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DISTRIBUTION OF ELEMENTS OF A FLOOR FUNCTION SET IN ARITHMETICAL PROGRESSIONS

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Abstract

Let $[t]$ be the integral part of the real number t. We study the distribution of the elements of the set $\mathcal{S}(x) := \{[x/n] : 1 \leqslant n \leqslant x\}$ in the arithmetical progression $\{a + dq\}_{d\geqslant 0}$. We give an asymptotic formula

$$S(x; q, a) := \sum_{\substack{m\in\mathcal{S}(x)\\ m\equiv a \pmod q}} 1 = \frac{2\sqrt{x}}{q} + O((x/q)^{1/3}\log x),$$

which holds uniformly for $x \geqslant 3$, $1 \leqslant q \leqslant x^{1/4}/(\log x)^{3/2}$ and $1 \leqslant a \leqslant q$, where the implied constant is absolute. The special case $S(x; q, q)$ confirms a recent numerical test of Heyman ['Cardinality of a floor function set', *Integers* **19** (2019), Article no. A67].

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1. Introduction

As usual, denote by $[t]$ the integral part of the real number t. Recently, Heyman [2] quantified the cardinality of the set

$$\mathcal{S}(x) := \left\{\left[\frac{x}{n}\right] : 1 \leqslant n \leqslant x\right\}.$$

By elementary arguments, he proved in [2, Theorems 1 and 2] that

$$S(x) := |\mathcal{S}(x)| = [\sqrt{4x+1}] - 1 = 2\sqrt{x} + O(1) \tag{1.1}$$

for $x \to \infty$. Subsequently [3, Theorem 1], he also investigated the number of primes in the set $\mathcal{S}(x)$, showing that as $x \to \infty$,

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$$\pi_{\mathcal{S}}(x) := \left|\left\{\left[\frac{x}{n}\right] : 1 \leqslant n \leqslant x \text{ and } \left[\frac{x}{n}\right] \text{ is prime}\right\}\right| = \frac{2\sqrt{x}}{\log\sqrt{x}} + O\Big(\frac{\sqrt{x}}{(\log x)^2}\Big). \tag{1.2}$$

This can be considered an analogue of the prime number theorem for the set $\mathcal{S}(x)$. Very recently, Ma and Wu [4] sharpened this result by proving the strong form of the prime number theorem for $\mathcal{S}(x)$:

$$\pi_{\mathcal{S}}(x) = \mathrm{Li}_{\mathcal{S}}(x) + O(\sqrt{x}\, e^{-c(\log x)^{3/5}(\log\log x)^{-1/5}}), \tag{1.3}$$

where $c > 0$ is a positive constant and

$$\mathrm{Li}_{\mathcal{S}}(x) := \int_2^{\sqrt{x}} \frac{dt}{\log t} + \int_2^{\sqrt{x}} \frac{dt}{\log(x/t)}.$$

Since $\mathcal{S}(x)$ is a very sparse subset of $[1, x] \cap \mathbb{N}$, the results (1.2) and (1.3) seem rather interesting.

In [2], Heyman also proposed a more general problem than (1.1), to study the asymptotic behaviour of the cardinality $S(x; q)$ of the set

$$\mathcal{S}(x; q) := \left\{\left[\frac{x}{n}\right] : 1 \leqslant n \leqslant x \text{ and } q \,\middle|\, \left[\frac{x}{n}\right]\right\}$$

for a fixed integer $q \geqslant 1$. Let $\psi(t) := t - [t] - \frac{1}{2}$ and $b := (\sqrt{4x+1} - 1)/2$. Heyman first showed in [2, Lemma 3] that

$$S(x; q) = \frac{4\sqrt{x}}{3q} + \sum_{r=1}^{[x/b]} \sum_{d=(x-r)/(qr)}^{x/(qr)} \left(\psi\Big(\frac{x}{dq+1}\Big) - \psi\Big(\frac{x}{dq}\Big)\right) + O(1)$$

for $x \to \infty$, and then wrote: 'Calculating various sums using Maple suggests that the double sum cannot successfully be bounded. In fact Maple suggests that the double sum is asymptotically equivalent to $2\sqrt{x}/(3q)$. If this argument is correct then

$$S(x; q) \sim \frac{2\sqrt{x}}{q} \qquad (x \to \infty), \tag{1.4}$$

as one would expect heuristically.'

The aim of this short note is to prove (1.4). In fact, we can consider a more general problem: for $1 \leqslant a \leqslant q$, study the distribution of elements of the set $\mathcal{S}(x)$ in the arithmetical progression $\{a + dq\}_{d \geqslant 0}$. For this, we define

$$\mathcal{S}(x; q, a) := \left\{\left[\frac{x}{n}\right] : 1 \leqslant n \leqslant x \text{ and } \left[\frac{x}{n}\right] \equiv a \pmod q\right\}$$

and

$$S(x; q, a) := |\mathcal{S}(x; q, a)|.$$

Our result is as follows.

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THEOREM 1.1. *Under the previous notation,*

$$S(x;q,a)=\frac{2\sqrt{x}}{q}+O((x/q)^{1/3}\log x) \tag{1.5}$$

uniformly for $x\geqslant 3$, $1\leqslant q\leqslant x^{1/4}/(\log x)^{3/2}$ *and* $1\leqslant a\leqslant q$, *where the implied constant is absolute.*

Since $S(x;q,q)=S(x;q)$, Heyman's expected result (1.4) is a special case of our (1.5) with fixed q. The result of Theorem 1.1 implies

$$S(x;q,a)\sim\frac{2\sqrt{x}}{q}$$

uniformly for $1\leqslant q=o(x^{1/4}/(\log x)^{3/2})$ and $1\leqslant a\leqslant q$. We did not try to get the best possible exponent and further improvements of the constant $\frac{1}{4}$ are possible.

It seems interesting to establish analogues of the Dirichlet theorem or more generally the Siegel–Walfisz theorem, the Brun–Titchmarsh theorem and the Bombieri–Vinogradov theorem for the set $\mathcal{S}(x)$. We shall leave these problems to another occasion.

2. Preliminary lemmas

In this section, we shall cite two lemmas, which will be needed in the next section. The first one is due to Vaaler (see [1, Theorem A.6]).

LEMMA 2.1. *Let* $\psi(t):=t-[t]-\frac{1}{2}$. *For* $x\geqslant 1$ *and* $H\geqslant 1$,

$$\psi(x)=-\sum_{1\leqslant|h|\leqslant H}\Phi\Big(\frac{h}{H+1}\Big)\frac{e(hx)}{2\pi ih}+R_H(x),$$

where $e(t):=e^{2\pi it}$, $\Phi(t):=\pi t(1-|t|)\cot(\pi t)+|t|$ *and the error term* $R_H(x)$ *satisfies*

$$|R_H(x)|\leqslant\frac{1}{2H+2}\sum_{0\leqslant|h|\leqslant H}\Big(1-\frac{|h|}{H+1}\Big)e(hx). \tag{2.1}$$

LEMMA 2.2. *For* $1\leqslant a\leqslant q$ *and* $\delta\in\{0,1\}$, *define*

$$\mathfrak{S}_\delta(D,D'):=\sum_{D<d\leqslant D'}\psi\Big(\frac{x}{dq+a+\delta}\Big).$$

If (κ,λ) *is an exponent pair, then*

$$\mathfrak{S}_\delta(D,D')\ll(x^\kappa D^{-\kappa+\lambda}q^{-\kappa})^{1/(1+\kappa)}+x^\kappa D^{-2\kappa+\lambda}q^{-\kappa}+x^{-1}D^2q \tag{2.2}$$

uniformly for $1\leqslant a\leqslant q$, $(\sqrt{x}-a)/q<D\leqslant(x/q)^{2/3}$ *and* $D<D'\leqslant 2D$, *where the implied constant depends on* (κ,λ) *at most.*

PROOF. Using Lemma 2.1, we can write

$$\mathfrak{S}_\delta(D,D')=-\frac{1}{2\pi i}(\mathfrak{S}_\delta^\flat(D,D')+\overline{\mathfrak{S}_\delta^\flat(D,D')})+\mathfrak{S}_\delta^\dagger(D,D'), \tag{2.3}$$

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where $H \leqslant D$ and

$$\mathfrak{S}_\delta^\flat(D, D') := \sum_{h \leqslant H} \frac{1}{h} \Phi\Big(\frac{h}{H+1}\Big) \sum_{D < d \leqslant D'} e\Big(\frac{hx}{dq+a+\delta}\Big),$$
$$\mathfrak{S}_\delta^\dagger(D, D') := \sum_{D < d \leqslant D'} R_H\Big(\frac{x}{dq+a+\delta}\Big).$$

Inverting the order of summations and applying the exponent pair (κ, λ) to the sum over d (see [1, page 31, Definition]), it follows that

$$\begin{aligned}\mathfrak{S}_\delta^\flat(D, D') &\ll \sum_{h \leqslant H} \frac{1}{h} \Big(\Big(\frac{xh}{D^2 q}\Big)^\kappa D^\lambda + \Big(\frac{xh}{D^2 q}\Big)^{-1}\Big) \\ &\ll x^\kappa D^{-2\kappa+\lambda} q^{-\kappa} H^\kappa + x^{-1} D^2 q. \end{aligned} \tag{2.4}$$

However, (2.1) of Lemma 2.1 allows us to derive

$$\begin{aligned}|\mathfrak{S}_\delta^\dagger(D, D')| &\leqslant \sum_{D < d \leqslant D'} \Big|R_H\Big(\frac{x}{dq+a+\delta}\Big)\Big| \\ &\leqslant \frac{1}{2H+2} \sum_{0 \leqslant |h| \leqslant H} \Big(1 - \frac{|h|}{H+1}\Big) \sum_{D < d \leqslant D'} e\Big(\frac{xh}{dq+a+\delta}\Big).\end{aligned}$$

When $h \neq 0$, as before, we apply the exponent pair (κ, λ) to the sum over d and obtain

$$\mathfrak{S}_\delta^\dagger(D, D') \ll DH^{-1} + x^\kappa D^{-2\kappa+\lambda} q^{-\kappa} H^\kappa + x^{-1} D^2 q. \tag{2.5}$$

Inserting (2.4) and (2.5) into (2.3), it follows that

$$\mathfrak{S}_\delta(D, D') \ll DH^{-1} + x^\kappa D^{-2\kappa+\lambda} q^{-\kappa} H^\kappa + x^{-1} D^2 q$$

for $H \leqslant D$. Optimising H on $[1, D]$, we obtain the required inequality (2.2). □

3. Proof of Theorem 1.1

If $[x/n] = m = dq + a$ with $0 \leqslant d \leqslant (x-a)/q$, then $x/(dq+a+1) < n \leqslant x/(dq+a)$. Thus we can write

$$\begin{aligned}S(x; q, a) &= \sum_{d \leqslant (x-a)/q} \mathbb{1}\Big(\Big[\frac{x}{dq+a}\Big] - \Big[\frac{x}{dq+a+1}\Big] > 0\Big) + O(1) \\ &= S_1(x; q, a) + S_2(x; q, a) + O(1),\end{aligned} \tag{3.1}$$

where $\mathbb{1} = 1$ if the statement is true and 0 otherwise, and

$$S_1(x; q, a) := \sum_{d \leqslant (\sqrt{x}-a)/q} \mathbb{1}\Big(\Big[\frac{x}{dq+a}\Big] - \Big[\frac{x}{dq+a+1}\Big] > 0\Big),$$
$$S_2(x; q, a) := \sum_{(\sqrt{x}-a)/q < d \leqslant (x-a)/q} \mathbb{1}\Big(\Big[\frac{x}{dq+a}\Big] - \Big[\frac{x}{dq+a+1}\Big] > 0\Big).$$

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For $d \leqslant (\sqrt{x} - a - 1)/q$,

$$\left[\frac{x}{dq+a}\right] - \left[\frac{x}{dq+a+1}\right] > \frac{x}{(dq+a)(dq+a+1)} - 1 > 0.$$

Thus,

$$S_1(x;q,a) = \frac{\sqrt{x}}{q} + O(1) \tag{3.2}$$

for $x \geqslant 3$, where the implied constant is absolute.

Next, we treat $S_2(x;q,a)$. Since $d > (\sqrt{x} - a)/q$,

$$0 < \frac{x}{dq+a} - \frac{x}{dq+a+1} = \frac{x}{(dq+a)(dq+a+1)} < 1.$$

Therefore, the quantity $[x/dq+a] - [x/dq+a+1]$ can only equal 0 or 1. However, for $d \geqslant (x/q)^{2/3}$, we have $dq + a = [x/n]$ for some $n \leqslant (x/q)^{1/3}$. Thus,

$$S_2(x;q,a) = \sum_{(\sqrt{x}-a)/q < d \leqslant (x/q)^{2/3}} \left(\left[\frac{x}{dq+a}\right] - \left[\frac{x}{dq+a+1}\right]\right) + O((x/q)^{1/3}).$$

Since

$$\left[\frac{x}{dq+a}\right] - \left[\frac{x}{dq+a+1}\right] = \frac{x}{dq+a} - \frac{x}{dq+a+1} - \psi\left(\frac{x}{dq+a}\right) + \psi\left(\frac{x}{dq+a+1}\right),$$

it follows that

$$S_2(x;q,a) = S_{2,1}(x;q,a) - S_{2,2}^{\langle 0\rangle}(x;q,a) + S_{2,2}^{\langle 1\rangle}(x;q,a) + O((x/q)^{1/3}), \tag{3.3}$$

where

$$S_{2,1}(x;q,a) := \sum_{(\sqrt{x}-a)/q < p \leqslant (x/q)^{2/3}} \left(\frac{x}{dq+a} - \frac{x}{dq+a+1}\right),$$

$$S_{2,2}^{\langle \delta\rangle}(x;q,a) := \sum_{(\sqrt{x}-a)/q < d \leqslant (x/q)^{2/3}} \psi\left(\frac{x}{dq+a+\delta}\right).$$

An elementary computation shows that, as $x \to \infty$,

$$S_{2,1}(x;q,a) = \sum_{(\sqrt{x}-a)/q < d \leqslant (x/q)^{2/3}} \frac{x}{d^2q^2} + O(1) = \frac{\sqrt{x}}{q} + O((x/q^4)^{1/3}). \tag{3.4}$$

It remains to bound $S_{2,2}^{\langle \delta\rangle}(x;q,a)$. According to [1, Theorem 3.10], $(\frac{1}{2}, \frac{1}{2})$ is an exponent pair (since $(0,1)$ is a trivial exponent pair). Thus we can take $(\kappa, \lambda) = (\frac{1}{2}, \frac{1}{2})$ in (2.2) of Lemma 2.2 to get

$$\mathfrak{S}_\delta(D, D') \ll (x/q)^{1/3} + (xD^{-1}q^{-1})^{1/2} + x^{-1}D^2q$$

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uniformly for $1 \leqslant a \leqslant q$, $(\sqrt{x}-a)/q < D \leqslant (x/q)^{2/3}$ and $D < D' \leqslant 2D$. Using the fact that $(\sqrt{x}-a)/q < D \leqslant (x/q)^{2/3}$, we easily see that the preceding inequality implies

$$\mathfrak{S}_\delta(D, D') \ll (x/q)^{1/3}$$

uniformly for $1 \leqslant a \leqslant q$, $(\sqrt{x}-a)/q < D \leqslant (x/q)^{2/3}$ and $D < D' \leqslant 2D$. From this,

$$S_{2,2}^{\langle\delta\rangle}(x; q) \ll (\log x) \max_{(\sqrt{x}-a)/q < D \leqslant (x/q)^{2/3}} |\mathfrak{S}_\delta(D, 2D)| \ll (x/q)^{1/3} \log x. \tag{3.5}$$

Inserting (3.4) and (3.5) into (3.3),

$$S_2(x; q, a) = \frac{\sqrt{x}}{q} + O((x/q)^{1/3} \log x) \tag{3.6}$$

for $x \to \infty$. Now the required result follows from (3.1), (3.2) and (3.6). □

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