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Classification of conditional measures along certain invariant one-dimensional foliations

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Abstract. Let $f : M \to M$ be a homeomorphism over a compact Riemannian manifold, ergodic with respect to a measure μ defined on the completion of the Borel σ-algebra, and $\mathcal{F}$ a f-invariant one-dimensional continuous foliation of M by C^1-leaves. Then, if f preserves a continuous $\mathcal{F}$-arc length system, then we only have three possibilities for the conditional measures of μ along $\mathcal{F}$, namely: (i) they are atomic for almost every leaf; or (ii) for almost every leaf, they are equivalent to the measure λ_x induced by the invariant arc-length system over $\mathcal{F}$; or (iii) for almost every leaf, their support is a nowhere dense, perfect subset of the leaf. Furthermore, we show that restricted to ergodic partially hyperbolic diffeomorphism with one-dimensional topological neutral center direction, we are able to eliminate the third case obtaining a dichotomy.

Key words: dynamical aspects of measure-preserving transformations, partially hyperbolic systems and dominated splittings, integration and disintegration of measures
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1. *Introduction*

Given a topological space X, $\mathcal{B}$ its Borel sigma-algebra, and μ a probability measure on X, it is well known that for any sub-sigma algebra $\mathcal{E} \subset \mathcal{B}$, μ may be disintegrated over the partition induced by $\mathcal{E}$ in the sense that we may find a system of probability measures $\{\mu_x\}_{x\in X}$ such that $\mu_x(\bigcap_{x\in E, E\in\mathcal{E}} E) = 1$, $x \mapsto \mu_x$ is Borel measurable, and $\mu(B) = \int \mu_x(B)\, d\mu$ for any $B \in \mathcal{B}$. This system is referred to as being the disintegration of μ along $\mathcal{E}$. This fact was later extended to the more general context of Lebesgue spaces, where V. Rokhlin (see [19]) proved that μ may be disintegrated over any measurable partition, that is, over any partition that is countably generated by elements of the σ-algebra of the Lebesgue space in consideration.

The mentioned theorem of Rokhlin is extensively used in dynamical systems and was further extended to more general contexts (see for example [16, 21]). In general, given a certain dynamics, in many contexts, this dynamics admits certain invariant partitions which are dynamically defined and the study of the disintegration of the invariant measures along these partitions usually yields important properties of the dynamics. In some cases, these partitions are naturally given by invariant foliations of the dynamics.

In smooth ergodic theory for example, Anosov and partially hyperbolic diffeomorphisms admit a pair of invariant foliations called stable and unstable foliations, which we will denote here by $\mathcal{F}^s$ and $\mathcal{F}^u$. In these contexts, measure disintegration techniques have been an essential tool to obtain ergodicity and rigidity of some properties such as regular conjugacy between certain C^1-close Anosov maps.

In his seminal work, Anosov [1] proved that the disintegration of the volume measure along the unstable (respectively stable) foliation of a volume preserving Anosov diffeomorphism is absolutely continuous with respect to the leaf measure. This result was generalized to the stable and unstable foliation of partially hyperbolic diffeomorphisms (see [3] for example). This is clearly not the general case for an arbitrary foliation, an example for which absolute continuity does not occur was given by Katok [12] and there are now several examples of other natures in the literature. More specifically, Katok's example shows a foliation by analytic curves of $(0, 1) \times \mathbb{R}/\mathbb{Z}$ such that there exists a full Lebesgue measure set which intersects each leaf in exactly one point. In this case, we say that the conditional measures along the leaves are atomic or that the foliation is atomic with respect to the reference measure, which in the example of Katok is the standard two-dimensional Lebesgue measure.

Atomicity and absolute continuity are two extremes among the possibilities that one could expect when studying the conditional measures along a foliation. The first one is equivalent to saying that the conditional measures are atomic measures and the last one implies that the conditional measures are absolutely continuous with respect to the Riemannian measure of the leaf, a property which is usually called leafwise absolute continuity or Lebesgue disintegration of measure. Although these are two extreme behaviors among, *a priori*, many possibilities for the disintegration of a measure, recent results have indicated that this dichotomy is more frequent than one would at first expect. In [20], Ruelle and Wilkinson proved that for certain skew product type of partially hyperbolic dynamics, if the fiberwise Lyapunov exponent is negative, then the disintegration of the preserved measure along the fibers is atomic. Later, Homburg [9] proved that for some

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examples treated in [20], one can actually prove that the disintegration is composed by only one atom per leaf. Avila *et al* [2] proved that for C^1-volume preserving perturbations of the time-1 map of geodesic flows on negatively curved surfaces, the disintegration of the volume measure along the center foliation is either atomic or absolutely continuous and that in the latter case, the perturbation should be itself the time-1 map of an Anosov flow. Although inside the class of derived from Anosov diffeomorphisms non-Lebesgue and non-atomic disintegration was given in [23], Ponce *et al* [15] exhibited an open class of volume preserving diffeomorphisms which have (mono) atomic disintegration along the center foliation and, recently, Tahzibi and Zhang [22] proved that non-hyperbolic measures of derived from Anosov diffeomorphisms on $\mathbb{T}^3$ also must have atomic disintegration along the center foliation, answering a question from [14].

In this paper, our main goal is to better understand the disintegration of an invariant measure along an invariant foliation for the dynamics without requiring hyperbolicity or partial hyperbolicity for f, but assuming that the invariant foliation has some type of metric rigidity with respect to f. In other words, we aim to investigate what are the possible characterizations of the conditional measures obtained when we disintegrate μ over a foliation $\mathcal{F}$, assuming that the behavior of f along $\mathcal{F}$ is very far from being hyperbolic.

1.1. *Statement of results.* Before stating our main result, we recall that a topological space X is called a *Cantor space* if it is homeomorphic to the standard ternary Cantor set. If X is a Cantor space with the topology induced by a certain space $N \supset X$, we will also say that X is a Cantor set or a Cantor subset of N.

It is well known that a non-empty metric space X is a Cantor space if, and only if, it is perfect, compact, and totally disconnected. We will say that a topological space X is a *σ-Cantor space* if there is a sequence of nested Cantor spaces $K_1 \subset K_2 \subset \cdots$ such that $X = \bigcup_{i=1}^{\infty} K_i$. Analogously, for $X \subset N$, we will say that X is a σ-Cantor set or σ-Cantor subset of N if it is a σ-Cantor space with the topology induced from N.

In particular, observe that in the case where $X \subset \mathbb{R}$, X is a σ-Cantor subset if and only if it is perfect and with empty interior, since the only connected sets in $\mathbb{R}$ are intervals.

Given a homeomorphism $f : M \to M$ over a compact Riemannian manifold M and $\mathcal{F}$ a f-invariant one-dimensional continuous foliation of M whose leaves are C^1 immersed submanifolds of M, an *$\mathcal{F}$-arc length system* is a continuous system of functions $\{l_x\}_{x \in M}$, where each l_x is a function defined on the equivalence classes of simple arcs of $\mathcal{F}(x)$ associating to each class a non-negative real number in a 'metric like fashion', that is, it vanishes only on degenerate arcs and is additive. Furthermore, this system is f-invariant in the sense that $l_x \circ f = l_{f(x)}$. The formal definition is presented in Definition 3.1.

The following is our first main result.

THEOREM A. *Let $f : M \to M$ be a homeomorphism over a compact Riemannian manifold, $\mathcal{F}$ be an f-invariant one-dimensional continuous foliation of M by C^1-leaves, and $\{l_x\}$ an $\mathcal{F}$-arc length system. If f is ergodic with respect to an f-invariant probability measure μ, then one of the following holds:*

(a) *the disintegration of μ along $\mathcal{F}$ is atomic;*

(b) *for almost every $x \in M$, the conditional measure on $\mathcal{F}(x)$ is equivalent to the measure λ_x defined on simple arcs of $\mathcal{F}(x)$ by*

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$$\lambda_x(\gamma([0,1])) = l_x(\gamma) \quad \textit{where } \gamma \textit{ is a simple arc;}$$

(c) *for almost every $x \in M$, the conditional measure on $\mathcal{F}(x)$ is supported in a σ-Cantor subset of $\mathcal{F}(x)$. Equivalently, in the leaf topology, this set is perfect and with empty interior.*

The existence of invariant systems of metrics was obtained in [5] for the context of transitive partially hyperbolic diffeomorphisms with topological neutral center, meaning that f and f^{-1} have Lyapunov stable center direction (see [17, §7.3.1]), i.e. given any $\varepsilon > 0$, there exists $\delta > 0$ for which, given any C^1 path γ tangent to the center direction, one has

$$\text{length}(\gamma) < \delta \Rightarrow \text{length}(f^n(\gamma)) < \varepsilon \quad \text{for all } n \in \mathbb{Z}.$$

For these diffeomorphisms, the center direction integrates to a continuous foliation $\mathcal{F}^c$ of M [18, Corollary 7.6]. In this context, we are able to improve the previous result.

THEOREM B. *Let $f : M \to M$ be a transitive C^1 partially hyperbolic diffeomorphism with one-dimensional topological neutral center direction over a compact Riemannian manifold. If f is ergodic with respect to an f-invariant probability measure μ with full support, then one of the following holds:*

(a) *the disintegration of μ along $\mathcal{F}^c$ is atomic;*

(b) *for almost every $x \in M$, the conditional measure on $\mathcal{F}^c(x)$ is equivalent to the measure λ_x defined on simple arcs of $\mathcal{F}^c(x)$ by*

$$\lambda_x(\gamma([0,1])) = l_x(\gamma) \quad \textit{where } \gamma \textit{ is a simple arc.}$$

1.2. *Organization of the paper.* In §2, we give some preliminaries on measure theory and disintegration of measures along a foliation. In §3, we introduce the definition of an $\mathcal{F}$-arc length system with respect to the dynamics in $(M, \mathcal{A}, \mu)$ and the construction of the measures induced by this system. In §4, we prove several technical lemmas concerning the continuity/measurability of certain functions such as the evaluation of conditional measures on certain balls inside the leaves of $\mathcal{F}$. Finally, in §§5 and 6, we prove Theorems A and B, respectively.

2. *Basics on conditional measures*

Throughout the paper, $(M, \mathcal{A}, \mu)$ will be a probability space, where M is a compact Riemannian manifold, with dimension at least two, μ is a non-atomic Borel measure, and $\mathcal{A}$ is a completion of the Borel σ-algebra $\mathcal{B}$ of M with respect to the measure μ. (We note that for simplicity, we are using a slight abuse of notation by also denoting the extension of the measure μ on $\mathcal{A}$ as μ.) In other words, $(M, \mathcal{A}, \mu)$ is measurably isomorphic to $([0,1], \mathcal{A}_{[0,1]}, \text{Leb}_{[0,1]})$, where $\text{Leb}_{[0,1]}$ is the standard Lebesgue measure on $[0,1]$ and $\mathcal{A}_{[0,1]}$ is the σ-algebra of Lebesgue measurable sets of $[0,1]$. In particular, whenever we state that a certain subset of M is measurable without mentioning the σ-algebra, it is implicitly understood that it is $\mathcal{A}$-measurable. We will denote by $\mu(\cdot|U)$ the restriction of μ to a subset $U \subset M$, that is, it denotes the measure given by $\mu(\cdot|U) = \mu(U)^{-1} \cdot \mu(B \cap U)$.

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Given a sub-σ-algebra $\mathcal{E} \subset \mathcal{B}$ generated by a countable family $\{E_n\}_{n\in\mathbb{N}}$, the *atom* of x is the set given by

$$[x] := \bigcap_{E\in\mathcal{E}:\, x\in E} E.$$

Since $\{E_n\}_{n\in\mathbb{N}}$ generates $\mathcal{E}$, we may also write

$$[x] = \bigcap_{x\in E_n} E_n.$$

Consequently, $[x]$ is a Borel set for every $x \in M$ and $\{[x] : x \in M\}$ is a partition of M.

Given $\mathcal{E} \subset \mathcal{B}$ a countably generated sub-σ-algebra, a family of measures $\{\mu_x\}_{x\in M}$ is called a *system of conditional measures of* μ *associated to* $\mathcal{E}$ if:

(i) the function $x \mapsto \mu_x(B)$ is $\mathcal{E}$-measurable in the sense that for every $B \in \mathcal{B}$, $x \to \mu_x(B)$ is $\mathcal{E}$-measurable;
(ii) for μ almost every $x \in X$, $\mu_x([x]) = 1$;
(iii) $\mu(B) = \int_{y\in B} \mu_y(B)\, d\mu(y)$.

As it is well known, every Borel measure μ admits a system of conditional measures with respect to any countably generated sub-σ algebra $\mathcal{E} \subset \mathcal{B}$ and such a system is essentially unique, see for example [8, Theorem 5.14].

In our context, it is usually convenient to consider systems of conditional measures along certain partitions by C^1 immersed submanifolds, as we detail in the following.

We say that $\mathcal{F}$ is a *continuous foliation of dimension m by* C^1*-leaves* if $\mathcal{F} = \{\mathcal{F}(x)\}_{x\in M}$ is a partition of M into C^1 immersed submanifolds of dimension m, such that for every $x \in M$, there exist open sets $U \subset M$, $V \subset \mathbb{R}^m$, $W \subset \mathbb{R}^{n-m}$ and a homeomorphism $\varphi : U \to V \times W$, called a *local chart*, such that for every $c \in W$, the set $\varphi^{-1}(V \times \{c\})$, which is called a *plaque* of $\mathcal{F}$, is a connected component of $L \cap U$ for a certain $L \in \mathcal{F}$. Given a foliation $\mathcal{F}$ of M, we denote by $\mathcal{F}(x)$ the element of $\mathcal{F}$ that contains x and call such elements the *leaves* of $\mathcal{F}$. Whenever U is a foliated chart, we denote by $\mathcal{F}|U$ the continuous foliation of U given by plaques of $\mathcal{F}$ restricted to U.

Given a one-dimensional continuous foliation $\mathcal{F}$ of M, consider a local chart of $\mathcal{F}$,

$$\varphi : U \to (0, 1) \times B_1^{n-1}(0),$$

we also call U a foliated box.

Let $\{\widetilde{E}_{q,k}\}$ be a countable collection of sets defined by

$$\widetilde{E}_{q,k} = (0, 1) \times B(q, 1/k) \subset \mathbb{R} \times \mathbb{R}^{n-1}, \quad q \in B_1^{n-1}(0) \cap \mathbb{Q}^{n-1},\ k \in \mathbb{N}.$$

Let $E_{q,k} = \varphi^{-1}(\widetilde{E}_{q,k})$ and $\mathcal{E} \subset \mathcal{B}$ be the sub-σ-algebra generated by the family of Borelian sets $E_{q,k}$. Notice that for every $y \in M$, the atom, $[y]$ is the connected component of $\mathcal{F}(y) \cap U$ that contains y, in this case, we also call the system of conditional measures of μ associated to $\mathcal{E}$, $\{\mu_y^U\}_{y\in U}$, *the disintegration of* μ *along* $\mathcal{F}$ *restricted to* U. We say that the disintegration of μ along $\mathcal{F}$ is *atomic* if for any local chart U, for almost every $y \in U$, there exists $a(y)$ in the plaque $\mathcal{F}|U(y)$ with $\mu_y^U(a(y)) > 0$.

As we may observe, the disintegration of μ along $\mathcal{F}$ is always done inside a local chart. However, the following well-known result states that if two local charts intersect,

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the disintegration of both local charts are equal except by a multiplicative constant when restricted to the intersection.

PROPOSITION 2.1. (See for example [7, Proposition 5.17]) *If U_1 and U_2 are domains of two local charts φ_1 and φ_2 of $\mathcal{F}$, then for almost every x, the conditional measures $\mu_x^{U_1}$ and $\mu_x^{U_2}$ coincide up to a multiplicative constant on $U_1 \cap U_2$.*

The above proposition provides a tool to extend a conditional measure (which is locally defined) to a measure on the full leaf that, when normalized on foliated charts, restores the disintegration. That is, assume x is a point that belongs to two different foliated charts U_1 and U_2, and let L_1 and L_2 be the respective plaques that contain the point x. Since by the above proposition $\mu_x^{U_1} = \alpha\mu_x^{U_2}$ at the intersection of both charts, we can extend $\mu_x^{U_1}$ to $L_1 \cup L_2$ in such a way that on L_2, the extension of $\mu_x^{U_1}$ is $\alpha\mu_x^{U_2}$. This process can be done inductively so that $\mu_x^{U_1}$ is extended to the whole leaf $\mathcal{F}(x)$. Therefore, $\mathcal{F}(x)$ have many measures that are constructed in such a way but using different plaques to start from, and there is no canonical way to choose one of them, but this is not a problem as we work with a class of measures instead.

As observed in [2], the property such as in Proposition 2.1 allows us to define a family of classes of measures $\{x \in M : \Omega_x\}$ such that:

- $\omega_x(M \setminus \mathcal{F}(x)) = 0$ for any representative ω_x of Ω_x;
- any two representatives of Ω_x are equal modulo multiplication by a constant;
- for any local chart U of M and $\{\mu_x^U\}_{x\in U}$, a disintegration of $\mu(\cdot|U)$ along $\mathcal{F}|U$, for almost every point $x \in U$,

$$\mu_x^U = \omega_x(\cdot|\mathcal{F}|U(x)),$$

where ω_x denotes a representative of Ω_x.

The family $\{\Omega_x\}_{x\in M}$ will be called a disintegration of μ along $\mathcal{F}$. Notice that considering such a family $\{\Omega_x\}_{x\in M}$ an atom in a certain leaf $\mathcal{F}(x)$ is now a well-defined concept independent of the local chart since it is a point $a \in \mathcal{F}(x)$ for which $\Omega_x(\{a\}) > 0$. Hence, in the context of an ergodic system, since the set of atoms is invariant, it has zero or full measure, which implies that, in the latter case, almost every measure of the family will be a sum of Dirac measures.

3. *Invariant arc-length and invariant metric systems*

Given $f : M \to M$ and a foliation $\mathcal{F}$ of M, we say that *f preserves $\mathcal{F}$*, or that *$\mathcal{F}$ is f-invariant* if for $x \in M$,

$$\mathcal{F}(f(x)) = f(\mathcal{F}(x)).$$

From now on, $\mathcal{F}$ will denote a continuous and f-invariant one-dimensional foliation.

By a *simple arc* γ on a leaf $\mathcal{F}(x)$, we mean a C^1 curve $\gamma : [0, 1] \to \mathcal{F}(x)$ for which $\gamma(t) \neq \gamma(s)$ for all $t \neq s$ with $(t, s) \notin \{(0, 1), (1, 0)\}$. In the space of simple arcs, we define an equivalence relation by saying that $\gamma \sim \sigma$ if σ is a reparameterization of γ. We say that a sequence of simple arcs γ_n *converges* to γ (in the C^0-topology) if γ_n converges pointwise to γ. By convention, by a *degenerate arc*, we mean a point.

The following definition is inspired by the concept of center metric given in [5].

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Definition 3.1. We will call $\{l_x\}$ an $\mathcal{F}$-arc length system if for $x \in M$, l_x is defined on the simple arcs on $\mathcal{F}(x)$ and l_x satisfies the following properties:

(1) strictly positive on the non-degenerate arcs and vanishing on degenerate arcs;
(2) $l_x(\gamma) = l_x(\sigma)$ if $\gamma \sim \sigma$;
(3) let $\gamma : [0, 1] \to \mathcal{F}(x)$ be a simple arc and $a \in (0, 1)$, then

$$l_x(\gamma[0, a]) + l_x(\gamma[a, 1]) = l_x(\gamma[0, 1]);$$

(4) let $\gamma : [0, 1] \to \mathcal{F}(x)$ be a simple arc, then

$$l_x(\gamma[0, 1]) = l_{f(x)}(f(\gamma[0, 1]));$$

(5) given a sequence of simple arcs $\gamma_n : [0, 1] \to \mathcal{F}(x_n)$, converging to a simple arc $\gamma : [0, 1] \to \mathcal{F}(x)$,

$$l_{x_n}(\gamma_n) \to l_x(\gamma) \quad \text{as } n \to +\infty.$$

In general, it is easy to give examples of systems preserving some continuous foliation of dimension one $\mathcal{F}$ and admitting some $\mathcal{F}$-arc length system.

Examples 3.2. (a) Let $M = \mathbb{T}^d$, $d \geq 2$, and $L : \mathbb{R}^d \to \mathbb{R}^d$ be a linear map given by a matrix with integer entries and for which 1 is an eigenvalue. Let v be an eigenvector associated to 1 and take $E = \mathbb{R} \cdot v$. The linear map L induces a linear function $f_L : \mathbb{T}^d \to \mathbb{T}^d$ and E induces a foliation $\mathcal{F}$ of $\mathbb{T}^d$ which is one-dimensional and f_L-invariant. Clearly, f_L is an isometry along $\mathcal{F}$. In particular, the family of standard arc-lengths on the leaves of $\mathcal{F}$ constitute an $\mathcal{F}$-arc length system. For this case, we obtain that any ergodic measure invariant by f_L must have either atomic conditionals, conditionals supported on σ-Cantor subsets of the leaves, or they must be equivalent to the Lebesgue measure on the leaves.

Question. Does there exists an ergodic measure μ preserved by some f_L whose conditional measures are supported on a σ-Cantor subset of the leaves?

(b) Let $\varphi : \mathbb{R} \times M \to M$ be any C^1 flow. The foliation $\mathcal{F}$ given by the orbits of φ is a φ_t-invariant C^1-foliation of M for any fixed $t \in \mathbb{R}$. There is a natural $\mathcal{F}$-arc length system, in this case, given by

$$l_x(\gamma) := |l| \quad \text{with } \varphi(l, \gamma(0)) = \gamma(1).$$

Assume that almost every $x \in M$ is not a periodic point of φ. Given any φ_t-ergodic invariant measure μ, it follows from [**11**, Example 7.4] that the disintegration of μ along $\mathcal{F}$ is either Lebesgue or atomic.

(c) Some skew-products also provide interesting examples. For example, take $f : \mathbb{T}^d \times S^1 \to \mathbb{T}^d \times S^1$ given by

$$f(x, y) = (g(x), R_\alpha(y)),$$

where $g : \mathbb{T}^d \to \mathbb{T}^d$ is any homeomorphism and $R_\alpha : S^1 \to S^1$ denotes a rotation of angle α. In this case, the foliation $\mathcal{F}$ whose leaves are $\{x\} \times S^1$, $x \in \mathbb{T}^d$, is f-invariant and by taking l_x on $\{x\} \times S^1$ to be given by the usual arc length on S^1, we conclude that $\{l_x\}$ is an

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$\mathcal{F}$-arc length system. In this example, it is easy to determine the measurable properties of $\mathcal{F}$ in the sense that, given a Borel g-invariant measure ν, the measure $\nu \times \lambda_{S^1}$ is f-invariant and a direct application of the Fubbini theorem shows that the disintegration of μ along $\mathcal{F}$ has the Lebesgue measures λ_{S^1} as its conditional measures. (Here, we denote by λ_{S^1} the standard Lebesgue measure on S^1.)

(d) Another more interesting case is provided by recent results of Bonatti and Zhang [5]. A C^1 diffeomorphism $f : M \to M$, on a compact Riemannian manifold M, is said to be partially hyperbolic if there is a non-trivial splitting

$$TM = E^s \oplus E^c \oplus E^u$$

such that

$$Df(x)E^\tau(x) = E^\tau(f(x)), \quad \tau \in \{s, c, u\}$$

and a Riemannian metric for which there are continuous positive functions μ, $\hat{\mu}$, ν, $\hat{\nu}$, γ, $\hat{\gamma}$ with

$$\nu(p), \hat{\nu}(p) < 1 \quad \text{and} \quad \mu(p) < \nu(p) < \gamma(p) < \hat{\gamma}(p)^{-1} < \hat{\nu}(p)^{-1} < \hat{\mu}(p)^{-1},$$

such that for any vector $v \in T_pM$,

$$\mu(p)\|v\| < \|Df(p) \cdot v\| < \nu(p)\|v\| \quad \text{if } v \in E^s(p),$$

$$\gamma(p)\|v\| < \|Df(p) \cdot v\| < \hat{\gamma}(p)^{-1}\|v\| \quad \text{if } v \in E^c(p),$$

$$\hat{\nu}(p)^{-1}\|v\| < \|Df(p) \cdot v\| < \hat{\mu}(p)^{-1}\|v\| \quad \text{if } v \in E^u(p).$$

We say that f has topological neutral center if, for any $\varepsilon > 0$, there exists $\delta > 0$ for which, given any smooth curve $\gamma : [0, 1] \to M$ with $\gamma'(t) \in E^c(\gamma(t))$, $0 \le t \le 1$, if length$(\gamma) < \delta$, then length$(f^n(\gamma)) < \varepsilon$ for all $n \in \mathbb{Z}$. For partially hyperbolic diffeomorphisms with neutral center, the center distribution E^c integrates to an f-invariant foliation $\mathcal{F}^c$ (see [18, Corollary 7.6]) called center foliation of f. In [5], the authors proved that if $f : M \to M$ is a C^1 partially hyperbolic diffeomorphism with neutral center direction, then f admits a continuous $\mathcal{F}$-arc length system. To understand the measurable properties of the center foliation preserved by such maps was one of the motivations of this work. As a consequence of our results, the disintegration of any f-invariant ergodic probability measure of such maps falls in three possible cases. When the conditional measures have full support, the second author proves in [13] the occurrence of an invariance principle. Further, if the measure is smooth, full support of the conditional measures imply the Bernoulli property for f. If, moreover, f is locally accessible, then $\mathcal{F}^c$ is as regular as the map f itself.

Definition 3.3. Given $y, z, w \in \mathcal{F}(x)$, we say that y is between z and w if there exists a simple arc $\gamma : [0, 1] \to \mathcal{F}(x)$ such that $\gamma(0) = z$, $\gamma(1) = w$, $\gamma(t) = y$ for some $t \in (0, 1)$, and

$$l_x(\gamma) = \min\{l_x(\alpha) \ : \ \alpha : [0, 1] \to \mathcal{F}(x), \ \alpha(0) = z, \ \alpha(1) = w\}.$$

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Definition 3.4. Let $\{l_x\}$ be an $\mathcal{F}$-arc length system. For $x \in M$, we define a metric d_x on $\mathcal{F}(x)$ by

$$d_x(y, z) := \min\{l_x(\gamma) \ : \ \gamma : [0, 1] \to \mathcal{F}(x) \text{ is simple with } \gamma(0) = y, \ \gamma(1) = z\}.$$

We call the family $\{d_x\}_x$ the $\mathcal{F}$-metric system associated to the $\mathcal{F}$-arc length system $\{l_x\}_x$.

Remark 3.5. By the definition of d_x, it is evident that d_x is an additive metric. Specifically, for any $y, z, w \in \mathcal{F}(x)$ such that y lies between z and w, the following holds:

$$d_x(z, w) = d_x(z, y) + d_x(y, w).$$

Moreover, the family $\{d_x\}$ is invariant under f, in the sense that

$$d_{f(x)}(f(z), f(y)) = d_x(z, y).$$

It is not true that $\{d_x\}_{x\in M}$ is continuous in the sense that we may have sequences $x_n \to x$, $y_n \to y$, with $y_n \in \mathcal{F}(x_n)$, $y \in \mathcal{F}(x)$, but $d_{x_n}(x_n, y_n) \nrightarrow d_x(x, y)$. Indeed, this happens, for example, for compact foliations where the leaves do not have uniformly bounded length. It is true, however, that this family of metrics is continuous when restricted to plaques inside local charts. We make this property more precise as follows.

Definition 3.6. Consider $\mathcal{F}$ a continuous foliation of M. A function $F : \bigcup_{x\in M} \mathcal{F}(x) \times \mathcal{F}(x) \to [0, \infty)$ will be called plaque-continuous if given any $p \in M$, there exists a local chart $p \in U$ of $\mathcal{F}$ such that for any sequences $x_n \to x$, $y_n \to y$ with $y_n \in \mathcal{F}|U(x_n)$, $x \in U$ and $y \in \mathcal{F}|U(x)$, we have

$$\lim_{n\to\infty} F(x_n, y_n) = F(x, y).$$

Any such local chart U will be called a continuity-domain of F.

Definition 3.7. We say that a family of metrics $\{d_x : x \in M\}$, where each d_x is defined on $\mathcal{F}(x)$, is plaque-continuous if $F : \bigcup_{x\in M} \mathcal{F}(x) \times \mathcal{F}(x) \to [0, \infty)$, defined by

$$F(x, y) := d_x(x, y),$$

is plaque continuous. In this case, if U is a continuity-domain of F, we will also say that U is a continuity-domain of $\{d_x\}$.

PROPOSITION 3.8. *The metric system given in Definition 3.4 is plaque-continuous.*

Proof. Let $\varphi : U \to (0, 1) \times V \subset \mathbb{R}^n$ be a local chart of $\mathcal{F}$, where $\varphi^{-1}((0, 1) \times \{c\})$, $c \in V$, are the plaques of $\mathcal{F}$ in U. For any $p \in U$, consider $\xi : W \subset U \to (0, 1) \times K \subset \mathbb{R}^n$ another local chart centered in p such that for any

$$z \in W \Rightarrow l_z(\mathcal{F}|U(z)) > 3 \cdot l_z(\mathcal{F}|W(z)).$$

This can be done by the continuity of $\{l_x\}$. In particular, for any $x \in W$, $y \in \mathcal{F}|W(x) = \xi^{-1}((0, 1), c')$, the simple curve $\gamma(t) = \xi^{-1}((1 - t)\xi(x) + t\xi(y), c')$ minimizes the l_x-length connecting x and y, that is, $d_x(x, y) = l_x(\gamma)$.

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Therefore, consider $x \in W$, $y \in \mathcal{F}|W(x) = \xi^{-1}((0,1), c')$ and sequences $x_n \in W$, $y_n \in \mathcal{F}|W(x_n) = \xi^{-1}((0,1), c_n)$ with $x_n \to x$ and $y_n \to y$. Let γ be defined as in the previous paragraph and $\gamma_n(t) := \xi^{-1}((1-t)\xi(x_n) + t\xi(y_n), c_n)$. By the convergence of the sequences, we have $\gamma_n \to \gamma$. However, by the previous discussion on the choice of the local chart W,

$$d_{x_n}(x_n, y_n) = l_{x_n}(\gamma_n) \quad \text{and} \quad d_x(x, y) = l_x(\gamma).$$

We then conclude by the continuity of $\{l_x\}$ that $\lim_{n\to\infty} d_{x_n}(x_n, y_n) = \lim_{n\to\infty} l_{x_n}(\gamma_n) = l_x(\gamma) = d_x(x, y)$.

Therefore, $\{d_x\}$ is plaque-continuous as we wanted to show. □

LEMMA 3.9. *Given any local open transversal T to $\mathcal{F}$, for any r small enough, the set*

$$S := \bigcup_{x\in T} B_{d_x}(x, r)$$

is open.

Proof. Assume that T is a local transversal associated to a local chart (U, φ). Let $r > 0$ be small enough that we have $B_{d_x}(x, r) \subset U$ for every $x \in T$. In particular, $U \setminus \overline{T}$ has two open connected components U_1 and U_2 with $\overline{U_1} \cap \overline{U_2} = \overline{T}$.

Since U is a local chart, we may consider an orientation on the $\mathcal{F}|U$-plaques. Assume that S is not open. Then, there exists $y \in S$ and a sequence $y_k \notin S$, with $y_k \to y$.

Consider $x \in T$ such that $y \in B_{d_x}(x, r)$ and denote by φ the flow on the $\mathcal{F}|U$-plaques induced by the orientation fixed before and such that

$$d_p(\varphi_t(p), p) = |t|,$$

whenever $\varphi_t(p)$ is defined. Let t_0 be such that $x = \varphi_{t_0}(y)$. As $y \in B_{d_x}(x, r)$, there exists $\delta > 0$ for which

$$\varphi_t(y) \in S, \quad t \in [t_0 - \delta, t_0 + \delta].$$

Now, by the plaque continuity and the fact that $y_k \to y$,

$$\varphi_{t_0-\delta}(y_k) \to \varphi_{t_0-\delta}(y), \quad \varphi_{t_0+\delta}(y_k) \to \varphi_{t_0+\delta}(y).$$

Observe that $\varphi_{t_0-\delta}(y)$ and $\varphi_{t_0+\delta}(y)$ belong to different connected components, thus, for k large enough, the same happens for $\varphi_{t_0-\delta}(y_k)$ and $\varphi_{t_0+\delta}(y_k)$. Since $\gamma_k := \{\varphi_t(y_k) : t \in [t_0 - \delta, t_0 + \delta]\}$ is an arc with points in the interior and in the exterior of U_1, it must intersect its boundary, namely T. This implies that $y_k \in S$ for large k, yielding a contradiction.

That is, S is open as we wanted to show. □

PROPOSITION 3.10. *Given a finite open cover $\mathcal{U}$ of M by local charts of $\mathcal{F}$, there exists $\mathfrak{r} > 0$ such that for all $x \in M$, there is $U \in \mathcal{U}$ with*

$$B_{d_x}(x, \mathfrak{r}) \subset U.$$

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Proof. For each $x \in M$, take any $U_x \in \mathcal{U}$ with $x \in U_x$. There exists $r_x > 0$ for which $B_{d_x}(x, r_x) \subset U_x$. By plaque continuity of $\{d_x\}$, there exists a neighborhood $x \in V_x \subset U_x$ for which

$$y \in V_x \Rightarrow B_{d_y}(y, r_x) \subset U_x.$$

Since M is compact, we may cover M with a finite number of neighborhoods V_{x_i}, $1 \leq i \leq l$. Take $\mathfrak{r} = \min\{r_{x_i} : 1 \leq i \leq l\}$. □

In the following, we will prove a technical result which will be used in the proof of the main theorem. Namely, we prove that the continuous translation of a measurable set along the foliation $\mathcal{F}$ is also a measurable set.

LEMMA 3.11. *There exists $t_0 > 0$ such that for every $0 \leq t \leq t_0$ and every Borel subset $A \subset M$, the set*

$$\Phi_t(A) := \{x \in M : d_x(x, A) < t\} \tag{1}$$

is a measurable set.

Proof. Let $\mathcal{U}$ be a finite cover of M by local charts that are continuity-domains of $\{d_x\}$. Consider $\mathfrak{r}$ the number given by Proposition 3.10. In particular, the family $\{U_{\mathfrak{r}/2} : U \in \mathcal{U}\}$, defined by

$$U_{\mathfrak{r}/2} = \{x \in U : d_x(x, \partial U) \geq \mathfrak{r}/2\},$$

is still a cover of M. Let $A \subset M$ be a Borel subset. Observe that

$$\Phi_t(A \cap U_{\mathfrak{r}/2}) \subset U, \quad U \in \mathcal{U},\ t < \mathfrak{r}/2.$$

We will prove that for $U \in \mathcal{U}$, the subset $\Phi_t(A \cap U_{\mathfrak{r}/2})$ is measurable.

Let $\varphi_U : U \to B_1^{n-1}(0) \times (0, 1)$ be a local chart of $\mathcal{F}$ and, inside U, consider the orientation in the plaques $\mathcal{F}(x)$ induced by the orientation in the line segments of the form $\{\overline{x}\} \times (0, 1) \subset \mathbb{R}^{n-1} \times \mathbb{R}$. This orientation induces, at each plaque, an order relation which we will denote by $\prec$ (the plaque being implicit in the context).

Now, for $s \in [-t, t]$, with $0 \leq t < \mathfrak{r}/2$ fixed, we define $\phi_s^U : U_{\mathfrak{r}/2} \to U$ by:

- for $s > 0$, $\phi_s^U(x)$ is the only point of the plaque $\mathcal{F}|U(x)$ such that $d_x(x, \phi_s^U(x)) = s$ and $x \prec \phi_s^U(x)$;
- for $s < 0$, $\phi_s^U(x)$ is the unique point of the plaque $\mathcal{F}|U(x)$ such that $d_x(x, \phi_s^U(x)) = -s$ and $\phi_s^U(x) \prec x$.

Observe that ϕ_s^U is continuous for every $|s| < t$ since U is a continuity-domain of $\{d_x\}$ and, consequently, it is a homeomorphism. Thus, $\phi_s^U(A \cap U_{\mathfrak{r}/2})$ is a measurable subset of M for every $s \in [-t, t]$.

Now, for each $1 \leq i \leq n$, take

$$\Phi_t^U(A) := \bigcup_{\substack{q \in \mathbb{Q} \\ q < t}} \phi_q^U(A \cap U_{\mathfrak{r}/2}), \quad 0 \leq t < \mathfrak{r}/2.$$

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Notice that $\Phi_t^U(A)$ is a measurable set, since each set in the countable union is measurable as we have proved before. Consequently,

$$\Phi_t(A) = \bigcup_{U \in \mathcal{U}} \Phi_t^U(A)$$

is a measurable set, as we wanted to show. □

Definition 3.12. Let $\{l_x\}$ be an $\mathcal{F}$-arc length system. Then, we have a well-defined homeomorphism

$$h_x : \mathcal{F}(x) \to F,$$

where $F = \mathbb{R}$ or $F = S^1$, $h_x(x) = 0$ (here, we are using the identification $S^1 = [0, 1]/\sim$ where $0 \sim 1$; thus, the point 0 stands for the equivalence class of 0 in S^1), and such that, for any simple arc $\gamma : [0, 1] \to \mathcal{F}(x)$,

$$l_x(\gamma[0, 1]) = \lambda(h_x(\gamma[0, 1])),$$

where λ denotes the Lebesgue measure on F. In particular, $\lambda(h_x(\gamma[0, 1]))$ is the size of the interval $h_x(\gamma[0, 1])$. We now define the measure λ_x on $\mathcal{F}(x)$ given by

$$\lambda_x = (h_x^{-1})_* \lambda.$$

Note that if $\gamma[0, 1]$ is a simple arc in $\mathcal{F}(x)$, then

$$\lambda_x(\gamma[0, 1]) = \lambda(h_x(\gamma[0, 1])) = l_x(\gamma[0, 1]).$$

Consequently, the measure λ_x is a doubling measure. (Recall that given a metric space (X, d), a measure ν on X is said to be a *doubling measure* if there exists a constant $\Omega > 0$ such that for any $x \in X$ and any $r > 0$, we have

$$\nu(B(x, 2r)) \leq \Omega \cdot \nu(B(x, r)).)$$

4. *Properties of non-atomic disintegrations over a continuous one-dimensional foliation*

The proof of the main result of this paper follows from understanding the topological structure of supp μ_x^U for a disintegration $\{\mu_x^U\}_{x \in U}$ of $\mu(\cdot|U)$ of μ on a local chart U. To this end, we first need to understand the behavior, in terms of measure theory, of the map

$$(x, r) \mapsto \mu_x^U(B_{d_x}(x, r)),$$

defined for a certain subset of $U \times \mathbb{R}$. This is the goal of this section.

For the rest of the paper, we assume the following:

- $\mathcal{F}$ is an f-invariant one-dimensional continuous foliation;
- $\mathcal{U}$ is a finite cover of M by local charts U of $\mathcal{F}$ such that $\overline{U}$ is still inside a local chart of $\mathcal{F}$;
- each $U \in \mathcal{U}$ is a continuity-domain of $\{d_x\}$;
- for each $U \in \mathcal{U}$, $\{\mu_x^U\}$ is a disintegration of $\mu(\cdot|U)$ along the plaques $\mathcal{F}|U$;
- the disintegration of μ along $\mathcal{F}$ is not atomic, in particular, for each $U \in \mathcal{U}$, there exists a subset $\mathcal{A}_U \subset U$ with $\mu(\mathcal{A}_U) = 0$ and for which

$$x \notin \mathcal{A}_U \Rightarrow \mu_x^U \text{ is not atomic;}$$

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- $\mathfrak{r} > 0$ is any constant given by Proposition 3.10;
- $\{\Omega_x\}_{x\in M}$ is a disintegration of μ along $\mathcal{F}$ and we denote ω_x any representative of Ω_x, $x \in M$.

We also fix the following notation: for any subset $X \subset M$, we denote by $\mathcal{B}_X$ the Borel σ-algebra of X given by the topology induced by that of M. It is important to observe that, by definition, for any $U \in \mathcal{U}$, the set $\mathcal{A}_U$ is $\mathcal{F}|U$-saturated in U.

LEMMA 4.1. *For each $0 < r < \mathfrak{r}$, $U \in \mathcal{U}$, $J \subset M$ Borel, and $x \in U \setminus \mathcal{A}_U$, the map*

$$y \mapsto \mu_x^U(J \cap B_{d_x}(y, r))$$

is continuous when restricted to the subset $F_r \subset \mathcal{F}|U(x)$ given by

$$F_r(x) = \{y \in \mathcal{F}|U(x) : \ B_{d_x}(y, r) \subset U\}.$$

In particular, for each $0 < r < \mathfrak{r}$, $U \in \mathcal{U}$, and $x \in U \setminus \mathcal{A}_U$, the map

$$y \mapsto \mu_x^U(B_{d_x}(y, r))$$

is continuous when restricted to $F_r \subset \mathcal{F}|U(x)$.

Proof. Take $x \in U$, $U \in \mathcal{U}$, and $J \subset M$ as in the statement. Let $y_n \to y$, $y_n, y \in F_r(x)$. We want to show that for $r > 0$,

$$\lim_{n\to\infty} \mu_x^U(J \cap B_{d_x}(y_n, r)) = \mu_x^U(J \cap B_{d_x}(y, r)).$$

Given any $k \in \mathbb{N}$, since μ_x^U is not atomic, we have that

$$\mu_x^U(\partial B_{d_x}(y, r)) = 0 \quad \text{and} \quad \mu_x^U(\partial B_{d_x}(y_n, r)) = 0 \quad \text{for all } n \in \mathbb{N},$$

where ∂B_{d_x} denotes the boundary of the set inside the leaf $\mathcal{F}(x)$. Now, let $B_n := (J \cap B_{d_x}(y_n, r))\Delta(J \cap B_{d_x}(y, r))$, where $Y\Delta Z$ denotes the symmetric difference of the sets Y and Z. From standard measure theory,

$$\limsup_{n\to\infty} \mu_x^U(B_n) \le \mu_x^U\Big(\limsup_{n\to\infty} B_n\Big).$$

Thus,

$$\limsup_{n\to\infty} \mu_x^U(B_n) \le \mu_x^U\Big(\bigcap_{m=1}^{\infty} \bigcup_{n\ge m} B_n\Big).$$

We know that $z \in \bigcap_{m=1}^{\infty} \bigcup_{n\ge m} B_n$ if, and only if, there exists $n_1 < n_2 < n_3 < \cdots$ such that $z \in B_{n_1} \cap B_{n_2} \cap \cdots$. Therefore, for each i, either:

- $z \in (J \cap B_{d_x}(y_{n_i}, r)) \setminus (J \cap B_{d_x}(y, r)) \Rightarrow z \in J, z \in B_{d_x}(y_{n_i}, r)$ and $z \notin B_{d_x}(y, r)$; or
- $z \in (J \cap B_{d_x}(y, r)) \setminus (J \cap B_{d_x}(y_{n_i}, r)) \Rightarrow z \in J, z \in B_{d_x}(y, r)$ and $z \notin B_{d_x}(y_{n_i}, r)$.

Observe that z cannot fall in the second case infinitely often since $\bigcap_i (B_{d_x}(y, r) \setminus B_{d_x}(y_{n_i}, r)) = \emptyset$. Therefore, passing to a subsequence if necessary,

$$z \in J, z \in B_{d_x}(y_{n_i}, r) \quad \text{and} \quad z \notin B_{d_x}(y, r),$$

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which implies, by taking $i \to \infty$, that $z \in J \cap \partial B_{d_x}(y, r) \subset \partial B_{d_x}(y, r)$. Consequently,

$$\limsup_{n\to\infty} \mu_x^U(B_n) \leq \mu_x^U(\partial B_{d_x}(y, r)) = 0.$$

Therefore,

$$\lim_{n\to\infty} \mu_x^U(J \cap B_{d_x}(y, r) \setminus J \cap B_{d_x}(y_n, r)) = \lim_{n\to\infty} \mu_x^U(J \cap B_{d_x}(y_n, r) \setminus J \cap B_{d_x}(y, r)) = 0$$

and, consequently,

$$\begin{aligned}\lim_{n\to\infty} \mu_x^U(J \cap B_{d_x}(y_n, r)) = \lim_{n\to\infty} (&\mu_x^U(J \cap B_{d_x}(y_n, r) \setminus J \cap B_{d_x}(y, r)) + \mu_x^U(J \cap B_{d_x}(y, r)) \\ &- \mu_x^U(J \cap B_{d_x}(y, r) \setminus J \cap B_{d_x}(y_n, r)) = \mu_x^U(J \cap B_{d_x}(y, r)),\end{aligned}$$

as we wanted to show. The second claim is a consequence of the first by taking $J = M$. □

PROPOSITION 4.2. *Let $U \in \mathcal{U}$ and $0 < r < \mathfrak{r}$. Consider (V, φ) a local chart inside U such that*

$$x \in V \Rightarrow B_{d_x}(x, r) \subset U,$$

and $J \subset M$ a Borel subset of M. Then, restricted to $V \setminus \mathcal{A}_U$, the map

$$x \mapsto \mu_x^U(J \cap B_{d_x}(x, r))$$

is $\mathcal{B}_{V\setminus\mathcal{A}_U}$-measurable and, consequently, $\mathcal{B}_{U\setminus\mathcal{A}_U}$-measurable as $V \subset U$. In particular, the map

$$x \in V \setminus \mathcal{A}_U \mapsto \mu_x^U(B_{d_x}(x, r))$$

is $\mathcal{B}_{U\setminus\mathcal{A}_U}$-measurable.

Proof. If x and y belong to the same $\mathcal{F}$-plaque in U, then $\mu_x^U = \mu_y^U$. We already know that for all Borel subset $W \subset M$,

$$x \in V \mapsto \mu_x^U(W \cap U)$$

is Borel measurable.

Consider the local chart be given by the homeomorphism $\varphi : V \to (0, 1) \times G$, where $G \subset \mathbb{R}^{n-1}$ is an open subset. Setting $g_r : V \to [0, \infty)$ as

$$g_r(x) = \mu_x^U(J \cap B_{d_x}(x, r)),$$

we have

$$g_r \circ \varphi^{-1}(x_1, x_2) = \mu^U_{\varphi^{-1}(x_1,x_2)}(J \cap B_{d_{\varphi^{-1}(x_1,x_2)}}(\varphi^{-1}(x_1, x_2), r)).$$

Let us prove that this function is continuous in x_1 and Borel measurable in x_2.

By Lemma 4.1, restricted to $V \setminus \mathcal{A}_U$, we already have the continuity of $g_r \circ \varphi^{-1}$ the first coordinate, since the second coordinate being fixed means we are evaluating the function on a single plaque where the conditional measure is non-atomic. Now, fix the first coordinate x_1 and consider the transversal $T = \{x_1\} \times B_1^{n-1}(0)$. By Lemma 3.9, the set

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$$S := \bigcup_{x \in \varphi^{-1}(T)} B_{d_x}(x, r)$$

is an open subset of M and, consequently, $S \cap J = \bigcup_{x \in \varphi^{-1}(T)} J \cap B_{d_x}(x, r)$ is Borel. Thus, $y \mapsto \mu_y^U(S \cap J)$ is a Borel measurable function in V, which implies that $g_r \circ \varphi^{-1}(x_1, \cdot)$ is $\mathcal{B}_T$-measurable. In particular, its restriction to $T \cap \varphi(V \setminus \mathcal{A}_U) = T \setminus \varphi(\mathcal{A}_U)$ is a $\mathcal{B}_{T \setminus \varphi(\mathcal{A}_U)}$-measurable map. However, observe that for $x \in T$,

$$\mu_x^U(S \cap J) = \mu_x^U(J \cap B_{d_x}(x, r)).$$

Therefore, for x_1 fixed, the map $x_2 \in G \setminus \pi_2(\varphi(\mathcal{A}_U)) \mapsto \mu^U_{\varphi^{-1}(x_1,x_2)}(J \cap B_{d_{\varphi^{-1}(x_1,x_2)}}(\varphi^{-1}(x_1, x_2), r))$ is $\mathcal{B}_{G \setminus \pi_2(\varphi(\mathcal{A}_U))}$-measurable, where $\pi_2 : (0, 1) \times G \mapsto G$ is the projection onto the second coordinate.

Consequently, $g_r \circ \varphi^{-1}$ restricted to $((0, 1) \times G) \setminus \varphi(\mathcal{A}_U)$ is a jointly measurable function with respect to the product σ-algebra $\mathcal{B}_{(0,1)} \times \mathcal{B}_{G \setminus \pi_2(\varphi(\mathcal{A}_U))}$ (see for example [6, Lemma 4.51]). As φ is a homeomorphism, we conclude that g_r is $\mathcal{B}_{V \setminus \mathcal{A}_U}$-measurable, as we wanted to show. □

In the following lemma, we prove that the subset of M, consisting of all points $x \in M$ for which there is a ball in $\mathcal{F}(x)$ with null μ_x^U measure, is a relatively Borel set.

LEMMA 4.3. *For each $U \in \mathcal{U}$, the set*

$$\mathcal{Z}_U = \bigcup_{x \in U \setminus \mathcal{A}_U} \mathcal{F}|U(x) \setminus \mathit{supp}\ \mu_x^U$$

is a $\mathcal{B}_{U \setminus \mathcal{A}_U}$-measurable set, meaning $\mathcal{Z}_U = V \cap (U \setminus \mathcal{A}_U)$ for some open set V.

Proof. First, let us give a better formulation for the definition of $\mathcal{Z}_U$. Observe that

$$\mathcal{Z}_U = \{x \in U \setminus \mathcal{A}_U : \mu_x^U(I) = 0 \text{ for some open ball } x \in I \subset \mathcal{F}(x)\}.$$

Consider $\{q_1, q_2, \ldots\}$ an enumeration of $\mathbb{Q} \cap [0, 1]$ and let $\mathcal{U}$ be the given finite family of local charts covering M. For each $U \in \mathcal{U}$ and $i \in \mathbb{N}$, define $\phi_i^U : U_i \setminus \mathcal{A}_U \to \mathbb{R}$ by

$$\phi_i^U(x) = \mu_x^U(B_{d_x}(x, q_i)),$$

where $U_i = \{x \in U : B_{d_x}(x, q_i) \subset U\}$. Observe that we may cover U_i with a countable number of local charts $V_i^j \subset U_i$, $j \in \mathbb{N}$ and, by Proposition 4.2, $\phi_i^U | V_i^j$ is $\mathcal{B}_{V_i^j \setminus \mathcal{A}_U}$-measurable for every j. In particular, ϕ_i^U is $\mathcal{B}_{U_i \setminus \mathcal{A}_U}$-measurable for every i. Therefore, we have that $\mathcal{Z}_i^U := (\phi_i^U)^{-1}(\{0\}) \subset M$ is a $\mathcal{B}_{U_i \setminus \mathcal{A}_U}$-measurable subset, in particular, a $\mathcal{B}_{U \setminus \mathcal{A}_U}$-measurable subset. It is not difficult to see that

$$\mathcal{Z}_U = \bigcup_{i=1}^{\infty} \mathcal{Z}_i^U. \tag{2}$$

Therefore, $\mathcal{Z}_U$ is a $\mathcal{B}_{U \setminus \mathcal{A}_U}$-measurable subset as we wanted to show. Moreover, $\mu(\mathcal{Z}_U) = 0$. □

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Consider $\mathcal{P}$ the set given by

$$\mathcal{P} = \{x : \text{there exists } U, V \in \mathcal{U},\ x \in U \cap V,\ \mu_x^U(\cdot|U \cap V) \nsim \mu_x^V(\cdot|U \cap V)\},$$

that is, $\mathcal{P}$ is the set of points x for which there exists two local charts U and V in $\mathcal{U}$, both containing x, where the respective conditional measures at the plaque of x, μ_x^U and μ_x^V, are not equivalent on the intersection $\mathcal{F}|U(x) \cap \mathcal{F}|V(x)$. In particular, this set has zero measure by Proposition 2.1. Set $\widetilde{M} := M \setminus (\bigcup_{U \in \mathcal{U}}(\mathcal{Z}_U \cup \mathcal{A}_U) \cup \mathcal{P})$. Let

$$M_0 := \bigcap_{n \in \mathbb{Z}} f^n(\widetilde{M}). \tag{3}$$

As $\mu(\widetilde{M}) = 1$, we have $\mu(M_0) = 1$. For each $x \in M_0$, we will denote by μ_x the measure on $B_{d_x}(x, \mathfrak{r})$ given by the conditional measure μ_x^U for some $U \in \mathcal{U}$ with $x \in U$, normalized to give weight exactly one to $B_{d_x}(x, \mathfrak{r})$, that is, for a measurable $F \subset \mathcal{F}|U(x)$,

$$\mu_x(F) = \mu_x^U(F|B_{d_x}(x, \mathfrak{r})). \tag{4}$$

Given any $y \in B_{d_x}(x, \mathfrak{r}) \cap M_0$, the measures μ_y and μ_x are proportional to each other at the intersection $B_{d_x}(x, \mathfrak{r}) \cap B_{d_y}(y, \mathfrak{r})$ by Proposition 2.1, that is, there exists a constant β for which $\mu_y = \beta \cdot \mu_x$ is restricted to $B_{d_x}(x, \mathfrak{r}) \cap B_{d_y}(y, \mathfrak{r})$.

In particular, evaluating both sides at $B_{d_x}(x, \mathfrak{r}) \cap B_{d_x}(y, \mathfrak{r})$,

$$\beta \cdot \mu_x(B_{d_x}(x, \mathfrak{r}) \cap B_{d_x}(y, \mathfrak{r})) = \mu_y(B_{d_x}(x, \mathfrak{r}) \cap B_{d_x}(y, \mathfrak{r}))$$

$$\Rightarrow \beta = \frac{\mu_y(B_{d_x}(x, \mathfrak{r}) \cap B_{d_x}(y, \mathfrak{r}))}{\mu_x(B_{d_x}(x, \mathfrak{r}) \cap B_{d_x}(y, \mathfrak{r}))}.$$

COROLLARY 4.4. *For each $0 < r < \mathfrak{r}$, $x \in M_0$, and $J \subset M$ Borel, the map*

$$y \in B_{d_x}(x, \mathfrak{r}) \cap M_0 \mapsto \mu_y(J \cap B_{d_x}(y, r))$$

is continuous.

Proof. For a certain $0 < r < \mathfrak{r}$ fixed, take any $x \in M_0$. Let $y \in B_{d_x}(x, \mathfrak{r}) \cap M_0$ and $U \in \mathcal{U}$ with $B_{d_x}(y, \mathfrak{r}) \subset U$, take $y_n \in B_{d_x}(x, \mathfrak{r}) \cap M_0$ with $y_n \to y$ as $n \to \infty$ and $B_{d_x}(y_n, \mathfrak{r}) \subset U$.

By definition,

$$\mu_y = \mu_y^U(\cdot|B_{d_x}(y, \mathfrak{r})),\ \mu_{y_n} = \mu_{y_n}^U(\cdot|B_{d_x}(y_n, \mathfrak{r})).$$

Therefore, for $n \in \mathbb{N}$ such that $B_{d_x}(y, r) \subset B_{d_x}(y_n, \mathfrak{r})$ and $B_{d_x}(y_n, r) \subset B_{d_x}(y, \mathfrak{r})$,

$$\mu_{y_n}(J \cap B_{d_x}(y_n, r)) = \frac{\mu_{y_n}^U(J \cap B_{d_x}(y_n, r))}{\mu_{y_n}^U(B_{d_x}(y_n, \mathfrak{r}))} = \frac{\mu_y^U(J \cap B_{d_x}(y_n, r))}{\mu_y^U(B_{d_x}(y_n, \mathfrak{r}))}. \tag{5}$$

By Lemma 4.1, we have $\mu_y^U(J \cap B_{d_x}(y_n, r)) \to \mu_y^U(J \cap B_{d_x}(y, r))$ and $\mu_y^U(B_{d_x}(y_n, \mathfrak{r})) \to \mu_y^U(B_{d_x}(y, \mathfrak{r}))$ as $n \to \infty$. Therefore, $\mu_{y_n}(J \cap B_{d_x}(y_n, r)) \to \mu_y(J \cap B_{d_x}(y, r))$ as we wanted to show. □

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COROLLARY 4.5. *For each $x \in M_0$ and $J \subset M$ Borel, the map*

$$r \in [0, \mathfrak{r}] \mapsto \mu_x(J \cap B_{d_x}(x, r))$$

is continuous. Furthermore, the map

$$(x, r) \in M_0 \times [0, \mathfrak{r}] \mapsto \mu_x(J \cap B_{d_x}(x, r)) \tag{6}$$

is jointly measurable. In particular,

$$(x, r) \in M_0 \times [0, \mathfrak{r}] \mapsto \mu_x(B_{d_x}(x, r)) \tag{7}$$

is jointly measurable.

Proof. Let $x \in M_0$. First, let us prove that $r \in [0, \mathfrak{r}] \mapsto \mu_x(J \cap B_{d_x}(x, r))$ is a continuous function. Let $0 = r < \mathfrak{r}$ and $r_n \in [0, \mathfrak{r}] \searrow r$ (if $r = \mathfrak{r}$, the argument is analogous). Hence,

$$\mu_x(J \cap B_{d_x}(x, r_n)) = \mu_x(J \cap B_{d_x}(x, r)) + \mu_x(J \cap B_{d_x}(x, r_n) \setminus J \cap B_{d_x}(x, r)).$$

As μ_x is non-atomic,

$$\lim_{n\to\infty} \mu_x(B_{d_x}(x, r_n) \setminus B_{d_x}(x, r)) = 0.$$

Then,

$$\mu_x(J \cap B_{d_x}(x, r_n) \setminus J \cap B_{d_x}(x, r)) = \mu_x(J \cap (B_{d_x}(x, r_n) \setminus B_{d_x}(x, r))) \to 0,$$

showing the first part of the statement.

Let us show the second part of the statement. For each $x \in M_0$, let $x \in V_x$ be a local chart with

$$y \in V_x \Rightarrow B_{d_y}(y, \mathfrak{r}) \subset U_x \quad \text{for some } U_x \in \mathcal{U}.$$

As M is compact, we may cover M with a finite number of such local charts, say $V_1, V_2, \ldots, V_l$ and call $U_1, U_2, \ldots, U_l$ the associated local charts in $\mathcal{U}$. For any j, consider

$$y \in V_j \mapsto \mu_y(J \cap B_{d_y}(y, r)).$$

Observe that

$$\mu_y(J \cap B_{d_y}(y, r)) = \frac{\mu_y^{U_j}(J \cap B_{d_y}(y, r))}{\mu_y^{U_j}(B_{d_y}(y, \mathfrak{r}))}.$$

Therefore, by Proposition 4.2, $y \in V_j \cap M_0 \mapsto \mu_y(J \cap B_{d_y}(y, r))$ is a $\mathcal{B}_{U_j \setminus \mathcal{A}_{U_j}}$-measurable map. As j is arbitrary, $y \in M_0 \mapsto \mu_y(J \cap B_{d_x}(y, r))$ is a $\mathcal{B}_{U \cap M_0}$-measurable map.

Thus, the map given by (6) is jointly measurable as it is continuous in the first coordinate and $\mathcal{B}_{M_0}$-measurable in the second. The second conclusion follows from taking $J = M$. □

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5. *Proof Theorem A*

First of all, we can assume that (M, μ) is an atom-less probability space and that the disintegration of μ along $\mathcal{F}$ is not atomic, otherwise, there would be nothing to do. We also consider the same objects fixed in the beginning of §4.

Now, we define the distortion of the disintegration measure of μ relative to the metric system in each leaf $\mathcal{F}(x)$ by the following definition.

Definition 5.1. Let (M, μ) be a probability space and $\mathcal{F}$ be a continuous one-dimensional foliation of M. Let $\{\mu_x\}$ be the system of conditional measures along $\mathcal{F}$ given by (4) for $x \in M_0$ (see Lemma 4.3 and recall that $\mu(M_0) = 1$), and let $d = \{d_x\}$ be the $\mathcal{F}$-metric system induced by the arc-length system $\{l_x\}$ as in Definition 3.4. We define the μ-distortion of the $\mathcal{F}$-metric system by

$$\Delta(x) = \begin{cases} \limsup_{\varepsilon \to 0} \dfrac{\mu_x(B_{d_x}(x,\varepsilon))}{2\varepsilon} := \lim_{n\to\infty} \left(\sup_{\varepsilon \le 1/n} \dfrac{\mu_x(B_{d_x}(x,\varepsilon))}{2\varepsilon} \right) & \text{if } x \in M_0, \\ 0 & \text{if } x \notin M_0. \end{cases}$$

Recall that $B_{d_x}(x, \varepsilon)$ is the ball inside $\mathcal{F}(x)$, centered in the point x and with radius ε with respect to the metric d_x.

LEMMA 5.2. *The map $x \in M \mapsto \Delta(x)$ is measurable.*

Proof. Clearly, it is sufficient to check the measurability of $\Delta(x)$ in M_0. By Corollary 4.5, for any fixed $x \in M_0$, the function $w(x, \cdot) : (0, \mathfrak{r}) \to (0, \infty)$ is continuous. Thus, for n small enough,

$$\sup_{\varepsilon \le 1/n} \frac{\mu_x(B_{d_x}(x,\varepsilon))}{2\varepsilon} = \sup_{r \in \mathbb{Q} \cap (0,1/n]} \frac{\mu_x(B_{d_x}(x,r))}{2r} \quad \text{for all } x \in M_0.$$

For each $n \in \mathbb{N}$, the map $L_n : M_0 \to \mathbb{R}$, $x \in M_0 \mapsto \sup_{r \in \mathbb{Q} \cap (0,1/n]} (\mu_x(B_{d_x}(x,r)))/2r$ is measurable, as it is defined by the supremum over a countable number of measurable maps. Consequently, since $\Delta(x) = \lim_{n\to\infty} L_n(x)$ for all $x \in M_0$, we conclude that $\Delta(x)$ is measurable as we wanted. □

Observe that, *a priori*, $\Delta(x)$ is a measurable function, but it is not immediately true that $\Delta(x) < \infty$ for μ-almost every x. Also note that

$$f_*\mu_x = \mu_{f(x)} \quad \text{and} \quad f(B_{d_x}(x,\varepsilon)) = B_{d_{f(x)}}(f(x),\varepsilon),$$

since

$$d_{f(x)}(f(x), f(y)) = d_x(x, y),$$

we conclude that $\Delta(x)$ is an f-invariant map. By ergodicity of f, it follows that $\Delta(x)$ is constant almost everywhere, let us call that constant by Δ, that is, for almost every x,

$$\Delta(x) = \Delta. \tag{8}$$

Let $\widehat{M} \subset M_0$ be a Borel f-invariant full measure set of points x for which (8) occurs.

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5.1. *Technical lemmas for the case* $\Delta = \infty$.

LEMMA 5.3. *If* $\Delta = \infty$, *there exists a sequence* $\varepsilon_k \to 0$, *as* $k \to +\infty$, *and a full measure subset* $R^\infty \subset \widehat{M}$ *such that:*

(i) R^∞ *is f-invariant;*

(ii) *for all* $x \in R^\infty$,

$$\frac{\mu_x(B_{d_x}(x, \varepsilon_k))}{2\varepsilon_k} \geq k. \tag{9}$$

Proof. Let $k \in \mathbb{N}^*$ arbitrary. Since $\Delta(x) = \Delta$ for every $x \in \widehat{M}$, define

$$\varepsilon_k(x) := \sup\left\{\varepsilon \leq 1 : \frac{\mu_x(B_{d_x}(x, \varepsilon))}{2\varepsilon} \geq k\right\}, \quad x \in \widetilde{M}.$$

CLAIM 1. *The function* $\varepsilon_k(x)$ *is measurable for all* $k \in \mathbb{N}$.

Proof. Define

$$w(x, \varepsilon) = \frac{\mu_x(B_{d_x}(x, \varepsilon))}{2\varepsilon}.$$

By Corollary 4.5, for any $x \in M_0$, the function $w(x, \cdot) : (0, \mathfrak{r}) \to (0, \infty)$ is continuous and, for $0 < \varepsilon < \mathfrak{r}$ fixed, the function $w(\cdot, \varepsilon) : M_0 \to (0, \infty)$ is a measurable function by Proposition 4.2. Given any $k \in \mathbb{N}$, $\beta > 0$, the continuity of $w(x, \cdot)$ implies that

$$\begin{aligned}\varepsilon_k^{-1}((0, \beta)) &= \{x : \varepsilon_k(x) \in (0, \beta)\} \\ &= \bigcap_{\beta \leq r \leq 1} w(\cdot, r)^{-1}([0, k)) \\ &= \bigcap_{\beta \leq r \leq 1, r \in \mathbb{Q}} w(\cdot, r)^{-1}([0, k)).\end{aligned}$$

Therefore, $\varepsilon_k^{-1}((0, \beta))$ is measurable, as it is a countable intersection of measurable subsets of M_0, and consequently, ε_k is a measurable function for every k. □

Note that $\varepsilon_k(x)$ is f-invariant. Thus, by ergodicity, for every $k \in \mathbb{N}$, the function ε_k is constant almost everywhere, let R_k^∞ be a full measure set such that $\varepsilon_k(x)$ is constant equal to ε_k. It is easy to see that the sequence ε_k goes to 0 as k goes to infinity. Take $\widetilde{R}^\infty := \bigcap_{k=1}^{+\infty} R_k^\infty$. Since each R_k^∞ has full measure, $\widetilde{R}^\infty$ has full measure and clearly satisfies what we want for the sequence $\{\varepsilon_k\}_k$. Finally, take $R^\infty = \bigcap_{i \in \mathbb{Z}} f^i(\widetilde{R}^\infty)$. The set R^∞ is f-invariant, has full measure and satisfies items (i) and (ii). □

We now set, for each $U \in \mathcal{U}$, $x \in U \setminus (\mathcal{Z}_U \cup \mathcal{A}_U)$,

$$\begin{aligned}&\Pi_{x,U}^\infty \\ &:= \left\{y \in \mathcal{F}|U(x) \setminus \mathcal{Z}_U : \frac{1}{2\varepsilon_k} \cdot \frac{\mu_y^U(B_{d_x}(y, \varepsilon_k))}{\mu_y^U(B_{d_x}(y, \mathfrak{r}))} \geq k \text{ for all } k \text{ with } B_{d_x}(y, \varepsilon_k) \subset U\right\}\end{aligned}$$

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and

$$\Pi_U^\infty := \bigcup_{x\in U\setminus(\mathcal{Z}_U\cup\mathcal{A}_U)} \Pi_{x,U}^\infty.$$

Observe that if $x \in R^\infty$, then $x \in \Pi_{x,U}^\infty$; therefore, $R^\infty \cap U \subset \Pi_U^\infty$. In particular, $U \setminus \Pi_U^\infty \subset U \setminus R^\infty$. Since $\mu(R^\infty) = 1$, then Π_U^∞ is measurable. Also, we can clearly assume that ε_k is strictly decreasing and $\varepsilon_1 < \mathfrak{r}$.

LEMMA 5.4. *For every $x \in R^\infty \cap U$, consider $\delta = \delta(x) > 0$ for which*

$$B_{d_x}[x, 2\cdot\delta + \mathfrak{r}] \subset U.$$

The set $\Pi_{x,U}^\infty \cap B_{d_x}[x,\delta]$ is a closed subset on the plaque $\mathcal{F}|U(x)$.

Proof. Let $y_n \to y$, $y_n \in \Pi_{x,U}^\infty \cap B_{d_x}[x,\delta]$, $y \in \mathcal{F}|U(x)$. In particular, $B_{d_x}(y_n,\mathfrak{r}) \subset B_{d_x}(x,\delta+\mathfrak{r}) \subset U$ and by taking the limit over n, we also have $B_{d_x}(y,\mathfrak{r}) \subset B_{d_x}(x,\delta+\mathfrak{r}) \subset U$. Furthermore, it is clear that $y \in B_{d_x}[x,\delta]$ since this is a closed set. By Lemma 4.1, for each $k \in \mathbb{N}$, the map

$$y \in B_{d_x}[x,\delta] \subset F_{\varepsilon_k}(x) \mapsto \mu_y^U(B_{d_x}(y,\varepsilon_k))$$

is continuous and the same holds for

$$y \in B_{d_x}[x,\delta] \subset F_{\mathfrak{r}}(x) \mapsto \mu_y^U(B_{d_x}(y,\mathfrak{r})).$$

Thus,

$$\lim_{n\to\infty} \frac{\mu_{y_n}^U(B_{d_x}(y_n,\varepsilon_k))}{\mu_{y_n}^U(B_{d_x}(y_n,\mathfrak{r}))} = \frac{\mu_y^U(B_{d_x}(y,\varepsilon_k))}{\mu_y^U(B_{d_x}(y,\mathfrak{r}))}, \quad k \geq 1,$$

which implies that for all $k \geq 1$,

$$\frac{\mu_y^U(B_{d_x}(y,\varepsilon_k))}{2\varepsilon_k\cdot\mu_y^U(B_{d_x}(y,\mathfrak{r}))} = \lim_{n\to\infty} \frac{\mu_{y_n}^U(B_{d_x}(y_n,\varepsilon_k))}{2\varepsilon_k\cdot\mu_{y_n}^U(B_{d_x}(y_n,\mathfrak{r}))} \geq k,$$

that is, $y \in \Pi_{x,U}^\infty$ as we wanted. □

Now, we consider the following sets:

$$D_U^\infty := \mathcal{F}|U(\Pi_U^\infty) \setminus (\mathcal{F}|U)(\mathcal{Z}_\mathcal{U}).$$

We claim that D_U^∞ is an $\mathcal{A}$-measurable subset. In fact, consider the natural projection $\pi : U \to U/\mathcal{F}$, as U is an open subset of a manifold (in particular, it is a Polish space), we have that $U/\mathcal{F}$ is a Polish space with the quotient topology. By Lemma 4.3, $\mathcal{Z}_U = V \cap (U \setminus \mathcal{A}_U)$ for some Borel set V. Since $U \setminus \mathcal{A}_U$ is $\mathcal{F}|U$-saturated,

$$\mathcal{F}|(\mathcal{Z}_U) = \pi^{-1}(\pi(V)) \cap (U \setminus \mathcal{A}_U).$$

Note that $\pi(V)$ is a Souslin set by [4, Corollary 1.10.9]. (A subset of a Polish space Y is called a Souslin set, or an analytical set, if it is the image of a Polish space X by a continuous map from X to Y.) Consequently, $\pi^{-1}(\pi(V))$ is measurable in the σ-algebra

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$\mathcal{A}$, which we recall is the completion of the Borel sigma algebra $\mathcal{B}$, implying that $\mathcal{F}|(\mathcal{Z}_U)$ is $\mathcal{A}$-measurable.

Since $R^\infty \cap U \subset \mathcal{F}|U(\Pi_U^\infty)$ and $\mu(R^\infty) = 1$, we have that $\mathcal{F}|U(\Pi_U^\infty)$ is an $\mathcal{A}$-measurable subset of U, this implies that $D_U^\infty = \mathcal{F}|U(\Pi_U^\infty) \setminus \mathcal{F}|U(\mathcal{Z}_U)$ is an $\mathcal{A}$-measurable set as we wanted to show.

Since D_U^∞ is an $\mathcal{A}$-measurable set, by ergodicity of f, the f-invariant set

$$D^\infty := \bigcup_{n\in\mathbb{Z}, U\in\mathcal{U}} f^n\Big(\bigcup D_U^\infty\Big)$$

must satisfy either $\mu(D^\infty) = 0$ or $\mu(D^\infty) = 1$.

5.2. *Technical lemmas for the case $\Delta < \infty$.*

LEMMA 5.5. *If $\Delta < \infty$, there exists a sequence $\varepsilon_k \to 0$, as $k \to +\infty$, and a full measure subset $R \subset \widehat{M}$ such that:*

(i) *R is f-invariant;*

(ii) *for every $x \in R$, then*

$$\left|\frac{\mu_x(B_{d_x}(x, \varepsilon_k))}{2\varepsilon_k} - \Delta\right| \le \frac{1}{k}. \tag{10}$$

Proof. The proof is similar to the proof of Lemma 5.3. Let $k \in \mathbb{N}^*$ be arbitrary. Since $\Delta(x) = \Delta$, for every $x \in \widehat{M}$,

$$\varepsilon_k(x) := \sup\left\{\varepsilon \le \frac{1}{k} : \left|\frac{\mu_x(B_{d_x}(x, \varepsilon))}{2\varepsilon} - \Delta\right| \le \frac{1}{k}\right\}.$$

Observe that for $x \in \widehat{M}$, such $\varepsilon_k(x)$ exists by the definition of Δ, $\varepsilon_l(x) \to 0$ as $l \to \infty$ and

$$\lim_{l\to\infty}\left|\frac{\mu_x(B_{d_x}(x, \varepsilon_l(x)))}{2\varepsilon_l(x)} - \Delta\right| = 0.$$

CLAIM 2. *The function $\varepsilon_k(x)$ is measurable for all $k \in \mathbb{N}$.*

Proof. Define

$$w(x, \varepsilon) = \frac{\mu_x(B_{d_x}(x, \varepsilon))}{2\varepsilon}.$$

As observed in the proof of Lemma 5.3, $r \mapsto w(x, r)$ is continuous and $x \mapsto w(x, r)$ is measurable. Given any $k \in \mathbb{N}$, $k > 0$, by the definition of ε_k, for all $\beta > 1/k$, we have $\varepsilon_k^{-1}((0, \beta)) = \widehat{M}$. Also, the continuity of $w(x, \cdot)$ implies that for every $\beta \le 1/k$,

$$\begin{aligned}\varepsilon_k^{-1}((0, \beta)) &= \{x : \varepsilon_k(x) \in (0, \beta)\}\\ &= \bigcap_{\beta\le r\le 1/k} w(\cdot, r)^{-1}\Big(\Big[\Delta + \frac{1}{k}, \infty\Big)\Big) \cup w(\cdot, r)^{-1}\Big(\Big[0, \Delta - \frac{1}{k}\Big)\Big)\\ &= \bigcap_{\beta\le r\le 1/k, r\in\mathbb{Q}} w(\cdot, r)^{-1}\Big(\Big[\Delta + \frac{1}{k}, \infty\Big)\Big) \cup w(\cdot, r)^{-1}\Big(\Big[0, \Delta - \frac{1}{k}\Big)\Big).\end{aligned}$$

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Therefore, $\varepsilon_k^{-1}((0,\beta))$ is measurable, as it is a countable intersection of measurable sets, and consequently, ε_k is a measurable function for every k. □

As $\varepsilon_k(x)$ is f-invariant, by ergodicity, we may take the full measure set R_k, where $\varepsilon_k(x)$ is constant equal to ε_k. The sequence ε_k goes to 0 as k goes to infinity, so we set $\tilde{R} := \bigcap_{k=1}^{+\infty} R_k$. Since each R_k has full measure, $\tilde{R}$ has full measure and clearly satisfies what we want for the sequence $\{\varepsilon_k\}_k$. The set $R = \bigcap_{i\in\mathbb{Z}} f^i(\widetilde{R})$ is f-invariant, has full measure, and satisfies items (i) and (ii) as we wanted. □

Similar to the definitions made in §5.1, we set

$$\Pi_U := \bigcup_{x\in U\setminus(\mathcal{Z}_U\cup\mathcal{A}_U)} \Pi_{x,U},$$

where

$$\Pi_{x,U} := \Big\{ y \in \mathcal{F}|U(x) \setminus \mathcal{Z}_U : \left| \frac{1}{2\varepsilon_k} \cdot \frac{\mu_y^U(B_{d_x}(y,\varepsilon_k))}{\mu_y^U(B_{d_x}(y,\mathfrak{r}))} - \Delta \right| \le \frac{1}{k} \text{ for all } k \text{ with } B_{d_x}(y,\mathfrak{r}) \subset U \Big\}.$$

LEMMA 5.6. *For every $x \in R \cap U$, consider $\delta(x) > 0$ for which*

$$B_{d_x}[x, 2\cdot\delta + \mathfrak{r}] \subset U.$$

The set $\Pi_{x,U} \cap B_{d_x}[x,\delta]$ is a closed subset on the plaque $\mathcal{F}|U(x)$.

Proof. Identical to the proof of Lemma 5.4. □

Similar to the definition made in §5.1, we consider the set

$$D_U := \mathcal{F}|U(\Pi_U) \setminus (\mathcal{F}|U)(\mathcal{Z}_{\mathcal{U}})$$

and

$$D := \bigcup_{n\in\mathbb{Z}, U\in\mathcal{U}} f^n\Big(\bigcup D_U\Big).$$

Similarly, D_U is $\mathcal{A}$-measurable for all $U \in \mathcal{U}$ and again by ergodicity, we have $\mu(D) = 0$ or $\mu(D) = 1$.

After proving the auxiliary lemmas for Δ^∞ (respectively Δ) and obtaining the sets D (respectively D^∞), we divide the next part of the proof into four cases.

5.3. *Case 1: $\Delta < \infty$ and $\mu(D) = 0$.* In this case, we will show that the support of μ_x is a Cantor subset for almost every $x \in M$.

As fixed in the beginning, consider $\{\omega_x\}_x$ the disintegration of μ along $\mathcal{F}$ and consider $\mathcal{G}$ a full measure $\mathcal{F}$-saturated set of points, where

$$f_*^j \Omega_x = \Omega_{f^j(x)} \quad \text{for all } j \in \mathbb{Z}.$$

Let $\mathcal{G}^U := \{x \in U \setminus D : \mu_x^U = \omega_x(\cdot|\mathcal{F}|U(x))\} \cap \{x : \mu_x^U \text{ is non-atomic}\}$.

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Consider:

- $\Phi^U_{1/n}(\mathcal{Z}_U) := \{x \in U : d_x(x, \mathcal{Z}_U) < 1/n\}$;
- $\mathcal{E}_n = \bigcup_j \mathcal{G} \cap f^j(\Phi^U_{1/n}(\mathcal{Z}_U) \cap \mathcal{G}^U)$.

As $\mathcal{Z}_U = \chi_U \cap (U \setminus \mathcal{A}_U)$,

$$\Phi^U_{1/n}(\mathcal{Z}_U) = \{x \in M : d_x(x, \mathcal{Z}_U) < 1/n\} \cap (U \setminus \mathcal{A}_U),$$

which is measurable for $n \geq \hat{n}$, for some $\hat{n} \in \mathbb{N}$ not depending on U, by Lemma 3.11 since χ_U is a Borel subset of U. Therefore, the set of the second item is an f-invariant measurable subset of M; thus, it either has full or null measure.

Now, again we separate two cases.

Case 1.1. Assume that for all $n \geq \hat{n}$, $\mu(\mathcal{E}_n) = 1$. Then,

$$\mathcal{E}^U_n = \mathcal{G}^U \cap \left(\bigcup_j \mathcal{G} \cap f^j(\Phi^U_{1/n}(\mathcal{Z}_U) \cap \mathcal{G}^U) \right)$$

has full measure in U for every $n \geq \hat{n}$. For $z \in \mathcal{E}^U = \bigcap_{n \geq \hat{n}} \mathcal{E}^U_n$, let $n_0 > 0$ such that for $n \geq n_0 \geq \hat{n}$, we have $B_{d_x}(z, 1/n_0) \subset U$. For $n \geq n_0$, let j with

$$f^{-j}(z) \in \Phi^U_{1/n}(\mathcal{Z}_U) \cap \mathcal{G}^U,$$

and $p \in \mathcal{Z}_U$ with $d_x(p, f^{-j}(z)) < 1/n$. Then, $d_x(f^j(p), z) < 1/n$, which implies $f^j(p) \in U$, and if $\mu^U_p(B_{d_x}(p, \delta)) = 0$, for δ small, then since $\mu^U_p \sim \omega_{f^{-j}(z)}$ (because $f^{-j}(z) \in \mathcal{G}^U$) and $z \in \mathcal{G}$, it follows that $\omega_z(f^{-j}(I_p)) = \omega_{f^j(z)}(I_p) = 0$ and $z \in \mathcal{G}^u$; thus, $\mu^U_z(f^{-j}(I_p)) = 0$. Therefore, $f^j(p) \in \mathcal{Z}_U$ and $z \in \Phi_{1/n}(\mathcal{Z}_U)$. That is, the set $\mathcal{Z}_U \cap \mathcal{F}|U(z)$ is dense in $\mathcal{F}|U(z)$ for almost every $z \in \mathcal{E}^U$.

CLAIM 3. *For $x \in \mathcal{E}^U$, $\mathcal{C}_x := \mathcal{F}|U(x) \setminus \mathcal{Z}_U$ is a Cantor subset in $\mathcal{F}|U(x)$.*

Proof. To prove that $\mathcal{C}_x$ is a Cantor set, we will show that this set is a nowhere dense and perfect set. Therefore, as it is bounded ($\mathcal{C}_x \subset \mathcal{F}|U(x)$), we can conclude that it is a compact, nowhere dense perfect set, that is, a Cantor set. Since $\mathcal{Z}_U \cap \mathcal{F}|U(x)$ is a dense and open set in $\mathcal{F}|U(x)$, we have that $\mathcal{C}_x$ is a nowhere dense and closed set. Now, let us see that $\mathcal{C}_x$ has no isolated points. Let $y \in \mathcal{C}_x$. Suppose that there exist $r > 0$ with $B_{d_x}(y, r) \subset U$ and $B_{d_x}(y, r) \cap \mathcal{C}_x = \{y\}$. Since $y \in \mathcal{C}_x$,

$$0 < \mu^U_x(B_{d_x}(y, r)) = \mu^U_x(B_{d_x}(y, r) \setminus \mathcal{C}_x) + \mu^U_x(\{y\}) = \mu^U_x(\{y\});$$

thus, $\mu^U_x(\{y\}) > 0$, which is a contradiction since $x \notin \mathcal{A}_U$.

Therefore, $\mathcal{C}_x$ is indeed a Cantor subset. □

Thus, for almost every $x \in M$ and any local chart U, the conditional measure μ^U_x is supported in the Cantor set $\mathcal{C}_x$. In particular, the measure ω_x on $\mathcal{F}(x)$ is supported on a σ-Cantor subset of the leaf. We remark that for the Claim, we did not use the fact that $\Delta = \infty$; thus, the same argument will work when $\Delta < \infty$.

Case 1.2. If, however, there exists $N_0 \in \mathbb{N}$ with $N_0 \geq \hat{n}$ such that $\mu(\mathcal{E}_{N_0}) = 0$, then $\mu(\mathcal{E}^U_{N_0}) = 0$ and, moreover, $\mu(\mathcal{E}^U_N) = 0$ for any $N \geq N_0$. Since

$$\mathcal{G}^U \cap \mathcal{G} \cap \varphi_{1/N}(\mathcal{Z}_U) \subset \mathcal{E}^U_{N_0},$$

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we have

$$\mu(\Phi^U_{1/N}(\mathcal{Z}_U)) = \mu(\mathcal{G}^U \cap \mathcal{G} \cap \Phi_{1/N}(\mathcal{Z}_U)) \leq \mu(\mathcal{E}^U_n) = 0 \quad \text{for all } N \geq N_0.$$

In particular, for almost every $x \in U$,

$$\mu^U_x(\Phi^U_{1/N}(\mathcal{Z}_U \cap \mathcal{F}|U(x))) = 0. \tag{11}$$

As $\Phi^U_{1/N}(\mathcal{Z}_U \cap \mathcal{F}|U(x))$ is an open subset of $\mathcal{F}|U(x)$, by (11), for almost every $x \in U$,

$$\Phi^U_{1/N}(\mathcal{Z}_U \cap \mathcal{F}|U(x)) \subset (\mathcal{Z}_U \cap \mathcal{F}|U(x)).$$

However, clearly the other continence holds; thus, $\mathcal{Z}_U \cap \mathcal{F}|U(x) = \Phi^U_{1/N}(\mathcal{Z}_U \cap \mathcal{F}|U(x))$ and this implies $\mathcal{Z}_U \cap \mathcal{F}|U(x) = \mathcal{F}|U(x)$. As this happens for almost every $x \in U$, we fall in a contradiction with the fact that $\mu(\mathcal{Z}_U \cap U) = 0$. Therefore, this case does not happen.

5.4. *Case 2:* $\Delta = \infty$ *and* $\mu(D^\infty) = 0$. In particular, for every $U \in \mathcal{U}$, we must have $\mu(D^\infty_U) = 0$, which implies $\mu(\mathcal{F}|U(\mathcal{Z}_U)) = \mu(U)$.

In this case, we will proceed very similarly to the previous case. For $U \in \mathcal{U}$, consider the sets $\mathcal{E}_n$ and $\mathcal{E}^U_n$ defined in Case 1.

Again, if $\mu(\mathcal{E}_n) = 1$ for every $n \in \mathbb{Z}$, then there exists a full measure subset of U, namely $\mathcal{E}^U$, such that $z \in \mathcal{E}^U$ implies $\mathcal{Z}_U \cap \mathcal{F}|U(z)$ is dense in $\mathcal{F}|U(z)$. Hence, as shown by the Claim in Case 1, it follows that the support of μ^U_x is a Cantor subset of the plaque $\mathcal{F}|U(x)$ for almost every $x \in U$.

Otherwise, if $\mu(\mathcal{E}_{N_0}) = 0$ for some $N_0 \in \mathbb{N}$, then as in Case 1, we conclude that $\mathcal{Z}_U \cap \mathcal{F}|U(x) = \mathcal{F}|U(x)$, which contradicts the fact that $\mu(\mathcal{Z}_U \cap U) = 0$; thus, this case does not occur.

5.5. *Case 3:* $\Delta = \infty$ *and* $\mu(D^\infty) = 1$. Let us prove that this case cannot occur. Since $\mu(D^\infty) = 1$, there exists $U \in \mathcal{U}$ with $\mu(D^\infty_U) > 0$. Since $\mu(\mathcal{F}|U(\Pi^\infty_U)) = \mu(U)$, for almost every point $\bar{x} \in D^\infty_U$,

$$\mu^U_{\bar{x}}(\Pi^\infty_U \cap \mathcal{F}|U(\bar{x})) = 1. \tag{12}$$

Take any such typical $\bar{x}$ and consider $x \in \mathcal{F}|U(\bar{x}) \cap \Pi^\infty_U$, in particular, $B_{d_x}(x, \mathfrak{r}) \subset U$. Also, $\Pi^\infty_{x,U} \cap B_{d_x}[x, \delta]$ is closed in $\mathcal{F}|U(x)$ for some $\delta > 0$ small if there exists $z \in B_{d_x}[x, \delta] \setminus \Pi^\infty_{x,U} \cap B_{d_x}[x, \delta]$, then for some $\delta_2 > 0$, we have $B_{d_x}(z, \delta_2) \subset B_{d_x}[x, \delta] \setminus \Pi^\infty_{x,U} \cap B_{d_x}[x, \delta]$ and $\mu^U_x(B_{d_x}(z, \delta_2)) = 0$ by (12). However, this cannot happen since this would imply $z \in \mathcal{Z}_U$ and, consequently, $\mathcal{F}|U(\mathcal{Z}_U) \cap D^\infty_U \neq \emptyset$, falling in a contradiction with the definition of D^∞_U. Therefore,

$$\Pi^\infty_{x,U} \cap B_{d_x}[x, \delta] = B_{d_x}[x, \delta].$$

Consider $0 < r_0 < \delta$ small enough so that $B_{d_x}(x, \mathfrak{r} + 2 \cdot r_0) \subset U$. By hypothesis, for k with $\varepsilon_k < r_0$, by Remark 3.5, we can take $\lfloor r_0/\varepsilon_k \rfloor$ disjoint balls of radius ε_k inside $B_{d_x}(x, r_0)$, say with center $a_1, a_2, \ldots, a_{\lfloor r_0/\varepsilon_k \rfloor}$. Then,

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$$\sum_{i=1}^{\lfloor r_0/\varepsilon_k\rfloor} \mu_x^U(B_{d_x}(a_i,\varepsilon_k)) \le \mu_x^U(B_{d_x}(x,r_0))$$

$$\Rightarrow \sum_{i=1}^{\lfloor r_0/\varepsilon_k\rfloor} \frac{\mu_x^U(B_{d_x}(a_i,\varepsilon_k))}{\mu_x^U(B_{d_x}(x,\mathfrak{r}))} \le \frac{\mu_x^U(B_{d_x}(x,r_0))}{\mu_x^U(B_{d_x}(x,\mathfrak{r}))}$$

$$\Rightarrow \sum_{i=1}^{\lfloor r_0/\varepsilon_k\rfloor} \frac{\mu_x^U(B_{d_x}(a_i,\mathfrak{r}))}{\mu_x^U(B_{d_x}(x,\mathfrak{r}))}\cdot\frac{\mu_x^U(B_{d_x}(a_i,\varepsilon_k))}{\mu_x^U(B_{d_x}(a_i,\mathfrak{r}))} \le \frac{\mu_x^U(B_{d_x}(x,r_0))}{\mu_x^U(B_{d_x}(x,\mathfrak{r}))}.$$

By Lemma 4.1,

$$w \in B_{d_x}[x,r_0] \subset F_{\mathfrak{r}} \mapsto \mu_x^U(B_{d_x}(w,\mathfrak{r}))$$

is continuous; hence, there exists $\eta > 0$ such that

$$\frac{\mu_x^U(B_{d_x}(w,\mathfrak{r}))}{\mu_x^U(B_{d_x}(x,\mathfrak{r}))} \ge \eta$$

for every $w \in B_{d_x}[x,r_0]$. Therefore, since $a_i \in B_{d_x}[x,\delta]$ for all i,

$$\begin{aligned}\eta\cdot(2\varepsilon_k\cdot k)\cdot\lfloor\frac{r_0}{\varepsilon_k}\rfloor &< \eta\cdot\sum_{i=1}^{\lfloor r_0/\varepsilon_k\rfloor}\frac{\mu_x^U(B_{d_x}(a_i,\varepsilon_k))}{\mu_x^U(B_{d_x}(a_i,\mathfrak{r}))}\\ &\le \sum_{i=1}^{\lfloor r_0/\varepsilon_k\rfloor}\frac{\mu_x^U(B_{d_x}(a_i,\mathfrak{r}))}{\mu_x^U(B_{d_x}(x,\mathfrak{r}))}\cdot\frac{\mu_x^U(B_{d_x}(a_i,\varepsilon_k))}{\mu_x^U(B_{d_x}(a_i,\mathfrak{r}))}\\ &\le \frac{\mu_x^U(B_{d_x}(x,r_0))}{\mu_x^U(B_{d_x}(x,\mathfrak{r}))}.\end{aligned}$$

Taking $k\to\infty$, the left side goes to infinity from where we conclude that $\mu_x^U(B_{d_x}(x,r_0)) = \infty$, falling in a contradiction. Thus, this case does not occur.

5.6. *Case 4:* $\Delta < \infty$ *and* $\mu(D) = 1$. We will prove that if this case occurs, then for almost every $x \in U$, $U \in \mathcal{U}$, the conditional measure in $\mathcal{F}|U(x)$ is equivalent to the measure λ_x given in Definition 3.12.

LEMMA 5.7. *The constant* Δ *is bounded away from zero and*

$$\mu_x \ll \lambda_x$$

for μ*-almost every* $x \in M$.

Proof. Let $y \in D$. Then, for some $n_0 \in \mathbb{Z}$ and $U \in \mathcal{U}$, we have $f^{n_0}(y) \in D_U$. Call $x = f^{n_0}(y)$. As $x \in \mathcal{F}|U(\Pi_U) \setminus \mathcal{F}|U(\mathcal{Z}_U)$, we have $\mu_x^U(\mathcal{F}|U(\Pi_U) \cap \mathcal{F}|U(x)) = 1$. Therefore, we conclude (by the same argument used in Case 3—§5.5) that

$$\Pi_{x,U} \cap B_{d_x}[x,\delta] = B_{d_x}[x,\delta]$$

for some δ small. Consequently, $\Pi_U \cap \mathcal{F}|U(x) \supset \{y \in \mathcal{F}|U(x) : d_x(y,\partial\mathcal{F}|U(x)) \ge \mathfrak{r}\}$.

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For any given $k \in \mathbb{N}^*$, $U \in \mathcal{U}$, and $x \in \Pi_U$,

$$\left| \frac{\mu_x^U(B_{d_x}(x, \varepsilon_k))}{2\varepsilon_k \cdot \mu_x^U(B_{d_x}(x, \mathfrak{r}))} - \Delta \right| \leq \frac{1}{k}. \tag{13}$$

Given $\varepsilon > 0$, take $k_0 \in \mathbb{N}$ such that $k_0^{-1} < \varepsilon$. Again, since $\{d_x\}$ is an $\mathcal{F}$-metric system, given a constant $r > 0$, we need at most $s(k) = \lfloor r/\varepsilon_k \rfloor + 1$ points, say $a_1, a_2, \ldots, a_{s(k)}$, to cover the ball $B_{d_x}(x, r)$ with balls of radius ε_k. Let $\alpha_i := (\mu_x^U(B_{d_x}(a_i, \mathfrak{r}))/\mu_x^U(B_{d_x}(x, \mathfrak{r})))$. Again, by continuity (see Lemma 4.1), there exists $\beta > 0$ such that $\alpha_i \leq \beta$ for all i. Therefore,

$$\begin{aligned}
\frac{\mu_x^U(B_{d_x}(x, r))}{\mu_x^U(B_{d_x}(x, \mathfrak{r}))} &\leq \sum_{i=1}^{s(k)} \frac{\mu_x^U(B_{d_x}(a_i, \varepsilon_k))}{\mu_x^U(B_{d_x}(x, \mathfrak{r}))} = \sum_{i=1}^{s(k)} \alpha_i \cdot \frac{\mu_x^U(B_{d_x}(a_i, \varepsilon_k))}{\mu_x^U(B_{d_x}(a_i, \mathfrak{r}))} \\
&\leq \beta \cdot \sum_{i=1}^{s(k)} \frac{\mu_x^U(B_{d_x}(a_i, \varepsilon_k))}{\mu_x^U(B_{d_x}(a_i, \mathfrak{r}))} \leq \beta \sum_{i=1}^{s(k)} \frac{2\varepsilon_k}{k} + \Delta 2\varepsilon_k \\
&= \beta \cdot s(k) \frac{2\varepsilon_k}{k} + \beta \cdot s(k) 2\varepsilon_k \cdot \Delta.
\end{aligned}$$

Since $\lim s(k)\varepsilon_k = r$, we have that $\beta \cdot s(k)(\varepsilon_k/k)$ goes to zero as $k \to \infty$, so

$$\frac{\mu_x^U(B_{d_x}(x, r))}{\mu_x^U(B_{d_x}(x, \mathfrak{r}))} \leq 2\Delta \cdot \beta \cdot r.$$

Therefore, $\mu_x^U \ll \lambda_x$ when restricted to $\{y \in \mathcal{F}|U(x) : d_x(y, \partial\mathcal{F}|U(x)) \geq \mathfrak{r}\}$ and $\Delta > 0$. As $\mathfrak{r}$ can be taken to be arbitrarily small in the beginning, it follows that $\mu_x^U \ll \lambda_x$ as we wanted to show. □

Next, we are able to conclude that μ_x^U is equivalent to the measure λ_x.

LEMMA 5.8. *For μ-almost every (a.e.) $x \in M$,*

$$\mu_x^U \sim \lambda_x.$$

Proof. By Lemma 5.7, we know that $\mu_x^U \ll \lambda_x$. Since λ_x is a doubling measure, we have that the Radon–Nikodyn derivative $d\mu_x^U/d\lambda_x$ exists and is given at λ_x-a.e. point $y \in \{y \in \mathcal{F}|U(x) : d_x(y, \partial\mathcal{F}|U(x)) \geq \mathfrak{r}\}$ by

$$\frac{d\mu_x^U}{d\lambda_x}(y) = \lim_{r \to 0} \frac{\mu_x^U(B_{d_x}(y, r))}{\lambda_x(B_{d_x}(y, r))}.$$

In particular, by taking the limit along the subsequence ε_k, $k \to \infty$, we conclude that

$$\frac{d\mu_x^U}{d\lambda_x}(y) = \lim_{k \to \infty} \frac{\mu_x^U(B_{d_x}(y, \varepsilon_k))}{\lambda_x(B_{d_x}(y, \varepsilon_k))}, \quad \lambda_x\text{-a.e. } y \in \{y \in \mathcal{F}|U(x) : d_x(y, \partial\mathcal{F}|U(x)) \geq \mathfrak{r}\},$$

which implies

$$\frac{d\mu_x^U}{d\lambda_x}(y) = \beta(y) \cdot \Delta, \quad \lambda_x\text{-a.e. } y \in \{y \in \mathcal{F}|U(x) : d_x(y, \partial\mathcal{F}|U(x)) \geq \mathfrak{r}\},$$

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where $\beta(y) = \mu_x^U(B_{d_x}(y, \mathfrak{r}))$. Since β is a continuous function, given any compact $I \subset \{y \in \mathcal{F}|U(x) : d_x(y, \partial\mathcal{F}|U(x)) \geq \mathfrak{r}\}$,

$$\beta_1\Delta \leq \frac{d\mu_x}{d\lambda_x}(y) \leq \beta_2\Delta,\ \lambda_x\text{-a.e. } y \in I.$$

Then,

$$\beta_1\Delta \cdot \lambda_x \leq \mu_x,\ \lambda_x\text{-a.e. } y \in I.$$

In particular, if $\mu_x^U(E) = 0$, then, for any compact subset $I \subset \{y \in \mathcal{F}|U(x) : d_x(y, \partial\mathcal{F}|U(x)) \geq \mathfrak{r}\}$, we have $\lambda_x(E \cap I) = 0$. Since $\mathcal{F}(x)$ may be written as a countable union of increasing compact subsets, we conclude that $\lambda_x(E \cap \{y \in \mathcal{F}|U(x) : d_x(y, \partial\mathcal{F}|U(x)) \geq \mathfrak{r}\}) = 0$. Again, since $\mathfrak{r}$ may be taken to be arbitrarily small, we conclude that $\lambda_x(E) = 0$; thus, $\lambda_x \ll \mu_x^U$ as we wanted to show. □

6. *Proof of Theorem B*

The proof of Theorem B employs arguments inspired by those used in [10], with several technical differences related to the fact that the balls inside the leaves here are generated by the plaque-continuous metric system $\{d_x\}_{x\in M}$.

Additionally, instead of affine maps, we define functions on $\mathcal{F}^c(x)$, as described in [5], referred to as the limit center map.

Consider the set $\mathcal{N} = \{x \in M : \alpha(x) = \omega(x) = M\}$. Since f is transitive and μ has full support, $\mathcal{N}$ is a full measure residual subset. The following definition is due to Bonatti and Zhang [5].

Definition 6.1. We say that a map $F : \mathcal{F}^c(x) \to \mathcal{F}^c(x)$ is a *limit center map* if there exists a sequence $\{n_i\} \subset \mathbb{Z}$ with $|n_i| \to \infty$ such that $\{f^{n_i}\}$ pointwise converges to F.

For each $x \in \mathcal{N}$, we consider

$$\mathcal{L}(\mathcal{F}^c(x)) := \{F : \mathcal{F}^c(x) \to \mathcal{F}^c(x) : F \text{ is a limit center map}\} \quad \text{and}$$
$$\mathcal{L}^+(\mathcal{F}^c(x)) := \{F \in \mathcal{L}(\mathcal{F}^c(x)) : F \text{ preserves the orientation of } \mathcal{F}^c(x)\}.$$

Remark 6.2. (See [5, Proposition 4.7] For every $x \in \mathcal{N}$, we have the following.

- If $L = \mathcal{F}^c(x)$ is not compact, then there is a homeomorphism $\psi_L : L \to \mathbb{R}$ such that
$$\mathcal{L}^+(L) = \{\psi_L^{-1} \circ T_t \circ \psi_L,\ t \in \mathbb{R}\},$$
where T_t is the translation $T_t : \mathbb{R} \to \mathbb{R},\ s \mapsto s + t$.
- If $L = \mathcal{F}^c(x)$ is compact, then there is a homeomorphism $\psi_L : L \to S^1$ such that
$$\mathcal{L}^+(L) = \{\psi_L^{-1} \circ R_t \circ \psi_L,\ t \in S^1 = \mathbb{R}/\mathbb{Z}\},$$
where R_t is the rotation $R_t : S^1 \to S^1,\ s \mapsto s + t (\text{mod } \mathbb{Z})$.
- $\mathcal{L}^+(\mathcal{F}^c(x))$ is a group whose action on $\mathcal{F}^c(x)$ is transitive.
- $\mathcal{L}(\mathcal{F}^c(x))$ either coincides with $\mathcal{L}^+(\mathcal{F}^c(x))$ or is the group generated by $\mathcal{L}^+(\mathcal{F}^c(x))$ and $-\text{Id}|\mathcal{F}^c(x)$.

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- If $F : \mathcal{F}^c(x) \to \mathcal{F}^c(x)$ is a limit center map having a fixed point $x \in \mathcal{F}^c(x)$, if F is orientation preserving, then F is the identity map of $\mathcal{F}^c(x)$. This is, the action on $\mathcal{F}^c(x)$ given by the group $\mathscr{L}^+(\mathcal{F}^c(x))$ is free.

Before we proceed, recall that the atlas $\mathcal{U}$ of $\mathcal{F} = \mathcal{F}^c$ was taken to have specific properties detailed in the beginning of §4. Furthermore, in what follows, we assume that the disintegration of μ along $\mathcal{F}^c$ is not atomic, otherwise, we already fall in one of the possibilities claimed by the statement.

For each point $x \in M$, consider $U_x \in \mathcal{U}$ and $V_x \subset U_x$ the closure of a neighborhood of x such that for every $y \in V_x$, we have $B_{d_y}(y, \mathfrak{r}) \subset U$.

LEMMA 6.3. *For μ-a.e. $x \in M$ and $\mu_x^{U_x}$-a.e. $y \in V_x \cap \mathcal{F}^c|U_x(x)$, there is a limit center map $F \in \mathscr{L}^+(\mathcal{F}^c(x))$ such that $F(x) = y$ and $F_*\mu_x = \mu_y$.*

Proof. Recall that for every $x \in M_0$ (see (3)), the measure μ_x is defined on $B_{d_x}(x, \mathfrak{r})$ by

$$\mu_x(J) = \mu_x^U(J|B_{d_x}(x, \mathfrak{r})) = \frac{\mu_x^U(J \cap B_{d_x}(x, \mathfrak{r}))}{\mu_x^U(B_{d_x}(x, \mathfrak{r}))}, \quad J \subset B_{d_x}(x, \mathfrak{r}).$$

Abusing notation, we may define $x \in M_0 \mapsto \mu_x(J)$ for any Borel set $J \subset M$ by taking $\mu_x(J) := \mu_x(J \cap B_{d_x}(x, \mathfrak{r}))$. By Corollary 4.5, the map $x \in M_0 \mapsto \mu_x$ is measurable in the sense that for any Borel subset $J \subset M$, the map $x \in M_0 \mapsto \mu_x(J \cap B_{d_x}(x, \mathfrak{r}))$ is measurable.

Using Lusin's theorem, we may take $\{K_i\}$ an increasing sequence of compact subsets of M_0 for which the map $x \mapsto \mu_x$ is continuous when restricted to each K_i. Once again, we emphasize that, in this case, it means that for any Borel subset $J \subset M$, the map $x \in K_i \mapsto \mu_x(J \cap B_{d_x}(x, \mathfrak{r}))$ is continuous. As μ is ergodic, for μ-a.e. point $x \in K_i$, the orbit of x is dense in a full measurable subset of K_i. Let $P_i = \mathcal{N} \cap K_i$.

Let $x \in P_i$ and $y \in V_x \cap \mathcal{F}^c|U_x(x) \cap P_i$. From Remark 6.2, we know that there exists $F \in \mathscr{L}^+(\mathcal{F}^c(x))$ such that $F(x) = y$, and since F is a limit center map, there exists a sequence $n_k \subset \mathbb{Z}$ with $|n_k| \to \infty$ so that the sequence $f^{n_k}|_{\mathcal{F}^c(x)}$ converges pointwise to F.

By the continuity of $x \in P_i \mapsto \mu_x$ and since $y \in P_i$,

$$F_*\mu_x = \lim_{k\to\infty} f_*^{n_k}\mu_x = \lim_{k\to\infty} \mu_{f^{n_k}(x)} = \mu_{F(x)} = \mu_y.$$

By taking the limit set of K_i, we conclude that for every point $x \in P$, where $P = \lim_i P_i$, and $y \in V_x \cap \mathcal{F}^c|U_x(x) \cap P$, there exists $F \in \mathscr{L}^+(\mathcal{F}^c(x))$ such that $F(x) = y$ and $F_*\mu_x = \mu_y$ as we wanted. □

Let P be the full measure subset of points for which the previous lemma holds. For $x \in P$, define

$$G_x^{U_x} = \{y \in V_x \cap \mathcal{F}^c|U_x(x) : \text{there exists } F \in \mathcal{L}^+(\mathcal{F}^c(x)) \text{ with } F(x) = y \text{ and } F_*\mu_x = \mu_y\}.$$

By Lemma 6.3, we know that $[V_x \cap \mathcal{F}^c|U_x(x)] \setminus G_x^{U_x}$ has zero $\mu_x^{U_x}$-measure.

LEMMA 6.4. *For $x \in P$, the set $G_x^{U_x}$ is closed in $\mathcal{F}^c|U_x(x)$.*

Proof. Assume that $\mathcal{F}^c(x)$ is not compact. The argument is identical for the other case.

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We know that $\mathcal{L}^+(\mathcal{F}^c(x))$ is a closed group as it is isomorphic to either group of translations (see Remark 6.2). Furthermore, for any $J \subset M$ Borel, the map $y \in B_{d_x}(x, \mathfrak{r}) \cap M_0 \mapsto \mu_y(J \cap B_{d_x}(y, \mathfrak{r}))$ is continuous by Corollary 4.4.

Let $y_n \in G_x^{U_x}$ with $y_n \to y \in V_x \cap \mathcal{F}^c|U(x)$ and consider $F_n \in \mathcal{L}^+(\mathcal{F}^c(x))$ be given by Lemma 6.3 satisfying $F_n(x) = y_n$.

Observe that the sequence $\{F_n\}_n$ converges pointwise to a map $F \in \mathcal{L}^+(\mathcal{F}^c(x))$ with $F(x) = y$. Indeed, by Remark 6.2, there are real numbers t_n for which

$$\psi_L^{-1} \circ T_{t_n} \circ \psi_L = F_n, \quad n \geq 1,$$

where $L = \mathcal{F}^c(x)$. In particular,

$$T_{t_n}(\psi_L(x)) = \psi_L(y_n).$$

Since $\psi_L(y_n)$ converges to $\psi_L(y)$, because ψ_L is a homeomorphism, it follows that $|\psi_L(x) - \psi_L(y_n)| \to |\psi_L(x) - \psi_L(y)|$. Since T_{t_n} are translations, we have $t_n = |\psi_L(x) - \psi_L(y_n)| \to |\psi_L(x) - \psi_L(y)|$. Consequently, the sequence of translations $\{T_{t_n}\}$ converges to the translation $T_t : \mathbb{R} \to \mathbb{R}$ with $t = |\psi_L(x) - \psi_L(y)|$. In particular, F_n converges to $F := \psi_L^{-1} \circ T_t \circ \psi_L$.

Finally, by the continuity mentioned previously,

$$F_*\mu_x = \lim_{n\to\infty} (F_n)_*\mu_x = \lim_{n\to\infty} \mu_{y_n} = \mu_y.$$

Thus, $y \in G_x^{U_x}$, showing that $G_x^{U_x}$ is closed in $\mathcal{F}^c|U_x(s)$ as we wanted. □

LEMMA 6.5. *For μ-a.e. $x \in P$, the support of $\mu_x^{U_x}$ cannot be a Cantor subset of $\mathcal{F}^c|U(x)$.*

Proof. Assume that for some $x \in P$, the support of $\mu_x^{U_x}$ is a Cantor subset of $\mathcal{F}^c|U(x)$. As V_x is compact, the intersection $V_x \cap \operatorname{supp}(\mu_x^{U_x})$ is also a Cantor subset and is the support of the restriction $\mu_x^{U_x}(\cdot|V_x \cap \mathcal{F}^c|U(x))$.

As we have seen, for μ-a.e. $x \in P$, the set $G_x^{U_x}$ is a full measure subset of $V_x \cap \mathcal{F}^c|U(x))$ and is closed by Lemma 6.4; therefore, it must contain the support $V_x \cap \operatorname{supp}(\mu_x^{U_x})$ which is a Cantor subset. Consider an ordering of $\mathcal{F}^c|U(x)$ and consider $a = \inf(V_x \cap \operatorname{supp}(\mu_x^{U_x}))$, $b = \sup(V_x \cap \operatorname{supp}(\mu_x^{U_x}))$, and γ the center arc connecting a and b inside $\mathcal{F}^c|U(x)$.

Let $\zeta : \gamma \to [0, 1]$ given by $\zeta(a) = 0$, $\zeta(b) = 1$, be increasing and linear with respect to d_x, that is, $\zeta(y) = d_x(a, y)/d_x(a, b)$. The set $\zeta(\gamma) \subset [0, 1]$ is a Cantor subset, in particular, there exists a strictly increasing homeomorphism $H_0 : \zeta(\gamma) \to \mathcal{C}_3$ with $H_0(0) = 0$, $H_0(1) = 1$, where $\mathcal{C}_3$ here denotes the standard ternary Cantor set in $[0, 1]$. Now, let us extend H_0 to a map of $[0, 1]$.

First, we fix some notation. Consider $I^0 = [0, 1]$, $I_1^1 = (1/3, 2/3) = (a_1^1, b_1^1)$, and $I_k^m = (a_k^m, b_k^m)$, with $1 \leq k \leq 2^{m-1}$ the open intervals removed at the mth stage of the construction of the ternary Cantor set with $a_k^m < b_k^m < a_{k+1}^m$ for any $m \geq 2$ and $1 \leq k \leq 2^{m-1} - 1$. Therefore,

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$$\mathcal{C}_3 = I_0 \setminus \left(\bigcup_{m=1}^{\infty} \bigcup_{k=1}^{2^{m-1}} I_k^m \right).$$

Let also $J_k^m := (H_0^{-1}(a_k^m), H_0^{-1}(b_k^m)) \subset [0,1]$. Then, $\zeta(\gamma) = I_0 \setminus (\bigcup_{m=1}^{\infty} \bigcup_{k=1}^{2^{m-1}} J_k^m)$. Consider $H : [0,1] \to [0,1]$ defined by:

- the restriction of H to each J_k^m is the linear map given by

$$H|J_k^m(y) = \left(1 - \frac{y - H_0^{-1}(a_k^m)}{H_0^{-1}(b_k^m) - H_0^{-1}(a_k^m)}\right) \cdot a_k^m + \frac{y - H_0^{-1}(a_k^m)}{H_0^{-1}(b_k^m) - H_0^{-1}(a_k^m)} \cdot b_k^m;$$

- the restriction of H to $\zeta(\gamma)$ is H_0, that is, $H|\zeta(\gamma) = H_0$.

Observe that H is continuous and strictly increasing; thus, it is a strictly increasing bijection and, consequently, a homeomorphism.

Consider now the homeomorphism $\widetilde{H} := H \circ \zeta : \gamma \to [0,1]$ and denote by E_3 the set of all points in $\mathcal{C}_3$ that are extremes of the removed intervals, that is, $E_3 = \{a_k^m,\ b_k^m : m \in \mathbb{Z},\ 1 \le k \le 2^{m-1}\}$, or equivalently,

$$E_3 = \left\{ \frac{k}{3^m}, \frac{k+1}{3^m} :\ m \ge 1,\ 0 < k < 3^m,\ k \text{ not divisible by } 3 \right\}.$$

Now, consider $y \in \operatorname{supp}(\mu_x^{U_x}(\cdot|V_x))$ such that $y \in \widetilde{H}^{-1}(E_3)$, that is, y is the preimage of an extreme of some open interval removed from the ternary Cantor set. Therefore, taking the pre-image by $\widetilde{H}$ of an open interval of $[0,1] \setminus \mathcal{C}_3$ having $\widetilde{H}(y)$ as one of its extremes, we obtain an open arc $\gamma_y \subset \mathcal{F}^c|U(x)$ with extreme in y and contained in the complement of $V_x \cap \operatorname{supp}(\mu_x^{U_x})$. In particular, γ_y has zero μ_y-measure and y as one of its extremes. As $y \in G_x^{U_x}$, there exists F such that $F(x) = y$ and $F_*\mu_x = \mu_y$; therefore, $F^{-1}(\gamma_y)$ is an open arc with zero μ_x-measure and extreme in x, that is, $x \in \widetilde{H}^{-1}(E_3)$. Analogously, for any other $z \in G_x^{U_x}$, there exists $F_{x,z} \in \mathcal{L}^+(\mathcal{F}^c(x))$ with $F_{x,z}(x) = z$ and $(F_{x,z})_*\mu_x = \mu_z$, which, by the same reasoning, implies that $z \in \widetilde{H}^{-1}(E_3)$. Therefore, $G_x^{U_x} = \widetilde{H}^{-1}(E_3)$, which is absurd since $E_3 \neq \mathcal{C}_3$. □

As $\mu(P) = 1$, then the previous lemma concludes that for μ-a.e. $x \in M$, the conditional measure $\mu_x^{U_x}$ cannot be supported on a Cantor subset of the plaque, which concludes the proof of Theorem B.

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