EQUALLY INCLINED SPHERES

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If five spheres in 3-space are such that each pair is inclined at the same non-zero angle θ, then

$$\left(\sum_{i=1}^{5} b_i\right)^2 = (4+\sec\theta)\sum_{i=1}^{5} b_i^2, \tag{1}$$

where $b_1, \ldots, b_5$ (the 'bends' (1) of the spheres) are the reciprocals of their radii. To prove this result, establish a system of rectangular cartesian coordinates (x, y, z) and let the spheres have centres (x_i, y_i, z_i) and radii $r_i = b_i^{-1}$, where $i = 1, \ldots, 5$. Then for x_5, y_5, z_5, r_5 we have the equations

$$(x_5-x_i)^2+(y_5-y_i)^2+(z_5-z_i)^2 = r_5^2+r_i^2-2r_5r_i\cos\theta \quad (i = 1, \ldots, 4), \tag{2}$$

which, on subtraction, yield three linear equations and one quadratic equation. Solving the three linear equations for x_5, y_5, z_5 and substituting, we see that the required relation is algebraic (indeed quadratic) in r_5 and hence in b_5. Since it is also symmetric in $b_1, \ldots, b_5$, it follows that it can be expressed as a polynomial relation in the elementary symmetric functions $p_1, \ldots, p_5$ in $b_1, \ldots, b_5$.

Next we consider the special case

$$b_1 = x, \quad b_2 = b_3 = b_4 = b_5 = y.$$

Then the centres of the spheres of radius y^{-1} are situated at the vertices of a regular tetrahedron of side $2y^{-1}\sin\frac{1}{2}\theta$ and of altitude $\frac{4}{3}(x^{-2}+y^{-2}-2x^{-1}y^{-1}\cos\theta)^{\frac{1}{2}}$, and these lengths are in the ratio $\sqrt{3}:\sqrt{2}$. Hence, in this case, the required relation must reduce to

$$4(x^2+y^2-2xy\cos\theta) = 3x^2(1-\cos\theta). \tag{3}$$

The quadratic character of (3) as an expression in x follows from the preceding general argument and is here irrelevant. From the fact that (3) is quadratic in x and y together, however, it follows that the required general relation, a polynomial in $p_1, \ldots, p_5$ which reduces to (3) in this special case, must be of weight 2 and so it is independent of p_i for $i > 2$. Since, in the case under consideration, we have

$$p_1 = x+4y, \quad p_2 = 4xy+6y^2, \tag{4}$$

the relation in question is obtained by first eliminating x and y from (3) and (4) and then substituting for p_1 and p_2 their general values. This yields (1) as desired.

When $\theta = \pi$ the result, first given by Soddy (1) in 1936, can be obtained even more easily by taking $b_1 = x$, $b_2 = b_3 = y$, $b_4 = b_5 = 0$, since (3) and (4) then take the form

$$(x-y)^2 = 0, \quad p_1 = x+2y, \quad p_2 = 2xy+y^2.$$

Another proof, for the two-dimensional case, can be found in § 1·5 of (2).

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Except for the case $\theta = \pi$ and the even better-known case $\theta = \frac{1}{2}\pi$, formula (1) seems not to have been noticed before. Its generalization to n dimensions, proved in the same way, is

$$\left(\sum_{i=1}^{n+2} b_i\right)^2 = (n+1+\sec\theta)\sum_{i=1}^{n+2} b_i^2, \tag{5}$$

and the subject is further generalized in (3) where it is shown, for example, that if $b_1, \ldots, b_5$ are any five real numbers satisfying (1), then $\cos\theta$ must be negative and real spheres of radius b_i^{-1} can be found which are mutually inclined at an angle θ. It is also shown that not more than $n+2$ spheres in n-space can be mutually inclined at an angle θ (real or not); the case $\theta = 0$ is a degenerate exceptional case which has been explicitly excluded.

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A NOTE ON THE VAGUE TOPOLOGY FOR MEASURES

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We refer for general background to N. Bourbaki, *Intégration*, chapters IV and V. We consider a locally compact Hausdorff space R and denote the set of continuous functions with compact support by $\mathscr{K}$. The Riesz–Markov theorem shows that there is a 1-1 correspondence between the set of regular Borel measures on R and the set of positive elements of $\mathscr{K}^*$, the topological dual of $\mathscr{K}$. Let $\{\mu_n\}$, μ be regular Borel probability measures on R. The sequence of measures $\{\mu_n\}$ is said to converge vaguely to μ if, for all $f \in \mathscr{K}$, $\int f d\mu_n \to \int f d\mu$. Thus the vague topology is that of simple convergence in $\mathscr{K}^*$. We shall call a μ-measurable set μ-quarrable if its boundary is a μ-null set.

Theorem. *Supposing R, $\{\mu_n\}$ and μ to be as above, the sequence of measures $\{\mu_n\}$ converges vaguely to μ as $n \to \infty$ if and only if $\mu_n(A)$ converges to $\mu(A)$ for all μ-quarrable sets A.*

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