of the depth.*

Some corollaries may also be deduced. We have

$$\begin{aligned}\frac{\partial}{\partial z}\overline{(u^2-w^2)} &= \frac{\partial}{\partial z}\frac{1}{\lambda}\int_0^\lambda \left[\left(\frac{\partial\phi}{\partial x}\right)^2 - \left(\frac{\partial\phi}{\partial z}\right)^2\right] dx \\ &= \frac{2}{\lambda}\int_0^\lambda \left[\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\,\partial z} - \frac{\partial\phi}{\partial z}\frac{\partial^2\phi}{\partial z^2}\right] dx \\ &= \frac{2}{\lambda}\int_0^\lambda \left[\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\,\partial z} + \frac{\partial\phi}{\partial z}\frac{\partial^2\phi}{\partial x^2}\right] dx = 0\end{aligned} \tag{12}$$

by equation (5). Therefore

$$\overline{u^2} - \overline{w^2} = C, \tag{13}$$

where C is a function of the time only. Since, at any instant, $\overline{u^2}$ and $\overline{w^2}$ differ by a constant, and their sum $\overline{q^2}$ is a decreasing function of z, each separately must be a decreasing function of z. Thus *the mean square of each component of velocity is a decreasing function of the depth.*

The total pressure p is given by the Bernoulli equation

$$\frac{p}{\rho} = -\frac{\partial\phi}{\partial t} - \tfrac{1}{2}q^2 + gz + F(t) \tag{14}$$

(see Lamb (2), § 20), where g denotes gravity and $F(t)$ is a function of the time t only. The dynamical part of the pressure, that is, the difference between the actual pressure and the pressure in the fluid at rest, is given by

$$\frac{p'}{\rho} = \frac{p}{\rho} - gz = -\frac{\partial\phi}{\partial t} - \tfrac{1}{2}q^2 + F(t). \tag{15}$$

ϕ is not yet completely determined, since the addition of an arbitrary function of t would not affect the velocities. Suppose then that ϕ and $F(t)$ are chosen so that the mean value of ϕ over a wave-length, at a certain depth z, is zero:

$$\overline{\phi} = \frac{1}{\lambda}\int_0^\lambda \phi\,dx = 0. \tag{16}$$

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Now
$$\frac{\partial\overline{\phi}}{\partial z} = \frac{1}{\lambda}\int_0^\lambda \frac{\partial\phi}{\partial z}dx = 0, \tag{17}$$

since the total flow across any horizontal plane is zero. Therefore (12) holds for all values of z. On taking mean values in equation (15) we have

$$\frac{\overline{p'}}{\rho} = -\tfrac{1}{2}\overline{q^2} + F(t). \tag{18}$$

It follows that $\overline{p'}$ is an increasing function of z: *the mean value of the dynamical part of the pressure increases with the depth.*

The function $F(t)$ may be evaluated explicitly in terms of the displacement ζ of the free surface. It has been shown (see Longuet-Higgins (3), equation (28)) that

$$\frac{\overline{p'}}{\rho} = \frac{\partial^2}{\partial t^2}(\tfrac{1}{2}\overline{\zeta^2}) - \overline{w^2}, \tag{19}$$

and so, on comparison with (18),

$$F(t) = \frac{\partial^2}{\partial t^2}(\tfrac{1}{2}\overline{\zeta^2}) + \tfrac{1}{2}(\overline{u^2} - \overline{w^2}). \tag{20}$$

2. *In any space-periodic motion in infinite depth, the motion tends to zero exponentially with depth (the frame of reference being suitably chosen).*

To prove this result a stronger method is required. From equations (1) and (2) above we have
$$u - iw = f(Z), \tag{21}$$

where $f(Z)$ is an analytic function of the complex variable

$$Z = x + iz. \tag{22}$$

$f(Z)$ is only defined when $0 \leqslant z \leqslant h^*$; but since w vanishes when $z = h$, $f(Z)$ may be continued analytically to the region $h \leqslant z \leqslant 2h$ by simple reflexion in the line $z = h$.

Consider a large rectangle $ABCD$ with vertices at the points $(R, 0)$, $(-R, 0)$, $(-R, 2h)$ and $(R, 2h)$ respectively. If $Z_0 = x_0 + iz_0$, is any point inside $ABCD$, we have by Cauchy's theorem

$$f(Z_0) = \frac{1}{2\pi i}\int_{ABCDA} \frac{f(Z)}{Z - Z_0}dZ. \tag{23}$$

Now $f(Z)$ is periodic in x and therefore bounded as x tends to infinity. Thus on the sides BC and DA the integrand is of order R^{-1}. As R tends to infinity the contributions from these sides of the rectangle therefore tend to zero. Hence we have

$$f(Z_0) = \frac{1}{2\pi i}\lim_{R\to\infty}\left[-\int_{-R}^{R}\frac{f(x)}{x - Z_0}dx + \int_{-R}^{R}\frac{f(x)}{x + 2ih - Z_0}dx\right] \tag{24}$$

$$= \frac{1}{2\pi i}[-I_1 + I_2], \tag{25}$$

say. Now since the integrand tends to zero for large x we have

$$I_1 = \lim_{N\to\infty}\sum_{n=-N}^{N}\int_0^\lambda \frac{f(x + n\lambda)}{x + n\lambda - Z_0}dx \tag{26}$$

$$= \lim_{N\to\infty}\sum_{n=-N}^{N}\int_0^\lambda \frac{f(x)}{x + n\lambda - Z_0}dx \tag{27}$$

* Strictly $f(Z)$ is only defined when $\zeta \leqslant z \leqslant h$; but we assume that f can be continued analytically up to the mean surface level $z = 0$.

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by the periodicity of f. When $z_0 > \epsilon > 0$ the above series is uniformly convergent, so that the order of summation and integration may be reversed, giving

$$I_1 = \int_0^\lambda f(x)\left[\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{x+n\lambda-Z_0}\right]dx \tag{28}$$

$$= \int_0^\lambda f(x)\frac{\pi}{\lambda}\cot\frac{\pi(x-Z_0)}{\lambda}\,dx \tag{29}$$

by a known result (see, for example, Titchmarsh (9)).

Now since the total flow of water across any horizontal plane is zero we have

$$\int_0^\lambda w\,dx = 0 \tag{30}$$

for all z and t. By imposing on the axes a suitable horizontal velocity we may also make

$$\int_0 u\,dx = 0 \tag{31}$$

for particular values of z and t. But

$$\frac{\partial}{\partial z}\int_0^\lambda u\,dx = \frac{\partial}{\partial z}[\phi]_0^\lambda = \left[\frac{\partial\phi}{\partial z}\right]_0^\lambda, \tag{32}$$

which vanishes by the periodicity, and from the Bernoulli equation (10) we have

$$\frac{\partial}{\partial t}\int_0^\lambda u\,dx = \left[\frac{\partial\phi}{\partial t}\right]_0^\lambda = \left[\frac{p}{\rho} - \tfrac{1}{2}q^2 + gz + F(t)\right]_0^\lambda, \tag{33}$$

which again vanishes by the periodicity. Hence (31) is valid for all values of z and t. From (30) and (31) we have then

$$\int_0^\lambda f(Z)\,dx = 0, \tag{34}$$

so that (29) may be written

$$I_1 = \int_0^\lambda f(x)\frac{\pi}{\lambda}\left[\cot\frac{\pi(x-Z_0)}{\lambda} - 1\right]dx \tag{35}$$

$$= \int_0^\lambda f(x)\frac{ik\exp\{ik(x-Z_0)\}}{1-\exp\{ik(x-Z_0)\}}\,dx, \tag{36}$$

where $k = 2\pi/\lambda$. Hence when $e^{-kz_0} < \frac{1}{2}$ (that is, when z_0 is greater than about $\frac{1}{4}\lambda$) we have

$$|I_1| < 4\pi e^{-kz_0}\max|f(x)|. \tag{37}$$

Similarly we may show that

$$I_2 = \int_0^\lambda f(x)\frac{ik\exp\{ik(x+2ih-Z_0)\}}{1-\exp\{ik(x+2ih-Z_0)\}}\,dx, \tag{38}$$

so that when $\exp\{-k(2h-z_0)\} < \frac{1}{2}$ (which is certainly satisfied if $e^{-kz_0} < \frac{1}{2}$ and $0 \leqslant z_0 \leqslant h$), we have

$$|I_2| < 4\pi\exp\{-k(2h-z_0)\}\max|f(x)|. \tag{39}$$

Therefore from equation (25)

$$|f(Z_0)| \leqslant \frac{1}{2\pi}(|I_1| + |I_2|) < 4e^{-kh}\cosh k(z_0-h)\max|f(x)|. \tag{40}$$

On dropping the suffix 0 we have

$$(u^2+w^2)^{\frac{1}{2}} < 4\,e^{-kh}\cosh k(z-h)\max[(u^2+w^2)^{\frac{1}{2}}]_{z=0}, \tag{41}$$

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and on letting kh tend to infinity we find that in infinitely deep water

$$(u^2+w^2)^{\frac{1}{2}} \leqslant 2\,e^{-kz} \max\,[(u^2+w^2)^{\frac{1}{2}}]_{z=0}. \tag{42}$$

This proves the result.

u and w are each less than or equal to $(u^2+w^2)^{\frac{1}{2}}$ in absolute value. Thus (42) implies that both components of velocity diminish exponentially.

If ϕ is chosen as in § 2, so that $\overline{\phi}$ vanishes, then we may show that ϕ also decreases exponentially. For since ϕ is continuous and its mean value when $0<x<\lambda$ is zero, it follows that ϕ itself must be zero somewhere in this interval, say when $x = x'$. We have then

$$|\,\phi\,| = \left|\int_{x'}^{x} \frac{\partial\phi}{\partial x}\,dx\right| \leqslant |\,x-x'\,|\,m, \tag{43}$$

where m is the maximum value of $|\,\partial\phi/\partial x\,|$ in the interval. But $|\,x-x'\,|<\lambda$, and m is exponentially small; hence ϕ also is exponentially small.

From equation (11) we have then, at great depths,

$$\frac{p'}{\rho} = F(t), \tag{44}$$

so that the pressure fluctuation is a function of the time t only, and therefore occurs uniformly and simultaneously at all points; and from equation (16) we see that in this case $F(t)$ is given by

$$F(t) = \frac{\partial^2}{\partial t^2}(\tfrac{1}{2}\overline{\zeta^2}). \tag{45}$$

The above expression has been evaluated in some particular cases (see Longuet-Higgins and Ursell (5), and Longuet-Higgins (3), § 2·3). In general, $F(t)$ represents a pressure fluctuation of double the fundamental frequency of the waves.

3. *In a progressive wave in water of uniform depth the mass-transport velocity diminishes with the mean depth of the particle.*

Following Ursell (10) we first express Rayleigh's geometrical argument (Rayleigh (7)) in analytical terms. Let the motion be reduced to a steady state by referring it to axes moving in the direction of the waves with the wave velocity c. In the present section x, z, ϕ and ψ will denote the coordinates, velocity potential and stream function in the *steady-state* motion. If the orbital motion of the particles does not exceed c it follows that the horizontal component of the steady-state velocity is negative:

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial z} < 0. \tag{46}$$

Now let ϕ and ψ be taken as fundamental coordinates, and x and z be expressed as functions of them. Let $s(\phi,\psi)$ denote the arc-length measured along a stream-line, ψ = constant, from the equipotential line $\phi = 0$. The motion being space-periodic, there will be another equipotential line $\phi = \phi_\lambda$ at a wave-length's distance from $\phi = 0$, where the motion is identical with that on $\phi = 0$. If the time taken for a particle to travel from $\phi = 0$ to $\phi = \phi_\lambda$ is denoted by $T(\psi)$, then the forward mass-transport velocity $U(\psi)$ may be defined by the equations

$$T = \frac{\lambda}{c-U}\,, \quad U = c - \frac{\lambda}{T}. \tag{47}$$

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Now the velocity q along a stream-line is given by the ratio of an increment $d\phi$ to the corresponding increment ds, ψ being kept constant; that is

$$q = \left(\frac{d\phi}{ds}\right)_{\psi \text{ constant}} = \left(\frac{\partial s}{\partial \phi}\right)^{-1}. \tag{48}$$

We have therefore $$T = \int_0^{\phi_\lambda} q^{-1} ds = \int_0^{\phi_\lambda} \frac{\partial s}{\partial \phi} ds = \int_0^{\phi_\lambda} \left(\frac{\partial s}{\partial \phi}\right)^2 d\phi, \tag{49}$$

that is, $$T = \int_0^{\phi_\lambda} \left[\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2\right] d\phi.$$

At this point Ursell (10) introduces a Fourier series expansion. However, since x, z are conjugate functions, $\partial z/\partial \phi$ can be replaced by $-\partial x/\partial \psi$, so that

$$T = \int_0^{\phi_\lambda} \left[\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial x}{\partial \psi}\right)^2\right] d\phi.$$

x satisfies Laplace's equation in the coordinates ϕ and ψ, and all its derivatives are periodic in ϕ with period ϕ_λ. Therefore, precisely as in § 1, we have

$$\frac{\partial T}{\partial \psi} = 4\int_0^{\phi_\lambda} \frac{\partial x}{\partial \phi} \frac{\partial^2 x}{\partial \phi \partial \psi} d\phi \tag{50}$$

and $$\frac{\partial^2 T}{\partial \psi^2} = 4\int_0^{\phi_\lambda} \left[\left(\frac{\partial^2 x}{\partial \phi \partial \psi}\right)^2 + \left(\frac{\partial^2 x}{\partial \phi^2}\right)^2\right] d\phi. \tag{51}$$

$\partial T/\partial \psi$ vanishes on the bottom $\psi = \psi_h$, since $\partial x/\partial \psi$, which equals $-\partial z/\partial \phi$, is zero. Since by (46) $\psi > \psi_h$ in general, it follows by an exactly similar argument that $\partial T/\partial \psi$ is in general positive. Thus U by (47) is an increasing function of ψ and so a decreasing function of the mean depth of the stream-line. This proves the result.

It may be asked whether the analytical proof given above could not be translated back into geometrical terms. This is probably so; but the geometrical version would almost certainly be more complicated; for it will be seen that the proof essentially involves the curvature of the velocity-profile, that is, the second derivative of the velocity. In a geometrical proof, therefore, small quantities of the second or higher orders would have to be taken into account.

4. *The solitary wave.* Similar results may be proved for the solitary wave. In this case it is assumed that when $x \to \pm\infty$ the motion tends to zero, or that, when a uniform velocity is superimposed, the motion tends to a uniform stream. Thus the velocity at $+\infty$ equals that at $-\infty$. Instead of the mean velocity $\overline{q^2}$, which over an infinite interval would be zero, we may consider the quantity

$$Q^2 = \int_{x_1}^{x_2} (u^2 + w^2)\, dx = \int_{x_1}^{x_2} \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] dx, \tag{52}$$

where x_1 and x_2 are made to tend to $+\infty$ and $-\infty$ respectively. In other words Q^2, in the limit, is the square of the velocity at a given depth z integrated with respect to x from $-\infty$ to $+\infty$. Since the values of $\partial\phi/\partial x$, $\partial\phi/\partial z$ and their derivatives at the two ends of the interval are equal in the limit it follows, as before, that

$$\frac{\partial Q^2}{\partial z} \leqslant 0, \quad \frac{\partial^2 Q^2}{\partial z^2} \geqslant 0, \tag{53}$$

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and hence (1) *the total density of kinetic energy, integrated from* $-\infty$ *to* $+\infty$, *is a decreasing function of the depth* z; (2) *the square of each component of velocity, integrated from* $-\infty$ *to* $+\infty$ *is a decreasing function of* z; (3) *the dynamical part of the total force on a horizontal plane* $z =$ *constant increases with* z.

To consider the mass transport we suppose, as in § 3, that the motion is reduced to a steady state by superposing on it a uniform horizontal velocity $-c$. The time T taken by a particle to travel along a stream-line $\psi =$ constant between the two equipotential lines $\phi = \phi_1$ and $\phi = \phi_2$ is given by

$$T(\psi) = \int_{\phi_1}^{\phi_2} \left[\left(\frac{\partial x}{\partial \phi} \right)^2 + \left(\frac{\partial x}{\partial \psi} \right)^2 \right] d\phi \tag{54}$$

(where x, ϕ and ψ now refer to the steady motion). The displacement of the particle with respect to the moving axes is $[x]_{\phi_1}^{\phi_2}$. Hence the displacement D with respect to stationary axes is given by

$$D = [x]_{\phi_1}^{\phi_2} + cT, \tag{55}$$

and so

$$\frac{\partial D}{\partial \psi} = \left[\frac{\partial x}{\partial \psi} \right]_{\phi_1}^{\phi_2} + c \frac{\partial T}{\partial \psi}. \tag{56}$$

Now as ϕ_1 and ϕ_2 tend to $-\infty$ and $+\infty$ respectively the flow, in the steady state, tends to a uniform stream, and so $\partial x/\partial \psi$ tends to zero. Thus, in the limit,

$$\frac{\partial D}{\partial \psi} = c \frac{\partial T}{\partial \psi}. \tag{57}$$

But since $\partial x/\partial \phi$, $\partial x/\partial \psi$, etc., tend, in the limit, to the same values at the two ends of the interval, we have from (54)

$$\frac{\partial T}{\partial \psi} \geqslant 0, \quad \frac{\partial^2 T}{\partial \psi^2} \geqslant 0. \tag{58}$$

Hence in the original motion: (4) *the total horizontal displacement of a particle during the passage of a solitary wave is a decreasing function of the mean depth of the particle.*

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