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Integration on complex Grassmannians, deformed monotone Hurwitz numbers, and interlacing phenomena

Xavier Coulter, Norman Do, and Ellena Moskovsky

Abstract. We introduce a family of polynomials, which arise in three distinct ways: in the large N expansion of a matrix integral, as a weighted enumeration of factorizations of permutations, and via the topological recursion. More explicitly, we interpret the complex Grassmannian $\mathrm{Gr}(M, N)$ as the space of $N \times N$ idempotent Hermitian matrices of rank M and develop a Weingarten calculus to integrate products of matrix elements over it. In the regime of large N and fixed ratio $\frac{M}{N}$, such integrals have expansions whose coefficients count factorizations of permutations into monotone sequences of transpositions, with each sequence weighted by a monomial in $t = 1 - \frac{N}{M}$. This gives rise to the desired polynomials, which specialise to the monotone Hurwitz numbers when $t = 1$. These so-called deformed monotone Hurwitz numbers satisfy a cut-and-join recursion, a one-point recursion, and the topological recursion. Furthermore, we conjecture on the basis of overwhelming empirical evidence that the deformed monotone Hurwitz numbers are real-rooted polynomials whose roots satisfy remarkable interlacing phenomena. An outcome of our work is the viewpoint that the topological recursion can be used to "topologise" sequences of polynomials, and we claim that the resulting families of polynomials may possess interesting properties.

1 Introduction

In this article, we introduce a family of polynomials, which arise in three distinct ways: in the large N expansion of a matrix integral, as a weighted enumeration of factorizations of permutations, and via the topological recursion. Our construction simultaneously generalizes the Narayana polynomials and the monotone Hurwitz numbers, both of which have garnered considerable attention in the literature [33, 46]. We prove or conjecture that the family of polynomials we introduce satisfies a number of remarkable properties concerning their coefficients and roots, such as symmetry, unimodality, real-rootedness, and interlacing.

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This work is inspired by the known relations between Weingarten calculus on unitary groups, Jucys–Murphy elements in the symmetric group algebra, and monotone Hurwitz numbers [33, 44]. Our starting point is the space of $N \times N$ idempotent Hermitian matrices of rank M, where $M < N$. This space admits the following three descriptions, where I_M denotes the $M \times M$ identity matrix and $I_{M,N}$ denotes the $N \times N$ matrix whose first M diagonal entries are 1 and whose remaining entries are 0.

$$\begin{aligned}\mathbf{S}(M,N) &= \{S \in \mathrm{Mat}_{N\times N}(\mathbb{C}) \mid S^2 = S, S = S^* \text{ and } \mathrm{rank}(S) = M\} \\ &= \{S = U^*U \mid U \in \mathrm{Mat}_{M\times N}(\mathbb{C}) \text{ and } UU^* = I_M\} \\ &= \{S = UI_{M,N}U^* \mid U \in \mathbf{U}(N)\}.\end{aligned}$$

The unitary group $\mathbf{U}(N)$ acts transitively on $\mathbf{S}(M,N)$ by conjugation, thus endowing it with the structure of a homogeneous space. Since the stabilizer of $I_{M,N}$ is $\mathbf{U}(M) \times \mathbf{U}(N-M)$, we may identify $\mathbf{S}(M,N)$ with the complex Grassmannian $\mathrm{Gr}(M,N) \cong \mathbf{U}(N) / \mathbf{U}(M) \times \mathbf{U}(N-M)$, which parametrizes M-dimensional subspaces of an N-dimensional complex vector space.

As a compact homogeneous space, $\mathbf{S}(M,N)$ inherits a normalized $\mathbf{U}(N)$-invariant Haar measure, which we denote succinctly by $\mathrm{d}S$. For $1 \leqslant i, j \leqslant N$, define the function $S_{ij} : \mathbf{S}(M,N) \to \mathbb{C}$ corresponding to a matrix element. Taking our cue from the general theory of Weingarten calculus, we consider integrals of the form

$$\int_{\mathbf{S}(M,N)} S_{i_1 j_1} S_{i_2 j_2} \cdots S_{i_k j_k} \, \mathrm{d}S,$$

where $1 \leqslant i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_k \leqslant N$. Our primary goal is to study such matrix integrals in the regime of large N and fixed ratio $\frac{M}{N}$.

In recent decades, Weingarten calculus has developed into a rich theory concerned with integration on compact groups and related objects, with respect to the Haar measure [14]. Following the usual paradigm, we define a Weingarten function that takes as input a permutation $\sigma \in S_k$ and outputs the following elementary integral, where our notation suppresses the dependence on M and N.

$$\mathrm{Wg}^{\mathbf{S}}(\sigma) = \int_{\mathbf{S}(M,N)} S_{1,\sigma(1)} S_{2,\sigma(2)} \cdots S_{k,\sigma(k)} \, \mathrm{d}S$$

Modern approaches to Weingarten calculus typically rely on abstract algebraic tools such as Schur–Weyl duality [12, 15]. These are not as amenable to the current setting as the ideas rooted in the pioneering work of Weingarten [49] and revisited more recently by Collins and Matsumoto [13], which we use as inspiration to obtain the following.[1]

Theorem A (Weingarten calculus)

- *Convolution formula (Theorem 2.3)*
  Arbitrary integrals of monomials in the matrix elements of $\mathbf{S}(M,N)$ *reduce to elementary integrals via the equation*

$$\int_{S(M,N)} S_{i_1 j_1} S_{i_2 j_2} \cdots S_{i_k j_k} \, \mathrm{d}S = \sum_{\sigma \in S_k} \delta_{i_{\sigma(1)}, j_1} \delta_{i_{\sigma(2)}, j_2} \cdots \delta_{i_{\sigma(k)}, j_k} \mathrm{Wg}^{\mathbf{S}}(\sigma).$$

[1] This introduction includes only a cursory discussion of our main results and conjectures. The reader is encouraged to consult the main text for the relevant definitions, precise statements, and further details.

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- *Orthogonality relations (Theorem 2.4)*
  For each permutation $\sigma \in S_k$, the Weingarten function satisfies the relation

$$\mathrm{Wg}^S(\sigma) = -\frac{1}{N}\sum_{i=1}^{k-1}\mathrm{Wg}^S(\sigma \circ (i\ k)) + \delta_{\sigma(k),k}\frac{M}{N}\mathrm{Wg}^S(\sigma^{\downarrow}) + \frac{1}{N}\sum_{i=1}^{k-1}\delta_{\sigma(i),k}\mathrm{Wg}^S([\sigma \circ (i\ k)]^{\downarrow}).$$

  Here, for any permutation $\rho \in S_k$ that fixes k, we write $\rho^{\downarrow} \in S_{k-1}$ to denote the permutation that agrees with ρ on the set $\{1, 2, \ldots, k-1\}$.

The orthogonality relations allow for at least two distinct approaches to calculate values of the Weingarten function. By solving the non-degenerate linear system provided by the orthogonality relations, one can explicitly obtain values of the Weingarten function, which are rational functions of M and N. Alternatively, by iteratively and infinitely applying the orthogonality relations, one obtains the following large N expansion for the Weingarten function, for fixed ratio $\frac{M}{N}$.

Theorem B (Large N expansion, Theorem 2.12) *For a permutation $\sigma \in S_k$, let $\vec{w}_r^t(\sigma)$ denote the weighted enumeration of tuples $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_r)$ of transpositions in S_k such that*

- *$\tau_1\tau_2\cdots\tau_r = \sigma$;*
- *the sequence $\boldsymbol{\tau}$ is monotone — that is, if we write $\tau_i = (a_i\ b_i)$ with $a_i < b_i$, then $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_r$; and*
- *the weight of $\boldsymbol{\tau}$ is $t^{|\{b_1, b_2, \ldots, b_r\}|}$.*

The Weingarten function has the following large N expansion, for fixed $t = 1 - \frac{N}{M}$.

$$\mathrm{Wg}^S(\sigma) = \frac{1}{(1-t)^k}\sum_{r=0}^{\infty}\vec{w}_r^t(\sigma)\left(\frac{-1}{N}\right)^r.$$

Monotone Hurwitz numbers count monotone sequences of transpositions with prescribed length and with product of a prescribed cycle type, usually with an additional transitivity assumption. They are known to arise in the Weingarten calculus for unitary groups and in the large N expansion of the HCIZ matrix integral [33]. The theorem above motivates a "deformed" version of the monotone Hurwitz numbers, in which each sequence $((a_1\ b_1), (a_2\ b_2), \ldots, (a_r\ b_r))$ of transpositions is weighted by the monomial $t^{|\{b_1, b_2, \ldots, b_r\}|}$. The resulting polynomials in t are referred to as *deformed monotone Hurwitz numbers* and denoted by $\vec{H}^t_{g,n}(\mu_1, \ldots, \mu_n)$. Their precise definition appears in Definition 3.1 and they are the central objects introduced and studied in the present work.

The deformed monotone Hurwitz numbers obey various recursions, from which known results for the usual monotone Hurwitz numbers are recovered when $t = 1$ [9, 18, 32].

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Theorem C (Recursions)

- *Cut-and-join recursion (Theorem 3.7)*
  The deformed monotone Hurwitz numbers can be computed from the base case $\vec{H}^t_{0,1}(1) = 1$ *and the recursion*

$$\begin{aligned}\mu_1 \vec{H}^t_{g,n}(\mu_1, \mu_S) &= \sum_{i=2}^{n} (\mu_1 + \mu_i)\, \vec{H}^t_{g,n-1}(\mu_1 + \mu_i, \mu_{S\setminus\{i\}}) \\ &\quad + (t-1)(\mu_1 - 1)\, \vec{H}^t_{g,n}(\mu_1 - 1, \mu_S) \\ &\quad + \sum_{\alpha+\beta=\mu_1} \alpha\beta \left[\vec{H}^t_{g-1,n+1}(\alpha, \beta, \mu_S) + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = S}} \vec{H}^t_{g_1,|I_1|+1}(\alpha, \mu_{I_1})\, \vec{H}^t_{g_2,|I_2|+1}(\beta, \mu_{I_2}) \right].\end{aligned}$$

- *One-point recursion (Theorem 3.12)*
  The one-point deformed monotone Hurwitz numbers—in other words, those with $n = 1$*—satisfy*

$$\begin{aligned}d^2\, \vec{H}^t_{g,1}(d) &= (d-1)(2d-3)(t+1)\, \vec{H}^t_{g,1}(d-1) \\ &\quad - (d-2)(d-3)(t-1)^2\, \vec{H}^t_{g,1}(d-2) + d^2(d-1)^2\, \vec{H}^t_{g-1,1}(d).\end{aligned}$$

- *Topological recursion (Theorem 4.1)*
  The deformed monotone Hurwitz numbers are governed by the topological recursion on the genus zero spectral curve $xy^2 + (t-1)xy - y + 1 = 0$.

These recursions are all effective and can be used to produce explicit data, some of which is contained in Appendix A. At the level of coefficients, each deformed monotone Hurwitz number is symmetric (the sequence of coefficients is palindromic) and unimodal (the sequence of coefficients increases to a point and then decreases). We prove this in Proposition 3.9 using the cut-and-join recursion and note that these properties are not immediate from the combinatorial deformation of the deformed monotone Hurwitz numbers.

At the level of roots, it appears that the deformed monotone Hurwitz numbers are real-rooted and that they exhibit interlacing phenomena. Two real-rooted polynomials are said to *interlace* if their degrees differ by one and their roots weakly alternate on the real number line. We have gathered overwhelming numerical evidence to support the following conjectures.

Conjecture D (Roots)

- *Real-rootedness (Conjecture 3.13)*
  The deformed monotone Hurwitz number $\vec{H}^t_{g,n}(\mu_1, \mu_2, \ldots, \mu_n)$ *is a real-rooted polynomial in* t.
- *Interlacing (Conjecture 3.14)*
  The polynomial $\vec{H}^t_{g,n}(\mu_1, \mu_2, \ldots, \mu_n)$ *interlaces each of the* n *polynomials*

$$\vec{H}^t_{g,n}(\mu_1 + 1, \mu_2, \ldots, \mu_n),\ \vec{H}^t_{g,n}(\mu_1, \mu_2 + 1, \ldots, \mu_n),\ \ldots,\ \vec{H}^t_{g,n}(\mu_1, \mu_2, \ldots, \mu_n + 1).$$

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In the case $(g,n) = (0,1)$, the deformed monotone Hurwitz numbers recover the sequence of Narayana polynomials via the equation

$$(\mu+1)\,\vec{H}^{t}_{0,1}(\mu+1) = \mathrm{Nar}_{\mu}(t) := \sum_{i=1}^{\mu} \frac{1}{\mu}\binom{\mu}{i}\binom{\mu}{i-1} t^{i}. \tag{1.1}$$

Thus, we consider the deformed monotone Hurwitz numbers to be a "topological generalization" of the Narayana polynomials. We propose the topological recursion as a mechanism to "topologise" sequences of polynomials more generally. In particular, we claim that doing so can preserve interesting behavior in the polynomials, such as symmetry, unimodality, real-rootedness, and interlacing properties. This is not only the case for the deformed monotone Hurwitz numbers, but we also observe these phenomena in the weighted enumeration of dessins d'enfant—in other words, bicolored maps–in which each black vertex in a dessin d'enfant is assigned a multiplicative weight t.

Our results concerning integration on complex Grassmannians and deformed monotone Hurwitz numbers suggest various avenues for further research. It would be natural to consider integration on real Grassmannians $\mathrm{Gr}(M,N)$, also in the regime of large N with fixed ratio $\frac{M}{N}$, and the first two authors are currently pursuing this line of investigation. Matsumoto considered Weingarten calculus on compact symmetric spaces [39] and the particular case of the symmetric space AIII bears a strong resemblance to the present work. It would be interesting to further develop the parallels between these two settings. The real-rootedness and interlacing conjectures for deformed monotone Hurwitz numbers (Conjectures 3.13 and 3.14) and the weighted dessin d'enfant enumeration (Conjecture 4.11) not only require proof, but also invite a deeper exploration of how common these phenomena might be.

The structure of the article is as follows.

- In Section 2, we develop the Weingarten calculus for integration over $\mathbf{S}(M,N)$. This includes a convolution formula (Theorem 2.3) and orthogonality relations (Theorem 2.4). We use the latter to express the Weingarten function of a permutation as a weighted enumeration of monotone sequences of transpositions (Theorem 2.12). As a consequence, the Weingarten function can be written succinctly in terms of Jucys–Murphy elements in the symmetric group algebra (Proposition 2.14).
- In Section 3, we define the notion of deformed monotone Hurwitz numbers, which are polynomials in the deformation parameter t, motivated by the results of the previous section. We prove "deformed" analogs of existing results concerning the usual monotone Hurwitz numbers, such as a character formula (Proposition 3.3), a cut-and-join recursion (Theorem 3.7), and a one-point recursion (Theorem 3.12). It follows from the cut-and-join recursion that the coefficients of deformed monotone Hurwitz numbers are symmetric and unimodal (Proposition 3.9). On the basis of extensive numerical evidence, we conjecture that these polynomials are real-rooted (Conjecture 3.13) and that their roots satisfy remarkable interlacing phenomena (Conjecture 3.14).
- In Section 4, we briefly introduce the topological recursion of Chekhov, Eynard, and Orantin [11, 25] and then use a powerful result of Bychkov, Dunin-Barkowski,

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Kazarian, and Shadrin [8] to prove that topological recursion on the spectral curve $xy^2 + (t-1)xy - y + 1 = 0$ governs the deformed monotone Hurwitz numbers (Theorem 4.1). We propose the topological recursion as a mechanism to produce topological generalizations of sequences of polynomials with interesting properties. As a case study, we consider the weighted enumeration of dessins d'enfant—in other words, bicolored maps—in which each black vertex receives a multiplicative weight t. This produces a family of polynomials that also satisfies a cut-and-join recursion (Proposition 4.8) and the topological recursion (Theorem 4.10). In analogy with the case of deformed monotone Hurwitz numbers studied in the previous section, we conjecture that these polynomials exhibit real-rootedness and interlacing phenomena (Conjecture 4.11).

2 Weingarten calculus

2.1 Convolution formula and orthogonality relations

As mentioned in the introduction, the present work is concerned with integration on the complex Grasmannian $\mathrm{Gr}(M, N)$ for $M < N$. We interpret this Grassmannian as the space of $N \times N$ idempotent Hermitian matrices of rank M, which admits three equivalent descriptions as per the following definition.

Definition 2.1 Let I_M denote the $M \times M$ identity matrix and $I_{M,N}$ denote the $N \times N$ matrix whose first M diagonal entries are 1 and whose remaining entries are 0. For $M < N$, define the space

$$\begin{aligned}\mathbf{S}(M,N) &= \{S \in \mathrm{Mat}_{N\times N}(\mathbb{C}) \mid S^2 = S, S = S^* \text{ and } \mathrm{rank}(S) = M\} \\ &= \{S = U^*U \mid U \in \mathrm{Mat}_{M\times N}(\mathbb{C}) \text{ and } UU^* = I_M\} \\ &= \{S = UI_{M,N}U^* \mid U \in \mathbf{U}(N)\}.\end{aligned}$$

The unitary group $\mathbf{U}(N)$ acts transitively on $\mathbf{S}(M,N)$ by conjugation, thus endowing it with the structure of a compact homogeneous space. Thus, the Haar measure on $\mathbf{U}(N)$ induces a $\mathbf{U}(N)$-invariant normalised Haar measure on $\mathbf{S}(M,N)$, which we denote succinctly by $\mathrm{d}S$. (See [16] for an introduction to Haar measures.) The fact that the stabilizer of $I_{M,N} \in \mathbf{S}(M,N)$ is $\mathbf{U}(M) \times \mathbf{U}(N-M)$ allows us to identify $\mathbf{S}(M,N)$ with the complex Grassmannian $\mathrm{Gr}(M,N) \cong \mathbf{U}(N) / \mathbf{U}(M) \times \mathbf{U}(N-M)$.

For $1 \leqslant i, j \leqslant N$, define the function $S_{ij} : \mathbf{S}(M,N) \to \mathbb{C}$ corresponding to the (i,j) matrix element. Our primary goal is to calculate integrals of the form

$$\int_{\mathbf{S}(M,N)} S_{i_1 j_1} S_{i_2 j_2} \cdots S_{i_k j_k} \, \mathrm{d}S,$$

where $1 \leqslant i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_k \leqslant N$. We impose the technical assumption that $k \leqslant N$ for future convenience. However, this assumption has little bearing on our work, since we will study these matrix integrals in the regime of large N and fixed ratio $\frac{M}{N}$. The following elementary integrals will be of particular importance.

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Definition 2.2 (Weingarten function) For each permutation $\sigma \in S_k$, define the integral

$$\mathrm{Wg}^{\mathbf{S}}(\sigma) = \int_{\mathbf{S}(M,N)} S_{1,\sigma(1)} S_{2,\sigma(2)} \cdots S_{k,\sigma(k)} \, \mathrm{d}S.$$

We refer to the function $\mathrm{Wg}^{\mathbf{S}} : S_0 \sqcup S_1 \sqcup S_2 \sqcup \cdots \to \mathbb{C}$ as the *Weingarten function* for $\mathbf{S}(M,N)$. Here, we include the symmetric group S_0, whose unique element is denoted ( ) and represents the empty permutation. In the notation for the Weingarten function, we suppress the dependence on M and N to avoid clutter.

The setup described above sits firmly in the realm of Weingarten calculus, which is broadly concerned with the calculation of integrals on compact groups and related objects with respect to the Haar measure [14]. Modern accounts of Weingarten calculus often rely on elegant algebraic approaches via Schur–Weyl duality [12, 15]. A direct use of such an argument is not immediately available for the case of integration over $\mathbf{S}(M,N)$. We instead follow the approach via orthogonality relations utilized by Collins and Matsumoto [13], which is in turn inspired by the ideas contained in the seminal paper of Weingarten [49].

In the remainder of this section, we develop the Weingarten calculus for integration on $\mathbf{S}(M,N)$ in three parts. First, we prove a convolution formula that reduces general integrals of monomials in the matrix elements to the elementary ones defined in Definition 2.2. Second, we prove so-called orthogonality relations that completely determine all values of the Weingarten function. Third, we solve the linear system provided by these orthogonality relations, which connects naturally to the representation theory of the symmetric groups, particularly to the Jucys–Murphy elements in the symmetric group algebra.

Theorem 2.3 (Convolution formula) *Arbitrary integrals of monomials in the matrix elements of* $\mathbf{S}(M,N)$ *reduce to elementary integrals via the equation*

$$\int_{\mathbf{S}(M,N)} S_{i_1 j_1} S_{i_2 j_2} \cdots S_{i_k j_k} \, \mathrm{d}S = \sum_{\sigma \in S_k} \delta_{i_{\sigma(1)}, j_1} \delta_{i_{\sigma(2)}, j_2} \cdots \delta_{i_{\sigma(k)}, j_k} \mathrm{Wg}^{S}(\sigma).$$

Proof Write $S \in \mathbf{S}(M,N)$ in the form $S = U I_{M,N} U^*$ for $U \in \mathbf{U}(N)$. Then the two sides of the desired equation can be equivalently expressed as integrals over $\mathbf{U}(N)$.

$$\mathrm{LHS} = \sum_{m_1,\ldots,m_k=1}^{N} \left(\prod_{i=1}^{k} [I_{M,N}]_{m_i m_i} \right) \int_{\mathbf{U}(N)} U_{i_1 m_1} \cdots U_{i_k m_k} U^*_{m_1 j_1} \cdots U^*_{m_k j_k} \, \mathrm{d}U$$

$$\mathrm{RHS} = \sum_{\sigma \in S_k} \left(\prod_{a=1}^{k} \delta_{i_{\sigma(a)}, j_a} \right) \sum_{m_1,\ldots,m_k=1}^{N} \left(\prod_{i=1}^{k} [I_{M,N}]_{m_i m_i} \right) \int_{\mathbf{U}(N)} U_{1 m_1} \cdots U_{k m_k} U^*_{m_1,\sigma(1)} \cdots U^*_{m_k,\sigma(k)} \, \mathrm{d}U.$$

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Thus, the result would follow from the equation

$$\int_{\mathrm{U}(N)} U_{i_1 m_1} \cdots U_{i_k m_k} U^*_{m_1 j_1} \cdots U^*_{m_k j_k} \, \mathrm{d}U$$
$$= \sum_{\sigma \in S_k} \delta_{i_{\sigma(1)}, j_1} \cdots \delta_{i_{\sigma(k)}, j_k} \int_{\mathrm{U}(N)} U_{1 m_1} \cdots U_{k m_k} U^*_{m_1, \sigma(1)} \cdots U^*_{m_k, \sigma(k)} \, \mathrm{d}U.$$

However, this is a direct consequence of applying the convolution formula for $\mathbf{U}(N)$ [15, Corollary 2.4] to both sides. ∎

The following orthogonality relations for Wg^{S} are inspired by the orthogonality relations obtained by Collins and Matsumoto in other settings for Weingarten calculus [13]. The proof crucially relies on the convolution formula of Theorem 2.3. For future reference, observe that we use the notational convention of composing permutations from right to left.

Theorem 2.4 (Orthogonality relations) *For each permutation $\sigma \in S_k$, the Weingarten function satisfies the relation*

$$\mathrm{Wg}^{\mathrm{S}}(\sigma) = -\frac{1}{N} \sum_{i=1}^{k-1} \mathrm{Wg}^{\mathrm{S}}(\sigma \circ (i\ k)) + \delta_{\sigma(k),k} \frac{M}{N} \mathrm{Wg}^{\mathrm{S}}(\sigma^{\downarrow})$$
$$+ \frac{1}{N} \sum_{i=1}^{k-1} \delta_{\sigma(i),k} \mathrm{Wg}^{\mathrm{S}}([\sigma \circ (i\ k)]^{\downarrow}).$$

Here, for any permutation $\rho \in S_k$ that fixes k, we write $\rho^{\downarrow} \in S_{k-1}$ to denote the permutation that agrees with ρ on the set $\{1, 2, \ldots, k-1\}$.

Proof Let $\sigma \in S_k$ and consider the cases $\sigma(k) = k$ and $\sigma(k) \neq k$ separately.

Case 1. Suppose $\sigma(k) = k$.

Consider the integral

$$(*) \qquad \sum_{i=1}^{N} \int_{\mathrm{S}(M,N)} S_{1,\sigma(1)} S_{2,\sigma(2)} \cdots S_{k-1,\sigma(k-1)} S_{i,i} \, \mathrm{d}S.$$

On the one hand, we can use $\sum_{i=1}^{N} S_{ii} = \mathrm{Tr}(S) = \mathrm{Tr}(U I_{M,N} U^*) = \mathrm{Tr}(I_{M,N}) = M$ and the definition of Wg^{S} to express the integral as $M\mathrm{Wg}^{\mathrm{S}}(\sigma^{\downarrow})$. On the other hand, we can apply the convolution formula directly to each summand. For the ith summand, where $1 \leqslant i < k$, the convolution formula yields

$$\int_{\mathrm{S}(M,N)} S_{1,\sigma(1)} \cdots S_{i,\sigma(i)} \cdots S_{k-1,\sigma(k-1)} S_{i,i} \, \mathrm{d}S = \mathrm{Wg}^{\mathrm{S}}(\sigma) + \mathrm{Wg}^{\mathrm{S}}(\sigma \circ (i\ k)).$$

For the ith summand, where $k \leqslant i \leqslant N$, the convolution formula yields

$$\int_{\mathrm{S}(M,N)} S_{1,\sigma(1)} S_{2,\sigma(2)} \cdots S_{k-1,\sigma(k-1)} S_{i,i} \, \mathrm{d}S = \mathrm{Wg}^{\mathrm{S}}(\sigma).$$

Adding these contributions over $i = 1, 2, \ldots, N$ and equating with the expression we previously obtained for the integral $(*)$ leads to

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$$M\mathrm{Wg}^{\mathrm{S}}(\sigma^{\downarrow}) = N\mathrm{Wg}^{\mathrm{S}}(\sigma) + \sum_{i=1}^{k-1} \mathrm{Wg}^{\mathrm{S}}(\sigma \circ (i\ k))$$

$$\Rightarrow \qquad \mathrm{Wg}^{\mathrm{S}}(\sigma) = -\frac{1}{N}\sum_{i=1}^{k-1} \mathrm{Wg}^{\mathrm{S}}(\sigma \circ (i\ k)) + \frac{M}{N}\mathrm{Wg}^{\mathrm{S}}(\sigma^{\downarrow}). \tag{2.1}$$

Case 2. Suppose $\sigma(k) \neq k$.
Let $j = \sigma^{-1}(k)$ and consider the integral

$$(**) \quad \sum_{i=1}^{N} \int_{\mathrm{S}(M,N)} \left(S_{1,\sigma(1)} \cdots S_{j-1,\sigma(j-1)}\right) S_{j,i} \left(S_{j+1,\sigma(j+1)} \cdots S_{k-1,\sigma(k-1)}\right) S_{i,\sigma(k)}\, \mathrm{d}S.$$

On the one hand, we have $S^2 = S$ for all $S \in \mathrm{S}(M,N)$, so it follows that $\sum_{i=1}^{N} S_{j,i}S_{i,\sigma(k)} = S_{j,\sigma(k)}$. Combining this observation with the convolution formula allows us to express the integral as follows:

$$\int_{\mathrm{S}(M,N)} S_{1,\sigma(1)} \cdots S_{j-1,\sigma(j-1)} S_{j,\sigma(k)} S_{j+1,\sigma(j+1)} \cdots S_{k-1,\sigma(k-1)}\, \mathrm{d}S = \mathrm{Wg}^{\mathrm{S}}([\sigma \circ (j\ k)]^{\downarrow}).$$

On the other hand, we can apply the convolution formula directly to each summand, resulting in a calculation analogous to that of Case 1. For the ith summand, where $1 \leqslant i < k$, the convolution formula yields

$$\int_{\mathrm{S}(M,N)} \left(S_{1,\sigma(1)} \cdots S_{j-1,\sigma(j-1)}\right) S_{j,i} \left(S_{j+1,\sigma(j+1)} \cdots S_{k-1,\sigma(k-1)}\right) S_{i,\sigma(k)}\, \mathrm{d}S = \mathrm{Wg}^{\mathrm{S}}(\sigma) + \mathrm{Wg}^{\mathrm{S}}(\sigma \circ (i\ k)).$$

For the ith summand, where $k \leqslant i \leqslant N$, the convolution formula yields

$$\int_{\mathrm{S}(M,N)} \left(S_{1,\sigma(1)} \cdots S_{j-1,\sigma(j-1)}\right) S_{j,i} \left(S_{j+1,\sigma(j+1)} \cdots S_{k-1,\sigma(k-1)}\right) S_{i,\sigma(k)}\, \mathrm{d}S = \mathrm{Wg}^{\mathrm{S}}(\sigma).$$

Adding these contributions over $i = 1, 2, \ldots, N$ and equating with the expression we previously obtained for the integral $(**)$ leads to

$$\mathrm{Wg}^{\mathrm{S}}([\sigma \circ (j\ k)]^{\downarrow}) = N\mathrm{Wg}^{\mathrm{S}}(\sigma) + \sum_{i=1}^{k-1} \mathrm{Wg}^{\mathrm{S}}(\sigma \circ (i\ k))$$

$$\Rightarrow \qquad \mathrm{Wg}^{\mathrm{S}}(\sigma) = -\frac{1}{N}\sum_{i=1}^{k-1} \mathrm{Wg}^{\mathrm{S}}(\sigma \circ (i\ k)) + \frac{1}{N}\mathrm{Wg}^{\mathrm{S}}([\sigma \circ (j\ k)]^{\downarrow}). \tag{2.2}$$

Finally, the desired result is obtained by writing the two expressions for $\mathrm{Wg}^{\mathrm{S}}(\sigma)$ obtained in equations (2.1) and (2.2) from the two separate cases in one formula, making use of the Kronecker delta notation. ∎

The orthogonality relations provide a non-degenerate linear system of equations that uniquely determines the Weingarten function. The example below shows that values of the Weingarten function can be computed explicitly and are rational functions of M and N.

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Example 2.5 By the orthogonality relations of Theorem 2.4 and the conjugacy invariance of Wg^{S}, we obtain the following equations.

$$\begin{aligned}
\mathrm{Wg}^{\mathrm{S}}((1)(2)(3)) &= -\frac{1}{N}\left[\mathrm{Wg}^{\mathrm{S}}((1\,3)(2)) + \mathrm{Wg}^{\mathrm{S}}((2\,3)(1))\right] + \frac{M}{N}\mathrm{Wg}^{\mathrm{S}}((1)(2)) \\
&= -\frac{2}{N}\mathrm{Wg}^{\mathrm{S}}((1\,2)(3)) + \frac{M}{N}\mathrm{Wg}^{\mathrm{S}}((1)(2)) \\
\mathrm{Wg}^{\mathrm{S}}((1\,2)(3)) &= -\frac{1}{N}\left[\mathrm{Wg}^{\mathrm{S}}((1\,3\,2)) + \mathrm{Wg}^{\mathrm{S}}((1\,2\,3)\right] + \frac{M}{N}\mathrm{Wg}^{\mathrm{S}}((1\,2)) \\
&= -\frac{2}{N}\mathrm{Wg}^{\mathrm{S}}((1\,2\,3)) + \frac{M}{N}\mathrm{Wg}^{\mathrm{S}}((1\,2)) \\
\mathrm{Wg}^{\mathrm{S}}((1\,2\,3)) &= -\frac{1}{N}\left[\mathrm{Wg}^{\mathrm{S}}((2\,3)(1)) + \mathrm{Wg}^{\mathrm{S}}((1\,2)(3)\right] + \frac{1}{N}\mathrm{Wg}^{\mathrm{S}}((1\,2)) \\
&= -\frac{2}{N}\mathrm{Wg}^{\mathrm{S}}((1\,2)(3)) + \frac{1}{N}\mathrm{Wg}^{\mathrm{S}}((1\,2)).
\end{aligned}$$

Using the values $\mathrm{Wg}^{\mathrm{S}}((1)(2)) = \frac{M(MN-1)}{N(N^2-1)}$ and $\mathrm{Wg}^{\mathrm{S}}((12)) = \frac{-M(M-N)}{N(N^2-1)}$, one can solve this linear system of equations to obtain the following unique solution.

$$\begin{aligned}
\mathrm{Wg}^{\mathrm{S}}((1)(2)(3)) &= \frac{-2(MN-2)}{N(N^2-4)}\mathrm{Wg}^{\mathrm{S}}((1\,2)) + \frac{M}{N}\mathrm{Wg}^{\mathrm{S}}((1)(2)) \\
&= \frac{M(M^2N^2 - 2M^2 - 3MN + 4)}{N(N^2-1)(N^2-4)} \\
\mathrm{Wg}^{\mathrm{S}}((1\,2)(3)) &= \frac{MN-2}{N^2-4}\mathrm{Wg}^{\mathrm{S}}((1\,2)) = \frac{-M(M-N)(MN-2)}{N(N^2-1)(N^2-4)} \\
\mathrm{Wg}^{\mathrm{S}}((1\,2\,3))) &= \frac{N-2M}{N^2-4}\mathrm{Wg}^{\mathrm{S}}((1\,2)) = \frac{M(M-N)(2M-N)}{N(N^2-1)(N^2-4)}.
\end{aligned}$$

In this way, one can begin with the base case $\mathrm{Wg}^{\mathrm{S}}(( )) = 1$ and inductively obtain $\mathrm{Wg}^{\mathrm{S}}(\sigma)$ for $\sigma \in S_k$ in terms of $\mathrm{Wg}^{\mathrm{S}}(\sigma')$ for $\sigma' \in S_{k-1}$. Further values can be found in Appendix A.

2.2 Large *N* expansion

A priori, computing the values of the Weingarten function Wg^{S} via the integral definition is far from straightforward. However, as evidenced by the calculations of Example 2.5, the orthogonality relations of Theorem 2.4 uniquely determine the Weingarten function and imply that its values are rational functions of M and N. We will shortly see that their structure also leads directly to a large N expansion for $\mathrm{Wg}^{\mathrm{S}}(\sigma)$, with coefficients that enumerate factorizations of σ into transpositions that satisfy a certain monotonicity condition. This combinatorial structure can be understood in terms of paths in the Weingarten graph, which encode the result of recursively applying the orthogonality relations ad infinitum. The notion of a Weingarten graph was originally introduced by Collins and Matsumoto [13]. In the setting of integration over unitary groups, the Weingarten graph $\mathcal{G}^{\mathrm{U}}$ has two types of edges, which reflects the fact that the orthogonality relations in that case express the Weingarten function of a permutation in terms of two types of terms. In the setting

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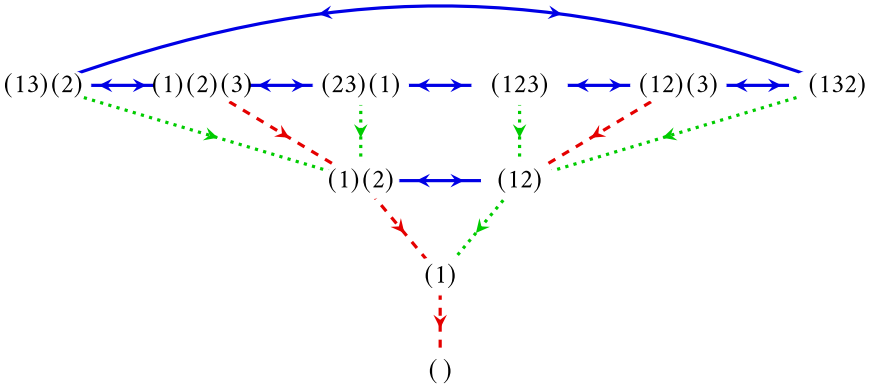


Figure 1: The part of the Weingarten graph $\mathcal{G}^{\mathbf{S}}$ induced by vertices belonging to $S_0 \sqcup S_1 \sqcup S_2 \sqcup S_3$.

of integration over $\mathbf{S}(M,N)$, we have three terms appearing on the right side of the orthogonality relations, motivating the following definition (Figure 1).

Definition 2.6 Define the *Weingarten graph* $\mathcal{G}^{\mathbf{S}}$ to be the infinite directed graph with vertex set $S = \bigsqcup_{i=0}^{\infty} S_i$ and edge set $E = E_A \sqcup E_B \sqcup E_C$, where:

- the set E_A comprises the "type A" edges, which are of the form

$$\sigma \longrightarrow \sigma \circ (i\ k)$$

  for $\sigma \in S_k$ and $1 \leqslant i < k$;
- the set E_B comprises the "type B" edges, which are of the form

$$\sigma \dashrightarrow \sigma^{\downarrow}$$

  for $\sigma \in S_k$ with $\sigma(k) = k$; and
- the set E_C comprises the "type C" edges, which are of the form

$$\sigma \dashrightarrow [\sigma \circ (i\ k)]^{\downarrow}$$

  for $\sigma \in S_k$ such that $\sigma(i) = k$ for some $1 \leqslant i < k$.

Remark 2.7. The Weingarten graph $\mathcal{G}^{\mathbf{U}}$ in the setting of integration over $\mathbf{U}(N)$ appears in [13] and is the subgraph of $\mathcal{G}^{\mathbf{S}}$ obtained by removing all type C edges.

Repeated application of the orthogonality relations leads to paths in the Weingarten graph $\mathcal{G}^{\mathbf{S}}$. In this way, one is motivated to enumerate paths in the Weingarten graph in which edges of types A, B, C receive multiplicative weights $-\frac{1}{N}$, $\frac{M}{N}$, $\frac{1}{N}$, respectively.

Definition 2.8 A *path* in $\mathcal{G}^{\mathbf{S}}$ is a sequence of permutations

$$\boldsymbol{\rho} = (\rho_0, \rho_1, \rho_2, \ldots, \rho_\ell) \in S^{\ell+1},$$

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where (ρ_{i-1}, ρ_i) is a directed edge of $\mathcal{G}^S$ for each $i = 1, 2, \ldots, \ell$. We call the integer $\ell = \ell(\boldsymbol{\rho})$ the *length* of the path and denote by $\mathcal{P}(\sigma, \sigma')$ the set of all paths from σ to σ'. Define the *weight* $w(\boldsymbol{\rho})$ of a path $\boldsymbol{\rho}$ by

$$\left(-\frac{1}{N}\right)^{\ell_A(\boldsymbol{\rho})} \left(\frac{M}{N}\right)^{\ell_B(\boldsymbol{\rho})} \left(\frac{1}{N}\right)^{\ell_C(\boldsymbol{\rho})},$$

where $\ell_K(\boldsymbol{\rho})$ denotes the number of edges of type $K \in \{A, B, C\}$ on the path $\boldsymbol{\rho}$.

By construction, the Weingarten graph is a combinatorial encoding of the orthogonality relations and their repeated application. Starting with $\sigma \in S_k$, iterating the orthogonality relations ℓ times produces the formula

$$\mathrm{Wg}^S(\sigma) = \sum_{\substack{\boldsymbol{\rho} \in \mathcal{P}(\sigma, (\,)) \\ \ell(\boldsymbol{\rho}) \leqslant \ell}} w(\boldsymbol{\rho}) + \sum_{\sigma' \in S_1 \sqcup \cdots \sqcup S_k} \mathrm{Wg}^S(\sigma') \sum_{\substack{\boldsymbol{\rho} \in \mathcal{P}(\sigma, \sigma') \\ \ell(\boldsymbol{\rho}) = \ell}} w(\boldsymbol{\rho}).$$

By sending the number of iterations ℓ to infinity, one obtains the following large N expansion of the Weingarten function.

Proposition 2.9 *For any permutation σ, we have the large N expansion*

$$\mathrm{Wg}^S(\sigma) = \sum_{\boldsymbol{\rho} \in \mathcal{P}(\sigma, (\,))} w(\boldsymbol{\rho}) = \sum_{\boldsymbol{\rho} \in \mathcal{P}(\sigma, (\,))} \left(-\frac{1}{N}\right)^{\ell_A(\boldsymbol{\rho})} \left(\frac{M}{N}\right)^{\ell_B(\boldsymbol{\rho})} \left(\frac{1}{N}\right)^{\ell_C(\boldsymbol{\rho})}.$$

At this point, let us consider the Weingarten function in the regime of large N with fixed ratio $q = \frac{M}{N}$.[2] For $\boldsymbol{\rho} \in \mathcal{P}(\sigma, (\,))$ and $\sigma \in S_k$, we have $\ell_B(\boldsymbol{\rho}) + \ell_C(\boldsymbol{\rho}) = k$, which leads to

$$\mathrm{Wg}^S(\sigma) = q^k \sum_{r=0}^{\infty} \left(-\frac{1}{N}\right)^r \sum_{\substack{\boldsymbol{\rho} \in \mathcal{P}(\sigma, (\,)) \\ \ell_A(\boldsymbol{\rho}) + \ell_C(\boldsymbol{\rho}) = r}} \left(-\frac{1}{q}\right)^{\ell_C(\boldsymbol{\rho})}.$$

To develop a clearer combinatorial description of the coefficients appearing in the expansion above, we consider the correspondence between paths in the Weingarten graph and monotone factorizations. This connection already appears in the Weingarten calculus for unitary groups [13].

Definition 2.10 A *monotone factorization* of $\sigma \in S_k$ is a sequence $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_r)$ of transpositions in S_k such that

- $\tau_1 \tau_2 \cdots \tau_r = \sigma$; and
- if we write $\tau_i = (a_i\ b_i)$ with $a_i < b_i$, then $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_r$.

Moreover, we call a monotone factorization *transitive* if $\tau_1, \tau_2, \ldots, \tau_r$ generate a transitive subgroup of S_k. Denote the set of monotone factorizations of σ by $\mathcal{M}(\sigma)$ and the length of $\boldsymbol{\tau}$ by $r(\boldsymbol{\tau})$. We refer to the number of distinct elements of the multiset $\{b_1, b_2, \ldots, b_r\}$ as the *hive number* of $\boldsymbol{\tau}$ and denote this quantity by $b(\boldsymbol{\tau})$.

Given a path in the Weingarten graph $\mathcal{G}^S$ from σ to $(\,)$, one can record the sequence of transpositions arising from the type A edges and the type C edges. The

[2] One could of course consider other regimes, such as fixed M, although this line of investigation did not appear to be as fruitful.

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composition of these transpositions in reverse order recovers the permutation σ. Thus, a path $\boldsymbol{\rho} \in \mathcal{P}(\sigma, (\,))$ gives rise to a monotone factorization $\boldsymbol{\tau} \in \mathcal{M}(\sigma)$ satisfying $\ell_A(\boldsymbol{\rho}) + \ell_C(\boldsymbol{\rho}) = r(\boldsymbol{\tau})$. Represent this construction via the map

$$\mathcal{F} : \mathcal{P}(\sigma, (\,)) \to \mathcal{M}(\sigma).$$

In the analogous construction for the unitary case, one obtains a one-to-one correspondence between paths and monotone factorizations, but that is not the case here. Given a monotone factorization $\boldsymbol{\tau}$ of σ, there exists a unique path in $\mathcal{F}^{-1}(\boldsymbol{\tau})$ containing only edges of types A and B. The number of pairs of consecutive A–B edges in the path is equal to the hive number $b(\boldsymbol{\tau})$. Any such pair can be replaced by a type C edge to obtain an element of $\mathcal{F}^{-1}(\boldsymbol{\tau})$ and every element of $\mathcal{F}^{-1}(\boldsymbol{\tau})$ can be obtained in this way. Thus, we have $|\mathcal{F}^{-1}(\boldsymbol{\tau})| = 2^{b(\boldsymbol{\tau})}$. Moreover, keeping track of the effect on edge weights, we have the equality

$$\sum_{\boldsymbol{\rho} \in \mathcal{F}^{-1}(\boldsymbol{\tau})} \left(-\frac{1}{q}\right)^{\ell_C(\boldsymbol{\rho})} = \left(1 - \frac{1}{q}\right)^{b(\boldsymbol{\tau})}.$$

These observations allow us to express the coefficients of the large N expansion of $\mathrm{Wg}^{\mathrm{S}}(\sigma)$ in terms of monotone factorizations via the equation

$$\mathrm{Wg}^{\mathrm{S}}(\sigma) = q^k \sum_{r=0}^{\infty} \left(-\frac{1}{N}\right)^r \sum_{\substack{\boldsymbol{\tau} \in \mathcal{M}(\sigma) \\ r(\boldsymbol{\tau}) = r}} \left(1 - \frac{1}{q}\right)^{b(\boldsymbol{\tau})}.$$

The discussion above motivates the following definition and theorem, whose proof is immediate after setting $t = 1 - \frac{1}{q}$ in the previous equation.

Definition 2.11 For $r \geqslant 0$ and σ a permutation, let $\vec{w}_r^{\,t}(\sigma)$ denote the weighted count of monotone factorizations of σ, where the weight of a monotone factorization $\boldsymbol{\tau}$ is $t^{b(\boldsymbol{\tau})}$. Let $\vec{h}_r^{\,t}(\sigma)$ denote the analogous count restricted to transitive monotone factorizations.

Theorem 2.12 (Large N expansion) *For a permutation $\sigma \in S_k$, we have the following large N expansion for fixed $t = 1 - \frac{N}{M}$.*

$$\mathrm{Wg}^{\mathrm{S}}(\sigma) = \frac{1}{(1-t)^k} \sum_{r=0}^{\infty} \vec{w}_r^{\,t}(\sigma) \left(-\frac{1}{N}\right)^r. \tag{2.3}$$

2.3 Representation-theoretic interpretation

The unitary invariance of the Haar measure implies that the Weingarten function Wg^{S} is a function of permutations that is constant on conjugacy classes. It is natural to ask whether it admits a natural description in the class algebra of the symmetric group. Such a description of the unitary Weingarten function Wg^{U} is well understood and given in terms of the Jucys–Murphy elements in the symmetric group algebra [40, 44]. We will show that the Weingarten function Wg^{S} can be expressed similarly.

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The *Jucys–Murphy elements* $J_1, J_2, \ldots, J_k$ are elements of the symmetric group algebra defined by

$$J_i = (1\ i) + (2\ i) + \cdots + (i-1\ i) \in \mathbb{C}[S_k],$$

where we interpret the formula for $i = 1$ as $J_1 = 0$. They were introduced independently by Jucys [36] and Murphy [41] and their seemingly simple definition belies their remarkable properties. For example, they commute with each other and indeed, generate a maximal commutative subalgebra of $\mathbb{C}[S_k]$. Any symmetric function of the Jucys–Murphy elements lies in the class algebra $Z\mathbb{C}[S_k]$ and the class expansions of such expressions are of significant interest, appearing in various contexts [30]. Furthermore, the Jucys–Murphy elements are essential elements of the Okounkov–Vershik approach to the representation theory of symmetric groups [45]. We will subsequently require the following results that date back to the seminal work of Jucys.

Proposition 2.13 (Jucys [36])

(a) *The Jucys–Murphy elements* $J_1, J_2, \ldots, J_k \in \mathbb{C}[S_k]$ *satisfy*

$$(x + J_1)(x + J_2)\cdots(x + J_k) = \sum_{\sigma \in S_k} x^{\#\text{cycles}(\sigma)}\, \sigma.$$

(b) *Let* λ *be a partition of* k *and let* χ^λ *the corresponding irreducible character of the symmetric group. We adopt the usual abuse of notation and consider* χ^λ *as an element of the class algebra* $Z\mathbb{C}[S_k]$. *If* f *is a symmetric polynomial in* n *variables, then*

$$f(J_1, J_2, \ldots, J_k)\, \chi^\lambda = f(\text{cont}(\lambda))\, \chi^\lambda,$$

where $\text{cont}(\lambda)$ *denotes the multiset of contents in the Young diagram for* λ. *(The content of a box in row* i *and column* j *of a Young diagram is the number* $j - i$.*)*

It was shown by Novak [44] that the unitary Weingarten function can be naturally expressed in terms of the Jucys–Murphy elements via the formula

$$\sum_{\sigma \in S_k} \text{Wg}^{\text{U}}(\sigma)\, \sigma = \prod_{i=1}^{k} \frac{1}{N + J_i}.$$

By considering the right side as a series in $-\frac{1}{N}$, one observes that the coefficients are homogeneous symmetric functions of the Jucys–Murphy elements. This leads directly to the notion of monotone Hurwitz numbers, which count monotone factorizations of a given permutation with a prescribed length. The analog of the equation above for the Weingarten function Wg^{S} is the following, which leads to a deformation of the monotone Hurwitz numbers, which are now weighted counts of monotone factorizations with weight equal to t to the power of the hive number. This idea was already briefly introduced in Definition 2.11, but will be studied in further detail in Section 3.

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Proposition 2.14 *For each positive integer k, we have the following equality in $Z\mathbb{C}[S_k]$, the centre of the symmetric group algebra.*

$$\sum_{\sigma\in S_k} \mathrm{Wg}^{\mathrm{S}}(\sigma)\,\sigma = \prod_{i=1}^{k} \frac{M+J_i}{N+J_i}.$$

Proof This is essentially an algebraic reformulation of the orthogonality relations of Theorem 2.4 and we will prove it by induction on k. The base case $k = 1$ corresponds to the fact that $\mathrm{Wg}^{\mathrm{S}}((1)) = \frac{M}{N}$, which is an immediate consequence of the orthogonality relation for $\sigma = (1)$.

It remains to show that, for $k \geqslant 2$,

$$\sum_{\sigma\in S_k} \mathrm{Wg}^{\mathrm{S}}(\sigma)\,\sigma = \frac{M+J_k}{N+J_k} \sum_{\sigma\in S_{k-1}} \mathrm{Wg}^{\mathrm{S}}(\sigma)\,\sigma^{\uparrow},$$

where for any permutation $\sigma \in S_{k-1}$, we write $\sigma^{\uparrow} \in S_k$ to denote the permutation that agrees with σ on the set $\{1, 2, \ldots, k-1\}$ and fixes k. Consider multiplying both sides of this equation by $N + J_k$ and use the definition of the Jucys–Murphy element J_k to show that

$$(N+J_k) \sum_{\sigma\in S_k} \mathrm{Wg}^{\mathrm{S}}(\sigma)\,\sigma = \sum_{\sigma\in S_k} \left(N\mathrm{Wg}^{\mathrm{S}}(\sigma) + \sum_{i=1}^{k-1} \mathrm{Wg}^{\mathrm{S}}((i\ k)\circ\sigma) \right)\sigma, \tag{2.4}$$

$$(M+J_k) \sum_{\sigma\in S_{k-1}} \mathrm{Wg}^{\mathrm{S}}(\sigma)\,\sigma^{\uparrow} = \sum_{\sigma\in S_k} \Bigg(\delta_{\sigma(k),k}\, M\mathrm{Wg}^{\mathrm{S}}(\sigma^{\downarrow}) \tag{2.5}$$

$$+ \sum_{i=1}^{k-1} \delta_{\sigma(i),k} \mathrm{Wg}^{\mathrm{S}}([\sigma\circ(i\ k)]^{\downarrow}) \Bigg)\sigma. \tag{2.6}$$

The second equality is more subtle than the first and relies on the observation that for $1 \leqslant i \leqslant k-1$,

$$\begin{aligned}\sum_{\sigma\in S_{k-1}} \mathrm{Wg}^{\mathrm{S}}(\sigma)\,(i\ k)\circ\sigma^{\uparrow} &= \sum_{\sigma\in S_k:\sigma(k)=k} \mathrm{Wg}^{\mathrm{S}}(\sigma^{\downarrow})\,(i\ k)\circ\sigma \\ &= \sum_{\sigma\in S_k:\sigma(i)=k} \mathrm{Wg}^{\mathrm{S}}([\sigma\circ(i\ k)]^{\downarrow})\,\sigma.\end{aligned}$$

Finally, the desired equality between equations (2.4) and (2.6) follows directly from the orthogonality relations of Theorem 2.4, which completes the proof. ∎

Proposition 2.14 implies the following character formula for the Weingarten function.

Corollary 2.15 (Character formula) *Consider a permutation $\sigma \in S_k$ and let* $\mathrm{id} \in S_k$ *be the identity. Then*

$$\mathrm{Wg}^{\mathrm{S}}(\sigma) = \frac{1}{k!} \sum_{\lambda\vdash k} \chi^{\lambda}(\mathrm{id})\,\chi^{\lambda}(\sigma) \prod_{\square\in\lambda} \frac{M+c(\square)}{N+c(\square)},$$

where $c(\square)$ denotes the content of the box $\square$ in a Young diagram of a partition. Here and throughout, we use the notation $\lambda \vdash k$ to denote that λ is a partition of k.

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Proof By the orthogonality of characters, the identity in $Z\mathbb{C}[S_k]$ can be expressed as $e = \frac{1}{k!}\sum_{\lambda\vdash k}\chi^\lambda(\mathrm{id})\,\chi^\lambda$. Use part (b) of Proposition 2.13 to write

$$\prod_{i=1}^{k}\frac{M+J_i}{N+J_i}\cdot e = \frac{1}{k!}\sum_{\lambda\vdash k}\left(\prod_{\square\in\lambda}\frac{M+c(\square)}{N+c(\square)}\right)\chi^\lambda(\mathrm{id})\,\chi^\lambda.$$

By Proposition 2.14, $\mathrm{Wg}^{\mathbf{S}}(\sigma)$ is the coefficient of σ in the above expression, and the result follows. ∎

In summary, integrals on the space $\mathbf{S}(M,N)$ of matrices admit a convolution formula involving the Weingarten function $\mathrm{Wg}^{\mathbf{S}}$. The Weingarten function is in turn determined by orthogonality relations, from which it follows that its values have a representation-theoretic interpretation in terms of Jucys–Murphy elements. An alternative perspective shows that the large N expansion of the Weingarten function for fixed $t = 1 - \frac{N}{M}$ has coefficients that are weighted counts of monotone factorisations. In the next section, we investigate the combinatorics of these enumerations in further detail.

3 Deformed monotone Hurwitz numbers

3.1 Deformed monotone Hurwitz numbers

Monotone Hurwitz numbers enumerate monotone factorizations of permutations and arise as coefficients in the large N expansion of the unitary Weingarten function [33]. Weingarten calculus on the space $\mathbf{S}(M,N)$ naturally motivates a deformation of the monotone Hurwitz numbers, obtained by counting with a weight equal to t to the power of the hive number. This construction produces a family of polynomials with remarkable properties.

Recall from Definition 2.11 that $\vec{w}_r^{\,t}(\sigma)$ denotes the weighted count of monotone factorizations of σ, while $\vec{h}_r^{\,t}(\sigma)$ restricts the count to transitive monotone factorizations. To fit the usual nomenclature concerning Hurwitz-type problems, we introduce a new notation to reindex by "topology," consider the enumeration as a function on cycle type, and normalise by the product of cycle lengths.

Definition 3.1 (Deformed monotone Hurwitz numbers) For $\mu_1,\dots,\mu_n \geqslant 1$, let σ be any permutation with n cycles of lengths $\mu_1,\dots,\mu_n$ and define

$$\vec{W}^t_{g,n}(\mu_1,\dots,\mu_n) = \frac{1}{\mu_1\cdots\mu_n}\,\vec{w}^{\,t}_{|\mu|+2g-2+n}(\sigma),$$
$$\vec{H}^t_{g,n}(\mu_1,\dots,\mu_n) = \frac{1}{\mu_1\cdots\mu_n}\,\vec{h}^{\,t}_{|\mu|+2g-2+n}(\sigma).$$

Here and throughout, we use the notation $|\mu|$ as a shorthand for the sum $\mu_1+\cdots+\mu_n$. We refer to $\vec{W}^t_{g,n}(\mu_1,\dots,\mu_n)$ as a *disconnected deformed monotone Hurwitz number* and $\vec{H}^t_{g,n}(\mu_1,\dots,\mu_n)$ as a *connected deformed monotone Hurwitz number*.

Remark 3.2. The pair (g,n) here can be interpreted as encoding the topology of a surface, with g denoting the genus and n the number of punctures, or marked points. A monotone factorization $(\tau_1,\tau_2,\dots,\tau_r)$ of σ can be interpreted as the

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monodromy of a branched cover $\mathcal{S} \to \mathbb{CP}^1$ of Riemann surfaces, with σ representing the monodromy over $\infty \in \mathbb{CP}^1$. So $\vec{W}^t_{g,n}(\mu_1, \ldots, \mu_n)$ and $\vec{H}^t_{g,n}(\mu_1, \ldots, \mu_n)$ enumerate certain branched covers of $\mathbb{CP}^1$ by a genus g surface with n preimages of $\infty \in \mathbb{CP}^1$, the latter restricting the enumeration to connected covers. Consistent with this interpretation, we have that $\vec{H}^t_{g,n}(\mu_1, \ldots, \mu_n) = 0$ unless g is a non-negative integer, while $\vec{W}^t_{g,n}(\mu_1, \ldots, \mu_n)$ can be non-zero when g is a negative integer. Note that the genus of a disconnected surface may be negative, since we consider the Euler characteristic $2 - 2g$ to be additive over disjoint union, rather than the genus itself.

The character formula of Corollary 2.15 translates into one for deformed monotone Hurwitz numbers as follows.

Proposition 3.3 (Character formula) *Let $[\hbar^r]F(\hbar)$ denote the coefficient of $\hbar^r$ in the series for $F(\hbar)$ at $\hbar = 0$ and let χ^λ_μ denote the value of the symmetric group character χ^λ on a permutation of cycle type μ. Then*

$$\vec{W}^t_{g,n}(\mu_1, \ldots, \mu_n) = \frac{1}{\mu_1 \cdots \mu_n} \left[\hbar^{|\mu|+2g-2+n}\right] \sum_{\lambda \vdash |\mu|} \frac{\chi^\lambda_{11\cdots 1}\, \chi^\lambda_\mu}{|\mu|!} \prod_{\square \in \lambda} \frac{1 - \hbar\, c(\square) + t\hbar\, c(\square)}{1 - \hbar\, c(\square)}.$$

Proof Equate the expressions for $\mathrm{Wg}^{\mathbf{S}}(\sigma)$ in terms of monotone factorizations (Theorem 2.12) and characters of the symmetric group (Corollary 2.15), using the notation $\hbar = -\frac{1}{N}$ and $t = 1 - \frac{N}{M}$.

$$\frac{1}{(1-t)^k} \sum_{r=0}^{\infty} \vec{w}^t_r(\sigma)\, \hbar^r = \frac{1}{k!} \sum_{\lambda \vdash k} \chi^\lambda(\mathrm{id})\, \chi^\lambda(\sigma) \prod_{\square \in \lambda} \frac{1 - \hbar\, c(\square) + t\hbar\, c(\square)}{(1-t)(1 - \hbar\, c(\square))}.$$

The desired result is obtained by multiplying through by $(1-t)^k$, extracting the coefficient of $\hbar^{|\mu|+2g-2+n}$ from both sides, and using Definition 3.1 for $\vec{W}^t_{g,n}(\mu_1, \ldots, \mu_n)$. ∎

The family of deformed monotone Hurwitz numbers is stored in the large N expansion of the following matrix integral, which we interpret as the partition function for the enumeration. As usual, the partition function naturally stores the disconnected enumeration, while its logarithm stores the connected enumeration.

Proposition 3.4 (Matrix integral) *The large N expansion of the formal matrix integral below is a partition function for the deformed monotone Hurwitz numbers. Here, we use the notation $\hbar = -\frac{1}{N}$ and p_i represents the ith power sum symmetric function, evaluated on the eigenvalues of the $N \times N$ complex matrix A. (It is common to see a power of $\hbar^{2g-2+n}$ in the partition function—the extra $\hbar^{|\mu|}$ here can be removed by rescaling.)*

$$\begin{aligned}\int_{S(M,N)} \exp \frac{N \operatorname{tr}(AS)}{M}\, \mathrm{d}S \\ &= 1 + \sum_{g=-\infty}^{\infty} \sum_{n=1}^{\infty} \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} \vec{W}^t_{g,n}(\mu_1, \ldots, \mu_n)\, \frac{\hbar^{|\mu|+2g-2+n}}{n!}\, p_{\mu_1} \cdots p_{\mu_n} \\ &= \exp\left[\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} \vec{H}^t_{g,n}(\mu_1, \ldots, \mu_n)\, \frac{\hbar^{|\mu|+2g-2+n}}{n!}\, p_{\mu_1} \cdots p_{\mu_n} \right].\end{aligned}$$

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Remark 3.5. Recall that the parameter t is encoded in the parameters of the space $\mathbf{S}(M,N)$ via the relation $t = 1 - \frac{N}{M}$. The usual monotone Hurwitz numbers are recovered from their deformed counterparts defined above by setting $t = 1$. Perhaps surprisingly, this particular value of t does not correspond to valid parameters and M and N in the matrix integral interpretation. It would be interesting to have a more direct connection between integration over $\mathbf{U}(N)$ and $\mathbf{S}(M,N)$.

Remark 3.6. It would also be natural to consider "double" deformed monotone Hurwitz numbers that take two partitions as arguments, rather than one. These enumerate monotone factorizations whose product with a permutation of cycle type given by the first partition produces a permutation of cycle type given by the second. Although this "double" enumeration deserves further attention, we refrain from developing this theory in the present work, since they do not give rise to polynomials with the same remarkable properties as the "single" enumeration.

3.2 Cut-and-join recursion

The notion of cut-and-join analysis was originally developed for single Hurwitz numbers, before being adapted to work in the monotone case [32, 34]. Deformed monotone Hurwitz numbers are amenable to a similar analysis, resulting in the following recursion.

Theorem 3.7 (Cut-and-join recursion) *Apart from the initial condition* $\vec{H}^t_{0,1}(1) = 1$, *the deformed monotone Hurwitz numbers satisfy*

$$\begin{aligned}\mu_1 \vec{H}^t_{g,n}(\mu_1,\mu_S) &= \sum_{i=2}^{n}(\mu_1+\mu_i)\,\vec{H}^t_{g,n-1}(\mu_1+\mu_i,\mu_{S\setminus\{i\}}) \\ &\quad + (t-1)(\mu_1-1)\,\vec{H}^t_{g,n}(\mu_1-1,\mu_S) \\ &\quad + \sum_{\alpha+\beta=\mu_1}\alpha\beta\left[\vec{H}^t_{g-1,n+1}(\alpha,\beta,\mu_S) + \sum_{\substack{g_1+g_2=g\\ I_1\sqcup I_2=S}}\vec{H}^t_{g_1,|I_1|+1}(\alpha,\mu_{I_1})\,\vec{H}^t_{g_2,|I_2|+1}(\beta,\mu_{I_2})\right],\end{aligned} \tag{3.1}$$

where we use the notation $S = \{2,3,\dots,n\}$ *and* $\mu_I = \{\mu_{i_1},\mu_{i_2},\dots,\mu_{i_k}\}$ *for* $I = \{i_1,i_2,\dots,i_k\}$.

Proof The proof is based on the case for the usual monotone Hurwitz numbers [32], with some additional care required to keep track of the deformation parameter t. Consider multiplying equation (3.1) by $\mu_2\cdots\mu_n$ and consider the right side to be composed of four terms in the obvious way.

Fix a permutation $\sigma \in S_k$ of cycle type $(\mu_1,\dots,\mu_n)$ and consider the quantity $\vec{h}^t_{r+1}(\sigma)$ for some $r \geqslant 0$. Note that a transitive monotone factorization $\boldsymbol{\tau} = (\tau_1,\tau_2,\dots,\tau_{r+1})$ of σ must have $\tau_{r+1} = (a\ \ k)$ for some $1 \leqslant a \leqslant k-1$. So a transitive monotone factorization $\boldsymbol{\tau}$ of σ with length $r+1$ is equivalent to a monotone factorization $\boldsymbol{\tau}'$ of $\sigma\circ(a\ k)$ with length r for some $1 \leqslant a \leqslant k-1$, where $\langle\boldsymbol{\tau}',(a\ k)\rangle \leqslant S_k$ is transitive. This can be expressed via the equation

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$$\vec{h}^t_{r+1}(\sigma) = \sum_{\substack{\boldsymbol{\tau}\in\mathcal{M}_{r+1}(\sigma)\\ \langle\boldsymbol{\tau}\rangle \text{ transitive}}} t^{b(\boldsymbol{\tau})} = \sum_{a=1}^{k-1} \sum_{\substack{\boldsymbol{\tau}'\in\mathcal{M}_r(\sigma\circ(a\ k))\\ \langle\boldsymbol{\tau}',(a\ k)\rangle \text{ transitive}}} t^{|\boldsymbol{b}(\boldsymbol{\tau}')\cup\{k\}|},$$

where we introduce the boldface notation $\boldsymbol{b}(\boldsymbol{\tau})$ to denote the set $\{b_1, b_2, \ldots, b_r\}$, excluding multiplicities, for $\boldsymbol{\tau} = ((a_1\ b_1), (a_2\ b_2), \ldots, (a_r\ b_r))$.

The sum on the right side can be broken down into further sums, depending on various cases. For a fixed value of a, either a is in the same cycle of σ as k, or a is in a different cycle. The transposition $(a\ k)$ is referred to as a cut or a join of σ depending on these two cases, respectively. Moreover, if $(a\ k)$ is a join of σ, then any monotone factorization $\boldsymbol{\tau}'$ of $\sigma\circ(a\ k)$, with $\langle\boldsymbol{\tau}', (a\ k)\rangle$ transitive, must be transitive itself. Based on these considerations, each term in the equation above falls into one of three cases: for each $1 \leqslant a \leqslant k-1$ and $\boldsymbol{\tau}' \in \mathcal{M}_r(\sigma\circ(a\ k))$ with $\langle\boldsymbol{\tau}', (a\ k)\rangle$ transitive, either

- $(a\ k)$ is a *join* of σ and $\langle\boldsymbol{\tau}'\rangle \leqslant S_k$ is transitive;
- $(a\ k)$ is a *cut* of σ and $\langle\boldsymbol{\tau}'\rangle \leqslant S_k$ is transitive; or
- $(a\ k)$ is a *cut* of σ and $\langle\boldsymbol{\tau}'\rangle \leqslant S_k$ is *not* transitive.

The first two cases can be expressed as one term to give the equation

$$\vec{h}^t_{r+1}(\sigma) = \sum_{a=1}^{k-1} \vec{h}^t_r(\sigma\circ(a\ k)) + \sum_{\substack{a=1\\ (a\ k) \text{ cuts } \sigma}}^{k-1} \sum_{\substack{\boldsymbol{\tau}'\in\mathcal{M}_r(\sigma\circ(a\ k))\\ \langle\boldsymbol{\tau}',(a\ k)\rangle \text{ transitive}\\ \langle\boldsymbol{\tau}'\rangle \text{ not transitive}}} t^{|\boldsymbol{b}(\boldsymbol{\tau}')\cup\{k\}|}. \tag{3.2}$$

In the usual notation for deformed monotone Hurwitz numbers, the first term here precisely gives rise to the first and third terms of equation (3.1), while the second term here *almost* gives rise to the fourth term of equation (3.1). The remainder of the proof explains the meaning of "almost" here and how the second term of equation (3.1) accounts correctly for this anomaly.

If $\boldsymbol{\tau}'$ falls into the third case for some value of $1 \leqslant a \leqslant k-1$, then the transitive orbit of $\langle\boldsymbol{\tau}', (a\ k)\rangle$ is split into two orbits of $\langle\boldsymbol{\tau}'\rangle$ — one containing a and one containing k. In fact, these orbits are given by a partition of the disjoint cycles of $\sigma\circ(a\ k)$ into two classes, thus giving rise to the quadratic terms in equation (3.1). The only instance that $b(\boldsymbol{\tau}')$ and $b(\boldsymbol{\tau}', (a\ k))$ differ is when k is in an orbit of size 1 under $\langle\boldsymbol{\tau}'\rangle$, which then requires an additional weight of t to be contributed. The $t-1$ factor in the second term of equation (3.1) precisely handles this adjustment, since we wish to remove one of the $\vec{H}^t_{0,1}(1)\,\vec{H}^t_{g,n}(\mu_1 - 1, \vec{\mu}_S)$ contributions appearing in the fourth term and weight it by an additional factor of t.

Thus, in the correct final accounting of all terms, we see that the first term of equation (3.2) corresponds to the first and third terms of equation (3.1), while the second term of equation (3.2) corresponds to the second and fourth terms of equation (3.1). ∎

The cut-and-join recursion provides an effective algorithm for calculating deformed monotone Hurwitz numbers, which is more efficient than naive approaches. With this improved computational power comes the ability to make large-scale empirical observations on the structure of the deformed monotone Hurwitz numbers

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and a recursive means to prove them. The most evident of these observations are the symmetric and unimodal nature of the coefficients of $\vec{H}^t_{g,n}(\mu_1, \ldots, \mu_n)$.

Adopting the terminology of Brenti [7] and Zeilberger [50], we define the following.

Definition 3.8 A polynomial $P(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$ with real coefficients is said to be:

- *symmetric* if the sequence $\ldots, 0, 0, a_0, a_1, a_2, \ldots, a_n, 0, 0, \ldots$ is palindromic;
- *unimodal* if there exists an integer k such that $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_k \geqslant a_{k+1} \geqslant \cdots \geqslant a_n$; and
- a *Λ-polynomial* if it is a non-zero polynomial with non-negative coefficients that is both palindromic and unimodal.

If P is a Λ-polynomial, then there exists a unique integer $\operatorname{dar}(P)$ such that $P(t) = t^{\operatorname{dar}(P)} P(t^{-1})$. The quantity $\operatorname{dar}(P)$ is called the *darga* of P.

Proposition 3.9 *For $g \geqslant 0$, $n \geqslant 1$ and $|\mu| \geqslant 2$, the deformed monotone Hurwitz number $\vec{H}^t_{g,n}(\mu_1, \ldots, \mu_n)$ is a Λ-polynomial of degree $|\mu| - 1$ and darga $|\mu|$.*

Proof We use strong induction on the value of $r = |\mu| + 2g - 2 + n \geqslant 1$. The only deformed monotone Hurwitz number corresponding to $r = 1$ is $\vec{H}^t_{0,1}(2) = \frac{1}{2}t$, which is indeed a Λ-polynomial of degree 1 and darga 2. Now consider a deformed monotone Hurwitz number $\vec{H}^t_{g,n}(\mu_1, \ldots, \mu_n)$ corresponding to $r \geqslant 2$ and rewrite the cut-and-join recursion of Theorem 3.7 as

$$\begin{aligned}
\mu_1 \vec{H}^t_{g,n}(\mu_1, \mu_S) &= \sum_{i=2}^{n} (\mu_1 + \mu_i)\, \vec{H}^t_{g,n-1}(\mu_1 + \mu_i, \mu_{S\setminus\{i\}}) \\
&\quad + (t+1)(\mu_1 - 1)\, \vec{H}^t_{g,n}(\mu_1 - 1, \mu_S) \\
&\quad + \sum_{\alpha+\beta=\mu_1} \alpha\beta \left[\vec{H}^t_{g-1,n+1}(\alpha, \beta, \mu_S) + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = S}}^{\text{no } \vec{H}^t_{0,1}(1)} \vec{H}^t_{g_1,|I_1|+1}(\alpha, \mu_{I_1})\, \vec{H}^t_{g_2,|I_2|+1}(\beta, \mu_{I_2}) \right].
\end{aligned}$$

Here, we have simply removed the two terms including $\vec{H}^t_{0,1}(1)$ from the final summation and incorporated them into the second term on the right side.

By the inductive hypothesis, all deformed monotone Hurwitz numbers appearing on the right side of the cut-and-join recursion are Λ-polynomials. To prove that $\vec{H}^t_{g,n}(\mu_1, \ldots, \mu_n)$ is also one, we rely on the following closure properties of Λ-polynomials, proofs of which can be found in [47].

- If P and Q are Λ-polynomials, then the product PQ is a Λ-polynomial with $\operatorname{dar}(PQ) = \operatorname{dar}(P) + \operatorname{dar}(Q)$.
- If P and Q are Λ-polynomials of the same darga d, then $P + Q$ is a Λ-polynomial of darga d.

It follows that each of the four terms on the right side of the equation above are Λ-polynomials of darga $|\mu|$, or possibly equal to the zero polynomial. So the entire right side, and hence $\vec{H}^t_{g,n}(\mu_1, \ldots, \mu_n)$, is a Λ-polynomial of darga $|\mu|$.

For any permutation $\sigma \in S_k$ for $k \geqslant 2$, one can construct a transitive monotone factorization of σ with hive number 1. So the polynomial $\vec{H}^t_{g,n}(\mu_1, \ldots, \mu_n)$ has a

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non-zero linear term, but no constant term. Combined with the fact that it is a Λ-polynomial of darga $|\mu|$, it follows that $\vec{H}^t_{g,n}(\mu_1, \ldots, \mu_n)$ is a polynomial of degree $|\mu| - 1$. ∎

Remark 3.10. We have thus far provided two interpretations for the polynomials that we refer to as deformed monotone Hurwitz numbers—one as weighted counts of monotone factorizations (Definition 3.1) and the other as coefficients coefficients in the large N expansion of a matrix integral (Proposition 3.4). The symmetry of these polynomials does not appear to be immediately evident from either of these manifestations. In the matrix integral the transformation $t \mapsto \frac{1}{t}$ encodes the transformation $M \mapsto N - M$. This is natural from the perspective of integration on the Grassmannian, since $\mathrm{Gr}(M, N) \cong \mathrm{Gr}(N - M, N)$, although further investigation of this symmetry is warranted. In the monotone factorization interpretation, the symmetry of the polynomials implies that the number of transitive monotone factorizations of a permutation $\sigma \in S_k$ with hive number h is equal to the number of transitive monotone factorizations of σ with hive number $k - h$. This does not appear to be immediately obvious, so it would be interesting to have a direct combinatorial proof of this fact. It is worth remarking that the disconnected enumeration $\vec{W}^t_{g,n}(\mu_1, \ldots, \mu_n)$ does not lead to symmetric polynomials, so the transitivity condition is important here.

Remark 3.11. There are various other properties of polynomial coefficients that are often mentioned alongside symmetry and unimodality. The deformed monotone Hurwitz numbers constitute a family of polynomials that appears to satisfy many of these. As examples, we pose the following questions. Do the coefficients of the deformed monotone Hurwitz numbers exhibit log-concavity? Do they exhibit asymptotic normality?

As we will see in Section 4, deformed monotone Hurwitz numbers belong to a large class of enumerative problems governed by the topological recursion. In many such instances, the one-point invariants—in other words, those with $n = 1$—are governed by a recursion that is linear, rather than quadratic like the cut-and-join recursion. In previous work of Chaudhuri and the second author [9], it was shown that such one-point recursions exist for "weighted Hurwitz numbers," to use the terminology of Alexandrov, Chapuy, Eynard, and Harnad [1]. Furthermore, they gave an explicit algorithmic approach to obtain these recursions that is effective in many examples. In the case of deformed monotone Hurwitz numbers, one obtains the following result.

Theorem 3.12 (One-point recursion) *The one-point deformed monotone Hurwitz numbers—in other words, those with $n = 1$—satisfy*

$$\begin{aligned} d^2\, \vec{H}^t_{g,1}(d) = {} & (d-1)(2d-3)(t+1)\, \vec{H}^t_{g,1}(d-1) \\ & - (d-2)(d-3)(t-1)^2\, \vec{H}^t_{g,1}(d-2) + d^2(d-1)^2\, \vec{H}^t_{g-1,1}(d). \end{aligned}$$

Observe that setting $t = 1$ above recovers the known one-point recursion for monotone Hurwitz numbers [9]. Furthermore, setting $g = 0$ and combining with equation (1.1) recovers the known linear recursion for Narayana polynomials [31].

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3.3 Interlacing phenomena

Root conjectures

The cut-and-join recursion of Theorem 3.7 can be used to effectively compute a large number of deformed monotone Hurwitz numbers, some of which are shown in Appendix A. We previously stated in Proposition 3.9 that the coefficients of these polynomials are symmetric and unimodal. We now turn our attention to their roots, which exhibit rather striking behavior that remains largely conjectural at present. The concerning roots of deformed monotone Hurwitz numbers is the following.

Conjecture 3.13 (Real-rootedness) *For all $g \geqslant 0$, $n \geqslant 1$ and $\mu_1, \ldots, \mu_n \geqslant 1$, the deformed monotone Hurwitz number $\vec{H}^t_{g,n}(\mu_1, \mu_2, \ldots, \mu_n)$ is a real-rooted polynomial in t.*

The roots of the deformed monotone Hurwitz numbers are not only real, but also possess interesting structure relative to each other. We say that a polynomial P *interlaces* a polynomial Q if

- the degree of P is n and the degree of Q is $n+1$ for some positive integer n;
- P has n real roots $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ and Q has $n+1$ real roots $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_{n+1}$, allowing for multiplicity; and
- $b_1 \leqslant a_1 \leqslant b_2 \leqslant a_2 \leqslant \cdots \leqslant b_n \leqslant a_n \leqslant b_{n+1}$.

If the inequalities above are strict, then we say that the polynomial P *strictly interlaces* the polynomial Q. By convention, we will also say that a polynomial P (strictly) interlaces a polynomial Q if P is constant and Q is affine.

It is known that the sequence of Narayana polynomials interlace in the sense that $\mathrm{Nar}_\mu(t)$ interlaces $\mathrm{Nar}_{\mu+1}(t)$ for every positive integer μ [31]. Given equation (1.1), one may wonder whether an analogous property persists more generally across the whole family of deformed monotone Hurwitz numbers. Overwhelming empirical evidence suggests that this is indeed the case and we have the following.

Conjecture 3.14 (Interlacing) *The polynomial $\vec{H}^t_{g,n}(\mu_1, \mu_2, \ldots, \mu_n)$ interlaces each of the n polynomials*

$$\vec{H}^t_{g,n}(\mu_1+1, \mu_2, \ldots, \mu_n), \quad \vec{H}^t_{g,n}(\mu_1, \mu_2+1, \ldots, \mu_n), \quad \ldots, \quad \vec{H}^t_{g,n}(\mu_1, \mu_2, \ldots, \mu_n+1).$$

Conjecture 3.14 states that for fixed (g, n), one obtains a lattice of interlacing polynomials. It has been checked computationally for many cases, including the following, amounting to 1,430 independent checks:

- $g \in \{0, 1\}$, $n \in \{1, 2, 3, 4, 5\}$, $|\mu| \leqslant 15$;
- $g \in \{2, 3\}$, $n \in \{1, 2, 3, 4, 5\}$, $|\mu| \leqslant 12$;
- $g \in \{4, 5\}$, $n \in \{1, 2, 3, 4, 5\}$, $|\mu| \leqslant 10$.

One can observe rich structure in the roots of the deformed monotone Hurwitz numbers and write down further conjectures. As an example, consider the following, which asserts that the roots of $\vec{H}^t_{g,n}(\mu_1, \ldots, \mu_n)$ behave well for fixed $\mu_1, \ldots, \mu_n$ and increasing g. A great deal of data can be generated to support this conjecture, some examples of which appear in the tables of Figures 2 and 3.

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$(\mu_1, \ldots, \mu_n)$	α_1	α_2	α_4	α_5
(7)	−5.001267941	−2.000328365	−0.499917922	−0.199949295
$(6,1)$	−5.001267941	−2.000328365	−0.499917922	−0.199949295
$(5,2)$	−5.001056435	−2.000273606	−0.499931607	−0.199957751
$(5,1,1)$	−5.001232684	−2.000319238	−0.499920203	−0.199950704
$(4,3)$	−5.001056438	−2.000273614	−0.499931605	−0.199957751
$(4,2,1)$	−5.001056436	−2.000273609	−0.499931607	−0.199957751
$(4,1,1,1)$	−5.001173928	−2.000304027	−0.499924004	−0.199953053
$(3,3,1)$	−5.001056438	−2.000273614	−0.499931605	−0.199957751
$(3,2,2)$	−5.000880235	−2.000227985	−0.499943010	−0.199964796
$(3,2,1,1)$	−5.001027065	−2.000266006	−0.499933507	−0.199958925
$(3,1,1,1,1)$	−5.001100492	−2.000285016	−0.499928756	−0.199955990
$(2,2,2,1)$	−5.000880235	−2.000227985	−0.499943010	−0.199964796
$(2,2,1,1,1)$	−5.000978117	−2.000253332	−0.499936675	−0.199960882
$(2,1,1,1,1,1)$	−5.001018906	−2.000263893	−0.499934035	−0.199959252
$(1,1,1,1,1,1,1)$	−5.001018906	−2.000263893	−0.499934035	−0.199959252

Figure 2: The roots $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_6$ of $\vec{H}^t_{20,n}(\mu_1, \ldots, \mu_n)$ for all $\mu_1, \ldots, \mu_n$ satisfying $|\mu| = 7$, to 9 decimal places. The roots $\alpha_3 = -1$ and $\alpha_6 = 0$ have been omitted. Observe the proximity of the numbers in each column to $-\frac{5}{1}, -\frac{4}{2}, -\frac{2}{4}, -\frac{1}{5}$, as predicted by Conjecture 3.15.

g	α_1	α_2	α_4	α_5
10	−5.04160471695	−2.01061201575	−0.49736099862	−0.19834954466
11	−5.02866367964	−2.00734550081	−0.49817034466	−0.19885998820
12	−5.01979225354	−2.00508876970	−0.49873103630	−0.19921143136
13	−5.01368866239	−2.00352761946	−0.49911964790	−0.19945394844
14	−5.00947834547	−2.00244657225	−0.49938910423	−0.19962158353
15	−5.00656851803	−2.00169741487	−0.49957600612	−0.19973760398
16	−5.00455473040	−2.00117796109	−0.49970568307	−0.19981797659
17	−5.00315968769	−2.00081762929	−0.49979567620	−0.19987369231
18	−5.00219259524	−2.00056759939	−0.49985814041	−0.19991233463
19	−5.00152183411	−2.00039406763	−0.49990150249	−0.19993914515
20	−5.00105643628	−2.00027360928	−0.49993160703	−0.19995775147

Figure 3: The roots $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_6$ of $\vec{H}^t_{g,3}(4,2,1)$ for $g = 10, 11, 12, \ldots, 20$, to 11 decimal places. The roots $\alpha_3 = -1$ and $\alpha_6 = 0$ have been omitted. Observe the monotonic convergence of the numbers in each column to $-\frac{5}{1}, -\frac{4}{2}, -\frac{2}{4}, -\frac{1}{5}$, as predicted by Conjecture 3.15.

Conjecture 3.15 *Fix positive integers $\mu_1, \ldots, \mu_n$ whose sum is d. As $g \to \infty$, the roots of the polynomial $\vec{H}^t_{g,n}(\mu_1, \ldots, \mu_n)$, in order from smallest to largest, respectively approach the values*

$$-\frac{d-2}{1}, \quad -\frac{d-3}{2}, \quad -\frac{d-4}{3}, \quad \ldots, \quad -\frac{2}{d-3}, \quad -\frac{1}{d-2}, \quad 0.$$

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Moreover, the convergence to a number less than -1 *is increasing from below, while the convergence to a non-zero number greater than* -1 *is decreasing from above.*

An interlacing result

There is a rich theory concerning interlacing polynomials, as evidenced by the tome of Fisk [31]. The following lemma is a slight adaptation of [31, Lemma 1.82], and will be used to prove the $(g,n) = (0,1)$ and $(g,n) = (1,1)$ cases of Conjectures 3.13 and 3.14.

Lemma 3.16 *Let* $f_1(t), f_2(t), \ldots$ *be a sequence of polynomials with non-negative coefficients, such that* $\deg f_i = i$. *Suppose that* $f_1(t)$ *strictly interlaces* $f_2(t)$, *and that the sequence of polynomials satisfies the relation*

$$f_{d+1}(t) = \ell_d(t)\, f_d(t) - q_d(t)\, f_{d-1}(t), \tag{3.3}$$

where $\ell_d(t)$ *is an affine function and* $q_d(t)$ *is a quadratic function that is positive for* $t \leqslant 0$. *Then all polynomials in the sequence are real-rooted and* $f_i(t)$ *strictly interlaces* $f_{i+1}(t)$ *for every positive integer i.*

Proof Suppose that $f_{d-1}(t)$ strictly interlaces $f_d(t)$. Since $f_d(t)$ has non-negative integer coefficients, its roots can be written as $0 \geqslant r_1 > r_2 > \cdots > r_d$. Then equation (3.3) evaluated at one of these roots yields

$$f_{d+1}(r_i) = -q_d(r_i)\, f_{d-1}(r_i),$$

which implies that $f_{d+1}(r_i)$ and $f_{d-1}(r_i)$ have different signs. Since $f_{d-1}(t)$ strictly interlaces $f_d(t)$, its values at $r_1, r_2, \ldots, r_d$ must alternate in sign, leading to the following table. The entries indicate the signs of $f_{d-1}(t)$ and $f_{d+1}(t)$, evaluated at $r_1, r_2, \ldots, r_d$ and in the limit $t \to \pm\infty$.

	∞	r_1	r_2	r_3	$\cdots$	r_d	$-\infty$
$f_{d-1}(t)$	$+$	$+$	$-$	$+$	$\cdots$	$(-)^{d-1}$	$(-)^{d-1}$
$f_{d+1}(t)$	$+$	$-$	$+$	$-$	$\cdots$	$(-)^{d}$	$(-)^{d+1}$

The intermediate value theorem implies that $f_{d+1}(t)$ has $d+1$ real roots and that $f_d(t)$ strictly interlaces $f_{d+1}(t)$. The result then follows by induction on d. ∎

Proposition 3.17 *Conjectures 3.13 and 3.14 hold for* $(g,n) = (0,1)$ *and* $(g,n) = (1,1)$.

Proof Setting $g = 0$ in the one-point recursion of Theorem 3.12 yields

$$d^2 \vec{H}^t_{0,1}(d) = (d-1)(2d-3)(t+1)\, \vec{H}^t_{0,1}(d-1) - (d-2)(d-3)(t-1)^2\, \vec{H}^t_{0,1}(d-2). \tag{3.4}$$

Divide both sides by $d^2 t$ to obtain a recursion governing the polynomials $\frac{\vec{H}^t_{0,1}(1)}{t}, \frac{\vec{H}^t_{0,1}(2)}{t}, \frac{\vec{H}^t_{0,1}(3)}{t}, \ldots$ of the form of equation (3.3). It follows from Lemma 3.16

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that this sequence is real-rooted and strictly interlacing. Therefore, the sequence $\vec{H}^t_{0,1}(1), \vec{H}^t_{0,1}(2), \vec{H}^t_{0,1}(3), \ldots$ is a sequence of real-rooted polynomials in which each polynomial interlaces the next.

For the genus 1 case, we argue in exactly the same way, using the as yet unproven claim that

(3.5)
$$d(d-2)\,\vec{H}^t_{1,1}(d) = (d-1)(2d-1)(t+1)\,\vec{H}^t_{1,1}(d-1) - (d-2)(d+1)(t-1)^2\,\vec{H}^t_{1,1}(d-2).$$

It remains to prove this relation, which we do using a generating function approach. Multiply both sides by x^{d-1} and sum over all positive integers d. Setting $w_{1,1}(x) = \sum_{d=1}^{\infty} d\,\vec{H}^t_{1,1}(d)\,x^{d-1}$, the claim is equivalent to

(3.6)
$$-x\left[(t-1)^2x^2 - 2(t+1)x + 1\right]\frac{\partial}{\partial x}w_{1,1}(x) = \left[4(t-1)^2x^2 - 3(t+1)x - 1\right]w_{1,1}(x).$$

We borrow Theorem 4.1 from the next section, which allows us to explicitly compute via $w_{1,1}(x) = \frac{\omega_{1,1}(z)}{\mathrm{d}x}$ that

(3.7)
$$w_{1,1}(x) = -\frac{tz(z-1)(1-z+tz)^4}{(tz^2-z^2+2z-1)^5},$$

where the coordinates x and z are related by $x = \frac{z(1-z)}{1-z+tz}$. One can now check that the expression for $w_{1,1}(x)$ of equation (3.7) satisfies the differential equation of equation (3.6) using a computer algebra system, thereby proving equation (3.5). ∎

Remark 3.18. It is natural to seek higher genus analogs of equations (3.4) and (3.5), in order to extend the result and proof of Proposition 3.17. However, one can prove that there exists no recursion of the form

$$dA(d)\,\vec{H}^t_{2,1}(d) = (d-1)B(d)f(t)\,\vec{H}^t_{2,1}(d-1) - (d-2)C(d)g(t)\,\vec{H}^t_{2,1}(d-2),$$

where $A(d), B(d), C(d)$ are affine functions of d, $f(t)$ is an affine function of t, and $g(t)$ is a quadratic function of t. So the exact argument used in the proof of Proposition 3.17 cannot extend to genus 2 and higher without some alteration.

4 Topological recursion

4.1 Topological recursion for deformed monotone Hurwitz numbers

The topological recursion arose in the mathematical physics literature as a formalism to capture loop equations in matrix models [11, 25]. There is now an extensive theory around topological recursion and it is known to govern a wide range of enumerative-geometric problems beyond the original matrix model context. These include: Hurwitz numbers and various generalizations [2, 4, 6, 19, 23, 27], monotone Hurwitz numbers [18], lattice points in moduli spaces of curves [10, 42], intersection theory on moduli spaces of curves [24, 25], Gromov–Witten theory of $\mathbb{CP}^1$ [43, 22], Gromov–Witten theory of toric Calabi–Yau threefolds [5, 26, 29], and cohomological

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field theories [22]. The topological recursion is also conjectured to govern certain quantum knot invariants, such as the large N asymptotics of colored Jones polynomials [3, 17] and colored HOMFLY-PT polynomials [35].

In its original formulation due to Chekhov, Eynard, and Orantin [11, 25], the topological recursion takes as input a spectral curve $(\mathcal{C}, x, y, \omega_{0,2})$ comprising a compact Riemann surface $\mathcal{C}$ equipped with two meromorphic functions $x, y : \mathcal{C} \to \mathbb{CP}^1$ and a bidifferential $\omega_{0,2}$ with a double pole along the diagonal. These are required to satisfy mild technical assumptions, which we do not mention here. Indeed, rather than fully define the topological recursion, we refer the reader to the many existing expositions in the literature [18, 28]. For the present discussion, it suffices to note that the topological recursion uses the initial data to define the base cases $\omega_{0,1}(z_1) = y(z_1)\,\mathrm{d}x(z_1)$ and $\omega_{0,2}(z_1, z_2)$, and then produces so-called correlation differentials via the following recursive formula.[3]

$$\omega_{g,n}(z_1,\dots,z_n) = \sum_{\alpha} \operatorname*{Res}_{z=\alpha} K(z_1,z)\Bigg[\omega_{g-1,n+1}(z,\sigma_\alpha(z),z_2,\dots,z_n) + \mathop{\sum_{g_1+g_2=g}}_{I_1\sqcup I_2=\{2,\dots,n\}}{}^{\circ} \omega_{g_1,|I_1|+1}(z,z_{I_1})\,\omega_{g_2,|I_2|+1}(\sigma_\alpha(z),z_{I_2})\Bigg].$$

In various settings, the correlation differential $\omega_{g,n}$ acts as a generating function for enumerative-geometric quantities, where g represents the genus of a surface and n its number of punctures, or boundaries.

It was previously shown that the monotone Hurwitz numbers are governed by the topological recursion in the following sense [18]. Topological recursion on the spectral curve[4]

$$\mathcal{C} = \mathbb{CP}^1, \quad x(z) = z(1-z), \quad y(z) = \frac{1}{1-z}, \quad \omega_{0,2}(z_1,z_2) = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1-z_2)^2}$$

produces correlation differentials satisfying

$$\omega_{g,n}(z_1,\dots,z_n) = \mathrm{d}_1\cdots\mathrm{d}_n \sum_{\mu_1,\dots,\mu_n=1}^{\infty} \vec{H}_{g,n}(\mu_1,\dots,\mu_n)\,x(z_1)^{\mu_1}\cdots x(z_n)^{\mu_n} + \delta_{g,0}\delta_{n,2}\,\frac{\mathrm{d}x(z_1)\,\mathrm{d}x(z_2)}{(x(z_1)-x(z_2))^2}.$$

Here, d_i represents the exterior derivative in the ith coordinate and $\vec{H}_{g,n}(\mu_1,\dots,\mu_n) = \left[\vec{H}^t_{g,n}(\mu_1,\dots,\mu_n)\right]_{t=1}$ denotes a monotone Hurwitz number.

[3] The reader should note that the various appearances of the topological recursion formula in the literature differ subtly, particularly in terms of the expression for $\omega_{0,1}$, the definition of the recursion kernel $K(z_1, z)$, and the choice of sign for y. Ultimately, the first two differences are inconsequential and a change of sign for y simply sends each correlation differential $\omega_{g,n}$ to $(-1)^n\omega_{g,n}$.

[4] The topological recursion is not sensitive to reparametrization of the underlying plane curve, whose global description in this case is $xy^2 - y + 1 = 0$. We have expressed the spectral curve here using a different rational parametrization to that appearing in [18], in order to align more closely with the statement and proof of Theorem 4.1.

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It is thus natural to consider whether the deformed monotone Hurwitz numbers satisfy the topological recursion on a deformation of the spectral curve above. Perhaps unsurprisingly, this is indeed the case and follows from the representation-theoretic interpretation for deformed monotone Hurwitz numbers of Corollary 2.15 and a powerful result of Bychkov, Dunin-Barkowski, Kazarian, and Shadrin [8], which builds on previous work of Alexandrov, Chapuy, Eynard, and Harnad [1].

Theorem 4.1 *Topological recursion on the spectral curve*

$$\mathcal{C} = \mathbb{CP}^1, \quad x(z) = \frac{z(1-z)}{1-z+tz}, \quad y(z) = \frac{1-z+tz}{1-z}, \quad \omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1-z_2)^2}$$

produces correlation differentials satisfying

$$\omega_{g,n}(z_1,\dots,z_n) = \mathrm{d}_1\cdots\mathrm{d}_n \sum_{\mu_1,\dots,\mu_n=1}^{\infty} \vec{H}^t_{g,n}(\mu_1,\dots,\mu_n)\, x(z_1)^{\mu_1}\cdots x(z_n)^{\mu_n} + \delta_{g,0}\delta_{n,2}\,\frac{\mathrm{d}x(z_1)\,\mathrm{d}x(z_2)}{\big(x(z_1)-x(z_2)\big)^2}.$$

Proof The main theorem of Bychkov, Dunin-Barkowski, Kazarian, and Shadrin in [8] states that the topological recursion governs the coefficients of certain KP tau functions of hypergeometric type. We present here a simplified version of their result that is fit for purpose.

Let $\tilde{y}(z) = \sum_{i=1}^{\infty} y_i\, z^i$ be the series expansion of a rational function satisfying $\tilde{y}(0) = 0$ and let $f(z)$ be a rational function satisfying $f(0) = 1$. From this data, define the generating function

$$Z(p_1, p_2, \dots; \hbar) = \sum_{\lambda\in\mathcal{P}} s_\lambda(p_1, p_2, \dots)\, s_\lambda\Big(\frac{y_1}{\hbar}, \frac{y_2}{\hbar}, \dots\Big) \prod_{\square\in\lambda} f(\hbar\, c(\square)),$$

where $\mathcal{P}$ is the set of partitions, s_λ represents the Schur function expressed in terms of power sum symmetric functions, and $c(\square)$ denotes the content of the box $\square$ in the Young diagram of a partition. This is a KP tau function of hypergeometric type and there exist "weighted Hurwitz numbers" $X_{g,n}(\mu_1,\dots,\mu_n)$ for $g \geqslant 0$, $n \geqslant 1$ and $\mu_1,\dots,\mu_n \geqslant 1$ such that

(4.1)
$$Z(p_1, p_2, \dots; \hbar) = \exp\left(\sum_{g=0}^{\infty}\sum_{n=1}^{\infty} \frac{\hbar^{2g-2+n}}{n!} \sum_{\mu_1,\dots,\mu_n=1}^{\infty} X_{g,n}(\mu_1,\dots,\mu_n)\, p_{\mu_1}\cdots p_{\mu_n}\right).$$

Bychkov, Dunin-Barkowski, Kazarian, and Shadrin [8] prove that topological recursion on the spectral curve

$$\mathcal{C} = \mathbb{CP}^1, \quad x(z) = \frac{z}{f(\tilde{y}(z))}, \quad y(z) = \frac{\tilde{y}(z)}{x(z)}, \quad \omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1-z_2)^2}$$

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produces correlation differentials satisfying

$$\omega_{g,n}(z_1,\dots,z_n) = \mathrm{d}_1\cdots\mathrm{d}_n \sum_{\mu_1,\dots,\mu_n=1}^{\infty} X_{g,n}\,(\mu_1,\dots,\mu_n)\,x(z_1)^{\mu_1}\cdots x(z_n)^{\mu_n} + \delta_{g,0}\delta_{n,2}\,\frac{\mathrm{d}x(z_1)\,\mathrm{d}x(z_2)}{(x(z_1)-x(z_2))^2}.$$

We simply specialise this result to $\tilde{y}(z) = z$ and $f(z) = \frac{1-z+tz}{1-z}$. It remains to check that the monotone deformed Hurwitz numbers agree with the weighted Hurwitz numbers with this particular choice of data.

The exponential of equation (4.1) transforms the connected generating function to the disconnected generating function, so it suffices to show that the disconnected deformed monotone Hurwitz numbers satisfy

$$\vec{W}^{t}_{g,n}(\mu_1,\dots,\mu_n) = |\mathrm{Aut}(\mu)|\left[\hbar^{2g-2+n}p_{\mu_1}\cdots p_{\mu_n}\right]Z(p_1,p_2,\dots;\hbar).$$

Here, $\mathrm{Aut}(\mu)$ denotes the subgroup of permutations in S_n that fix $(\mu_1,\dots,\mu_n)$. This automorphism factor arises because the summation in equation (4.1) is over all tuples of positive integers, rather than partitions.

Now use $s_\lambda(p_1,p_2,\dots) = \sum_{\mu\vdash|\lambda|}\frac{\chi^\lambda_\mu}{|\mathrm{Aut}(\mu)|}\prod\frac{p_{\mu_i}}{\mu_i}$ in the expression

$$Z(p_1,p_2,\dots;\hbar) = \sum_{\lambda\in\mathcal{Y}} s_\lambda(p_1,p_2,\dots)\,s_\lambda(\tfrac{1}{\hbar},0,0,\dots)\prod_{\square\in\lambda}\frac{1-\hbar\,c(\square)+t\hbar\,c(\square)}{1-\hbar\,c(\square)}$$

to obtain

$$|\mathrm{Aut}(\mu)|\left[\hbar^{2g-2+n}p_{\mu_1}\cdots p_{\mu_n}\right]Z(p_1,p_2,\dots;\hbar) = \frac{1}{\mu_1\cdots\mu_n}\left[\hbar^{|\mu|+2g-2+n}\right]\sum_{\lambda\vdash|\mu|}\frac{\chi^\lambda_{11\cdots1}\chi^\lambda_\mu}{|\mu|!}\prod_{\square\in\lambda}\frac{1-\hbar\,c(\square)+t\hbar\,c(\square)}{1-\hbar\,c(\square)}.$$

That this is equal to $\vec{W}^{t}_{g,n}(\mu_1,\dots,\mu_n)$ is precisely the content of Proposition 3.3, and this concludes the proof. ∎

Remark 4.2. The meromorphic functions $x, y : \mathbb{CP}^1 \to \mathbb{CP}^1$ of Theorem 4.1 satisfy

$$xy^2 + (t-1)xy - y + 1 = 0,$$

which is the global form of the spectral curve. The general theory of topological recursion asserts that, for $(g,n) \neq (0,1)$ or $(0,2)$, the correlation differential $\omega_{g,n}$ is a rational multidifferential on this curve with poles only at the zeroes of $\mathrm{d}x$ and further conditions on the pole structure [25]. This in turn leads to a polynomiality structure theorem for the deformed monotone Hurwitz numbers. Furthermore, the relation between topological recursion and cohomological field theories should lead to an interpretation of deformed monotone Hurwitz numbers as intersection numbers on moduli spaces of curves [22, 24]. However, we do not pursue this investigation further in the present work.

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4.2 Topologising sequences of polynomials

From Catalan curves to Narayana curves

As previously mentioned in equation (1.1), the $(g,n) = (0,1)$ case of the deformed monotone Hurwitz numbers recovers the Narayana polynomials in the following way.

$$(\mu+1)\,\vec{H}^t_{0,1}(\mu+1) = \mathrm{Nar}_\mu(t) := \sum_{i=1}^{\mu} \frac{1}{\mu}\binom{\mu}{i}\binom{\mu}{i-1} t^i.$$

Hence, we consider the deformed monotone Hurwitz numbers to be a "topological generalization" of the Narayana polynomials. This viewpoint appears throughout the literature, such as in Dumitrescu and Mulase's exposition on generalized Catalan numbers [21].

We propose that the topological recursion can be used as a mechanism to "topologise" sequences of polynomials. One would apply topological recursion to a spectral curve with a deformation parameter t to obtain a family of polynomials $X^t_{g,n}(\mu_1,\dots,\mu_n)$ for $g \geqslant 0$, $n \geqslant 1$ and $\mu_1,\dots,\mu_n \geqslant 1$ such that $X^t_{0,1}(1), X^t_{0,1}(2),\dots$ recovers the original sequence of polynomials in t. Furthermore, we claim that interesting properties of the original sequence of polynomials can persist into the topological generalization. Examples of such properties are the symmetry and unimodality of coefficients as well as the interlacing of roots. These both hold for Narayana polynomials and are respectively proven (Proposition 3.9) and conjectured (Conjecture 3.14) to hold for deformed monotone Hurwitz numbers more generally.

There is more than one way to topologise a given sequence of polynomials. In order to obtain a topological generalization of the Narayana numbers, it is natural to consider spectral curves that store Catalan numbers in the correlation differential $\omega_{0,1}$ and to consider a suitable t-deformation. The literature suggests three natural candidates for such "Catalan spectral curves" and these curves govern monotone Hurwitz numbers, map enumeration, and dessin d'enfant enumeration. This leads to the following three "Narayana spectral curves" with $\mathcal{C} = \mathbb{CP}^1$, $\omega_{0,2} = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1-z_2)^2}$, and the meromorphic functions $x, y : \mathcal{C} \to \mathbb{CP}^1$ satisfying the following.

- *Monotone Hurwitz numbers.* We have

$$y(x) = \sum_{\mu=0}^{\infty} \mathrm{Cat}_\mu\, x^\mu \quad \Rightarrow \quad xy^2 - y + 1 = 0,$$

  which is the global form of the spectral curve for monotone Hurwitz numbers [18]. Promoting the Catalan numbers to the Narayana polynomials, one obtains

$$y(x) = \sum_{\mu=0}^{\infty} \mathrm{Nar}_\mu(t)\, x^\mu \quad \Rightarrow \quad xy^2 + (t-1)xy - y + 1 = 0.$$

  This is the global form of the spectral curve for deformed monotone Hurwitz numbers, which appeared above in Theorem 4.1.
- *Map enumeration.* We have

$$y(x) = \sum_{\mu=0}^{\infty} \mathrm{Cat}_\mu\, x^{-2\mu-1} \quad \Rightarrow \quad y^2 - xy + 1 = 0,$$

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which is the global form of the spectral curve for the enumeration of maps [11, 25]. Promoting the Catalan numbers to the Narayana polynomials, one obtains

$$y(x) = \sum_{\mu=0}^{\infty} \mathrm{Nar}_\mu(t)\, x^{-2\mu-1} \quad \Rightarrow \quad xy^2 - x^2 y + (t-1)y + x = 0.$$

To the best of our knowledge, this is the global form of a spectral curve that has not yet been studied in the literature. We do not investigate it further in the present work, largely due to the fact that it has genus 1 for generic values of t. The topological recursion on spectral curves of positive genus is much less straightforward than on those of genus 0. Nevertheless, it would be interesting to understand this spectral curve and whether it governs a natural enumerative problem.

- *Dessin d'enfant enumeration.* We have

$$y(x) = \sum_{\mu=0}^{\infty} \mathrm{Cat}_\mu\, x^{-\mu-1} \quad \Rightarrow \quad xy^2 - xy + 1 = 0,$$

which is the global form of the spectral curve for the enumeration of dessins d'enfant [20, 37]. Promoting the Catalan numbers to the Narayana polynomials, one obtains

$$y(x) = \sum_{\mu=0}^{\infty} \mathrm{Nar}_\mu(t)\, x^{-\mu-1} \quad \Rightarrow \quad xy^2 - xy + (t-1)y + 1 = 0.$$

This is the global form of a spectral curve with genus 0. It leads to a topological generalisation of the Narayana polynomials that differs from the deformed monotone Hurwitz numbers. We will consider its properties in the next section.

Remark 4.3. We certainly do not claim that the spectral curves above provide the only natural topological generalizations of the Catalan numbers or the Narayana polynomials. Indeed, one could study spectral curves arising from $y(x) = \sum_{\mu=0}^{\infty} \mathrm{Cat}_\mu\, x^{a\mu+b}$ or $y(x) = \sum_{\mu=0}^{\infty} \mathrm{Nar}_\mu(t)\, x^{a\mu+b}$ for judicious choices of a and b, and it is not unlikely that these would lead to interesting enumerative problems.

Deforming the dessin d'enfant enumeration

The previous discussion motivates the study of the spectral curve whose global form is $xy^2 - xy + (t-1)y + 1 = 0$. In parametrized form, this corresponds to the spectral curve

$$\mathcal{C} = \mathbb{CP}^1, \quad x(z) = \frac{1 - z + tz}{z(1-z)}, \quad y(z) = z, \quad \omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1\, \mathrm{d}z_2}{(z_1 - z_2)^2}.$$

By construction, this spectral curve should govern a t-deformation of the usual dessin d'enfant enumeration. In the following, we prove that this enumeration arises by attaching a multiplicative weight t to each black vertex of a dessin d'enfant. Moreover, we observe that the polynomials arising from this construction empirically satisfy real-rootedness and interlacing properties analogous to those observed for deformed monotone Hurwitz numbers in Conjectures 3.13 and 3.14. Although the weighted enumeration of dessins d'enfant has been studied previously [37], these real-rootedness and interlacing properties have not been observed in the literature, to the best of our

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knowledge. Our discussion of the weighted enumeration of dessins d'enfant should be considered as a further case study promoting the viewpoint that topological recursion produces interesting topological generalizations of sequences of polynomials.

First, let us define dessins d'enfant—in other words, bicolored maps—and their enumeration.

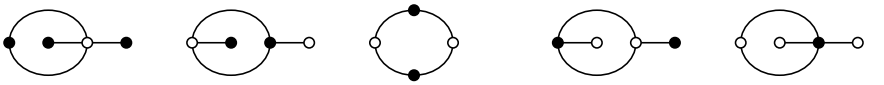
Definition 4.4 A *map* is a finite graph—possibly with loops or multiple edges—embedded in a compact oriented surface such that the complement of the graph is a disjoint union of topological disks. We refer to these disks as *faces* and require that they are labeled $1, 2, 3, \ldots, n$, where n denotes the number of faces.

A *dessin d'enfant* is a map whose vertices have been colored black or white such that each edge is incident to one black vertex and one white vertex. The *degree* of a face is defined to be half the number of edges incident to it.

An *equivalence* between two maps (or dessins d'enfant) is a bijection between their respective vertices, edges and faces that is realised by an orientation-preserving homeomorphism of their underlying surfaces and preserves all adjacencies, incidences, labels (and colors). An *automorphism* of a map (or dessin d'enfant) is an equivalence from the map (or dessin d'enfant) to itself.

Definition 4.5 For $g \geqslant 0$, $n \geqslant 1$ and $\mu_1, \ldots, \mu_n \geqslant 1$, let $D^t_{g,n}(\mu_1, \ldots, \mu_n) \in \mathbb{Q}[t]$ denote the weighted enumeration of connected genus g dessins d'enfant with n labeled faces of degrees $\mu_1, \ldots, \mu_n$. The weight attached to a dessin d'enfant Γ is $\frac{t^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}$, where $b(\Gamma)$ denotes the number of black vertices of Γ and $\mathrm{Aut}(\Gamma)$ denotes the automorphism group of Γ. Let $D^{t\bullet}_{g,n}(\mu_1, \ldots, \mu_n) \in \mathbb{Q}[t]$ denote the analogous count, including possibly disconnected dessins d'enfant.

Example 4.6 Consider the dessin d'enfant enumeration $D^t_{0,2}(2,2)$. The only dessins d'enfant that contribute are the five pictured below. We have represented these as plane graphs, with the face labeled 1 corresponding to the bounded region and the face labeled 2 corresponding to the unbounded region.



The central dessin d'enfant has two automorphisms, while the remaining dessins d'enfant have one, so we have $D^t_{0,2}(2,2) = \left(\frac{t^3}{1}\right) + \left(\frac{t^2}{1} + \frac{t^2}{2} + \frac{t^2}{1}\right) + \left(\frac{t^1}{1}\right) = t^3 + \frac{5}{2}t^2 + t$.

Proposition 4.7 *If $|\mu| < 2g + n$, then $D^t_{g,n}(\mu_1, \ldots, \mu_n) = 0$. If $|\mu| \geqslant 2g + n$, then $D^t_{g,n}(\mu_1, \ldots, \mu_n) \in \mathbb{Q}[t]$ is a polynomial of degree $|\mu| + 1 - 2g - n$ whose coefficients are symmetric.*

Proof Consider a dessin d'enfant with genus g and n faces with degrees $\mu_1, \ldots, \mu_n$. By an Euler characteristic calculation, the number of vertices is $|\mu| + 2 - 2g - n$. Since there must be at least one black vertex and at least one white vertex, we must have have $|\mu| \geqslant 2g + n$.

We will show that, under the assumption $|\mu| \geqslant 2g + n$, there exists a dessin d'enfant with exactly one white vertex. Take a polygon with $4g + 2$ sides, whose vertices are alternately colored black and white. By identifying opposite edges, one obtains a

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dessin d'enfant on a genus g surface, with one black vertex, one white vertex, and one face with degree $2g+1$. By adding $n-1$ appropriate edges from the black vertex to the white vertex, one can obtain a genus g dessin d'enfant with one black vertex, one white vertex, and n faces with any degrees $\mu_1, \ldots, \mu_n$ satisfying $|\mu| = 2g+n$. We can relax this condition to $|\mu| \geqslant 2g+n$ by adding appropriate black vertices with degree 1 that are adjacent to the unique white vertex. Such a dessin d'enfant contributes to degree $|\mu|+1-2g-n$ in $D^t_{g,n}(\mu_1, \ldots, \mu_n)$, and there can be no contributions in higher degree.

Finally, observe that switching the colors of the vertices of a dessin d'enfant changes the number of black vertices from b to $(|\mu|+2-2g-n)-b$, leading to the desired symmetry in the coefficients of $D^t_{g,n}(\mu_1, \ldots, \mu_n)$. ■

There is an analog of the cut-and-join recursion for deformed monotone Hurwitz numbers of Theorem 3.7 in the context of the dessin d'enfant enumeration. We omit the proof, since it can be found in the literature [37].

Proposition 4.8 (Cut-and-join recursion) *Apart from the initial condition $D^t_{0,1}(1) = 1$, the dessin d'enfant enumeration satisfies*

$$\begin{aligned}\mu_1 D^t_{g,n}(\mu_1, \mu_S) = &\sum_{i=2}^{n} (\mu_1 + \mu_i - 1)\, D^t_{g,n-1}(\mu_1 + \mu_i - 1, \mu_{S\setminus\{i\}}) \\ &+ (t+1)(\mu_1 - 1)\, D^t_{g,n}(\mu_1 - 1, \mu_S) \\ &+ \sum_{\alpha+\beta=\mu_1-1} \alpha\beta \left[D^t_{g-1,n+1}(\alpha, \beta, \mu_S) + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = S}} D^t_{g_1,|I_1|+1}(\alpha, \mu_{I_1})\, D^t_{g_2,|I_2|+1}(\beta, \mu_{I_2}) \right],\end{aligned}$$

where we use the notation $S = \{2, 3, \ldots, n\}$ and $\mu_I = \{\mu_{i_1}, \mu_{i_2}, \ldots, \mu_{i_k}\}$ for $I = \{i_1, i_2, \ldots, i_k\}$.

It is well-known that dessins d'enfant correspond to pairs of permutations acting on the edges—one permutation rotates edges around black vertices and the other rotates edges around white vertices. (Equivalent descriptions in the literature often use triples of permutations that compose to give the identity.) This leads to an expression for the disconnected enumeration of dessins d'enfant in terms of characters of the symmetric group, via Frobenius's formula. For details, we recommend the book of Lando and Zonkin [38]. To incorporate the parameter t, we observe that its exponent should equal the number of cycles in the permutation that rotates edges around black vertices. As a consequence of Jucys's results described in part (a) of Proposition 2.13, we obtain the following result, for which we omit the proof.

Proposition 4.9 *The disconnected enumeration of dessins d'enfant is given by the formula*

$$D^{t\bullet}_{g,n}(\mu_1, \ldots, \mu_n) = \frac{1}{\mu_1 \cdots \mu_n} \left[\hbar^{2g-2+n}\right] \sum_{\lambda \in \mathcal{P}} \chi^\lambda_\mu\, s_\lambda\left(\tfrac{1}{\hbar}, \tfrac{1}{\hbar}, \ldots\right) \prod_{\square \in \lambda} \left(t + \hbar\, c(\square)\right).$$

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We are now in a position to prove that topological recursion governs the weighted enumeration of dessins d'enfant.[5]

Theorem 4.10 *Topological recursion on the spectral curve*

$$\mathcal{C} = \mathbb{CP}^1, \quad x(z) = \frac{1 - z + tz}{z(1-z)}, \quad y(z) = z, \quad \omega_{0,2} = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1 - z_2)^2}$$

produces correlation differentials satisfying

$$\omega_{g,n}(z_1, \ldots, z_n) = \mathrm{d}_1 \cdots \mathrm{d}_n \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} D^t_{g,n}(\mu_1, \ldots, \mu_n)\, x(z_1)^{-\mu_1} \cdots x(z_n)^{-\mu_n} + \delta_{g,0}\delta_{n,2} \frac{\mathrm{d}x(z_1)\,\mathrm{d}x(z_2)}{(x(z_1) - x(z_2))^2}.$$

Proof We again invoke the result of Bychkov, Dunin-Barkowski, Kazarian, and Shadrin [8], noting that the more restricted result of Alexandrov, Chapuy, Eynard, and Harnad [1] would suffice in this particular context. See the proof of Theorem 4.1 above for a statement of the result.

The representation-theoretic interpretation of the dessin d'enfant enumeration given in Proposition 4.9 implies that they are weighted Hurwitz numbers with the choice of data $\tilde{y}(z) = \sum_{i=1}^{\infty} z^i = \frac{z}{1-z}$ and $f(z) = t + z$. Hence, topological recursion on the spectral curve

$$\mathcal{C} = \mathbb{CP}^1, \quad x(z) = \frac{z(1-z)}{t - tz + z}, \quad y(z) = \frac{t - tz + z}{(1-z)^2}, \quad \omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1 - z_2)^2}$$

produces correlation differentials satisfying

$$\omega_{g,n}(z_1, \ldots, z_n) = \mathrm{d}_1 \cdots \mathrm{d}_n \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} D^t_{g,n}(\mu_1, \ldots, \mu_n)\, x(z_1)^{\mu_1} \cdots x(z_n)^{\mu_n} + \delta_{g,0}\delta_{n,2} \frac{\mathrm{d}x(z_1)\,\mathrm{d}x(z_2)}{(x(z_1) - x(z_2))^2}.$$

This does not yet match our spectral curve, since the enumeration is stored as an expansion at $x(z) = 0$ rather than at $x(z) = \infty$. We obtain the desired spectral curve by performing the following three transformations in order:

$$(x, y, t) \mapsto (x^{-1}, x^2 y, t), \qquad (x, y, t) \mapsto (x, y + tx, t), \qquad (x, y, t)(tx, y, t^{-1}).$$

The first transformation changes the expansion at $x(z) = 0$ to an expansion at $x(z) = \infty$; the second transformation preserves the correlation differentials [28, Section 4.2]; and the third transformation preserves the coefficients of the expansion, due to the symmetry of the coefficients of $D^t_{g,n}(\mu_1, \ldots, \mu_n)$ and the homogeneity property of topological recursion [28, Section 4.1]. Thus, we arrive at the desired result. ∎

[5] Topological recursion for a more general weighted enumeration of dessins d'enfant was previously studied by Kazarian and Zograf [37], although they obtain a different form for the spectral curve.

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Although the enumeration of dessins d'enfant has been studied previously [37], the following conjectural properties have not yet been observed in the literature, to the best of our knowledge.

Conjecture 4.11 (Real-rootedness and interlacing) *For all* $g \geqslant 0$, $n \geqslant 1$ *and* $\mu_1, \ldots, \mu_n \geqslant 1$, *the dessin d'enfant enumeration* $D^t_{g,n}(\mu_1, \mu_2, \ldots, \mu_n)$ *is a real-rooted polynomial in* t. *Furthermore, the polynomial* $D^t_{g,n}(\mu_1, \mu_2, \ldots, \mu_n)$ *interlaces each of the* n *polynomials*

$$D^t_{g,n}(\mu_1 + 1, \mu_2, \ldots, \mu_n), \quad D^t_{g,n}(\mu_1, \mu_2 + 1, \ldots, \mu_n), \quad \ldots, \quad D^t_{g,n}(\mu_1, \mu_2, \ldots, \mu_n + 1).$$

As with the deformed monotone Hurwitz numbers, we can effectively calculate $D^t_{g,n}(\mu_1, \ldots, \mu_n)$ using the cut-and-join recursion of Proposition 4.8 and explicitly check for real-rootedness and interlacing. Again, one finds a wealth of data to support Conjecture 4.11. We consider this as evidence that the real-rootedness and interlacing observed for deformed monotone Hurwitz numbers is not simply a sporadic artefact, but may be a more widespread phenomenon with a deep reason underlying it.

Remark 4.12. As previously mentioned, dessins d'enfant correspond to pairs of permutations acting on the edges, or equivalently, triples of permutations that compose to give the identity. It follows from Proposition 2.13 that each permutation has a unique *strictly monotone factorization*, defined analogously to a monotone factorization in Definition 2.10, but with the stronger monotonicity requirement that $b_1 < b_2 < \cdots < b_r$. Using this result on the permutation that rotates edges around black vertices leads to the fact that the enumeration of dessins d'enfant can be interpreted as strictly monotone Hurwitz numbers.

A Data

A.1 Deformed monotone Hurwitz numbers

The deformed monotone Hurwitz numbers can be computed using the cut-and-join recursion or the topological recursion. These were both implemented in SageMath to produce the following table [48].

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$(\mu_1, \ldots, \mu_n)$	$\mu_1 \cdots \mu_n \vec{H}^t_{0,n}(\mu_1, \ldots, \mu_n)$	$\mu_1 \cdots \mu_n \vec{H}^t_{1,n}(\mu_1, \ldots, \mu_n)$
(1)	1	0
(2)	t	t
(3)	$t^2 + t$	$5t^2 + 5t$
(4)	$t^3 + 3t^2 + t$	$15t^3 + 40t^2 + 15t$
(5)	$t^4 + 6t^3 + 6t^2 + t$	$35t^4 + 175t^3 + 175t^2 + 35t$
(6)	$t^5 + 10t^4 + 20t^3 + 10t^2 + t$	$70t^5+560t^4+1050t^3+560t^2+70t$
(7)	$t^6 + 15t^5 + 50t^4 + 50t^3 + 15t^2 + t$	$126t^6 + 1470t^5 + 4410t^4 + 4410t^3 + 1470t^2 + 126t$
(1, 1)	t	t
(2, 1)	$2t^2 + 2t$	$10t^2 + 10t$
(3, 1)	$3t^3 + 9t^2 + 3t$	$45t^3 + 120t^2 + 45t$
(2, 2)	$4t^3 + 10t^2 + 4t$	$50t^3 + 128t^2 + 50t$
(4, 1)	$4t^4 + 24t^3 + 24t^2 + 4t$	$140t^4 + 700t^3 + 700t^2 + 140t$
(3, 2)	$6t^4 + 30t^3 + 30t^2 + 6t$	$168t^4 + 792t^3 + 792t^2 + 168t$
(5, 1)	$5t^5 + 50t^4 + 100t^3 + 50t^2 + 5t$	$350t^5 + 2800t^4 + 5250t^3 + 2800t^2 + 350t$
(4, 2)	$8t^5 + 68t^4 + 128t^3 + 68t^2 + 8t$	$448t^5 + 3348t^4 + 6128t^3 + 3348t^2 + 448t$
(3, 3)	$9t^5 + 72t^4 + 138t^3 + 72t^2 + 9t$	$462t^5 + 3432t^4 + 6312t^3 + 3432t^2 + 462t$
(1, 1, 1)	$4t^2 + 4t$	$20t^2 + 20t$
(2, 1, 1)	$10t^3 + 28t^2 + 10t$	$140t^3 + 368t^2 + 140t$
(3, 1, 1)	$18t^4 + 102t^3 + 102t^2 + 18t$	$588t^4 + 2892t^3 + 2892t^2 + 588t$
(2, 2, 1)	$24t^4 + 120t^3 + 120t^2 + 24t$	$672t^4 + 3168t^3 + 3168t^2 + 672t$
(4, 1, 1)	$28t^5+268t^4+528t^3+268t^2+28t$	$1848t^5 + 14548t^4 + 27128t^3 + 14548t^2 + 1848t$
(3, 2, 1)	$42t^5+348t^4+660t^3+348t^2+42t$	$2268t^5 + 16908t^4 + 31008t^3 + 16908t^2 + 2268t$
(2, 2, 2)	$56t^5+424t^4+768t^3+424t^2+56t$	$2688t^5 + 19128t^4 + 34416t^3 + 19128t^2 + 2688t$

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$(\mu_1, \ldots, \mu_n)$	$\mu_1 \cdots \mu_n \vec{H}^t_{2,n}(\mu_1, \ldots, \mu_n)$	$\mu_1 \cdots \mu_n \vec{H}^t_{3,n}(\mu_1, \ldots, \mu_n)$
(1)	0	0
(2)	t	t
(3)	$21t^2 + 21t$	$85t^2 + 85t$
(4)	$161t^3 + 413t^2 + 161t$	$1555t^3 + 3930t^2 + 1555t$
(5)	$777t^4 + 3612t^3 + 3612t^2 + 777t$	$14575t^4 + 65505t^3 + 65505t^2 + 14575t$
(6)	$2835t^5 + 20538t^4 + 37338t^3 + 20538t^2 + 2835t$	$91960t^5 + 633160t^4 + 1132340t^3 + 633160t^2 + 91960t$
(7)	$8547t^6 + 88473t^5 + 251328t^4 + 251328t^3 + 88473t^2 + 8547t$	$443872t^6 + 4303728t^5 + 11851268t^4 + 11851268t^3 + 4303728t^2 + 443872t$
(1, 1)	t	t
(2, 1)	$42t^2 + 42t$	$170t^2 + 170t$
(3, 1)	$483t^3 + 1239t^2 + 483t$	$4665t^3 + 11790t^2 + 4665t$
(2, 2)	$504t^3 + 1278t^2 + 504t$	$4750t^3 + 11956t^2 + 4750t$
(4, 1)	$3108t^4 + 14448t^3 + 14448t^2 + 3108t$	$58300t^4+262020t^3+262020t^2+58300t$
(3, 2)	$3402t^4 + 15450t^3 + 15450t^2 + 3402t$	$61116t^4+271764t^3+271764t^2+61116t$
(5, 1)	$14175 * t^5 + 102690t^4 + 186690t^3 + 102690t^2 + 14175t$	$459800t^5 + 3165800t^4 + 5661700t^3 + 3165800t^2 + 459800t$
(4, 2)	$16296t^5+114256t^4+205376t^3+114256t^2+16296t$	$499136t^5 + 3377976t^4 + 6005536t^3 + 3377976t^2 + 499136t$
(3, 3)	$16443t^5+115344t^4+207666t^3+115344t^2+16443t$	$500544t^5 + 3389664t^4 + 6029784t^3 + 3389664t^2 + 500544t$
(1, 1, 1)	$84t^2 + 84t$	$340t^2 + 340t$
(2, 1, 1)	$1470t^3 + 3756t^2 + 1470t$	$14080t^3 + 35536t^2 + 14080t$
(3, 1, 1)	$12726t^4 + 58794t^3 + 58794t^2 + 12726t$	$236016t^4 + 1057824t^3 + 1057824t^2 + 236016t$
(2, 2, 1)	$13608t^4 + 61800t^3 + 61800t^2 + 13608t$	$244464t^4 + 1087056t^3 + 1087056t^2 + 244464t$
(4, 1, 1)	$72996t^5+525016t^4+952136t^3+525016t^2+72996t$	$2338336t^5 + 16041176t^4 + 28652336t^3 + 16041176t^2 + 2338336t$
(3, 2, 1)	$81774t^5 + 573456t^4 + 1031460t^3 + 573456t^2 + 81774t$	$2498496t^5 + 16913256t^4 + 30076176t^3 + 16913256t^2 + 2498496t$
(2, 2, 2)	$90552t^5 + 619728t^4 + 1104624t^3 + 619728t^2 + 90552t$	$2658656t^5 + 17760016t^4 + 31432800t^3 + 17760016t^2 + 2658656t$

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A.2 Dessin d'enfant enumeration

The weighted dessin d'enfant enumeration can be computed using the cut-and-join recursion or the topological recursion. These were both implemented in SageMath to produce the following table [48].

$(\mu_1, \ldots, \mu_n)$	$\mu_1 \cdots \mu_n D^t_{0,n}(\mu_1, \ldots, \mu_n)$	$\mu_1 \cdots \mu_n D^t_{1,n}(\mu_1, \ldots, \mu_n)$
(1)	t	0
(2)	$t^2 + t$	0
(3)	$t^3 + 3t^2 + t$	t
(4)	$t^4 + 6t^3 + 6t^2 + t$	$5t^2 + 5t$
(5)	$t^5 + 10t^4 + 20t^3 + 10t^2 + t$	$15t^3 + 40t^2 + 15t$
(6)	$t^6 + 15t^5 + 50t^4 + 50t^3 + 15t^2 + t$	$35t^4 + 175t^3 + 175t^2 + 35t$
(7)	$t^7+21t^6+105t^5+175t^4+105t^3+21t^2 + t$	$70t^5+560t^4+1050t^3+560t^2+70t$
(1, 1)	t	0
(2, 1)	$2t^2 + 2t$	0
(3, 1)	$3t^3 + 9t^2 + 3t$	$3t$
(2, 2)	$4t^3 + 10t^2 + 4t$	$2t$
(4, 1)	$4t^4 + 24t^3 + 24t^2 + 4t$	$20t^2 + 20t$
(3, 2)	$6t^4 + 30t^3 + 30t^2 + 6t$	$18t^2 + 18t$
(5, 1)	$5t^5 + 50t^4 + 100t^3 + 50t^2 + 5t$	$75t^3 + 200t^2 + 75t$
(4, 2)	$8t^5 + 68t^4 + 128t^3 + 68t^2 + 8t$	$80t^3 + 200t^2 + 80t$
(3, 3)	$9t^5 + 72t^4 + 138t^3 + 72t^2 + 9t$	$75t^3 + 198t^2 + 75t$
(6, 1)	$6t^6+90t^5+300t^4+300t^3+90t^2+6t$	$210t^4 + 1050t^3 + 1050t^2 + 210t$
(5, 2)	$10t^6 + 130t^5 + 400t^4 + 400t^3 + 130t^2 + 10t$	$250t^4 + 1150t^3 + 1150t^2 + 250t$
(4, 3)	$12t^6 + 144t^5 + 444t^4 + 444t^3 + 144t^2 + 12t$	$240t^4 + 1140t^3 + 1140t^2 + 240t$
(1, 1, 1)	$2t$	0
(2, 1, 1)	$6t^2 + 6t$	0
(3, 1, 1)	$12t^3 + 36t^2 + 12t$	$12t$
(2, 2, 1)	$16t^3 + 40t^2 + 16t$	$8t$
(4, 1, 1)	$20t^4 + 120t^3 + 120t^2 + 20t$	$100t^2 + 100t$
(3, 2, 1)	$30t^4 + 150t^3 + 150t^2 + 30t$	$90t^2 + 90t$
(2, 2, 2)	$40t^4 + 176t^3 + 176t^2 + 40t$	$80t^2 + 80t$
(5, 1, 1)	$30t^5+300t^4+600t^3+300t^2+30t$	$450t^3 + 1200t^2 + 450t$
(4, 2, 1)	$48t^5+408t^4+768t^3+408t^2+48t$	$480t^3 + 1200t^2 + 480t$
(3, 3, 1)	$54t^5+432t^4+828t^3+432t^2+54t$	$450t^3 + 1188t^2 + 450t$
(3, 3, 2)	$72t^5+528t^4+960t^3+528t^2+72t$	$480t^3 + 1176t^2 + 480t$

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A.3 Weingarten functions

The value of $\mathrm{Wg}^{\mathbf{S}}(\sigma)$ can be calculated by inverting the orthogonality relations or by the character formula. The value of $\mathrm{Wg}^{\mathbf{U}}(\sigma)$ is the leading coefficient of $\mathrm{Wg}^{\mathbf{S}}(\sigma)$, considered as a polynomial in M.

σ	$\mathrm{Wg}^{\mathbf{U}}(\sigma)$	$\mathrm{Wg}^{\mathbf{S}}(\sigma)$
$()$	1	1
(1)	$\frac{1}{N}$	$\frac{M}{N}$
$(1)(2)$	$\frac{1}{N^2-1}$	$\frac{M(MN-1)}{N(N^2-1)}$
(12)	$\frac{-1}{N(N^2-1)}$	$\frac{-M(M-N)}{N(N^2-1)}$
$(1)(2)(3)$	$\frac{N^2-2}{N(N^2-1)(N^2-4)}$	$\frac{M(M^2N^2-2M^2-3MN+4)}{N(N^2-1)(N^2-4)}$
$(12)(3)$	$\frac{-1}{(N^2-1)(N^2-4)}$	$\frac{-M(M-N)(MN-2)}{N(N^2-1)(N^2-4)}$
(123)	$\frac{2}{N(N^2-1)(N^2-4)}$	$\frac{M(M-N)(2M-N)}{N(N^2-1)(N^2-4)}$
$(1)(2)(3)(4)$	$\frac{N^4-8N^2+6}{N^2(N^2-1)(N^2-4)(N^2-9)}$	$\frac{M(M^3N^4-8M^3N^2+6M^3)}{N^2(N^2-1)(N^2-4)(N^2-9)} + \frac{M(-6M^2N^3+24M^2N+19MN^2-6M-30N)}{N^2(N^2-1)(N^2-4)(N^2-9)}$
$(12)(3)(4)$	$\frac{-1}{N(N^2-1)(N^2-9)}$	$\frac{-M(M-N)(M^2N^2-4M^2-5MN+10)}{N(N^2-1)(N^2-4)(N^2-9)}$
$(12)(34)$	$\frac{N^2+6}{N^2(N^2-1)(N^2-4)(N^2-9)}$	$\frac{M(M-N)(M^2N^2+6M^2-MN^3-6MN+4N^2-6)}{N^2(N^2-1)(N^2-4)(N^2-9)}$
$(123)(4)$	$\frac{2N^2-3}{N^2(N^2-1)(N^2-4)(N^2-9)}$	$\frac{M(M-N)(2M^2N^2-3M^2-MN^3-6MN+3N^2+3)}{N^2(N^2-1)(N^2-4)(N^2-9)}$
(1234)	$\frac{-5}{N(N^2-1)(N^2-4)(N^2-9)}$	$\frac{-M(M-N)(5M^2-5MN+N^2+1)}{N(N^2-1)(N^2-4)(N^2-9)}$

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