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Mixed Spectral Method for Heat Transfer Using Generalised Hermite Functions and Legendre Polynomials

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Abstract. We propose a mixed spectral method for heat transfer in unbounded domains, using generalised Hermite functions and Legendre polynomials. Some basic results on the mixed generalised Hermite-Legendre orthogonal approximation are established, which plays important roles in spectral methods for various problems defined on unbounded domains. As an example, the mixed generalised Hermite-Legendre spectral scheme is constructed for anisotropic heat transfer. Its convergence is proven, and some numerical results demonstrate the spectral accuracy of this approach.

AMS subject classifications: 33C45, 76M22, 65N12, 35K15, 35K20

Key words: The scaled generalised Hermite functions, mixed spectral method, anisotropic heat transfer, unbounded domains.

1. Introduction

Spectral methods have developed rapidly over the past three decades. The main merit is high accuracy, but earlier spectral methods were only applicable to bounded domains — cf. [1, 3–9] and references therein. However, increasingly attention turned to problems on unbounded domains, where direct approaches have been based upon certain orthogonal approximations — e.g. the Hermite and Laguerre spectral method now commonly used for differential equations on unbounded domains [1, 10–20, 22–30, 32–34]. Hermite polynomials have previously been used, but for some applications the solutions decay to zero at infinity — e.g. in heat transfer problems or when the differential equations involve variable coefficients. Since the weight function e^{-x^2} (or e^{x^2}) destroys crucial the symmetry and positive definiteness of bilinear operators [1, 12–14, 19], in such cases it is more appropriate to exploit Hermite functions with weight $\chi(x) \equiv 1$ — e.g. in the recent numerical simulation

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of the Dirac equation [16]. However, Tang [27] pointed out that standard Hermite functions sometimes work poorly in practice because too many Hermite functions are required, and consequently introduced the scaled generalised Hermite functions

$$H_n^\alpha(x) = \frac{1}{\sqrt{2^n n!}} H_n(\alpha x) e^{-\frac{1}{2}\alpha^2 x^2}, \quad \alpha > 0\,,$$

as basis functions that are mutually orthogonal with weight $\chi(x) \equiv 1$. Xiang & Wang [34] considered the spectral method for the generalised Ginzburg-Landau equation, using these scaled generalised Hermite functions. Tang [27] also considered how to choose a proper scaling factor for a class of functions that decay at infinity at least like $e^{-\gamma x^2}$ (with $\gamma > 0$).

We consider anisotropic heat transfer in an infinite plate, where the domain is $\Omega = \Lambda \times I$ where $\Lambda = \{x \mid -\infty < x < \infty\}$ and $I = \{y \mid |y| < 1\}$ and the boundary is denoted by $\Gamma = \{(x,y) \mid -\infty < x < \infty,\ |y| = 1\}$. For brevity, we write $\partial_z W = \partial W/\partial z$, etc. Then the anisotropic heat transfer in Ω is governed by

$$\begin{cases} \partial_t W(x,y,t) + a\partial_x W(x,y,t) + b\partial_y W(x,y,t) \\ \quad -\nu\partial_x^2 W(x,y,t) - \mu\partial_y^2 W(x,y,t) = F(x,y,t)\,, & (x,y) \in \Omega\,,\ 0 < t \le T\,, \\ W(x,y,t) = W_1(x,y,t)\,, & (x,y) \text{ on } \Gamma\,,\ 0 < t \le T\,, \\ W(x,y,0) = W_0(x,y)\,, & (x,y) \in \bar{\Omega}\,, \end{cases} \tag{1.1}$$

where $W(x,y,t)$ denotes the temperature, positive constants ν and μ are the conductivities in the respective directions x and y, the parameters a and b are convective constants, and the specified functions $F(x,y,t)$ and $W_0(x,y)$ describe the heat source and the initial state, respectively. Guo & Wang [19] used Hermite functions as the orthogonal basis functions (associated with the weight function e^{x^2}), and since $W(x,y,t)$ decays very fast as $|x| \to \infty$ it is proved better to adopt the generalised Hermite functions [13, 16, 19]. However, it is notable that problem (1.1) is not well posed in the weighted Sobolev space. Our aim here is to develop a mixed generalised Hermite-Legendre approximation with application to the numerical solution of problem (1.1) using the above scaled generalised Hermite functions $H_n^\alpha(x)$ and Legendre polynomials, associated with simple alternative Galerkin variational formulations. Since such a formulation can approximate the solution directly and be implemented easily, invoking the scaled generalised Hermite functions leads to a much simplified analysis and more precise error estimates [34]. Moreover, the adjustable parameter α offers great flexibility to match the asymptotic behaviour of the exact solutions at infinity [31].

In Section 2, we establish some results on the mixed generalised Hermite-Legendre approximation, which play important roles in the numerical analysis of spectral methods for various differential equations in unbounded domains. In Section 3, we construct our mixed generalised Hermite-Legendre spectral scheme for the problem (1.1) and prove its convergence. Some numerical results indicating the high accuracy of proposed algorithm in Section 4, and our concluding remarks are in Section 5.

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2. Mixed Generalised Hermite-Legendre Approximation

2.1. Hermite approximation

We first recall some results on the Hermite approximation. Let $\chi(x)$ be a certain weight function. For any integer $r \geq 0$, we define the space $H^r_\chi(\Lambda)$ with the inner product $(u,v)_{r,\chi,\Lambda}$, the semi-norm $|u|_{r,\chi,\Lambda}$ and the norm $\|u\|_{r,\chi,\Lambda}$. In particular, $H^0_\chi(\Lambda) = L^2_\chi(\Lambda)$, with the inner product $(u,v)_{\chi,\Lambda}$, and the norm $\|u\|_{\chi,\Lambda}$. For any $r > 0$, we define the space $H^r_\chi(\Lambda)$ and its norm $||u||_{r,\chi,\Lambda}$ by space interpolation, as in Ref. [2]. We omit the subscript χ in the notations whenever $\chi(x) \equiv 1$.

Let $H_n(x)$ be the usual Hermite polynomials – viz.

$$H_n(x) = (-1)^n e^{x^2} \partial_x^n (e^{-x^2}) \,,$$

which are mutually orthogonal with the weight function e^{-x^2}. The scaled generalised Hermite function of degree n is defined by [27, 34]

$$H_n^\alpha(x) = \frac{1}{\sqrt{2^n n!}} H_n(\alpha x) e^{-\frac{1}{2}\alpha^2 x^2} \,, \quad n \geq 0 \,, \tag{2.1}$$

where $\alpha > 0$ is a constant. The set of generalised Hermite functions $H_n^\alpha(x)$ forms a complete $L^2(\Lambda)$-orthogonal system with the weight function $\chi(x) \equiv 1$ — i.e.

$$\int_\Lambda H_m^\alpha(x) H_n^\alpha(x)\, dx = \frac{\sqrt{\pi}}{\alpha} \delta_{m,n} \,, \tag{2.2}$$

where $\delta_{m,n}$ is the Kronecker function [21]. The function $H_n^\alpha(x)$ is the n-th eigenfunction of the Sturm-Liouville problem where

$$e^{\frac{1}{2}\alpha^2 x^2} \partial_x \left(e^{-\alpha^2 x^2} \partial_x (e^{\frac{1}{2}\alpha^2 x^2} u(x)) \right) + \lambda_n u(x) = 0 \,, \tag{2.3}$$

with the corresponding eigenvalue $\lambda_n = \lambda_n(n,\alpha) := 2\alpha^2 n$.

Given $H_n^\alpha(x) \equiv 0$ for $n < 0$, for $n \geq 0$ it can be verified readily that

$$\partial_x H_n^\alpha(x) = -\frac{1}{2}\sqrt{\lambda_{n+1}} H_{n+1}^\alpha(x) + \frac{1}{2}\sqrt{\lambda_n} H_{n-1}^\alpha(x) \,, \tag{2.4}$$

$$\int_\Lambda \partial_x H_{n+2}^\alpha(x) \partial_x H_n^\alpha(x)\, dx = -\frac{\alpha}{2}\sqrt{\pi(n+1)(n+2)} \,, \tag{2.5}$$

$$\int_\Lambda (\partial_x H_n^\alpha(x))^2 dx = \sqrt{\pi}\alpha \left(n + \frac{1}{2} \right) \,, \tag{2.6}$$

$$\int_\Lambda \partial_x H_n^\alpha(x) \partial_x H_m^\alpha(x)\, dx = 0 \quad \text{if } n \neq m, n \neq m \pm 2 \,, \tag{2.7}$$

hence for any $u \in L^2(\Lambda)$ we can write

$$u(x) = \sum_{n=0}^{\infty} u_n H_n^\alpha(x) \,, \quad u_n = \frac{\alpha}{\sqrt{\pi}} \int_\Lambda u(x) H_n^\alpha(x)\, dx \,. \tag{2.8}$$

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For an integer $N > 0$, $\mathscr{P}_N(\Lambda)$ denotes the set of all polynomials of degree at most N; and

$$\mathscr{Q}_{N,\alpha}(\Lambda) = \text{ span } \left\{H_0^\alpha(x), \cdots, H_N^\alpha(x)\right\} = \left\{\psi e^{-\frac{1}{2}\alpha^2 x^2} \mid \psi \in \mathscr{P}_N(\Lambda)\right\} .$$

According to Lemma 2.1 of Ref. [34], for any $\phi \in \mathscr{Q}_{N,\alpha}(\Lambda)$

$$\|\partial_x^r \phi\|_\Lambda \le c\alpha^r N^{\frac{r}{2}} \, ||\phi||_\Lambda . \tag{2.9}$$

In order to obtain the approximation results, for any integer $r \ge 0$ we define the normed spaces [34]

$$H_{A,\alpha}^r(\Lambda) = \left\{ u \mid \|u\|_{r,A,\alpha,\Lambda} < \infty \right\} , \qquad \|u\|_{r,A,\alpha,\Lambda} = \left(\sum_{k=0}^{r} \left\| (\alpha^4 x^2 + \alpha^2)^{\frac{r-k}{2}} \partial_x^k u \right\|_\Lambda^2 \right)^{\frac{1}{2}} .$$

The $L^2(\Lambda)$-orthogonal projection $P_{N,\alpha,\Lambda} : L^2(\Lambda) \to \mathscr{Q}_{N,\alpha}(\Lambda)$ is defined by

$$(P_{N,\alpha,\Lambda} u - u, \phi)_\Lambda = 0 , \qquad \phi \in \mathscr{Q}_{N,\alpha}(\Lambda) .$$

According to Theorem 2.1 of Ref. [34], we also have the following result.

Lemma 2.1. *If $u \in H_{A,\alpha}^r(\Lambda)$ and integers $0 \le s \le r$, then*

$$\|u - P_{N,\alpha,\Lambda} u\|_{s,\Lambda} \le c\alpha^{s-r} N^{\frac{s-r}{2}} \|u\|_{r,A,\alpha,\Lambda} . \tag{2.10}$$

2.2. Legendre approximation

We now consider the Legendre approximation. For $r \ge 0$, we define the space $H^r(I)$ and its norm $\|u\|_{r,I}$ as usual, and in particular take $L^2(I) = H^0(I)$ with the inner product $(u, v)_I = (u, v)_{0,I}$ and the norm $\|u\|_I = \|u\|_{0,I}$. The Legendre polynomial of degree l is

$$L_l(y) = \frac{(-1)^l}{2^l l!} \partial_y^l (1 - y^2)^l ,$$

and the set of Legendre polynomials is a complete $L^2(I)$-orthogonal system.

Now, let M be any positive integer, and let $\mathscr{P}_M(I)$ denote the set of all polynomials of degree at most M. In the forthcoming discussion an unusual projection is needed, where [20]

$$\mathscr{F}(I) = L^2(I) \cap \{ u \mid \text{ there exists finite trace of } u \text{ at } y = \pm 1\} ,$$
$$\mathscr{F}_0(I) = \{ u \mid u \in \mathscr{F}(I) \text{ and } u(\pm 1) = 0\} , \qquad \mathscr{P}_M^0(I) = \mathscr{P}_M(I) \cap \mathscr{F}_0(I) .$$

Then the new orthogonal projection $P_{M,I}^0 : \mathscr{F}_0(I) \to \mathscr{P}_M^0(I)$ is defined by

$$(P_{M,I}^0 u - u, \phi)_I = 0 , \quad \forall \phi \in \mathscr{P}_M^0(I) .$$

According to Lemma 2.3 of Ref. [20], if $u \in \mathscr{F}_0(I), (1-y^2)^{\frac{q-1}{2}} \partial_y^q u \in L^2(I)$ and integer $q \ge 1$ then

$$\left\| P_{M,I}^0 u - u \right\|_I \le cM^{-q} \left\| (1-y^2)^{\frac{q-1}{2}} \partial_y^q u \right\|_I . \tag{2.11}$$

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Let $H_0^1(I) = \{u \in H^1(I) \text{ and } u(\pm 1) = 0\}$ and $\mathscr{P}_M^0(I) = H_0^1(I) \cap \mathscr{P}_M(I)$. The orthogonal projection $P_{M,I}^{1,0} : H_0^1(I) \to \mathscr{P}_M^0(I)$ is defined by

$$\left(\partial_y(P_{M,I}^{1,0}u - u), \partial_y\phi\right)_I = 0\,, \quad \forall \phi \in \mathscr{P}_M^0(I)\,.$$

According to Theorem 3.4 of Ref. [9], if $u \in H_0^1(I), (1-y^2)^{\frac{q-1}{2}}\partial_y^q u \in L^2(I)$ and integer $q \geq 1$ then

$$\left\|\partial_y^s(P_{M,I}^{1,0}u - u)\right\|_I \leq cM^{s-q}\left\|(1-y^2)^{\frac{q-1}{2}}\partial_y^q u\right\|_I\,, \quad s = 0, 1\,. \tag{2.12}$$

2.3. Mixed generalised Hermite-Legendre approximation

We now proceed to discuss the mixed Hermite-Legendre approximation. The space $L_\chi^2(\Omega)$ is defined in the usual way, with the inner product and norm

$$(u,v)_{\chi,\Omega} = \int\int_\Omega u(x,y)v(x,y)\chi(x,y)\,dxdy\,, \quad \|u\|_{\chi,\Omega} = (u,u)_{\chi,\Omega}^{\frac{1}{2}}\,,$$

and we omit the subscript χ whenever $\chi(x,y) \equiv 1$. Letting $\mathscr{F}_0(\Omega) = \mathscr{F}_0(I; L^2(\Lambda))$ and $V_{M,N,\alpha}^0(\Omega) = \mathscr{Q}_{N,\alpha}(\Lambda) \otimes \mathscr{P}_M^0(I)$, we consider the orthogonal projection $P_{M,N,\alpha,\Omega}^0 : \mathscr{F}_0(\Omega) \to V_{M,N,\alpha}^0(\Omega)$ defined by

$$(P_{M,N,\alpha,\Omega}^0 u - u, \phi)_\Omega = 0\,, \quad \forall \phi \in V_{M,N,\alpha}^0(\Omega)\,.$$

For estimating $||P_{M,N,\alpha,\Omega}^0 u - u||_\Omega$, we use

$$\mathbb{A}_{\alpha,\Omega}^{q,r}(u) = \int_\Lambda \left\|(1-y^2)^{\frac{q-1}{2}}\partial_y^q u\right\|_I^2 dx + \int_I \|u\|_{r,A,\alpha,\Lambda}^2 dy\,,$$

where $q \geq 1$ and $r \geq 0$ are integers.

Theorem 2.1. *For any $u \in \mathscr{F}_0(\Omega)$ and integers $q \geq 1, r \geq 0$,*

$$\left\|P_{M,N,\alpha,\Omega}^0 u - u\right\|_\Omega^2 \leq c(M^{-2q} + \alpha^{-2r}N^{-r})\mathbb{A}_{\alpha,\Omega}^{q,r}(u) \tag{2.13}$$

provided that $\mathbb{A}_{\alpha,\Omega}^{q,r}(u)$ is finite.

Proof. From the projection theorem,

$$\left\|P_{M,N,\alpha,\Omega}^0 u - u\right\|_\Omega^2 \leq \|\phi - u\|^2\,, \quad \forall \phi \in V_{M,N,\alpha}^0(\Omega)\,.$$

We take $\phi = (P_{M,I}^0 \cdot P_{N,\alpha,\Lambda})u$. From inequalities (2.10) and (2.11),

$$\begin{aligned}\|\phi - u\|^2 &\leq 2\left\|P_{M,I}^0(P_{N,\alpha,\Lambda}u) - P_{N,\alpha,\Lambda}u\right\|_\Omega^2 + 2\left\|P_{N,\alpha,\Lambda}u - u\right\|_\Omega^2 \\ &\leq cM^{-2q}\int_\Lambda \left\|(1-y^2)^{\frac{q-1}{2}}\partial_y^q P_{N,\alpha,\Lambda}u\right\|_I^2 dx + c\alpha^{-2r}N^{-r}\int_I \|u\|_{r,A,\alpha,\Lambda}^2 dy\,.\end{aligned}$$

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Moreover, from (2.10) with $r=s=0$ we have that

$$\int_\Lambda \left\| (1-y^2)^{\frac{q-1}{2}} \partial_y^q P_{N,\alpha,\Lambda} u \right\|_I^2 dx \le c \int_\Lambda \left\| (1-y^2)^{\frac{q-1}{2}} \partial_y^q u \right\|_I^2 dx .$$

The desired result follows immediately. □

In the numerical analysis of mixed spectral methods for boundary value problems, we need another orthogonal projection. For this purpose, we introduce the bilinear form

$$a_{\beta,\gamma,\Omega}(u,v) = \beta(\partial_x u, \partial_x v)_\Omega + \gamma(\partial_y u, \partial_y v)_\Omega + (u,v)_\Omega , \quad \beta,\gamma > 0 , \ \forall u, v \in H_0^1(\Omega) .$$

The orthogonal projection $P^{1,0}_{M,N,\alpha,\beta,\gamma,\Omega} : H_0^1(\Omega) \to V^0_{M,N,\alpha}(\Omega)$ is defined by

$$a_{\beta,\gamma,\Omega}(P^{1,0}_{M,N,\alpha,\beta,\gamma,\Omega} u - u, \phi) = 0 , \quad \forall \phi \in V^0_{M,N,\alpha}(\Omega) . \tag{2.14}$$

In describing approximation results, for integers $r \ge 1, q \ge 2$ we use

$$\begin{aligned}
&\mathbb{D}^{q,r}_{\alpha,\beta,\gamma,\Omega}(u) \\
&= \int_I \left(\beta ||u||^2_{r,A,\alpha,\Lambda} + ||u||^2_{r-1,A,\alpha,\Lambda} + \beta(1-y^2)^{q-2} ||\partial_y^{q-1} u||^2_{1,A,\alpha,\Lambda} + \gamma ||\partial_y u||^2_{r-1,A,\alpha,\Lambda} \right) dy \\
&\quad + \int_\Lambda \left(\beta ||(1-y^2)^{\frac{q-2}{2}} \partial_x \partial_y^{q-1} u||_I^2 + ||(1-y^2)^{\frac{q-2}{2}} \partial_y^{q-1} u||_I^2 + \gamma ||(1-y^2)^{\frac{q-1}{2}} \partial_y^q u||_I^2 \right) dx .
\end{aligned}$$

Theorem 2.2. *For any $u \in H_0^1(\Omega)$ and integers $r \ge 1, q \ge 2$*

$$\begin{aligned}
&\beta \left\| \partial_x (P^{1,0}_{M,N,\alpha,\beta,\gamma,\Omega} u - u) \right\|_\Omega^2 + \gamma \left\| \partial_y (P^{1,0}_{M,N,\alpha,\beta,\gamma,\Omega} u - u) \right\|_\Omega^2 + \left\| P^{1,0}_{M,N,\alpha,\beta,\gamma,\Omega} u - u \right\|_\Omega^2 \\
&\le c \left(M^{2-2q} + (\alpha^2 N)^{1-r} \right) \mathbb{D}^{q,r}_{\alpha,\beta,\gamma,\Omega}(u) ,
\end{aligned} \tag{2.15}$$

provided that $\mathbb{D}^{q,r}_{\alpha,\beta,\gamma,\Omega}(u)$ is finite.

Proof. From the projection theorem,

$$\begin{aligned}
&\beta \left\| \partial_x (P^{1,0}_{M,N,\alpha,\beta,\gamma,\Omega} u - u) \right\|_\Omega^2 + \gamma \left\| \partial_y (P^{1,0}_{M,N,\alpha,\beta,\gamma,\Omega} u - u) \right\|_\Omega^2 + \left\| P^{1,0}_{M,N,\alpha,\beta,\gamma,\Omega} u - u \right\|_\Omega^2 \\
&\le \beta \left\| \partial_x (\phi - u) \right\|_\Omega^2 + \gamma \left\| \partial_y (\phi - u) \right\|_\Omega^2 + \left\| \phi - u \right\|_\Omega^2 , \ \forall \phi \in V^0_{M,N,\alpha}(\Omega) .
\end{aligned} \tag{2.16}$$

Taking $\phi = P_{N,\alpha,\Lambda}(P^{1,0}_{M,I} u)$, we have

$$\beta \left\| \partial_x (\phi - u) \right\|_\Omega^2 + \gamma \left\| \partial_y (\phi - u) \right\|_\Omega^2 + \left\| \phi - u \right\|_\Omega^2 \le F_1(u) + F_2(u) + F_3(u) + F_4(u) ,$$

where

$$\begin{aligned}
F_1(u) &= 2\beta ||\partial_x (P_{N,\alpha,\Lambda} u - u)||_\Omega^2 + 2 \|P_{N,\alpha,\Lambda} u - u\|_\Omega^2 , \\
F_2(u) &= 2\beta ||\partial_x (P^{1,0}_{M,I}(P_{N,\alpha,\Lambda} u) - P_{N,\alpha,\Lambda} u)||_\Omega^2 , \\
F_3(u) &= 2\gamma ||\partial_y (P^{1,0}_{M,I}(P_{N,\alpha,\Lambda} u) - P_{N,\alpha,\Lambda} u)||_\Omega^2 + 2 ||P^{1,0}_{M,I}(P_{N,\alpha,\Lambda} u) - P_{N,\alpha,\Lambda} u||_\Omega^2 , \\
F_4(u) &= 2\gamma ||\partial_y (P_{N,\alpha,\Lambda} u - u)||_\Omega^2 .
\end{aligned}$$

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From inequality (2.10) with $s = 1, 0$ we have

$$F_1(u) \le c\alpha^{2-2r} N^{1-r} \int_I \left(\beta ||u||^2_{r,A,\alpha,\Lambda} + ||u||^2_{r-1,A,\alpha,\Lambda}\right) dy\,,$$

so inequalities (2.12) with $s = 0$ and (2.10) with $s = 1$, $r = 1$ imply

$$\begin{aligned} F_2(u) \le & c\beta M^{2-2q} \int_\Lambda \left\| (1-y^2)^{\frac{q-2}{2}} \partial_x \partial_y^{q-1} (P_{N,\alpha,\Lambda} u) \right\|_I^2 dx \\ \le & c\beta M^{2-2q} \int_\Lambda \left\| (1-y^2)^{\frac{q-2}{2}} \partial_x \partial_y^{q-1} u \right\|_I^2 dx \\ & + c\beta M^{2-2q} \int_I (1-y^2)^{q-2} \left\| \partial_y^{q-1} u \right\|^2_{1,A,\alpha,\Lambda} dy\,. \end{aligned}$$

Using inequality (2.12) with $s = 1, 0$ and (2.10) with $s = 0$, $r = 0$ again, we then obtain

$$\begin{aligned} F_3(u) & \le cM^{2-2q} \int_\Lambda \left(\gamma \left\| (1-y^2)^{\frac{q-1}{2}} \partial_y^q (P_{N,\alpha,\Lambda} u) \right\|_I^2 + \left\| (1-y^2)^{\frac{q-2}{2}} \partial_y^{q-1} (P_{N,\alpha,\Lambda} u) \right\|_I^2 \right) dx \\ & \le cM^{2-2q} \int_\Lambda \left(\gamma \left\| (1-y^2)^{\frac{q-1}{2}} \partial_y^q u \right\|_I^2 + \left\| (1-y^2)^{\frac{q-2}{2}} \partial_y^{q-1} u \right\|_I^2 \right) dx\,, \end{aligned}$$

and inequality (2.10) with $s = 0$ yields

$$F_4(u) \le c\gamma\alpha^{2-2r} N^{1-r} \int_I ||\partial_y u||^2_{r-1,A,\alpha,\Lambda}\, dy\,.$$

The desired result comes from a combination of the previous statements. □

3. Mixed Hermite-Legendre Spectral Scheme

We now consider the mixed Hermite-Legendre spectral method for (1.1), together with the convergence analysis.

3.1. Semi-discrete scheme

Letting

$$W_1(x, 1, t) = g_1(x, t)\,, \qquad W_1(x, -1, t) = g_2(x, t)\,,$$

we define the function corresponding to the boundary Γ by

$$W_B(x, y, t) = \frac{1}{2}\big((1+y)g_1(x, t) + (1-y)g_2(x, t)\big)\,. \tag{3.1}$$

Obviously, $W_B(x, y, t) = W(x, y, t)$ on Γ, and $W_B(x, y, t) \to 0$ when $|x| \to \infty$. Now, we make the variable transformation

$$W(x, y, t) = U(x, y, t) + W_B(x, y, t)\,,$$

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and

$$\begin{aligned} f(x,y,t) =& F(x,y,t) - \partial_t W_B(x,y,t) - a\partial_x W_B(x,y,t) \\ & - b\partial_y W_B(x,y,t) + \nu\partial_x^2 W_B(x,y,t) + \mu\partial_y^2 W_B(x,y,t)\,. \end{aligned}$$

Then the problem (1.1) can be rewritten

$$\begin{cases} \partial_t U(x,y,t) + a\partial_x U(x,y,t) + b\partial_y U(x,y,t) & \\ \quad -\nu\partial_x^2 U(x,y,t) - \mu\partial_y^2 U(x,y,t) = f(x,y,t)\,, & (x,y)\in\Omega, 0<t\le T\,, \\ U(x,y,t) = 0\,, & (x,y) \text{ on } \Gamma, 0<t\le T\,, \\ U(x,y,t)\to 0\,, & |x|\to\infty, 0<t\le T\,, \\ U(x,y,0) = U_0(x,y)\,, & (x,y)\in\Omega\cup\Gamma\,, \end{cases} \tag{3.2}$$

where $U_0(x,y) = W_0(x,y) - W_B(0)$ with $W_B(0) = W_B(x,y,0)$.

A weak formulation of (3.2) is to seek solution $U\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$ such that

$$\begin{cases} (\partial_t U(t),v)_\Omega + a(\partial_x U(t),v)_\Omega + b(\partial_y U(t),v)_\Omega + \nu(\partial_x U(t),\partial_x v)_\Omega & \\ \quad +\mu(\partial_y U(t),\partial_y v)_\Omega = (f(t),v)_\Omega\,,\ \forall v\in H_0^1(\Omega), 0<t\le T\,, \\ U(0) = U_0\,. \end{cases} \tag{3.3}$$

By a standard compactness proof, it can be demonstrated that if $U_0\in L^2(\Omega)$, then (3.3) has a unique solution. The mixed spectral scheme for (3.3) is to find $u_{M,N}\in V_{M,N,\alpha}^0(\Omega)$, such that

$$\begin{cases} (\partial_t u_{M,N}(t),\phi)_\Omega + a(\partial_x u_{M,N}(t),\phi)_\Omega + b(\partial_y u_{M,N}(t),\phi)_\Omega + \nu(\partial_x u_{M,N}(t),\partial_x\phi)_\Omega & \\ \quad +\mu(\partial_y u_{M,N}(t),\partial_y\phi)_\Omega = (f(t),\phi)_\Omega\,,\ \forall\phi\in V_{M,N,\alpha}^0(\Omega),\ 0<t\le T\,, \\ u_{M,N}(0) = P_{M,N,\alpha,\Omega}^0 U_0\,. \end{cases} \tag{3.4}$$

The numerical solution of problem (1.1) is given by

$$w_{M,N}(x,y,t) = u_{M,N}(x,y,t) + W_B(x,y,t)\,. \tag{3.5}$$

We now discuss the convergence of scheme (3.4). Letting $U_{M,N} = P_{M,N,\alpha,\nu,\mu\Omega}^{1,0} U$, from (3.3) we have

$$\begin{cases} (\partial_t U_{M,N}(t),\phi)_\Omega + a(\partial_x U_{M,N}(t),\phi)_\Omega + b(\partial_y U_{M,N}(t),\phi)_\Omega & \\ \quad +\nu(\partial_x U_{M,N}(t),\partial_x\phi)_\Omega + \mu(\partial_y U_{M,N}(t),\partial_y\phi)_\Omega + \sum_{j=1}^4 G_j(t,\phi) & \\ \quad = (f(t),\phi)_\Omega\,,\ \forall\phi\in V_{M,N,\alpha}^0(\Omega)\,,\ 0<t\le T\,, \\ U_{M,N}(0) = P_{M,N,\alpha,\nu,\mu,\Omega}^{1,0} U_0\,, \end{cases} \tag{3.6}$$

where

$$\begin{aligned} G_1(t,\phi) &= \big(\partial_t(U(t)-U_{M,N}(t)),\phi\big)_\Omega\,, & G_2(t,\phi) &= a\big(\partial_x(U(t)-U_{M,N}(t)),\phi\big)_\Omega\,, \\ G_3(t,\phi) &= b\big(\partial_y(U(t)-U_{M,N}(t)),\phi\big)_\Omega\,, & G_4(t,\phi) &= \big(U_{M,N}(t)-U(t),\phi\big)_\Omega\,. \end{aligned}$$

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Letting $\widetilde{U}_{M,N} = u_{M,N} - U_{M,N}$ and subtracting (3.6) from (3.4) yields

$$\begin{cases} (\partial_t \widetilde{U}_{M,N}(t), \phi)_\Omega + a(\partial_x \widetilde{U}_{M,N}(t), \phi)_\Omega + b(\partial_y \widetilde{U}_{M,N}(t), \phi)_\Omega \\ \quad + \nu(\partial_x \widetilde{U}_{M,N}(t), \partial_x \phi)_\Omega + \mu(\partial_y \widetilde{U}_{M,N}(t), \partial_y \phi)_\Omega \\ \quad = \sum_{j=1}^{4} G_j(t, \phi)\,,\ \forall \phi \in V^0_{M,N,\alpha}(\Omega)\,,\ 0 < t \le T\,, \\ \widetilde{U}_{M,N}(0) = P^0_{M,N,\alpha,\Omega} U_0 - P^{1,0}_{M,N,\alpha,\nu,\mu,\Omega} U_0\,. \end{cases} \tag{3.7}$$

Taking $\phi = 2\widetilde{U}_{M,N}$ in (3.7), we deduce that

$$\begin{aligned} &\partial_t \left\|\widetilde{U}_{M,N}(t)\right\|^2_\Omega + 2\nu \left\|\partial_x \widetilde{U}_{M,N}(t)\right\|^2_\Omega + 2\mu \left\|\partial_y \widetilde{U}_{M,N}(t)\right\|^2_\Omega \\ =&2\sum_{j=1}^{4} G_j(t, \widetilde{U}_{M,N})\,, \quad 0 < t \le T\,, \end{aligned} \tag{3.8}$$

therefore it suffices to estimate the terms $|\, G_j(t, \widetilde{U}_{M,N})\,|$. We first use the Cauchy inequality and inequality (2.15) to verity that for integers $r \ge 1, q \ge 2$

$$\begin{aligned} 2\left|G_1(t, \widetilde{U}_{M,N}(t))\right| =& 2\left|(\partial_t U(t) - \partial_t U_{M,N}(t), \widetilde{U}_{M,N}(t))_\Omega\right| \\ \le& \left\|\partial_t (U(t) - U_{M,N}(t))\right\|^2_\Omega + \left\|\widetilde{U}_{M,N}(t)\right\|^2_\Omega \\ \le& c(M^{2-2q} + (\alpha^2 N)^{1-r}) \mathbb{D}^{q,r}_{\alpha,\nu,\mu,\Omega}(\partial_t U(t)) + \left\|\widetilde{U}_{M,N}(t)\right\|^2_\Omega\,. \end{aligned} \tag{3.9}$$

Then using (2.15) again, we have

$$\begin{aligned} &2\left|G_2(t, \widetilde{U}_{M,N}(t)) + G_3(t, \widetilde{U}_{M,N}(t)) + G_4(t, \widetilde{U}_{M,N}(t))\right| \\ \le& a\left\|\partial_x (U(t) - U_{M,N}(t))\right\|^2_\Omega + b\left\|\partial_y (U(t) - U_{M,N}(t))\right\|^2_\Omega + \left\|U(t) - U_{M,N}(t)\right\|^2_\Omega \\ &+ (a + b + 1)\left\|\widetilde{U}_{M,N}(t)\right\|^2_\Omega \\ \le& c \max\left(\frac{a}{\nu}, \frac{b}{\mu}, 1\right)\left(M^{2-2q} + \left(\alpha^2 N\right)^{1-r}\right) \mathbb{D}^{q,r}_{\alpha,\nu,\mu,\Omega}(U(t)) + (a + b + 1)\left\|\widetilde{U}_{M,N}(t)\right\|^2_\Omega\,. \end{aligned} \tag{3.10}$$

Furthermore, from inequalities (2.13) and (2.15) we obtain

$$\begin{aligned} \left\|\widetilde{U}_{M,N}(0)\right\|^2_\Omega \le& 2\left\|P^0_{M,N,\alpha,\Omega} U_0 - U_0\right\|^2_\Omega + 2\left\|U_0 - P^{1,0}_{M,N,\alpha,\nu,\mu,\Omega} U_0\right\|^2_\Omega \\ \le& c\left(M^{2-2q} + (\alpha^2 N)^{1-r}\right)\left(\mathbb{A}^{q-1,r-1}_{\alpha,\Omega}(U_0) + \mathbb{D}^{q,r}_{\alpha,\nu,\mu,\Omega}(U_0)\right)\,. \end{aligned} \tag{3.11}$$

For simplicity, let us write

$$E_{\nu,\mu}(u(t)) = \|u(t)\|^2_\Omega + 2\int_0^t \left(\nu\|\partial_x u(\xi)\|^2_\Omega + \mu\|\partial_y u(\xi)\|^2_\Omega\right) d\xi\,. \tag{3.12}$$

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Invoking (3.9) and (3.10) in Eq. (3.8), we get

$$\partial_t E_{\nu,\mu}(\widetilde{U}_{M,N}(t)) \le c_1 \|\widetilde{U}_{M,N}(t)\|_{\Omega}^2 + c_2 \left(M^{2-2q} + (\alpha^2 N)^{1-r}\right) \cdot \left(\mathbb{D}^{q,r}_{\alpha,\nu,\mu,\Omega}(\partial_t U(t)) + \mathbb{D}^{q,r}_{\alpha,\nu,\mu,\Omega}(U(t))\right), \tag{3.13}$$

where

$$c_1 = a + b + 2, \qquad c_2 = c\left(1 + \max\left(\frac{a}{\nu}, \frac{b}{\mu}, 1\right)\right).$$

Then

$$\begin{aligned} &\partial_t E_{\nu,\mu}(\widetilde{U}_{M,N}(t)) \\ \le& c_1 E_{\nu,\mu}\left(\widetilde{U}_{M,N}(t)\right) + c_2 \left(M^{2-2q} + (\alpha^2 N)^{1-r}\right)\left(\mathbb{D}^{q,r}_{\alpha,\nu,\mu,\Omega}(\partial_t U(t)) + \mathbb{D}^{q,r}_{\alpha,\nu,\mu,\Omega}(U(t))\right), \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\partial_t \left(E_{\nu,\mu}\left(\widetilde{U}_{M,N}(t)\right) e^{-c_1 t}\right) \\ \le& c_2 e^{-c_1 t}\left(M^{2-2q} + (\alpha^2 N)^{1-r}\right)\left(\mathbb{D}^{q,r}_{\alpha,\nu,\mu,\Omega}(\partial_t U(t)) + \mathbb{D}^{q,r}_{\alpha,\nu,\mu,\Omega}(U(t))\right). \end{aligned} \tag{3.14}$$

Integrating (3.14) with respect to t and using inequality (3.11), we obtain

$$E_{\nu,\mu}\left(\widetilde{U}_{M,N}(t)\right) \le c\left(M^{2-2q} + (\alpha^2 N)^{1-r}\right) R_{\Omega}(U, U_0, q, r, \alpha, \nu, \mu, t), \tag{3.15}$$

where

$$\begin{aligned} R_{\Omega}(U, U_0, q, r, \alpha, \nu, \mu, t) =& e^{c_1 t}\Bigg(\bar{c}_2 \int_0^t e^{-c_1 \xi}\left(\mathbb{D}^{q,r}_{\alpha,\nu,\mu,\Omega}(\partial_\xi U(\xi)) + \mathbb{D}^{q,r}_{\alpha,\nu,\mu,\Omega}(U(\xi))\right) d\xi \\ &+ \mathbb{A}^{q-1,r-1}_{\alpha,\Omega}(U_0) + \mathbb{D}^{q,r}_{\alpha,\nu,\mu,\Omega}(U_0)\Bigg), \end{aligned}$$

with $\bar{c}_2 = 1 + \max(a/\nu, b/\mu, 1)$. A combination of (2.15) with $\beta = \nu, \gamma = \mu$ and (3.15) yields

$$\begin{aligned} E_{\nu,\mu}\left(U(t) - u_{M,N}(t)\right) \le& c\left(M^{2-2q} + (\alpha^2 N)^{1-r}\right)\Bigg(R_{\Omega}(U, U_0, q, r, \alpha, \nu, \mu, t) \\ &+ \mathbb{D}^{q,r}_{\alpha,\nu,\mu,\Omega}(U(t)) + \int_0^t \mathbb{D}^{q,r}_{\alpha,\nu,\mu,\Omega}(U(\xi))\, d\xi\Bigg). \end{aligned} \tag{3.16}$$

From Eq. (3.1) and inequality (3.16), for $0 \le t \le T$ we then finally obtain

$$\begin{aligned} &E_{\nu,\mu}(W(t) - w_{M,N}(t)) \\ \le& c\left(M^{2-2q} + (\alpha^2 N)^{1-r}\right)\Bigg(R_{\Omega}(W, W_0, q, r, \alpha, \nu, \mu, t) + \mathbb{D}^{q,r}_{\alpha,\nu,\mu,\Omega}(W(t)) \\ &+ \int_0^t \mathbb{D}^{q,r}_{\alpha,\nu,\mu,\Omega}(W(\xi))\, d\xi + R_{\Omega}(W_B, W_B(0), q, r, \alpha, \nu, \mu, t) + \mathbb{D}^{q,r}_{\alpha,\nu,\mu,\Omega}(W_B(t)) \\ &+ \int_0^t \mathbb{D}^{q,r}_{\alpha,\nu,\mu,\Omega}(W_B(\xi))\, d\xi\Bigg). \end{aligned} \tag{3.17}$$

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3.2. Fully-discrete scheme

Let $[T]$ denote the integer part of any fixed positive constant T and τ the mesh size of the variable t, and write

$$R_\tau(T)=\left\{\ t=k\tau \mid k=1,2,\cdots,\left[\frac{T}{\tau}\right]\right\},\quad \bar{R}_\tau(T)=R_\tau(T)\cup\{0\}\,,$$
$$D_\tau u(t)=\frac{1}{\tau}\big(u(t+\tau)-u(t)\big),\qquad \hat{u}(t)=\frac{1}{2}\big(u(t+\tau)+u(t)\big)\,.$$

To show the convergence with respect to time t, we consider the fully-discrete scheme of (3.3) — i.e. seek $u_{M,N}(t)\in V^0_{M,N,\alpha}(\Omega)$ such that [30]

$$\begin{cases}(D_\tau u_{M,N}(t),\phi)_\Omega+a(\partial_x\hat{u}_{M,N}(t),\phi)_\Omega+b(\partial_y\hat{u}_{M,N}(t),\phi)_\Omega\\ \quad+\nu(\partial_x\hat{u}_{M,N}(t),\partial_x\phi)_\Omega+\mu(\partial_y\hat{u}_{M,N}(t),\partial_y\phi)_\Omega=(\hat{f}(t),\phi)_\Omega\,,\\ \quad\forall\ \phi\in V^0_{M,N,\alpha}(\Omega)\,,\ t\in R_\tau(T-\tau)\,,\\ u_{M,N}(x,y,0)=P^0_{M,N,\alpha,\Omega}U_0(x,y)\,.\end{cases}\tag{3.18}$$

We now consider the convergence of scheme (3.18). Let $U_{M,N}=P^{1,0}_{M,N,\alpha,\nu,\mu\Omega}U$. From (3.4),

$$\begin{cases}(D_\tau U_{M,N}(t),\phi)_\Omega+a(\partial_x\hat{U}_{M,N}(t),\phi)_\Omega+b(\partial_y\hat{U}_{M,N}(t),\phi)_\Omega+\nu(\partial_x\hat{U}_{M,N}(t),\partial_x\phi)_\Omega\\ \quad+\mu(\partial_y\hat{U}_{M,N}(t),\partial_y\phi)_\Omega=\sum_{j=j}^{5}\bar{G}_j(t,\phi)+(\hat{f}(t),\phi)_\Omega\,,\\ \quad\forall\ \phi\in V^0_{M,N,\alpha}(\Omega)\,,\ t\in R_\tau(T-\tau)\,,\\ U_{M,N}(x,y,0)=P^{1,0}_{M,N\,,\alpha,\nu,\mu,\Omega}U_0(x,y)\,,\end{cases}\tag{3.19}$$

where

$$\bar{G}_1(t,\phi)=\Big(D_\tau U_{M,N}(t)-\partial_t U(t+\frac{\tau}{2}),\phi\Big)_\Omega\,,\quad \bar{G}_2(t,\phi)=a\Big(\partial_x(\hat{U}_{M,N}(t)-U(t+\frac{\tau}{2})),\phi\Big)_\Omega,$$
$$\bar{G}_3(t,\phi)=b\Big(\partial_y(\hat{U}_{M,N}(t)-U(t+\frac{\tau}{2})),\phi\Big)_\Omega\,,\ \bar{G}_4(t,\phi)=\Big(U(t+\frac{\tau}{2})-\hat{U}_{M,N}(t),\phi\Big)_\Omega\,,$$
$$\bar{G}_5(t,\phi)=\Big(f(t+\frac{\tau}{2})-\hat{f}(t),\phi\Big)_\Omega\,.$$

Writing $\widetilde{U}_{M,N}=u_{M,N}-U_{M,N}$ and subtracting (3.19) from (3.18), we obtain

$$\begin{cases}(D_\tau\widetilde{U}_{M,N}(t),\phi)_\Omega+a(\partial_x\hat{\widetilde{U}}_{M,N}(t),\phi)_\Omega+b(\partial_y\hat{\widetilde{U}}_{M,N}(t),\phi)_\Omega+\nu(\partial_x\hat{\widetilde{U}}_{M,N}(t),\partial_x\phi)_\Omega\\ \quad+\mu(\partial_y\hat{\widetilde{U}}_{M,N}(t),\partial_y\phi)_\Omega=-\sum_{j=1}^{5}\bar{G}_j(t,\phi)\,,\ \forall\ \phi\in V^0_{M,N,\alpha}(\Omega)\,,\ \ t\in R_\tau(T-\tau)\,,\\ \widetilde{U}_{M,N}(x,y,0)=P^0_{M,N,\alpha,\Omega}U_0(x,y)-P^{1,0}_{M,N,\alpha,\nu,\mu,\Omega}U_0(x,y)\,.\end{cases}\tag{3.20}$$

Taking $\phi=2\hat{\widetilde{U}}_{M,N}(t)$ in (3.20), we obtain

$$D_\tau\|\widetilde{U}_{M,N}(t)\|^2_\Omega+2\nu\|\partial_x\hat{\widetilde{U}}_{M,N}(t)\|^2_\Omega+2\mu\|\partial_y\hat{\widetilde{U}}_{M,N}(t)\|^2_\Omega=-2\sum_{j=1}^{7}\bar{G}_j(t,\phi)\,,\tag{3.21}$$

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where

$$\bar{G}_6(t,\phi)=a(\partial_x\hat{\tilde{U}}_{M,N}(t)\,,\hat{\tilde{U}}_{M,N}(t))_\Omega\,,\quad \bar{G}_7(t,\phi)=b(\partial_y\hat{\tilde{U}}_{M,N}(t),\hat{\tilde{U}}_{M,N}(t))_\Omega\,.$$

Following Ref. [30] as in the derivation of (3.27), for $s\in\bar{R}_\tau(t-\tau)$ we obtain

$$\begin{aligned}&\left(1-\tau\left(2T+1+\frac{a^2}{\nu}+\frac{b^2}{\mu}\right)\right)\|\widetilde{U}_{M,N}(s)\|_\Omega^2+\tau\sum_{s\in\bar{R}_\tau(t-\tau)}\left(\nu\|\partial_x\hat{\tilde{U}}_{M,N}(s)\|_\Omega^2+\mu\|\partial_y\hat{\tilde{U}}_{M,N}(s)\|_\Omega^2\right)\\ \le&\tau\sum_{s\in\bar{R}_\tau(t-\tau)}\left(2T+1+\frac{a^2}{\nu}+\frac{b^2}{\mu}\right)\|\widetilde{U}_{M,N}(s)\|_\Omega^2+c\tau^4\left(\|U\|^2_{H^3(0,T;L^2(\Omega))}+\|f\|^2_{H^2(0,T;L^2(\Omega))}\right)\\ &+cc^*\left(M^{2-2q}+(\beta N)^{1-r}\right)\\ &\cdot\sum_{s\in\bar{R}_\tau(t-\tau)}\left(\mathbb{D}^{q,r}_{\beta,\Omega}(\hat{U}(s))+\mathbb{D}^{q,r}_{\alpha,\nu,\mu,\Omega}(\partial_s\hat{U}(s))+\mathbb{D}^{q,r}_{\alpha,\nu,\mu,\Omega}(\hat{U}_0)+\mathbb{A}^{q-1,r-1}_{\alpha,\Omega}(U_0)\right),\end{aligned}\tag{3.22}$$

with $c^*=\nu+\mu+1$. Let

$$\begin{aligned}E(v,t)&=\|v(t)\|_\Omega^2+\tau\sum_{s\in\bar{R}_\tau(t-\tau)}\left(\nu\|\partial_x\hat{v}(s)\|_\Omega^2+\mu\|\partial_y(\hat{v}(s)\|_\Omega^2\right),\\ \rho(t)&=c^*\sum_{s\in\bar{R}_\tau(t-\tau)}\left(\mathbb{D}^{q,r}_{\alpha,\nu,\mu,\Omega}(\hat{U}(s))+\mathbb{D}^{q,r}_{\alpha,\nu,\mu,\Omega}(\partial_s\hat{U}(s))+\mathbb{D}^{q,r}_{\alpha,\nu,\mu,\Omega}(\hat{U}_0)+\mathbb{A}^{q-1,r-1}_{\alpha,\Omega}(U_0)\right)\\ &\quad+\|U\|^2_{H^3(0,T;L^2(\Omega))}+\|f\|^2_{H^2(0,T;L^2(\Omega))}\,.\end{aligned}$$

Finally, we apply Lemma 3.2 of Ref. [7] to (3.22) to obtain the following result.

If $\tau<(2T+1+a^2/\nu+b^2/\mu)^{-1}$, then for all $t\in\bar{R}_\tau(T)$ we have

$$E(U_{M,N}-u_{M,N},t)\le c\rho(t)e^{(2T+\frac{a^2}{\nu}+\frac{b^2}{\mu})t}\left(\tau^4+M^{2-2q}+(\alpha^2N)^{1-r}\right)\tag{3.23}$$

provided that $\rho(t)$ is finite.

4. Numerical Results

We now proceed to implementations and some numerical results that illustrate our theoretical analysis in the previous section. We use the Crank-Nicolson discretisation in time t with the step size τ, and for simplicity write

$$a_\Omega(z,\phi)=a(\partial_x z,\phi)_\Omega+b(\partial_y z,\phi)_\Omega+\nu(\partial_x z,\partial_x\phi)_\Omega+\mu(\partial_y z,\partial_y\phi)_\Omega\,,$$

and thus express the fully discrete scheme (3.18) as

$$\begin{cases}\dfrac{1}{\tau}(u_{M,N}(t+\tau)-u_{M,N}(t),\phi)_\Omega+\dfrac{1}{2}a_\Omega(u_{M,N}(t+\tau)+u_{M,N}(t),\phi)\\ \qquad=\dfrac{1}{2}(f(t+\tau)+f(t),\phi)_\Omega\,,\ t=0,\tau,2\tau,\cdots,T-\tau\,,\ \forall\phi\in V^0_{M,N,\alpha}(\Omega)\,,\\ u_{M,N}(0)=P^0_{M,N,\alpha,\Omega}U_0\,.\end{cases}\tag{4.1}$$

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Letting

$$A_\Omega(z,\phi)=\frac{1}{2}\tau a_\Omega(z,\phi)+(z,\phi)_\Omega\,,\quad G_\Omega(z,\phi)=-\frac{1}{2}\tau a_\Omega(z,\phi)+(z,\phi)_\Omega\,,$$

at each time step we solve the equation

$$A_\Omega(u_{M,N}(t),\phi)=G_\Omega(u_{M,N}(t-\tau),\phi)+\frac{1}{2}\tau(f(t)+f(t-\tau),\phi)_\Omega\,,\ \forall\phi\in V^0_{M,N,\alpha}(\Omega)\,. \quad (4.2)$$

For notational convenience, we also write

$$\zeta_k(y)=L_k(y)-L_{k+2}(y)\,,\quad 0\le k\le M-2\,.$$

In the computation, we write the numerical solution as

$$u_{M,N,\tau}(x,y,t)=\sum_{l=0}^{N}\sum_{k=0}^{M-2}a_{k,l}(t)H_l^\alpha(x)\zeta_k(y)\,.$$

Letting

$$f_{k'l'}=\big(f(t),H_{l'}^\alpha(x)\zeta_{k'}(y)\big)_\Omega\,,\quad 0\le k'\le M-2\,,\ 0\le l'\le N\,,$$

and defining the vectors

$$\begin{aligned}X(t)=\Big(&a_{0,0}(t),a_{1,0}(t),\cdots,a_{M-2,0}(t),a_{0,1}(t),a_{1,1}(t),\cdots,a_{M-2,1}(t),\\&\cdots,a_{0,N}(t),a_{1,N}(t),\cdots,a_{M-2,N}(t)\Big)^T,\\F_1(t)=\Big(&f_{0,0}(t),f_{1,0}(t),\cdots,f_{M-2,0}(t),f_{0,1}(t),f_{1,1}(t),\cdots,f_{M-2,1}(t)\,,\\&\cdots,f_{0,N}(t),f_{1,N}(t),\cdots,f_{M-2,N}(t)\Big)^T.\end{aligned}$$

On taking $\phi=H_{l'}^\alpha(x)\zeta_{k'}(y)$ in Eq. (4.2) for $0\le k'\le M-2$, $0\le l'\le N$, we find that the scheme (3.4) can be written as the matrix form

$$C_{11}X(t)=\widetilde{C}_{11}X(t-\tau)+\frac{1}{2}\tau(F_1(t)+F_1(t-\tau))\,, \quad (4.3)$$

where

$$\begin{aligned}C_{11}&=A_{11}\otimes B_{11}+\frac{1}{2}\tau\big(a(A_{12}\otimes B_{12})+b(A_{13}\otimes B_{13})+\nu(A_{14}\otimes B_{14})+\mu(A_{15}\otimes B_{15})\big)\,,\\\widetilde{C}_{11}&=A_{11}\otimes B_{11}-\frac{1}{2}\tau\big(a(A_{12}\otimes B_{12})+b(A_{13}\otimes B_{13})+\nu(A_{14}\otimes B_{14})+\mu(A_{15}\otimes B_{15})\big)\,,\end{aligned}$$

and the matrixes

$$A_{1q}=(a_{l'l}^{(1,q)})\ \text{and}\ B_{1q}=(b_{k'k}^{1,q})\,,\quad 1\le q\le 5\,,$$

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have the entries

$$a_{l'l}^{(1,1)} = a_{l'l}^{(1,3)} = a_{l'l}^{(1,5)} = \int_\Lambda H_{l'}^\alpha(x) H_l^\alpha(x)\,dx\,, \quad a_{l'l}^{(1,2)} = \int_\Lambda H_{l'}^\alpha(x)\partial_x H_l^\alpha(x)\,dx\,,$$
$$a_{l'l}^{(1,4)} = \int_\Lambda \partial_x H_{l'}^\alpha(x)\partial_x H_l^\alpha(x)\,dx\,, \quad 0 \le l', l \le N\,,$$

and

$$b_{k'k}^{(1,1)} = b_{k'k}^{(1,2)} = b_{k'k}^{(1,4)} = \int_I \zeta_{k'}(y)\zeta_k(y)\,dy\,, \quad b_{k'k}^{(1,3)} = \int_I \zeta_{k'}(y)\partial_y\zeta_k(y)\,dy\,,$$
$$b_{k'k}^{(1,5)} = \int_I \partial_y\zeta_{k'}(y)\partial_y\zeta_k(y)\,dy\,, \quad 0 \le k', k \le M-2\,.$$

Now to compute the entries of A_{1q} and B_{1q}, $1 \le q \le 5$, from Eqs. (2.2) and (2.5)-(2.8) for $0 \le l', l \le N$ we have

$$a_{l'l}^{(1,1)} = \begin{cases} \sqrt{\pi}/\alpha\,, & l' = l\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \qquad a_{l'l}^{(1,2)} = \begin{cases} \sqrt{2l\pi}/2\,, & l' = l-1\,, \\ -\sqrt{2\pi(l+1)}/2\,, & l' = l+1\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

$$a_{l'l}^{(1,4)} = \begin{cases} -\alpha\sqrt{\pi l(l-1)}\,/\,2\,, & l' = l-2\,, \\ \sqrt{\pi}\alpha(l+\frac{1}{2})\,, & l' = l\,, \\ -\alpha\sqrt{\pi(l+1)(l+2)}\,/\,2\,, & l' = l+2\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Next, we use (3.2) and (3.3) of Ref. [19] such that for $0 \le k', k \le M-2$ we have

$$b_{k',k}^{(1,1)} = \begin{cases} -\frac{2}{2k+1}\,, & k' = k-2\,, \\ \frac{2}{2k+1} + \frac{2}{2k+5}\,, & k' = k\,, \\ -\frac{2}{2k+5}\,, & k' = k+1\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

$$b_{k',k}^{(1,3)} = \begin{cases} 2\,, & k' = k-1\,, \\ -2\,, & k' = k+1\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \qquad b_{k',k}^{(1,5)} = \begin{cases} 4k+6\,, & k' = k\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

We take the test function

$$W(x,y,t) = \tanh\left(\frac{\sqrt{6}}{12}\left(x + y - \frac{5\sqrt{6}}{6}t\right)\right)e^{-x^2}\,.$$

Let $a = 1$, $b = 1$, $\nu = 1$ and $\mu = 1$ in (4.1). To describe the error $W - w_{M,N}$, let $\zeta_{M,I,k}$ be the zeros of the polynomials $(1-x^2)\partial_x L_M(x)$, let $\rho_{M,I,k}$ denote the Christoffel numbers of the corresponding Legendre interpolations, and let $\sigma_{N,\Lambda,l}$ and $\omega_{N,\Lambda,l}$ be the nodes and

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Table 1: Numerical errors using the scheme (4.1) and (5.3) of Ref. [19] with $N = M^2$.

τ	$M = 5$	$M = 6$	$M = 7$	$M = 8$
0.0001(MLHM)	3.21e-04	2.89e-5	8.01e-06	7.76e-06
0.001(MSMGHF)	1.49e-03	1.47e-04	1.22e-05	8.63e-07

weights of the standard Hermite-Gauss interpolation. The nodes and weights of the *scaled* Hermite-Gauss interpolation are then taken to be

$$\sigma^{\alpha}_{N,\Lambda,l} = \frac{\sigma_{N,\Lambda,l}}{\alpha}, \qquad \omega^{\alpha}_{N,\Lambda,l} = \frac{1}{\alpha}\omega_{N,\Lambda,l} e^{\sigma^2_{N,\Lambda,l}},$$

and we write

$$\|u\|_{M,N,\Omega} = \left(\sum_{k=0}^{M} \sum_{l=0}^{N} u^2(\sigma^{\alpha}_{N,\Lambda,l}, \zeta_{M,I,k}) \rho_{M,I,k} \omega^{\alpha}_{N,\Lambda,l} \right)^{\frac{1}{2}}.$$

The numerical errors are measured by the discrete norm

$$E_{M,N,\tau}(t) = \|W(t) - w_{M,N,\tau}(t)\|_{M,N,\Omega} \approx \left(\int \int_{\Omega} (W(t) - w_{M,N,\tau}(t))^2 dx dy \right)^{\frac{1}{2}}.$$

In Fig. 1, we plot the errors $\log_{10} E_{M,N,\tau}(t)$ with $N = 2M$, $t = 1$ and $\alpha = 1.5$. Clearly, the errors rapidly decay when M, N increase and τ decreases. For the fixed time step size $\tau = 0.02$ and $M \leq 7$, the total numerical errors are dominated by the approximation errors in space, so they rapidly decay as M increases. However, for $M \geq 7$ the total numerical errors are dominated by the approximation errors in time t, so the numerical results keep the same accuracy even if M and N increase further. The results are similar for $\tau = 0.002$. In contrast, for $\tau \leq 0.0002$ the total numerical errors are dominated by the approximation errors in space, so they decay very rapidly as M and N increase. The above results coincide very well with our theoretical analysis in Section 3. In particular, they show the spectral accuracy in space of scheme (3.4), and convergence as M, N increase and τ decreases. Fig. 2 is a plot of $\log_{10} E_{M,N,\tau}(t)$ with $t = 1$, $N = 2M$, $\tau = 0.002$ and two different values of the parameter α, where the errors when $\alpha = 1.5$ are clearly smaller than the errors when $\alpha = 1$. However, how to choose the best parameter α is still an open question.

Finally, we compared numerical results from the mixed spectral method for heat transfer using generalised Hermite functions and Legendre polynomials (MSMGHF) and the mixed Legendre-Hermite spectral method (MLHM) [19], for the test function [19]

$$W(x, y, t) = (1 - y^2) e^{y - \frac{1}{2}t - x^2},$$

with $a = 1$, $b = \sqrt{2}$, $\nu = 1$, $\mu = 1/2$ and $\alpha = 1$ in (4.1). In Table 1, the values $E_{M,N,\tau}$ vs. the mode M with $N = M^2$ show that results using the MSMGHF are accurate even for large τ, and for $M = 8$ they are also better.

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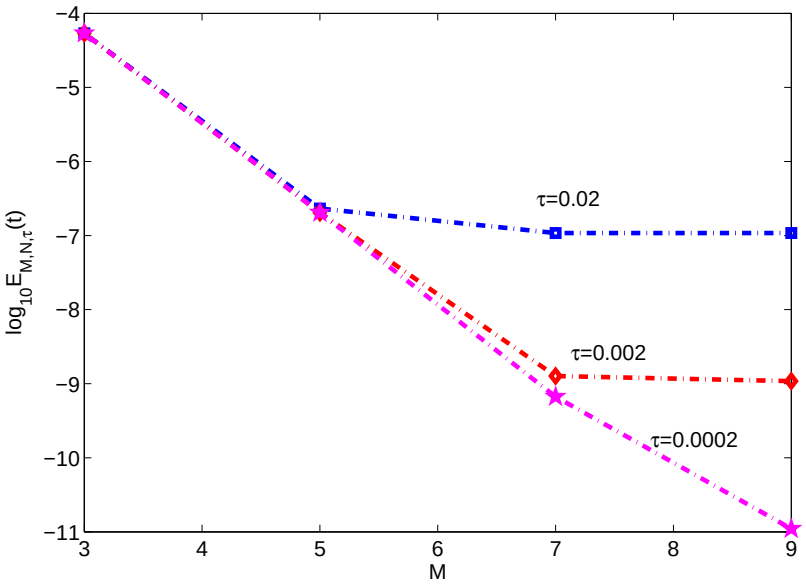


Figure 1: $\log_{10} E_{M,N,\tau}(1)$ vs M using the scheme (3.4) and (3.5) for (3.2) with three different τ.

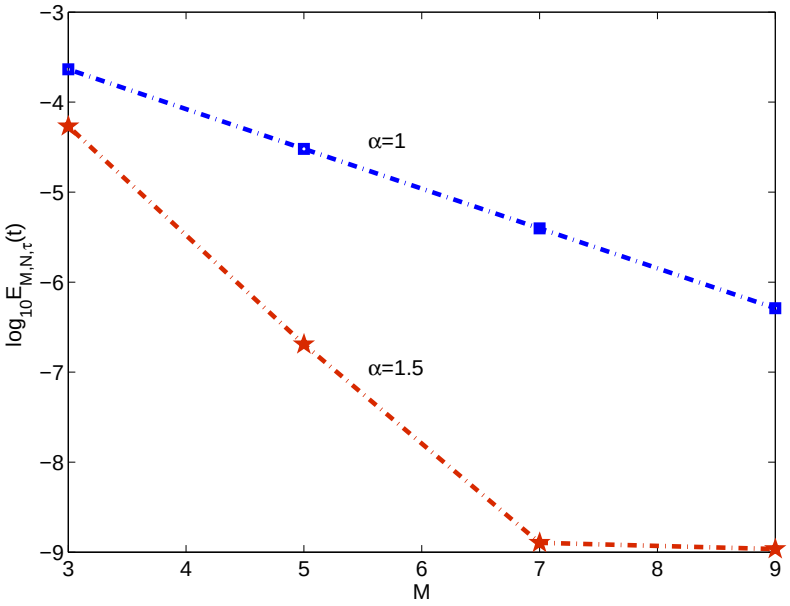


Figure 2: $\log_{10} E_{M,N,\tau}(1)$ vs M using the scheme (3.4) and (3.5) for (3.2) with two different α.

5. Concluding Remarks

Anisotropic heat transfer in an infinite plate with Dirichlet boundary conditions has been simulated using the generalised Hermite functions $H_n^{\alpha}(x)$ as the basis functions at the

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$x-$ direction, since the solution decays exponentially as $|x| \to \infty$. The generalised Hermite functions lead to much simpler analysis and more precise error estimates. We managed to improve the algorithm scheme and the numerical results for problem (1.1) by using the generalised Hermite functions, and the adjustable parameter α offers great flexibility to match various asymptotic solution behaviour at infinity. However, the choice of the optimal scaling factor for a given differential equation is still an open problem. We also established some results on the mixed generalised Hermite-Legendre orthogonal approximation, which provided useful tools in developing and analying generalised Hermite-Legendre spectral methods for problems involving unbounded domains.

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