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Units in real cyclic quartic fields

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1. *Introduction*

Let F be a totally real quartic field. For any α in F, let $\alpha, \alpha', \alpha'', \alpha'''$ or $\alpha^{(0)} = \alpha, \alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}$ denote the conjugates of α. Define the function $T(\alpha)$ by

$$T(\alpha) = \text{trace}\,(\alpha^2) = \alpha^2 + \alpha'^2 + \alpha''^2 + \alpha'''^2.$$

We define a triple of units $\epsilon_1, \epsilon_2, \epsilon_3$ in F as follows. Let ϵ_1 be a unit which gives the least value of $T(\epsilon)$ for any unit $\epsilon \neq \pm 1$ in F. Let ϵ_2 be a unit which gives the least value of $T(\epsilon)$ for any unit $\epsilon \neq \pm \epsilon_1^m$ with m a rational integer. Let ϵ_3 be a unit which gives the least value of $T(\epsilon)$ for $\epsilon_3 \neq \pm \epsilon_1^m \epsilon_2^n$ with m and n rational integers. We call $\epsilon_1, \epsilon_2, \epsilon_3$ the *successive unit minima* for $T(\epsilon)$.

In an earlier paper ([**2**], p. 192) we conjectured that the successive unit minima are a fundamental triple of units for any totally real quartic field F. This will be referred to as the 'unit conjecture' in the rest of the paper. The corresponding result for a fundamental pair of units in a totally real cubic field is true ([**2**], theorem 1, p. 191) and we did prove that ϵ_1 and ϵ_2 can always be taken to be two units in a fundamental triple for F ([**2**], theorem 2, p. 192). Now, however, we prove that, at least for real cyclic quartic fields, it can happen that $\epsilon_1, \epsilon_2, \epsilon_3$ is not a fundamental triple. This occurs only when the field F satisfies some very special conditions, which are completely characterized below; in these exceptional cases, the successive unit minima generate a subgroup of index 2 in the group of units of F. In addition, two specific numerical counter-examples are given, by using the table of cyclic quartic fields provided by Gras [**3**]. It seems certain that such counter-examples, though rare, are infinite in number. The status of the unit conjecture for non-cyclic real quartic fields is still unclear.

From now on, the totally real quartic field F will always be assumed to be cyclic.

2. *Unit lattices*

We shall use various classical results on real cyclic quartic fields which were first proved by Hasse [**4**].

We let k denote the unique quadratic subfield of F and we let η denote the fundamental unit greater than 1 of k. Given α in F, we let $N(\alpha)$ denote the norm from the rationals $\mathbb{Q}$ to F and we let $n(\alpha) = \alpha\alpha'' = \alpha'\alpha'''$ denote the relative norm from k to F.

Let U_F and U_k denote the unit groups of F and k, respectively. Let $U_{F/k}$ denote the group of *relative units*, that is, units ϵ in F such that $n(\epsilon) = \pm 1$. The groups $U_{F/k}$ and

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U_k have only ± 1 in common and we define the unit index Q to be the index of their direct product in U_F:

$$Q = [U_F : U_{F/k}\, U_k].$$

LEMMA 1. *The unit index Q is 1 or 2, and $Q = 2$ if and only if there exists a unit E_* in U_F such that $n(E_*) = \pm\eta$.*

Proof. This is Satz 16, p. 37 in Hasse [**4**].

We define the logarithm space $L(F)$ to be the set of all four-dimensional log vectors

$$\boldsymbol{\alpha} = (\log|\alpha|, \log|\alpha'|, \log|\alpha''|, \log|\alpha'''|),$$

where α runs through all non-zero elements of F. We define the unit lattice $\Lambda(F)$ to be the 3-dimensional lattice in $L(F)$ made up of all vectors $\boldsymbol{\varepsilon}$, where ϵ is a unit in F.

Hasse ([**4**], Satz 22, p. 40) proved that there exists a unit E such that E, E' and ± 1 generate the relative unit group $U_{F/k}$. By choosing our field conjugates appropriately and taking a reciprocal if necessary, we may suppose

$$E > |E'| > 1.$$

We call this uniquely determined unit E the *relative fundamental unit*. Since we have $EE'' = E'E''' = \pm 1$, the three vectors

$$\left.\begin{aligned} \mathbf{E} &= (\log E, \log|E'|, -\log E, -\log|E'|) \\ \mathbf{E}' &= (\log|E'|, -\log E, -\log|E'|, \log E) \\ \boldsymbol{\eta} &= (\log\eta, -\log\eta, \log\eta, -\log\eta) \end{aligned}\right\} \tag{1}$$

are mutually orthogonal in $\Lambda(F)$. A characterization of the relative fundamental unit is the following:

LEMMA 2. *The units $\pm E^{(i)}$ $(i = 1, 2, 3)$ give the least value for $T(\epsilon)$ where ϵ runs through all of the relative units $\epsilon \neq \pm 1$; these eight units are the only relative units $\neq \pm 1$ for which $T(\epsilon)$ has its minimal value.*

Proof. This is Satz 23, p. 41 in Hasse [**4**].

If the unit index Q is equal to 1, then it follows from Lemma 1 that the three vectors in (1) form a basis of mutually orthogonal vectors for $\Lambda(F)$. Thus when $Q = 1$ the unit lattice $\Lambda(F)$ is a rectangular lattice containing a square sublattice made up of the relative unit log vectors (see Figure 1).

If the unit index Q is equal to 2, then the unit E_* of Lemma 1 is uniquely determined by

$$E_* > 0, \tag{2}$$

$$E_* E_*'' = \pm\eta \tag{3}$$

and

$$E_* E_*' = \pm E \tag{4}$$

(see Hasse [**4**], p. 45). It follows from (3) and (4) that

$$E_*^2 = \pm\eta E/E' \tag{5}$$

and

$$E_*' E_*''' = \pm\eta^{-1}. \tag{6}$$

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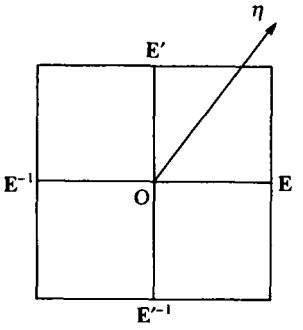


Fig. 1. The unit lattice $\Lambda(F)$ when $Q = 1$.

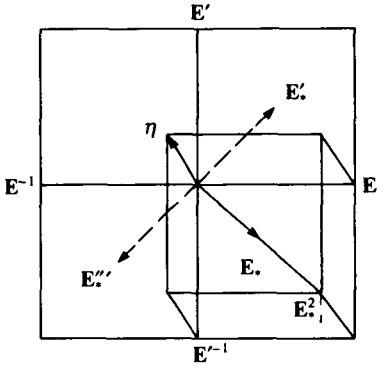


Fig. 2. Typical unit lattice $\Lambda(F)$ when $Q = 2$.

It turns out that the unit E_* is a *Minkowski unit* for F, i.e. a unit ϵ with the property that $-1, \epsilon$ and two of its conjugates generate the whole unit group U_F:

LEMMA 3. *If $Q = 2$, then there exists a unit E_*, uniquely determined by* (2), (3) *and* (4), *such that the unit group U_F is generated by -1 and the conjugates of E_*.*

Proof. This is Satz 28, p. 45 in Hasse[**4**].

It follows from Lemma 2 that when $Q = 2$ the unit lattice $\Lambda(F)$ is the lattice of Figure 1 augmented with the centres of each of its fundamental cells (see Figure 2).

We shall call the unit E_* which is uniquely determined by (2), (3) and (4) the *basic unit* of F. Of course F has a basic unit if and only if $Q = 2$.

We shall make extensive use of the above results on the lattices $\Lambda(F)$ in our study of the unit conjecture.

3. *The case $Q = 1$*

Let $\mathbf{x} = (x_1, x_2, x_3, x_4)$ be any point in 4-dimensional space and define the function

$$f(\mathbf{x}) = f(x_1, \ldots, x_4) = \sum_{i=1}^{3} e^{2x_i} + \exp\left(-2\sum_{i=1}^{3} x_i\right).$$

We shall use the function $f(\mathbf{x})$ only for $\mathbf{x}$ in the hyperplane $x_1 + \ldots + x_4 = 0$ which contains the unit lattice $\Lambda(F)$. Clearly if $\mathbf{x} = \boldsymbol{\varepsilon}$ for some unit ϵ in F, then $f(\mathbf{x}) = T(\epsilon)$,

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so $f(\mathbf{x})$ coincides with $T(\epsilon)$ at points in $\Lambda(F)$. We shall require the following two lemmas from [2].

LEMMA 4. *Given any straight line in 4-space, $f(\mathbf{x})$ has a unique minimum value for $\mathbf{x}$ on the line and increases monotonically as the point $\mathbf{x}$ moves away from the minimizing point along the line in either direction.*

Proof. This is lemma 2, p. 192 in Cusick [2].

The following evident corollary will be used several times in what follows.

COROLLARY. *If $f(\mathbf{x})$ takes on two equal values at two different points on any given line in 4-space, then the minimum of $f(\mathbf{x})$ on that line occurs between the two points and $f(\mathbf{x})$ increases as $\mathbf{x}$ moves away from the minimizing point along the line in either direction.*

LEMMA 5. *If $\mathbf{a}$ is any non-zero vector, then $f(t\mathbf{a})$ increases as t increases.*

Proof. This is lemma 3, p. 192 in Cusick [2].

For the rest of this section, we suppose that $Q = 1$, so that the units E, E' and η are a fundamental triple of units in F. We shall prove that the unit conjecture is true whenever $Q = 1$.

THEOREM 1. *If $Q = 1$, then the successive unit minima $\epsilon_1, \epsilon_2, \epsilon_3$ are a fundamental triple of units for F.*

Given any real number r and any unit ϵ in F, we define the vector $\boldsymbol{\varepsilon}^r$ by

$$\boldsymbol{\varepsilon}^r = (r \log|\epsilon|, r \log|\epsilon'|, r \log|\epsilon''|, r \log|\epsilon'''|).$$

This extends our earlier notation, which agrees with the new one when r is a rational integer.

Given any real number r, we shall refer to the hyperplane parallel to the relative unit plane and containing the point $\boldsymbol{\eta}^r$ as the η^r-plane.

Given any unit $\delta = E^a E'^b \eta^c$ with a, b, c rational integers, we have

$$\delta' = E^{-b} E'^a \eta^{-c}, \quad \delta'' = E^{-a} E'^{-b} \eta^c, \quad \delta''' = E^b E'^{-a} \eta^{-c}. \tag{7}$$

As a supplement to Lemma 2, we have

LEMMA 6. *When $Q = 1$, the least value for $T(\epsilon)$ when E is a unit but not a relative unit occurs when ϵ is $\pm\eta$ or $\pm\eta^{-1}$.*

Proof. Given a unit $\delta = E^a E'^b \eta^c$ which is not a relative unit (so $c \neq 0$), it follows from (7) that the points $\boldsymbol{\delta}$, $\boldsymbol{\delta}''$ and $\boldsymbol{\eta}^c$ lie on a line L in the η^c-plane. The lemma now follows from the inequalities

$$T(\delta) \geqslant T(\eta^c) \geqslant T(\eta) = T(\eta^{-1}).$$

Here the first inequality follows from applying Lemma 4 to the line L and the second inequality follows from applying Lemma 5 to the line through $\mathbf{0}$ and $\boldsymbol{\eta}$.

If now we have $T(E) \leqslant T(\eta)$, then since $T(E) = T(E')$ and E, E' are independent units, by Lemmas 2 and 6 we can put $E_1 = E, E_2 = E', E_3 = \eta$. This proves Theorem 1 in this case.

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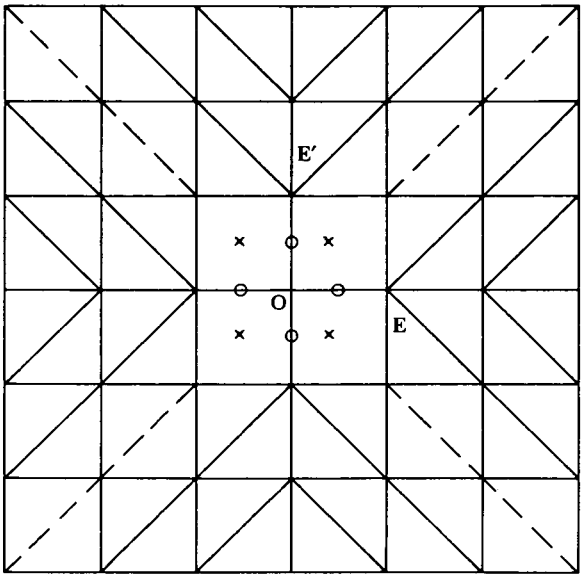


Fig. 3. The relative unit plane with special points marked by × and ○.

On the other hand, it may happen that $T(\eta) < T(E)$. In this case, $\epsilon_1 = \eta$ by Lemmas 2 and 6. If E is the next successive unit minimum for $T(\epsilon)$, then we can take $\epsilon_2 = E, \epsilon_3 = E'$ and we have Theorem 1 as before. Otherwise there is a unit $\delta = \epsilon_2 = E^a E'^b \eta^c$, independent of η and with $c \neq 0$, such that $T(\eta) \leqslant T(\delta) < T(E)$, and we can put $\epsilon_3 = \delta'$ since δ and δ' are independent units. Now a computation using $\delta' = E^{-b} E'^a \eta^{-c}$ shows that η, δ, δ' is a fundamental triple (so Theorem 1 is true) if and only if $a^2 + b^2 = 1$. Our next lemma establishes this.

LEMMA 7. *Suppose* $Q = 1$ *and* $T(\eta) < T(E)$. *If* $\delta = E^a E'^b \eta^c$ *is a unit such that* $(a, b) \neq (0, 0)$, $c \neq 0$ *and* $T(\eta) \leqslant T(\delta) < T(E)$, *then either* $a = \pm 1$, $b = 0$, *or* $a = 0$, $b = \pm 1$.

Proof. Given a number $\mu = E^r E'^s \eta^t$, where r, s, t are real numbers, we let $\boldsymbol{\mu} : (r, s, t)$ denote the *unit coordinates* (with respect to E, E', η) of the corresponding point $\boldsymbol{\mu}$ in the hyperplane $x_1 + \ldots + x_4 = 0$. Thus $\mathbf{E} : (1, 0, 0)$, $\mathbf{E}' : (0, 1, 0)$ and $\boldsymbol{\eta} : (0, 0, 1)$ are unit coordinates for the fundamental triple E, E', η.

By (7), the line connecting $\boldsymbol{\delta}$ and $\boldsymbol{\delta}'$ intersects the relative unit plane in the point $\mathbf{P}$ with unit coordinates $\mathbf{P} : (\frac{1}{2}(a-b), \frac{1}{2}(a+b), 0)$. Applying the Corollary to Lemma 4 to this line gives

$$f(\mathbf{P}) \leqslant T(\delta) = T(\delta') < T(E). \tag{8}$$

We shall prove that (8) is possible only for $a = \pm 1, b = 0$ or $a = 0, b = \pm 1$.

For $j = 1, 2, \ldots$, the four pairs of points $\mathbf{E}^j, \mathbf{E}'^j; \mathbf{E}'^j, \mathbf{E}^{-j}; \mathbf{E}^{-j}, \mathbf{E}'^{-j}$; and $\mathbf{E}'^{-j}, \mathbf{E}^j$ give equal values of the function $T(\epsilon)$. Applying the Corollary to Lemma 4 to the lines defined by all of these pairs of points, we see that all points $\mathbf{x}$ in the relative unit plane which lie on one of the dashed diagonal lines in Figure 3 satisfy $f(\mathbf{x}) \geqslant T(E)$. (Here we are also using the simple fact that

$$T(E) < T(E^2) < \ldots < T(E^j) < \ldots,$$

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which follows from Lemma 5 or can easily be proved directly.) Using Lemma 5 and the fact that for $\epsilon = EE'$ we have $T(\epsilon^{(j)}) > T(E)$ for $0 \leqslant j \leqslant 3$ (the inequality is immediate from Lemma 2), we see that all points $\mathbf{x}$ which lie on the solid diagonal lines in Figure 3 satisfy $f(\mathbf{x}) > T(E)$.

The point $\mathbf{P}: (\frac{1}{2}(a-b), \frac{1}{2}(a+b), 0)$ has first two coordinates either both integers or both halves of odd integers, so $\mathbf{P}$ in Figure 3 either lies on a lattice point or at the centre of one of the lattice cells. Since $\mathbf{P} \neq \mathbf{0}$, the results in the previous paragraph contradict (8) unless $\mathbf{P}$ is one of the four points marked with a $\times$ in Figure 3. These four points are precisely the allowable cases $a = \pm 1, b = 0$ and $a = 0, b = \pm 1$ in the lemma.

Remark 1. If $\mathbf{P}$ is a lattice point in Figure 3, then the corresponding real number $P = E^{\frac{1}{2}(a-b)} E'^{\frac{1}{2}(a+b)}$ is a relative unit and Lemma 2 gives $f(\mathbf{P}) \geqslant T(E)$. This is another way to obtain a contradiction to (8) in this case.

Remark 2. It is easy to see from the proof of Lemma 7 that Lemmas 4 and 5 can be used to give a short proof of Lemma 2. This proof of Lemma 2 is simpler than the original one of Hasse ([**4**], pp. 41–42).

Remark 3. The situation in Lemma 7 does occur. The cyclic quartic field of discriminant 1125 defined by the roots of

$$q(x) = x^4 - x^3 - 4x^2 + 4x + 1$$

furnishes an example. If these roots are $\rho^{(i)}$ $(0 \leqslant i \leqslant 3)$, then $\rho, \rho-1, \rho-2$ are all units of norm 1 and $T(\rho) = 9, T(\rho-1) = 11, T(\rho-2) = 21$. We have $\eta = \frac{1}{2}(1+\sqrt{5})$ and $T(\eta) = 6$. Computations show that if

$$\rho = -\tfrac{1}{2}(\tfrac{1}{2}(-1+\sqrt{5}) + (\tfrac{1}{2}(15+3\sqrt{5}))^{\frac{1}{2}}) = -1{\cdot}956\ldots$$

is the root of $q(x)$ of largest absolute value, then $E = 1-\rho, T(E) = 11$ and we can take $\epsilon_1 = \eta, \epsilon_2 = \rho, \epsilon_3 = \rho'$. Thus $\delta = \rho$ is an example of the situation in Lemma 7. In such cases the successive unit minima vectors ϵ_i $(i = 1, 2, 3)$ do not correspond to the three shortest independent vectors (clearly $\boldsymbol{\eta}$, $\mathbf{E}$ and $\mathbf{E}'$) in the unit lattice $\Lambda(F)$.

4. *The case* $Q = 2, \epsilon_1 \neq \eta$

In this section we shall prove that the unit conjecture is true when $Q = 2$ provided that ϵ_1 can be chosen different from $\pm\eta$ and $\pm\eta^{-1}$.

THEOREM 2. *If* $Q = 2$ *and there is a unit* δ *not* $\pm\eta$ *or* $\pm\eta^{-1}$ *such that* $T(\delta) \leqslant T(\eta)$, *then the successive unit minima* $\epsilon_1, \epsilon_2, \epsilon_3$ *are a fundamental triple of units for* F.

To prove Theorem 2, we shall require

LEMMA 8. *If* $Q = 2$ *and* $\eta < E$, *then* $T(E) > T(E_*)$.

Proof. We first observe that if $|E'_*| \leqslant 1$, then (4) and (5) give

$$E_*^2 = |\eta E/E'| \geqslant E^2,$$

whence we have $E > \eta \geqslant |EE'|$, contradicting the fact that $|E'| > 1$. Hence we may define c by

$$|E'_*| = c^{-1} > 1.$$

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Now we define $g_c(x)$ by

$$g_c(x) = x^2 + x^{-2} + (\eta c^{-2}x^{-1})^2 + (c^2 x\eta^{-1})^2 - (cx)^2 - c^{-2} - (\eta c^{-1}x^{-1})^2 - (c\eta^{-1})^2.$$

It follows from (3), (4), (5) and (6) that

$$|E_*| = cE, \quad |E''_*| = |\eta c^{-1}E^{-1}|, \quad |E'''_*| = c\eta^{-1} \tag{9}$$

and that

$$|E'| = |E'_* E''_*| = \eta/(E_* c) = \eta c^{-2}E^{-1}. \tag{10}$$

Now (9) and (10) imply that $T(E) > T(E_*)$ is equivalent to

$$g_c(E) > 0. \tag{11}$$

We define $h_c(x) = x^2 g_c(x)$. If we define $x_0 > 0$ by

$$x_0^2 = (2c^2)^{-1}(1 + c^4\eta^{-2})/(1 + c^4\eta^{-2} - c^2)$$

then calculation shows

$$h'_c(x_0) = 0, \quad h'_c(x) > 0 \quad \text{for} \quad x > x_0. \tag{12}$$

Since $c < 1$ we have $\frac{1}{2}(\eta^2 + 1) > x_0^2$, so that $0 < x_0 < \eta$; thus (12) and our hypothesis $E > \eta$ imply $E^2 g_c(E) > \eta^2 g_c(\eta)$. Thus in order to prove (11) and with it the lemma, we only need to verify that

$$g_c(\eta) > 0. \tag{13}$$

To prove this, we observe that

$$g'_c(x) = 2(1 - c^2)(x - x^{-3}) > 0 \quad \text{for} \quad x > 1$$

and

$$g_c(1) = c^{-4}(c^2 - 1)^2(c^4 + 1) > 0.$$

Now (13) follows since $\eta > 1$, and this completes the proof of the lemma.

In proving Theorem 2, we first consider the case where ϵ_1 lies in the relative unit plane. It follows from Lemma 2 that in this case we may assume $\epsilon_1 = E$ and $\epsilon_2 = E'$. If now $T(E_*) \leqslant T(E)$ then we can take $\epsilon_3 = E_*$ and Theorem 2 is true (the fact that $\epsilon_1, \epsilon_2, \epsilon_3$ is a fundamental triple in this case follows from Figure 2 or from (4) and the corresponding formula $E'_* E''_* = \pm E'$). Thus we may further assume $T(E) < T(E_*)$ and so

$$\eta \geqslant E \tag{14}$$

by Lemma 8. In order to prove that E, E', ϵ_3 is a fundamental triple, it suffices to show that

$$T(\eta) > T(E^*) \tag{15}$$

because of

LEMMA 9. *If $Q = 2$, then the minimum of $T(\epsilon)$ for units ϵ not in the relative unit plane occurs either for $\epsilon = \eta$ or for $\epsilon = E^a E'^b E_*^c$ with $c = \pm 1$.*

Proof. It is easily verified that the function $f(\mathbf{x})$, defined in Section 3 above, is increasing as $\mathbf{x}$ moves away from the point $\boldsymbol{\eta}^r$, with r any real number, in any direction parallel to the relative unit plane. Thus the minimum value of $f(\mathbf{x})$ in any η^k-plane, with k a rational integer, is $f(\boldsymbol{\eta}^k) = T(\eta^k)$.

Let $\delta = E^a E'^b E_*^c$ be any unit with $c \neq 0$, so that $\delta'' = E^{-a-c}E'^{-b+c}E_*^c$. The line segment through $\boldsymbol{\delta}$ and $\boldsymbol{\delta}''$ contains the point $\boldsymbol{\mu}$ for $\mu = E^{-\frac{1}{2}c}E'^{\frac{1}{2}c}E_*^c$ and, by (5),

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$|\mu| = \eta^{\frac{1}{2}c}$. Applying Lemma 4 to the line through $\boldsymbol{\delta}$ and $\boldsymbol{\delta}''$ gives $T(\delta) \geqslant f(\boldsymbol{\eta}^{\frac{1}{2}c})$ with equality only if $a = -\frac{1}{2}c$ and $b = \frac{1}{2}c$. For $|c| \geqslant 2$, Lemma 5 implies $T(\delta) \geqslant f(\boldsymbol{\eta}) = T(\eta)$ with equality only if $\delta = \eta^{\pm 1}$. In view of the property of $f(\mathbf{x})$ given in the previous paragraph, this proves Lemma 9.

Of course (15) does not guarantee that we can take $\epsilon_3 = E_*$, but it does imply that $\epsilon_3 = E^a E'^b E_*^c$ for some a, b and $c = \pm 1$. Any such ϵ_3 forms a fundamental triple with E and E', so we need only prove (15) to settle this case of Theorem 2. We do this by first observing that since $x + x^{-1}$ increases for $x \geqslant 1$ and $E > |E'| > 1$, (14) implies

$$\eta(|E/E'| + |E'/E|) \leqslant \eta(\eta + \eta^{-1}) = \eta^2 + 1 \tag{16}$$

and

$$\eta^{-1}(|EE'| + |EE'|^{-1}) \leqslant \eta^{-1}(\eta^2 + \eta^{-2}) = \eta + \eta^{-3}. \tag{17}$$

Combining (16) and (17) and using (5) (with the conjugate formulas for $E_*^{(i)2}$ for $1 \leqslant i \leqslant 3$) gives

$$\eta^2 + \eta + 1 + \eta^{-3} \geqslant T(E_*). \tag{18}$$

Now

$$T(\eta) = 2(\eta^2 + \eta^{-2}) > \eta^2 + \eta + 1 + \eta^{-3} \tag{19}$$

holds, because $\eta > 1$ and the function

$$x^2 - x - 1 + 2x^{-2} - x^{-3} = x^{-3}(x-1)^2(x^3 + x^2 - 1)$$

is non-negative for $x \geqslant 0{\cdot}8$. Combining (18) and (19) proves (15).

Now we turn to the case of Theorem 2 where ϵ_1 does not lie in the relative unit plane. By Lemma 9, we may assume $\epsilon_1 = E^a E'^b E_*^c$ with $c = \pm 1$. Now $\delta = \epsilon_1$, $\delta' = E^{-b+c} E'^a E_*^{-c}$ and $\delta'' = E^{-a-c} E'^{-b+c} E_*^c$ are independent units, and so we can take $\epsilon_2 = \delta', \epsilon_3 = \delta''$. For $c = \pm 1$ we have

$$\det \begin{bmatrix} a & b & c \\ -b+c & a & -c \\ -a-c & -b+c & c \end{bmatrix} = 2(a^2+b^2) + c(2a-2b) + 1 = u(a,b,c), \quad \text{say.}$$

Thus this case of Theorem 2 will be proved if we show that $\epsilon_1 = E^a E'^b E_*^{\pm 1}$ implies $u(a, b, \pm 1) = 1$. We shall do this using an argument similar to that in the proof of Lemma 7.

Given a number $\mu = E^r E'^s E_*^t$, where r, s, t are real numbers, we let $\boldsymbol{\mu}: (r, s, t)$ denote the unit coordinates (with respect to E, E', E_*) of the corresponding point $\boldsymbol{\mu}$ in the hyperplane $x_1 + \ldots + x_4 = 0$. Thus

$$\boldsymbol{\delta}: (a, b, c), \quad \boldsymbol{\delta}': (-b+c, a, -c), \quad \boldsymbol{\delta}''': (b, -a-c, -c)$$

are unit coordinates for $\boldsymbol{\delta} = \epsilon_1$.

A slight extension of the argument given in the proof of Lemma 7 shows that $f(\mathbf{x}) \geqslant T(E) > T(\epsilon_1)$ for all $\mathbf{x}: (\frac{1}{2}m, \frac{1}{2}n, 0)$ with m, n rational integers in the relative unit plane except those $\mathbf{x}$ with $|m| \leqslant 1$ and $|n| \leqslant 1$. These are the eight points marked with a $\times$ or 0 in Figure 3, together with $\mathbf{0}$; we call these nine points the *special points* in the relative unit plane.

We first suppose $c = 1$ and consider the line connecting $\boldsymbol{\delta}$ and $\boldsymbol{\delta}'''$. This line intersects the relative unit plane in the point $\mathbf{Q}$ with unit coordinates $\mathbf{Q}: (\frac{1}{2}(a+b), \frac{1}{2}(b-a-1), 0)$. Applying Lemma 4 as in the proof of Lemma 7 we find that

$$f(\mathbf{Q}) \leqslant T(\delta) = T(\epsilon_1) < T(E).$$

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Table 1. *Allowable choices for* $\epsilon_1 = E^a E'^b E_*^c$

c	a	b	Corresponding special point	$\epsilon_1 = E^a E'^b E_*^c$
1	0	1	$\frac{1}{2}, 0, 0$	$1/E'''_*$
1	−1	0	$-\frac{1}{2}, 0, 0$	$1/E'_*$
1	0	0	$0, -\frac{1}{2}, 0$	E_*
1	−1	1	$0, \frac{1}{2}, 0$	E''_*
−1	0	−1	$0, -\frac{1}{2}, 0$	E'''_*
−1	1	0	$0, \frac{1}{2}, 0$	E'_*
−1	0	0	$-\frac{1}{2}, 0, 0$	$1/E_*$
−1	1	−1	$\frac{1}{2}, 0, 0$	$1/E''_*$

This is a contradiction unless **Q** is a special point. Next we suppose $c = -1$ and consider the line connecting **δ** and **δ′**. This line intersects the relative unit plane in the point **R** with unit coordinates $\mathbf{R}: (\frac{1}{2}(a-b-1), \frac{1}{2}(a+b), 0)$. Reasoning as above, we reach a contradiction unless **R** is a special point.

Simple calculations show that for $c = 1$ there are only four possible pairs a, b for which **Q** is a special point and for $c = -1$ there are only four possible pairs a, b such that **R** is a special point. These eight possible choices for a, b, c are tabulated above.

We have $u(a, b, c) = 1$ for all of these eight allowable choices for a, b, c. Thus the proof of Theorem 2 is complete.

Remark. The last column of Table 1 shows that if $Q = 2$ and $\epsilon_1 = E^a E'^b E_*^{\pm 1}$, then ϵ_1 must be one of the conjugates of E_* or of $1/E_*$. This is an analogue for units in the $\eta^{\frac{1}{2}}$-plane or $\eta^{-\frac{1}{2}}$-plane of Lemma 2 for the case of relative units.

5. *The case* $Q = 2, \epsilon_1 = \eta$

In this section we assume that $T(\epsilon) = T(\epsilon_1)$ only for $\epsilon = \pm\eta$ or $\pm\eta^{-1}$, so that we can choose $\epsilon_1 = \eta$. We first observe that when $\epsilon_1 = \eta$, the unit ϵ_2 cannot lie in the relative unit plane because of Lemma 2 and the following result:

LEMMA 10. *If* $Q = 2$ *and* $\epsilon_1 = \eta$, *then* $T(E) > T(E_*)$.

Proof. Since $\epsilon_1 = \eta$ we have $T(\eta) < T(E)$, that is

$$2(\eta^2 + \eta^{-2}) < E^2 + E^{-2} + E'^2 + E'^{-2} < 2(E^2 + E^{-2}),$$

where the last inequality follows since $|E'| < E$. Since $x^2 + x^{-2}$ is an increasing function, this implies $\eta < E$ and the lemma follows from Lemma 8.

Thus we may assume

$$\epsilon_2 = E^a |E'|^b \eta^c, \quad c \neq 0, \tag{20}$$

where a, b, c are of the form $\frac{1}{2}m$ with m in $\mathbb{Z}$. Of course, since ϵ_2 is a unit, it follows from (5) that a and b are integers if c is, and when $c = \frac{1}{2}m$ is not an integer, then $a = \frac{1}{2}n_1$ and $b = \frac{1}{2}n_2$ for odd integers n_1 and n_2. Given a number $\mu = E^r E'^s \eta^t$, as in the proof of Lemma 7 we shall let $\boldsymbol{\mu}: (r, s, t)$ denote the unit coordinates with respect to E, E', η of the point **μ**.

Let $\delta = E^a E'^b \eta^c$. As in the proof of Lemma 7, the line connecting **δ** and **δ′** intersects

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Table 2. *Possible candidates for* $\epsilon_2 = E^a|E'|^b\eta^c, c \neq 0$

$a,\ b$	Corresponding special point	Candidates for value of ϵ_2
$0,\ 0$	$0,\ 0,\ 0$	η^c
$\frac{1}{2},\ -\frac{1}{2}$	$\frac{1}{2},\ 0,\ 0$	$\lvert E/E'\rvert^{\frac{1}{2}}\eta^c, c = \frac{1}{2}m$
$\frac{1}{2},\ \frac{1}{2}$	$0,\ \frac{1}{2},\ 0$	$\lvert EE'\rvert^{\frac{1}{2}}\eta^c,\ c = \frac{1}{2}m$
$-\frac{1}{2},\ \frac{1}{2}$	$-\frac{1}{2},\ 0,\ 0$	$\lvert E'/E\rvert^{\frac{1}{2}}\eta^c, c = \frac{1}{2}m$
$-\frac{1}{2},\ -\frac{1}{2}$	$0,\ -\frac{1}{2},\ 0$	$\lvert EE'\rvert^{-\frac{1}{2}}\eta^c, c = \frac{1}{2}m$
$1,\ 0$	$\frac{1}{2},\ \frac{1}{2},\ 0$	$E\eta^c$
$0,\ 1$	$-\frac{1}{2},\ \frac{1}{2},\ 0$	$\lvert E'\rvert\eta^c$
$-1,\ 0$	$-\frac{1}{2},\ -\frac{1}{2},\ 0$	$E^{-1}\eta^c$
$0,\ -1$	$\frac{1}{2},\ -\frac{1}{2},\ 0$	$\lvert E'^{-1}\rvert\eta^c$

the relative unit plane in the point **P** with unit coordinates $\mathbf{P}:(\frac{1}{2}(a-b), \frac{1}{2}(a+b), 0)$. By the same reasoning that was used in Section 4, we see that δ cannot be ϵ_2 unless **P** is one of the nine special points. The nine possibilities for the pair a, b are tabulated above.

Of course $\epsilon_2 = \eta^c$ (the first line in Table 2) is impossible since $\epsilon_1 = \eta$. If ϵ_2 is one of the candidates on the next four lines of Table 2, then we can take $\epsilon_3 = |\epsilon_2'|$ and we have (using (5))

$$\epsilon_1 = \eta = E^{-1}|E'|E_*^2,$$

$$\epsilon_2 = E^a|E'|^b\eta^c = E^{a-c}|E'|^{b+c}E_*^{2c},$$

$$\epsilon_3 = E^{-b}|E'|^a\eta^c = E^{-b+c}|E'|^{a-c}E_*^{-2c}.$$

Now since for any c

$$\det \begin{vmatrix} -1 & 1 & 2 \\ a-c & b+c & 2c \\ -b+c & a-c & -2c \end{vmatrix} = 2(a^2+b^2)$$

(note that the entries in the matrix are integers for any allowable value of c) and $a^2+b^2 = \frac{1}{2}$ for the four lines of Table 2 under consideration, we see that $\epsilon_1, \epsilon_2, \epsilon_3$ are a fundamental triple of units for F. Thus we have proved

THEOREM 3. *If* $Q = 2, T(\epsilon) = T(\epsilon_1)$ *only for* $\epsilon = \pm\eta$ *or* $\pm\eta^{-1}$ *and* $\epsilon_2 = E^aE'^b\eta^{\frac{1}{2}m}$ *with* $a = \pm\frac{1}{2}, b = \pm\frac{1}{2}$ *and* m *an odd integer, then the successive unit minima* $\epsilon_1, \epsilon_2, \epsilon_3$ *are a fundamental triple of units for* F.

If ϵ_2 is one of the candidates on the last four lines of Table 2, then c is an integer and we can take $\epsilon_3 = \epsilon_2'$. Since $E^{-1} = \pm E'''$, we can find a value for ϵ_2 in any of these four cases by choosing the integer c to minimize $T(E\eta^c)$. So in all four cases we can put $\epsilon_2 = E\eta^c$. This gives

$$\epsilon_1 = \eta = E^{-1}E'E_*^2$$

$$\epsilon_2 = E\eta^c = E^{-c+1}E'^cE_*^{2c}$$

$$\epsilon_3 = E'\eta^{-c} = E^cE'^{-c+1}E_*^{-2c}$$

and for any c

$$\det \begin{vmatrix} -1 & 1 & 2 \\ -c+1 & c & 2c \\ c & -c+1 & -2c \end{vmatrix} = 2.$$

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Thus $\epsilon_1, \epsilon_2, \epsilon_3$ generate a subgroup of index 2 in the unit group U_F and the conjecture that $\epsilon_1, \epsilon_2, \epsilon_3$ form a fundamental triple is false.

Thus there are two possible situations for the value of ϵ_2 in the case being analysed in this section: the situation in Theorem 3, where $\epsilon_1, \epsilon_2, \epsilon_3$ is a fundamental triple; and the situation just discussed, where $\epsilon_1, \epsilon_2, \epsilon_3$ generate a subgroup of index 2. We have already seen that in the latter situation we can choose $\epsilon_2 = E\eta^c$, with c an integer such that $T(E\eta^c)$ is minimal. In the former situation, we see from (5) that in lines 2 to 5 of Table 2 the candidates for the value of ϵ_2 have the form $\pm E_*^{(i)}\eta^d$, with d some integer, for $i = 0, 1, 2, 3$, respectively. Hence in this situation we can choose $\epsilon_2 = E_*\eta^d$ with d an integer such that $T(E_*\eta^d)$ is minimal.

Thus in order to decide which of the above two situations occurs for a given field F, we need to determine when the least value of $\{T(E\eta^c); c \in \mathbb{Z}\}$ exceeds the least value of $\{T(E_*\eta^d); d \in \mathbb{Z}\}$. This is a matter of calculation for any given field F, but the next two lemmas indicate the circumstances under which we can expect the second situation above to be the outcome of this calculation.

LEMMA 11. *For any real number c, define*

$$\mathbf{c} = (\log E\eta^c, \log|E'\eta^{-c}|, \log|E''\eta^c|, \log|E'''\eta^{-c}|).$$

The function of c given by

$$f(\mathbf{c}) = \eta^{2c}(E^2 + E^{-2}) + \eta^{-2c}(E'^2 + E'^{-2})$$

is concave up for all c and has minimum value $2T(EE')^{\frac{1}{2}}$ at

$$c = C = (4\log\eta)^{-1}\log\left(\frac{E'^2 + E'^{-2}}{E^2 + E^{-2}}\right).$$

Proof. Differentiate $f(\mathbf{c})$ with respect to c.

LEMMA 12. *For any real number d, define*

$$\mathbf{d} = (\log E_*\eta^d, \log|E'_*\eta^{-d}|, \log|E''_*\eta^d|, \log|E'''_*\eta^{-d}|).$$

The function of d given by

$$f(\mathbf{d}) = \eta^{2d}(E_*^2 + E_*''^2) + \eta^{-2d}(E_*'^2 + E_*'''^2)$$

is concave up for all d and has minimum value $2T(E)^{\frac{1}{2}}$ at

$$d = D = (4\log\eta)^{-1}\log\left(\frac{E_*'^2 + E_*'''^2}{E_*^2 + E_*''^2}\right).$$

Proof. Differentiate $f(\mathbf{d})$ with respect to d and observe that the least value of $f(\mathbf{d})$ is

$$2((E_*^2 + E_*''^2)(E_*'^2 + E_*'''^2))^{\frac{1}{2}} = 2T(E)^{\frac{1}{2}}$$

where equality holds because of (4).

Since $|E'| > 1$ we always have $T(EE') > T(E)$, and so if we extend the definition of $T(\epsilon)$ by letting $T(E\eta^c) = f(\mathbf{c})$ for any real number c and $T(E_*\eta^d) = f(\mathbf{d})$ for any real number d then Lemmas 11 and 12 give

$$\min_{c\,\text{real}} T(E\eta^c) > \min_{d\,\text{real}} T(E_*\eta^d).$$

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However, we are interested in the minima for *integers* c and d, so if $|E'| \approx 1$ it seems possible that

$$\min_{c \in \mathbf{Z}} T(E\eta^c) < \min_{d \in \mathbf{Z}} T(E_* \eta^d), \tag{21}$$

which would give the second of the two situations discussed above.

In fact, if $|E'| \approx 1$ then by (5)

$$\eta^{4D} = \frac{E_*'^2 + E_*'''^2}{E_*^2 + E_*''^2} \approx \frac{E\eta^{-1} + (\eta E)^{-1}}{\eta E + \eta E^{-1}} = \eta^{-2},$$

so that $D \approx -\frac{1}{2}$. Thus when $|E'| \approx 1$ we have

$$\min_{d \in \mathbf{Z}} T(E_* \eta^d) \approx T(E_* \eta^{D+\frac{1}{2}}) = (\eta + \eta^{-1})\, T(E)^{\frac{1}{2}} \approx (\eta + \eta^{-1})\, E. \tag{22}$$

We also have

$$\eta^{4c} = \frac{E'^2 + E'^{-2}}{E^2 + E^{-2}} \approx 2E^{-2}$$

and so

$$\min_{c \text{ real}} T(E\eta^c) \approx 2\sqrt{2}\, E. \tag{23}$$

Therefore if $|E'| \approx 1, \eta + \eta^{-1} > 2\sqrt{2}$ and C is near an integer, then (22) and (23) will give (21).

There is a class of fields for which E and η are easy to compute. These are the quartic fields F_a generated by roots of the polynomials

$$p_a(x) = x^4 - ax^3 - 6x^2 + ax + 1,$$

where a is an integer different from 0 and ± 3. If $p_a(\theta) = 0$, then the other roots of $p_a(x)$ are $-\theta^{-1}$, $(\theta-1)/(\theta+1)$ and $(\theta+1)/(-\theta+1)$, and so all of these numbers are relative units in F_a. These fields were studied by Gras ([3], pp. 15–17), who gave a simple sufficient condition for the equality $\theta = E$. The fields F_a were further considered by Cornell and Washington [1] and by Lehmer [5], who called these fields the 'simplest quartic fields'.

If $\theta = E$ in F_a and a is large, then $E \approx a$ and $|E'| = (\theta+1)/(\theta-1) \approx 1$. If in addition $\eta + \eta^{-1} > 2\sqrt{2}$ and the number C defined in Lemma 11 is near an integer, then by our remarks above F_a will be a field for which (21) holds. Gras ([3], pp. 26–78) gave a table of all real cyclic quartic fields with conductor less than 4000. This table of some 1500 fields includes a number of fields F_a with $Q = 2$ and $\theta = E$. Examination of these entries in the table led to the following two examples of fields F_a for which (21) holds.

Example 1. F_{169}: conductor $= 697$ $\eta = 4 + \sqrt{17}$

$$T(\eta) = 132, \quad \min T(E\eta^c) = T(E\eta^{-1}) = 565, \quad \min T(E_* \eta^d) = T(E_*) = 1388$$
$$C = -1{\cdot}14\ldots, \quad 2\sqrt{2}E = 478{\cdot}09\ldots, \quad (\eta + \eta^{-1})\, T(E)^{\frac{1}{2}} = 1393{\cdot}90\ldots.$$

Example 2. F_{6796}: conductor $= 1649$ $\eta = 4 + \sqrt{17}$

$$T(\eta) = 132, \quad \min T(E\eta^c) = T(E\eta^{-2}) = 19325, \quad \min T(E_* \eta^d) = T(E_*) = 56050$$
$$C = -2{\cdot}02\ldots, \quad 2\sqrt{2}E = 19230{\cdot}48\ldots, \quad (\eta + \eta^{-1})\, T(E)^{\frac{1}{2}} = 56066{\cdot}00\ldots.$$

We can summarize the above results as follows:

THEOREM 4. *If $Q = 2, T(\epsilon) = T(\epsilon_1)$ only for $\epsilon = \pm\eta$ or $\pm\eta^{-1}$ and (21) holds (so that $\epsilon_2 = E\eta^c$ for some integer c), then the successive unit minima generate a subgroup of*

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index 2 in the group of units. This situation will occur if $|E'| \approx 1, \eta + \eta^{-1} > 2\sqrt{2}$ *and the number C defined in Lemma* 11 *is near a rational integer. The conditions* $|E'| \approx 1$ *and* $\eta + \eta^{-1} > 2\sqrt{2}$ *are necessary for this situation to occur.*

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