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ARTICLE

Generalizations of risk-weighted expected utility

Kenny Easwaran

Logic and Philosophy of Science, University of California, Irvine, CA, USA
Email: easwaran@gmail.com

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Abstract

Buchak's risk-weighted expected utility considers not just the probability of an outcome, but also the probability of getting a strictly better outcome, when weighting the contribution that outcome gives to the evaluation of a gamble. It uses a risk-weighting function R sending probabilities in $[0, 1]$ to decision weights $[0, 1]$. I adapt this to allow weights in any real interval. Finite intervals yield nothing new, but if the interval is infinite, then the resulting rule can incorporate maximin or maximax preferences (or both!) while still satisfying stochastic dominance. There are advantages to working with marginal risk-weighting, R's derivative, r.

Keywords: risk-weighted expected utility; comparative utility; infinite utility; maximin; lexicographic preferences

This paper considers the "risk-weighted expected utility" (*REU*) introduced by Lara Buchak (2013) (formally identical to the "anticipated utility" introduced by John Quiggin (1982), but with a normative rather than a descriptive interpretation), which is based on a risk function R that is required to be a continuous, strictly increasing function from the unit interval $[0, 1]$ to itself. Buchak motivates her generalization of risk-neutral expected utility (*EU*) by arguing that means-ends reasoning requires an improvement in any possible outcome of a gamble to contribute positively to its evaluation, but not necessarily in direct proportion to its probability. On her framework, this corresponds to the requirement that R be strictly increasing.

Buchak notes (2013: 68–70) that maximin and maximax preferences could be said to be particularly extreme forms of risk sensitivity that can be accommodated in a generalization of this approach by relaxing the requirement that R be continuous and strictly increasing. But she notes that this involves giving up her means-ends understanding of decision theory. Thus, rather than relaxing strict monotonicity, I

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relax the requirement that the range of this function be the unit interval. I show that when the interval is infinite, the resulting decision rule still has the maximin or maximax property of prioritizing the best or worst outcome over all others. I observe that the resulting rule is interestingly different in other ways, but satisfies more of the axioms of her representation theorem than the classic maximin and maximax rules, as well as others I call "leximin" and "leximax".

An important running theme throughout involves the relationship between Buchak's risk-weighting function R, and the *marginal* risk-weighted contributions of individual outcomes, which I will notate with "r". I show that in many contexts it is simpler and more natural to calculate by means of r than by R. I argue that r does a better job of representing Buchak's means-ends philosophical motivations. This is especially clear in the broadest generalization I reach at the end, which is the condition that r be a measurable, real-valued function with at most finitely many singularities, that is everywhere non-negative, and for which there is no interval over which it is almost always 0. I claim that this condition on r properly represents Buchak's means-ends motivation, but when phrased in terms of R requires that it be locally strictly increasing with finitely many singularities.

In section 1, I formally define finite gambles, as well as the "cumulative" parameters U and P, and the "marginal" or "incremental" parameters u and p. In section 2 I review the standard definition of expected utility (EU), and show how to calculate it with vertical rectangles using U and p, or horizontal rectangles using u and P. In section 3, I review Buchak's definition of risk-weighted expected utility (REU), using a risk-weighting function R that is continuous, strictly increasing, and sends the $[0, 1]$ interval of probabilities to the $[0, 1]$ interval of decision weights. I include both the definition of REU she emphasizes using horizontal rectangles with u and R, and an equivalent definition using vertical rectangles with U and r. In section 4 I show that her use of the interval $[0, 1]$ for decision weights is an inessential convention. In section 5, I show that relaxing to other finite intervals yields a pleasing symmetry in that both utility and decision weight end up being unique only up to affine transformation. In section 6 I introduce the most significant generalization of Buchak's risk-weighted utility, by allowing R to send the $[0, 1]$ interval to infinite intervals, and allowing r to give infinite decision weight to one or more positions in the scale (such as the maximum, minimum, or even median). In section 7, I compare the motivation for this generalization to the motivations for two other types of decision rule that are sometimes said to be infinitely extreme versions of risk-sensitivity, namely "classic maximin" (and maximax) and a less-familiar "leximin" (and leximax). But I argue that my generalization does a better job of preserving Buchak's motivation for risk-sensitive decision theory as a general kind of means-ends rationality.

I also include two appendices. In Appendix 1, I show how the informal motivations I give for my extended decision rule correspond to modifications of the axiomatic structure of Buchak's representation theorem. My extended decision rule satisfies all but two of her axioms, while leximin and classic maximin each violate some additional axioms. In Appendix 2, I develop the entire apparatus for continuous gambles as well as for finite ones. This appendix mostly follows the same structure as sections 1–6 of the main body of the paper. However, Appendix 2.2 develops more precisely a proposal of Colyvan (2008) to define decision theory in

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terms of *comparisons* of pairs of gambles, rather than by providing a numerical evaluation of each gamble individually. This is essential even for standard *EU* because of cases Colyvan cites (due to Nover and Hájek 2004), in which there are contributions from unboundedly positive or negative utility, but it becomes more pressing when the decision weights can become infinite as well. A few aspects of this comparative proposal, and of the advantages of working with *r* over *R*, are clearer in the continuous case than the finite case. All footnotes in the main text are references to relevant points in this appendix. Even for readers who are sceptical of the value of decision rules that put infinite weight at some points in the scale, the motivations it gives for the comparative formulation of risk-weighted expected utility (for continuous gambles), and the formulation in terms of *r* rather than *R* (both for finite and continuous gambles), should be of value.

1. Finite Gambles

Represent a finite gamble G as a finite set of pairs $G = \{\ldots, (U_i, p_i), \ldots\}$, where each pair represents the utility and probability of one possible outcome of the gamble. (I assume that the U_i and p_i are real numbers, and that the p_i are strictly positive, with $\sum p_i = 1$.) It will be helpful to adopt a convention whereby the values i range from 0 to n, and where $U_{i+1} < U_i$. (This can always be arranged by merging outcomes with the same utility, and then re-numbering the remaining outcomes from highest to lowest.)

It will also be helpful to define the gamble by another set of ordered pairs. Let $P_i = \sum_{j=0}^{i} p_j$ be the total probability of the $i + 1$ outcomes with greatest utility. Then $p_i = P_i - P_{i-1}$, with the special convention that $P_{-1} = 0$. Similarly, define $u_i = U_i - U_{i+1}$, with the special convention that $U_{n+1} = 0$. Then, for $0 \leq i < j \leq n$, $P_i < P_j$ and $U_i > U_j$ (though this generally fails if $i = -1$ or $j = n + 1$). I say that lower case p_i and u_i represent the *marginal* (incremental) probability or utility of the *i*th best outcome, while capital P_i and U_i represent the *cumulative* (total) probability down to or utility up to that outcome.

Figure 1 illustrates a simple case where $n = 2$ and all the U_i are positive.

2. Standard Expected Utility: Vertical or Horizontal Rectangles

Standard expected utility theory defines the expected utility of a gamble by $EU(G) = \sum_{i=0}^{n} U_i \cdot p_i$ and says that G_1 should be preferred to G_2 iff $EU(G_1) > EU(G_2)$. This standard formula can be visualized as taking the area under the utility-probability steps as seen in Figure 2.

Because the p_i are non-negative, increasing the utility of one or more outcomes without changing that of any others increases *EU*. This can be generalized. We say that one gamble *stochastically dominates* another iff for every utility value, the probability that the first has a utility at least that high is at least as great as for the second. If one finite gamble stochastically dominates another, then the dominated one can be transformed into the dominating one by first splitting some outcomes into multiple outcomes of the same utility (to ensure that the sequence of outcomes of the two gambles has the same sequence of p_i, without changing *EU*) and then

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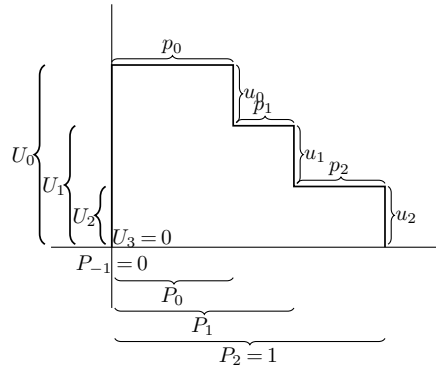


Figure 1. Illustration of the cumulative and marginal utilities U_i and u_i, and the cumulative and marginal probabilities P_i and p_i.

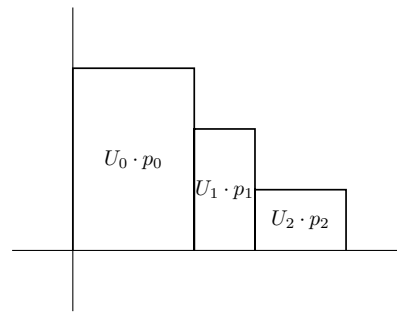


Figure 2. Expected utility calculated with vertical rectangles.

increasing the utility of some outcomes without changing that of any others. Thus, if one gamble stochastically dominates another, it has a higher *EU*, and is thus preferred.

Buchak notes that this corresponds to the idea of decision theory as a codification of means-ends reasoning. Any motivation for preferring one gamble over another must be based in some possibility of receiving a better outcome, and any possibility of receiving a better outcome provides a decisive reason to prefer that outcome, unless there is also some possibility of receiving a worse outcome (in which case some way of measuring the tradeoff is needed). Tarsney (2020) argues that stochastic dominance might be the *only* property shared by all reasonable decision theories, but as we will see later, Buchak focuses on the way some better outcomes are traded off against some worse outcomes in particular decision theories.

Because $P_n = \sum_{i=0}^{n} p_i = 1$, the area under this step function is somewhere between that of a rectangle with width 1 and height U_n and a rectangle with width 1 and height U_0. That is, the standard *EU* of a gamble is somewhere between the minimum and maximum utilities that the gamble can achieve, i.e. $U_n \leq EU(G) \leq U_0$. Thus, we can think of $EU(G)$ as in some sense an estimate of the outcome of gamble G. I claim that this is a formal convenience, which is useful for some applications of *EU*, but is not essential to a way of comparing gambles the way that agreement with stochastic dominance is.

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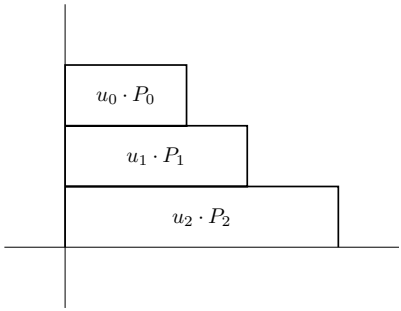


Figure 3. Expected utility calculated with horizontal rectangles.

This way of calculating the area uses the marginal probabilities and the cumulative utilities. However, we can equally define $EU(G) = \sum_{i=0}^{n} u_i \cdot P_i$, using marginal utilities and cumulative probabilities. The two definitions correspond to measuring the same area under the utility-probability steps, but breaking the area into horizontal rectangles instead of vertical ones, as seen in Figure 3. This summation of horizontal rectangles rather than vertical ones gives another way to see that *EU* is compatible with stochastic dominance. It is conceptually less familiar than the summation of vertical rectangles, but is helpful in giving a presentation of Buchak's risk-weighted expected utility.

3. Risk-weighted Expected Utility: Horizontal or Vertical Rectangles

Risk-weighted expected utility adds a further consideration – a risk-sensitivity function $R : [0, 1] \rightarrow [0, 1]$ with the requirements that $R(0) = 0$, $R(1) = 1$, and R is continuous and strictly increasing, i.e. $R(x) < R(y)$ whenever $x < y$. This function is used to "stretch" the importance of various outcomes based on their cumulative probability. That is, we don't take the cumulative probability P_i itself to measure how much the ith marginal utility contributes to the evaluation of a gamble, but instead use $R(P_i)$. Standard *EU* assumes that the importance of an outcome is directly proportional to its probability, but under risk-weighting, this is adjusted by the function R. Buchak motivates this adjustment by arguing that the concept of means-ends rationality doesn't require direct proportionality, but just requires the idea that all possible outcomes contribute positively. Figure 4 graphs utility against risk-sensitivity, instead of probability.

The simplest definition of risk-weighted utility parallels the "horizontal" definition of standard *EU*, using the risk-weightings instead of the probabilities, as illustrated in Figure 5. $REU(G) = \sum_G u_i \cdot R(P_i)$. But just as we had marginal and cumulative utility, and marginal and cumulative probability, we can call $R(P_i)$ the cumulative risk-weighting, and define $r_i = R(P_i) - R(P_{i-1})$ (again with the convention that $P_{-1} = 0$) as the marginal risk weighting. Using the marginal risk-weightings instead of the marginal probabilities, we see that the sum $REU(G) = \sum_G U_i \cdot r_i$ calculates this "vertically", as in Figure 6.

Although the r_i are more complicated to define directly in terms of the ordered pairs that make up the gamble, they make certain features of *REU* easier to understand. To the extent that the vertical calculation is more familiar in the case of

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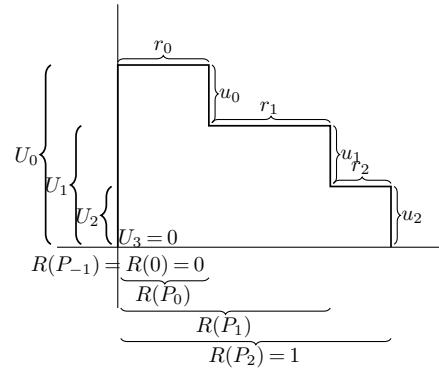


Figure 4. Illustration of the cumulative and marginal risk-weightings $R(P_i)$ and r_i, along with U_i and u_i.

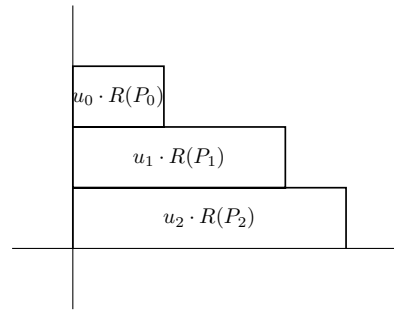


Figure 5. Risk-weighted expected utility calculated with horizontal rectangles.

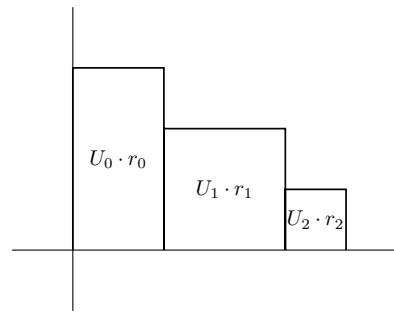


Figure 6. Risk-weighted expected utility calculated with vertical rectangles.

EU, the version of *REU* defined in terms of the r_i may be more intuitively meaningful than the version defined in terms of the $R(P_i)$. Furthermore, the requirement that $R(x) < R(y)$ whenever $x < y$ corresponds to the requirement that each r_i is strictly positive. This means that every term in this vertical sum contributes with positive weight. This again ensures agreement with stochastic dominance, and Buchak's means-ends motivation. (If some r_i were zero or negative, then two gambles with the same probabilities, and same utilities for every outcome other than U_i, would not be strictly ranked in the order of these U_is by *REU*.) Since

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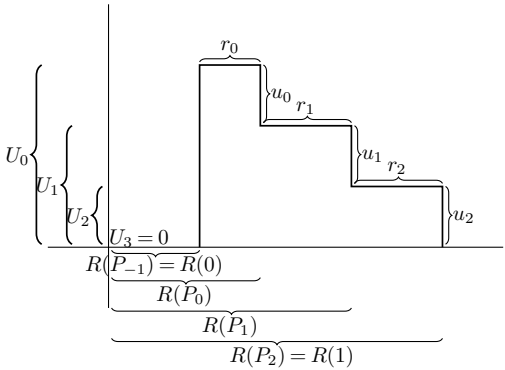


Figure 7. Illustration of non-$[0, 1]$ risk-weighting and utility.

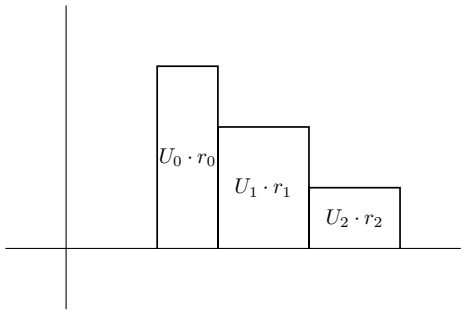


Figure 8. Risk-weighted expected utility calculated with vertical rectangles, with non-$[0, 1]$ risk function.

$$\sum_{i=0}^{n} r_i = \sum_{i=0}^{n} (R(P_i) - R(P_{i-1})) = R(P_n) - R(P_{-1}) = R(1) - R(0),$$

the requirement that $R(0) = 0$ and $R(1) = 1$ corresponds to the requirement that the sum of the r_i, and thus the width of the above diagram, is 1. This again yields the convenience that $U_n \leq REU(G) \leq U_0$.

4. Rescaling R Beyond $[0, 1]$

From looking at these calculations geometrically, it is clear that the requirement that $R : [0, 1] \rightarrow [0, 1]$ is in some sense an arbitrary convention. In this section I explore how the decision rule as defined above works when R sends $[0, 1]$ to some other finite interval, as illustrated in Figure 7. I show that the resulting decision rule is reasonable as long as $R(w)$ is continuous and strictly increasing so that all the r_i are strictly positive. The project of this section is merely a technical warm-up to the more interesting generalizations of the later sections.

To calculate the area under the steps using the vertical rectangles is fairly straightforward even when $R(0) \neq 0$ and $R(1) \neq 1$, as shown in Figure 8. Recall that if the P_i are the cumulative probabilities $P_i = \sum_{j=0}^{i} p_j$, we define $r_i = R(P_i) - R(P_{i-1})$ as the marginal risk-weightings, with $P_{-1} = 0$. Then the risk-weighted expected utility is given by $REU(G) = \sum_{i=0}^{n} r_i \cdot U_i$.

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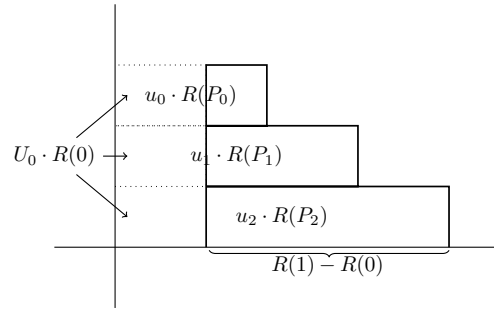


Figure 9. Risk-weighted expected utility calculated with horizontal rectangles, with non-$[0, 1]$ risk function.

For the horizontal rectangles, we use the marginal utilities (that is, $u_i = U_i - U_{i+1}$, with the special convention that $U_{n+1} = 0$) and the risk-weighted cumulative probabilities. However, as shown in Figure 9, we need to apply a correction to deal with the fact that $R(0) \neq 0$. Thus, we get $REU(G) = \sum_{i=0}^{n} u_i \cdot R(P_i) - U_0 \cdot R(0)$. When $R(0) \neq 0$, we need a correction term when using the method of horizontal rectangles. But no correction terms were needed when using the method of vertical rectangles, based on r rather than R. These correction terms will often show up when using R rather than r.[1]

Whenever R is strictly increasing and the p_i are all positive, the r_i will all be positive, so every cumulative utility will appear positively in the calculation, so the rule will still satisfy stochastic dominance. However, we can no longer guarantee the convenience of *REU* lying between the minimum and maximum utilities of the gamble. Instead, we have a correction term given by the width of the relevant shape, which is $R(1) - R(0)$. That is, instead of $U_n \leq REU(G) \leq U_0$, we now get that $U_n \cdot (R(1) - R(0)) \leq REU(G) \leq U_0 \cdot (R(1) - R(0))$.

5. Affine Transformations

Let a, b be real numbers, with $a > 0$. Then we say that $f(x) = ax + b$ is an affine transformation. It is familiar that the utility scale is, in a sense, only meaningful up to affine transformation.

If G is any gamble, let $f(G)$ be the gamble that replaces each cumulative utility U by $f(U)$. From the formulation of *EU* in terms of vertical rectangles, it is straightforward to see that $EU\big(f(G)\big) = f(EU(G))$. Similarly, from the formulation of *REU* in terms of vertical rectangles, it is straightforward to see that $REU\big(f(G)\big) = f(REU(G))$. Thus, changing the utility scale by an affine transformation doesn't change any pairwise evaluation of gambles.

Now that we have considered R that send the $[0, 1]$ interval to some other interval, we can also consider what happens with an affine transformation of R rather than U. Let $f(x) = ax + b$ be an affine transformation, let $R'(x) = f(R(x))$, and let $R'EU(G)$ be calculated with R' instead of R. Notice that $r'_i = a \cdot r_i$. Thus, using the vertical

[1]In fact, as we will see in Appendix 2.1, these correction terms when using horizontal rectangles rather than vertical show up for continuous gambles even for standard *EU*, and become even more significant for *REU*. However, in Appendix 2.2 they cancel out for the comparative versions.

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calculations in terms of these marginal risk weightings, it is easy to see that $R'EU(G) = a \cdot REU(G)$. Since a is positive, changing the scale of risk-weighting by an affine transformation again doesn't change any pairwise evaluations of gambles. Since the finite interval $[R_0, R_1]$ is related to the interval $[0, 1]$ by the affine transformation $f(x) = (R_1 - R_0)x + R_0$, this means that, for every risk-weighting function with any finite interval, there is a unique risk-weighting function with the interval $[0, 1]$ that yields all the same pairwise comparisons of gambles.

6. Infinite Risk-weighting

However, if we allow R to send $[0, 1]$ to an *infinite* interval, we get some essentially new ways of being sensitive to risk, that aren't equivalent to any risk-weighting function that uses the $[0, 1]$ interval. As in the finite case, it will be easier to do the calculations with the vertical rectangle calculation, based on marginal risk-weighting and cumulative utility, while the horizontal calculations with cumulative risk-weighting and marginal utility require a correction term.

Some candidate continuous, strictly increasing risk functions from $[0, 1]$ to infinite intervals of the extended real line $[-\infty, +\infty]$ are:

- $R_1(w) = \tan(\pi w/2)$, which has range $[0, +\infty]$
- $R_2(w) = -\log(1 - w)$, which has range $[0, +\infty]$
- $R_3(2) = \log(w)$, which has range $[-\infty, 0]$
- $R_4(w) = -1/w$, which has range $[-\infty, -1]$
- $R_5(w) = \tan\pi\big(w - 1/2\big)$, which has range $[-\infty, +\infty]$

Recall that U_i is the ith best cumulative utility and $r_i = R(P_i) - R(P_{i-1})$ is the marginal risk-weighting of this ith best outcome. When G is a finite gamble, we said

$$REU(G) = \sum_{i=0}^{n} r_i \cdot U_i = \sum_{i=0}^{n} u_i \cdot R(P_i) - U_0 \cdot R(0).$$

6.1 R(0), is finite, R(1) = $+\infty$

Let us first consider what happens when $R(1) = +\infty$, like with $R_1(w) = \tan(\pi w/2)$ and $R_2(w) = -\log(1 - w)$. In these cases, $R(P_n) = +\infty$, so $r_n = R(P_n) - R(P_{n-1}) = +\infty$. Thus, each of these formulas will have an infinite term, unless $u_n = U_n = 0$, as in Figure 10. For now, we use the convention that $\infty \cdot 0 = 0$. In this case, $REU(G)$ will be finite. But if $U_n > 0$, then $REU(G) = +\infty$, and if $U_n < 0$, then $REU(G) = -\infty$. These risk functions where $R(1) = +\infty$ encode a preference for whichever gamble has higher minimum utility, if these minimum utilities are of different signs.

When the minimum utility of the two gambles has the same sign (i.e. both positive, both negative, or both 0), the situation is slightly more complicated. If they are both positive, or both negative, our official definition of *REU* gives them both the same infinite value. However, if we think of utility as only defined up to affine transformation, it is natural to argue that we shouldn't necessarily think of gambles

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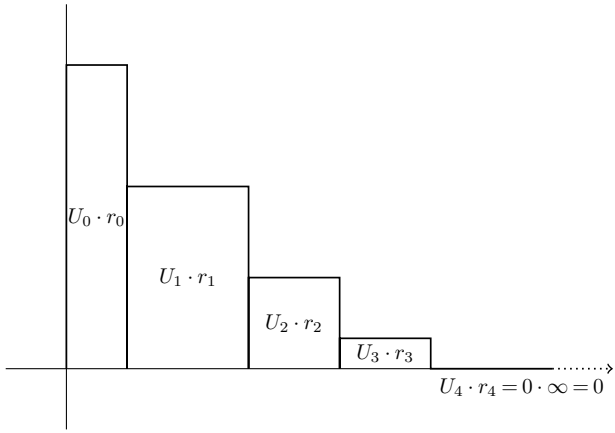


Figure 10. *REU* when $R(0) = 0$, $R(1) = +\infty$, and $U_n = 0$.

with the same infinite *REU* as equivalent. Instead, we should apply an affine transformation to the utility scale that sends the minimum utility of one of these gambles to 0, so that this gamble has finite *REU*. If the minimum utility of the other is distinct, then that other gamble has infinite *REU* on this scale (positive if its minimum utility is higher, negative if lower).[2]

Thus, this risk function encodes a strict preference for whichever gamble has a less bad worst possible outcome, regardless of any other outcomes of either gamble. I will say that any such preferences *have the maximin property*. There is no strictly increasing risk-weighting function on the $[0, 1]$ interval that encodes preferences with the maximin property.

Buchak (p. 68) mentions that if one were to allow *non*-strictly increasing risk-weighting functions, then the function given by $R_{min}(1) = 1$ and $R_{min}(x) = 0$ when $x < 1$, which I call "classic maximin", would have the maximin property. But there are some important differences when we use one of these strictly increasing functions that goes to $+\infty$. For one thing, since $R_{min}(x) = 0$ whenever $x < 1$, we have $R_{min}EU(G) = U_n$, and thus that decision rule is strictly indifferent between any two gambles with the same minimum utility. But since R_1 and R_2 are strictly increasing, they satisfy stochastic dominance. Effectively, they lexicographically prefer gambles on the basis of their minimum utility, but among gambles with the same minimum utility, they give some continuous risk-weighted comparison of the gambles based on their other outcomes. I will discuss the differences between classic maximin and strictly increasing functions to the $[0, +\infty]$ interval in section 7.

Among these strictly increasing risk functions with the maximin property, we can investigate the difference between R_1 and R_2, by seeing how they evaluate two different standardized gambles. Let G_1 be a gamble with $1/2$ probability of outcome 2, and $1/2$ probability of outcome 0, as shown in Figure 11, and let G_2 be a gamble with

[2]When discussing continuous gambles, this issue of cancelling out infinities is pressing even with standard *EU*. In Appendix 2.2, I develop a comparative version of *EU*, which I later extend to *REU*. A version of that comparative theory could be applied here as well, though it is more trouble than it is worth to state precisely in the finite case when the gambles involved have different sequences of p_i.

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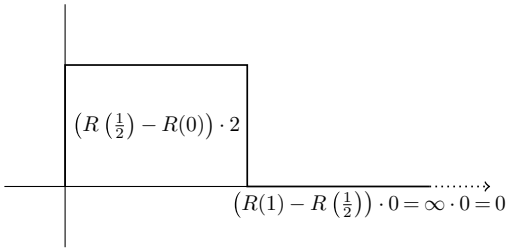


Figure 11. *REU* of G_1, with probability $1/2$ of outcomes 2 or 0, when $R(1) = +\infty$.

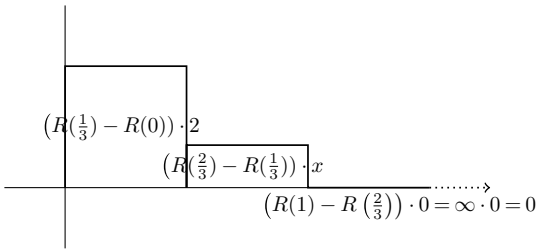


Figure 12. *RU* of G_2, with probability $1/3$ of outcomes 2, x, or 0, when $R(1) = +\infty$.

$1/3$ probability of three different outcomes, one with utility 2, one with utility 0, and one with utility $0 < x < 2$, as illustrated in Figure 12.

For a risk-neutral decision-maker, these gambles would be equal when $x = 1$. But both of these risk functions are risk-averse, and would thus strictly prefer the gamble G_2 with lower chance of utility 0 in the case where the risk-neutral decision-maker judges them as equivalent. We can get one measure of how risk-averse they are by seeing how far we have to reduce x below 1 to make this decision rule judge the two gambles to be equivalent.

Using $R_1(w) = \tan(\pi w/2)$, for G_1 we get:

$$\begin{aligned} R_1EU(G_1) &= \left(R_1\left(\tfrac{1}{2}\right) - R_1(0)\right) \cdot 2 \\ &= 2\tan\tfrac{\pi}{4} = 2 \end{aligned}$$

For G_2 we get:

$$\begin{aligned} R_1EU(G_2) &= \left(R_1\left(\tfrac{1}{3}\right) - R_1(0)\right) \cdot 2 + \left(R_1\left(\tfrac{2}{3}\right) - R_1\left(\tfrac{1}{3}\right)\right) \cdot x \\ &= 2\tan\tfrac{\pi}{6} + x\left(\tan\tfrac{\pi}{3} - \tan\tfrac{\pi}{6}\right) \\ &= \tfrac{2}{\sqrt{3}} + x\left(\sqrt{3} - 1/\sqrt{3}\right) \end{aligned}$$

These values are equal when $\frac{2}{\sqrt{3}} + x\left(\sqrt{3} - 1/\sqrt{3}\right) = 2$, which occurs when $2 + x(3-1) = 2\sqrt{3}$, so that $x = \sqrt{3} - 1 \approx 0.732$.

Using $R_2(2) = -\log(1-w)$, for G_1 we get:

$$\begin{aligned} R_2EU(G_1) &= \left(R_2\left(\tfrac{1}{2}\right) - R_2(0)\right) \cdot 2 \\ &= \left(-\log\tfrac{1}{2} + \log 1\right) \cdot 2 = 2\log 2 \end{aligned}$$

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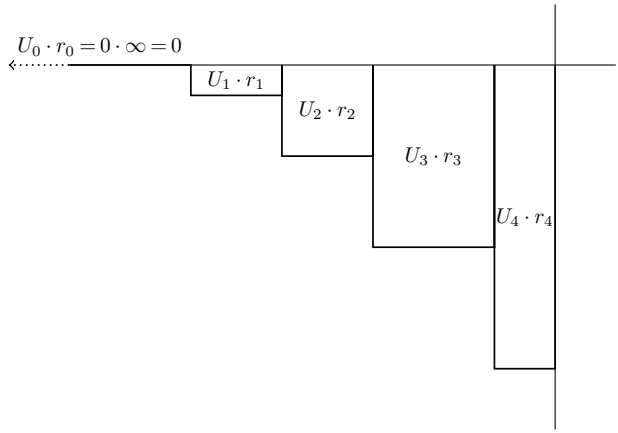


Figure 13. *REU* calculated when $R(0) = -\infty$, $R(1) = 0$, and $U_0 = 0$.

For G_2 we get:

$$\begin{aligned} R_2EU(G_2) &= \left(R_2\left(\tfrac{1}{3}\right) - R_2(0)\right) \cdot 2 + \left(R_2\left(\tfrac{2}{3}\right) - R_2\left(\tfrac{1}{3}\right)\right) \cdot x \\ &= \left(-\log\tfrac{2}{3} + \log 1\right) \cdot 2 + \left(-\log\tfrac{1}{3} + \log\tfrac{2}{3}\right) \cdot x \\ &= 2\log\tfrac{3}{2} + x\log 2 \end{aligned}$$

These values are equal when $2\log\frac{3}{2} + x\log 2 = 2\log 2$, or when $(2-x)\log 2 = \log(3/2)$, so that $1 - x/2 = \log_2(3/2)$, so that $x = 2\left(1 - \log_2(3/2)\right) \approx 0.830$.

Thus, we can see that distinct risk functions that encode preferences with the maximin property by having $R(1) = +\infty$ can yield different comparisons of gambles with the same minimum utility. For this particular comparison, it appears that R_1 is more risk-averse, even though both have the maximin property.

6.2 R(0) = −∞, R(1) is finite

A similar set of considerations applies when $R(0) = -\infty$, as with $R_3(w) = \log(w)$ and $R_4(w) = -1/w$. In these cases, a finite gamble will have finite *REU* iff its *maximum* utility, $U_0 = 0$, as illustrated in Figure 13. (Note that since U_0 is the highest utility, all other outcomes of this gamble must have negative utility.)

If U_0 is positive, then $REU(G) = +\infty$, and if U_0 is negative, then $REU(G) = -\infty$. Since U_0 is the highest utility, applying an affine transformation to send the highest utility of one of a pair of gambles to 0 indicates that this R will encode preferences with what I call *the maximax property*. But again, such a risk function will encode some sort of continuous preference relation that obeys stochastic dominance among gambles with the same highest utility, unlike classic maximax, given by the risk function R_{max} with $R_{max}(0) = 0$ and $R_{max}(x) = 1$ whenever $x > 0$.

We can again compare risk-weighting functions R_3 and R_4 on the same standardized pair of reference gambles, and see that R_3 says they are equal when $x = 3 - \sqrt{3} \approx 1.268$, and R_4 does when $x = 2\log_2(3/2) \approx 1.170$. Both are risk-seeking, but R_3 is more so for this particular pair.

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6.3 R(0) = $-\infty$, R(1) = $+\infty$

When R goes off to infinity in both directions, as with $R_5(w) = \tan\pi\left(w - 1/2\right)$, every gamble where some outcome has both non-zero utility and non-zero probability will have an infinite term somewhere in the calculation. However, there are still some meaningful ways that this function can be used to make risk-sensitive decisions among finite gambles.

In the case of risk functions that go off to infinity at just one end, we could apply an affine transformation to the utility scale to send the utility of one of the gambles at that end to 0. If the other gamble is non-zero at that point, it has (positive or negative) infinite *REU*, and we can compare that way. But if both have the same utility at this infinitely weighted end-point, then sending one to 0 there sends the other to 0 there as well. This leaves both gambles with finite *REU*, and we can compare these numbers to see which is preferred.

But another way to achieve the same effect is to make the comparison by comparing the infinitely weighted endpoint, if the gambles differ there, and to just *ignore* that endpoint if the two gambles agree there, *without* changing the utility scale. For gambles that agree at this endpoint, calculating the finite area under the parts where they *differ* is sufficient to yield the comparison.[3] For a risk function that goes off to infinity at *both* ends, we can compare finite gambles similarly, by first looking at the infinitely weighted ends, and then looking elsewhere if the gambles agree there.

When the maximum and minimum utilities of two gambles aren't both the same, then there are some infinite regions in which the utilities of the two gambles differ. If G_1 has higher maximum *and* higher minimum than G_2, as illustrated in Figure 14, then both of those infinite regions count towards it, and the other can only get at most a finite boost in between, so G_1 is strictly preferred. But if G_1 has higher maximum and G_2 has higher minimum, then the two infinite regions count towards the different gambles, and the decision rule gives no way to compare them, as illustrated in Figure 15.

When the gambles agree at *both* ends (as with our two-outcome G_1 and three-outcome G_2 from earlier), we just compare the middle area where they differ, and ignore the infinite areas on each end where they agree, as shown in Figure 16. These two gambles are evaluated equally when $U_0 \cdot (R(P_1) - R(P_0)) + U_2 \cdot (R(P_2) - R(P_1)) = U_1 \cdot (R(P_2) - R(P_0)$. If we use $R_5(w) = \tan\pi\left(w - 1/2\right)$, using the same numbers from earlier (so $P_0 = 1/3$, $P_1 = 1/2$, $P_2 = 2/3$, $U_0 = 2$, $U_1 = x$, $U_2 = 0$) then we see that the gambles are equal when

$$2\left(\tan 0 - \tan\frac{-\pi}{6}\right) + 0\left(\tan\frac{\pi}{6} - \tan 0\right) = x\left(\tan\frac{\pi}{6} - \tan\frac{-\pi}{6}\right).$$

Since $\tan 0 = 0$ and $\tan -x = -\tan x$, this simplifies to $2\tan\frac{\pi}{6} = 2\,x\tan\frac{\pi}{6}$, so that $x = 1$. It is no surprise that we get the same value as the risk-neutral function, because the gambles we are considering here are symmetric about probability 1/2, as is the risk function.

[3]In Appendix 2.2, I work this out in greater detail for continuous gambles.

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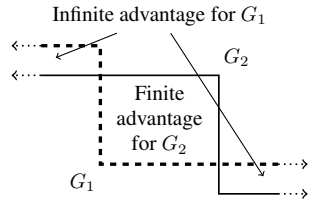


Figure 14. *REU* calculated when $R(0) = -\infty$, $R(1) = +\infty$, and G_1 has both higher max and min than G_2.

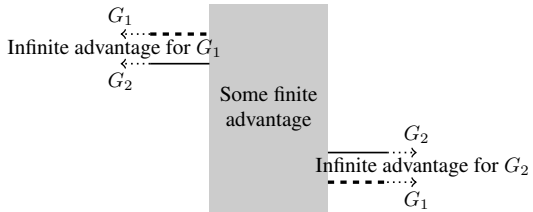


Figure 15. *REU* calculated when $R(0) = -\infty$, $R(1) = +\infty$, and G_1 has higher max while G_2 has higher min.

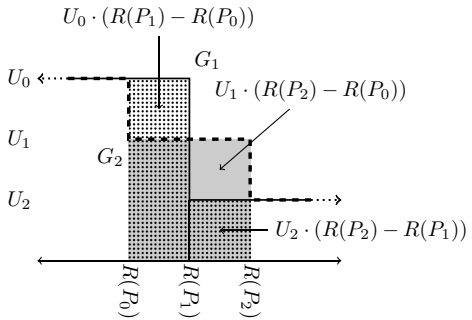


Figure 16. Comparative *REU* calculated when $R(0) = -\infty$, $R(1) = +\infty$, where G_1 and G_2 have the same max (U_0) and min (U_2), but G_1 has probability P_1 of U_0, while G_2 has probability P_0 of U_0, and $P_2 - P_0$ of U_1.

If instead we use gambles where $P_0 = 1/4$, $P_1 = 1/3$, $P_2 = 1/2$ (and keep $U_0 = 2, U_1 = x, U_2 = 0$), then the gambles are equal when

$$2\left(\tan\frac{-\pi}{6} - \tan\frac{-\pi}{4}\right) + 0\left(\tan 0 - \tan\frac{-\pi}{6}\right) = x\left(\tan 0 - \tan\frac{-\pi}{4}\right).$$

This simplifies to $2\tan\frac{\pi}{4} - 2\tan\frac{\pi}{6} = x\tan\frac{\pi}{4}$. Since $\tan\frac{\pi}{4} = 1$ and $\tan\frac{\pi}{6} = 1/\sqrt{3}$, this simplifies to $x = 2\left(1 - 1/\sqrt{3}\right) \approx 0.845$. Under standard *EU*, these would have been equal when $x = 2/3$, so this behaviour is somewhat risk-seeking (as would be expected from the fact that the risk function is concave down in the region below $1/2$).

When the risk function $R(w)$ goes both to $+\infty$ as w goes to 1, and to $-\infty$ as w goes to 0, the decision is lexicographically sensitive to both the maximum and the minimum utility of the gamble. If one gamble has better maximum *and* minimum than another, then it is preferred. If two gambles have the same maximum and minimum, then the behaviour of R on intermediate probabilities matters. If one gamble has higher maximum and the other has higher minimum, then this risk function yields no clear way to decide between them.

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Note that in the case of bounded risk functions that aren't required to be strictly increasing, we could get lexicographic sensitivity to both endpoints by using $R(0) = 0$, $R(1) = 1$, and $R(x) = k$ otherwise. As Buchak notes (pp. 68–70), this is formally equivalent to the use of the Hurwicz criterion, treating the worst outcome of the gamble as precisely $k/(1-k)$ times as important as the best outcome of the gamble, and giving zero weight to all other outcomes of the gamble. There might be some advantage to such a risk function over the ones I discuss that go to infinity on both sides, in being able to compare the importance of the two endpoints. But just as with R_{min} and R_{max}, the fact that this risk function is not strictly increasing means that it doesn't respect Buchak's means-ends motivation – intermediate outcomes make *no* contribution to the evaluation of the gamble.

6.4 Infinite risk weighting at other points on the scale

It is hard to put infinite risk-weighting at other points in the scale with a strictly increasing function R. However, we can do it with a *locally* strictly increasing R that has a singularity, if we use the convention that r_i is ∞ for any outcome that spans the singularity.[4] For instance let $R(w) = \tan(\pi w)$, which is locally strictly increasing from finite values toward $+\infty$ as P_i increases *to* $1/2$, and then is locally strictly increasing from $-\infty$ to finite values as P_i increases *above* $1/2$. In this case, we can conventionally stipulate that $r_i = \infty$ if $P_{i-1} \leq 1/2 \leq P_i$, and $R(P_i) - R(P_{i-1})$ otherwise. This yields something like a "maximedian" decision rule – it ranks one gamble above another if the median utility for the first is strictly higher than the median utility for the second, and there are no other sources of unbounded value.[5]

Additionally, just as it was possible to infinitely prioritize *both* the top and bottom end of the probability interval with an R that went to infinity at both ends, it is possible to infinitely prioritize multiple points within the probability interval by using this same trick with a function R that is everywhere locally strictly increasing, but has finitely many singularities. For instance, use $R(p_i) = \tan 3\pi P_i$, and let $r_i = \infty$ whenever $P_{i-1} \leq 1/6 \leq P_i$ or $P_{i-1} \leq 1/2 \leq P_i$ or $P_{i-1} \leq 5/6 \leq P_i$.[6] It's not

[4]In the continuous case, this is easier to state precisely as the requirement that the *marginal* risk-weighting function r, which is the derivative of R, be measurable, everywhere non-negative, and not precisely 0 almost everywhere in an interval. Singularities of R correspond to intervals over which r has infinite integral.

[5]This corresponds to $r(w) = \pi \sec^2 \pi w$. The decision weights r_i given above are just $\int_{w=P_{i-1}}^{P_i} r(w)dw$, which is infinite when the interval spans $1/2$, and equal to the difference in the $\tan \pi P_i$ function otherwise. We can use the formula $CREU(G, G') = \int_{w=0}^{1} [U(w) - U'(w)]r(w)dw$ to compute a meaningful generalization of comparison by *REU*. In particular, if $U(1/2) > U'(1/2)$, then because U and U' are both continuous, there must be some interval around $1/2$ for which $U - U' > \varepsilon$. The integral of $[U(w) - U'(w)]r(w)$ over this interval will then be infinite. Assuming that $U - U'$ is bounded on the rest of the interval, $CREU(G, G') = +\infty$, and thus this does produce a "maximedian" rule, as claimed.

[6]In the continuous case, this corresponds to using $r(w) = 3\pi \sec^2 3\pi w$. This risk-weighting function gives infinite weight to intervals around $w = 1/6, 3/6, 5/6$. If G and G' are such that for the associated U and U', $U(1/6) \geq U'(1/6)$, $U(3/6) \geq U'(3/6)$ and $U(5/6) \geq U'(5/6)$, with at least one of the inequalities being strict, and if $U(w) - U'(w)$ is bounded as w goes to 0 and to 1, then G is strictly preferred to G', regardless of what happens at other points of the interval. But if the difference between the gambles goes to 0 quickly enough around each of those points, then the finite weights at other points of the probability scale can make a difference.

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at all obvious what particular application this sort of risk-weighting function could be useful for, but it is a natural formal extension of Buchak's *REU*. Importantly, it shows one more way that r enables a more direct definition of the risk-weighting rule than R.

7. Three Ways of Achieving the Maximin and Maximax Properties

As we can see from the calculation with vertical rectangles, r_i is effectively a measure of how important the ith best outcome is to the comparison.[7] This helps explain how the maximin and maximax properties arise for finite gambles – the relevant r assigned infinite weight to one of the two endpoints.[8] But this also helps explain some differences between the preferences encoded by these risk functions, and other preferences with the maximin or maximax properties.

The non-strictly increasing risk functions R_{min} and R_{max} encode classic maximin or maximax preferences that give *full* weight to intervals around one endpoint, and *no* weight to any other interval.[9] R functions that are locally strictly increasing can give infinitely much weight to some points in the scale (that is, enough weight to outweigh any finite differences elsewhere), while still giving non-zero weight elsewhere. This distinction is essential to the preservation of stochastic dominance, and Buchak's means-ends motivation.

Another decision rule with the maximin (or maximax) properties involves not just looking at the worst (or best) outcome to compare the gambles, but rather looking at the worst (or best) *different* outcome of the gambles. This gives rise to what we might call *leximin* (or leximax) preferences. In the finite case, we say that G is preferred to G' if the greatest i with $\left(U_i, p_i\right) \neq \left(U'_i, p'_i\right)$ (or least such i for leximax) has $U_i > U'_i$, or has $U_i = U'_i$ and $p_i > p'_i$.[10] Note that under these preferences, the agent is never strictly indifferent between gambles unless they are identical.[11] There is no way to encode leximin with risk-weighting by real-valued R or r functions. (Such a function would have to give infinitely more weight to probabilities around 1 than to probabilities around 0.99, but would *also* have to give infinitely more weight to probabilities around 0.99 than to probabilities around 0.98, and so on.)

These three types of decision rule, leximin, *REU* with a locally strictly increasing function that sends the interval $[0, 1]$ to $[0, +\infty]$, and classic maximin, reflect three different motivations that all happen to lead to the maximin property. As a referee

[7]For continuous gambles, as described in Appendix 2.3, integration using vertical rectangles shows more generally that the marginal risk-weighting function $r(w)$ is a scaling factor for the importance of the outcome at the w mark of probability.

[8]For continuous gambles, it matters not just that r goes to infinity, but that it does so quickly enough that any interval containing that endpoint has infinite integral. There can be a point where $r(w)$ is infinite even though every interval has finite weight. As an example, consider $R(w) = \sqrt{w}$, which has $r(w) = \frac{1}{2\sqrt{w}}$, which is infinite when $w = 0$, even though $\int_0^w r(x)dx = R(w) - R(0)$ is finite for any $0 \leq w \leq 1$.

[9]In the continuous case, we can understand this more precisely by treating r as a Dirac delta function, which gives weight 1 to any interval containing one endpoint, and zero weight everywhere else.

[10]In the continuous case, leximin says that G is preferred to G' iff there exists $0 < w_0 < 1$ with $U(w_0) > U'(w_0)$, while $U(w) \geq U'(w)$ for all $w > w_0$ (or for all $w < w_0$ for leximax).

[11]In the continuous case, there are still incomparable pairs of gambles. For instance, with $U(w) = -1/(1-w)$ and $U'(w) = -1/(1-w) + \sin\left(1/(1-w)\right)$, for any $w_0 < 1$, there is $w > w_0$ with $U(w) > U'(w)$ and there is $w > w_0$ with $U'(w) > U(w)$.

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for this journal has pointed out (as well as Buchak, in a footnote on p. 68), classic maximin and maximax are most naturally motivated in contexts where an agent has no thoughts about the probability of various outcomes, and wants to use a single feature of each gamble to compare them. But these give up Buchak's means-ends motivation. Leximin arises from a single-minded focus on avoiding bad outcomes. If there is some utility x such that one gamble has higher probability of being below x than the other, then this can't be outweighed by any probability of *better* outcomes – it can only be outweighed by the behaviour of these gambles on *worse* outcomes than x. But the *size* of these differences doesn't matter – only ordinal relations among utilities are relevant for either classic maximin or for leximin.

However R functions with singularities at one or more points in the scale are instead naturally thought of as particularly extreme versions of R functions that give large amounts of weight to those points, while still having an idea of the relative importance of larger or smaller differences between outcomes at other points of the probability scale. For any particular comparison of two finite gambles made by R that goes to infinity at one endpoint, whether it is a preference based entirely in a higher utility at the relevant endpoint, or a preference based on equality at that endpoint and some comparison of the finite remainder, there is a truncation of the risk function to some large finite value that gives rise to the same preference, for the same reason. (And by applying an affine transformation, as described in section 5, this can be done with one of Buchak's risk functions on the $[0, 1]$ interval.) On this view, the maximin property arises just as a limiting property that can emerge from risk functions that put more and more weight on low positions in the scale. But no matter how much weight is put on the lowest positions in the scale, even so much that it can outweigh any difference at any other position, this motivation still requires non-zero weight on other positions in the scale. This guarantees that any improvement to an outcome at any point in the scale yields an improvement in the evaluation of the gamble, and also ensures that the amount of improvement depends on the size of the improvement to the outcome.

In Appendix 1, I investigate these motivations more precisely by looking at the axioms that go into Buchak's representation theorem. Because the maximin property and maximax property require putting infinite weight on some positions in the scale, all of these decision rules falsify two of Buchak's axioms that are meant to ensure that no outcome has infinite weight. Locally strictly increasing R satisfies all the rest of her axioms, even when some weights are infinite. But because leximin and leximax, and classic maximin and maximax, completely ignore the cardinal structure of utility, they all violate two more of her axioms that are connected to this cardinal structure. And because classic maximin and maximax violate stochastic dominance, they violate an additional two axioms connected to that. Thus, we can see formally that my generalization allowing locally strictly increasing R to put infinite weight at some points is closer to her axiomatic characterization, as well as the informal motivation.

8. Conclusion

I have shown how to generalize Buchak's risk-weighted expected utility to allow for a risk-weighting function that sends the interval of probabilities from $[0, 1]$ to an

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interval of weights other than $[0, 1]$. When this interval is finite, the possible preferences are the same as the ones given by her method, though this formalization establishes a symmetry between these weights and utilities, in that either are now only meaningful up to affine transformation. When this interval is infinite, we can get decision rules that encode preferences with the maximin or maximax properties, but satisfy more of Buchak's axioms and motivation than leximin, leximax, or classic maximin or maximax. We can also get substantively new types of preferences that put infinite weight at both ends of the utility scale, or points in the middle of the scale, rather than just one endpoint.

Each of these rules can be calculated either by multiplying *marginal* (incremental) utility by the risk-weighting of the *total* probability of all states with utility greater than that utility, or by multiplying *total* utility against the *marginal* (incremental) risk-weighting. The latter formulation bears a closer resemblance to familiar treatments of standard expected utility, and allows for more flexible generalizations. As described in Appendix 2.2, the formulation in terms of marginal risk-weighting has advantages for the evaluation of continuous gambles whose utilities go to infinity too quickly to have an *EU*, and this is valuable even independent of whether risk-weighting itself can be infinite.

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Appendix 1. Axiomatic Analysis

To state Buchak's axioms, I need to introduce some of her terminology. While I treat (finite) gambles as *sets* of ordered pairs of probabilities and utilities, she treats gambles as *functions* from states to outcomes that have only a finite set of outcomes in their range. An *event* is any set of states that has a probability. In the main text, I have ignored distinctions between states of the same probability, and outcomes of the same utility, and represented relevant events by ordered pairs. But in this appendix, I will follow her terminology.

For gambles f, g, Buchak writes "$f \geq g$" to indicate that f is (weakly) preferred to g, and uses "$f > g$" as an abbreviation for "$f \geq g$ and $g \not\geq f$", and "$f \sim g$" as abbreviation for "$f \geq g$ and $g \geq f$". These relations also apply to outcomes by identifying each outcome with the constant gamble that yields that outcome in every state.

For any outcome x and event E, she writes "$x_E f$" for the gamble that has the same outcome as f on all states outside of E, and yields outcome x on all states in E.

A *comoncone* is a set F of gambles such that for any f, g in F, and any two states s, s', if $f(s) > f(s')$ then $g(s) \geq g(s')$, and if $g(s) > g(s')$ then $f(s) \geq f(s')$. A maximal comoncone implicitly defines an ordering on the states, by saying that $s > s'$ iff there exists f in F with $f(s) > f(s')$. An *upper interval* of states for a maximal comoncone is a set of states that includes every state that is higher in this ordering than any state it contains, and conversely for a *lower interval* for a maximal comoncone. Within a comoncone, an event E

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always occurs at the same position(s) in this ordering, so any risk-weighting function R gives a specific weight to that E. An event E is *null in a comoncone* F iff there are a gamble f, and outcomes $x > y$ such that $x_E f$ and $y_E f$ are both in the comoncone, but $x_E f \sim y_E f$. It is straightforward to see that for leximin and leximax, and for any decision rule based on a strictly increasing R, an event is non-null in a comoncone iff it has non-zero probability, while for classic maximin (or maximax), an event is non-null in a comoncone iff it contains some lower (upper) interval of non-zero probability.

With this terminology established, I consider each axiom in turn, and show whether or not it is satisfied for classic maximin or maximax, by locally strictly increasing R (which may give infinite weight to one or more points on the scale), and by leximin and leximax. In a few cases, I note that the axiom as stated is technically satisfied by one of these rules in a trivial way, but that strengthening it with the inclusion of some other complementary feature of Buchak's risk-weighted decision rules makes the result non-trivial.

1.1 Ordering

Her axiom Ordering (A1) states that the preference ordering is a reflexive, transitive, total relation.

Reflexivity and transitivity are straightforward for all of the decision rules under consideration here. Totality is straightforward for classic maximin and maximax, and locally strictly increasing R that have only one point of infinite weight, as well as for leximin and leximax when gambles are finite. When there is infinite weight at two or more points in the risk spectrum, there can be incomparability between pairs of gambles, one of which has better outcome at one point of infinite weight, and the other of which has better outcome at a distinct point of infinite weight.

When gambles are continuous, even standard *EU* violates totality, because the probabilities of large positive outcomes and large negative outcomes can make infinite contributions to the evaluation with opposite signs (as shown by Nover and Hájek 2004). This seems analogous to the kinds of violations of totality that occur for locally strictly increasing R that place infinite weight at two or more points of the scale. The failures of totality for leximin and leximax (noted in footnote 11) are a bit harder to generate, but seem somewhat different from these others.

1.2 Statewise Dominance

Her axiom Statewise Dominance (A2) states that if $f(s) \geq g(s)$ for all states s, then $f \geq g$. Furthermore, if additionally, $f(s) > g(s)$ for all s in some non-null event E, then $f > g$.

All decision rules considered here satisfy the first half. For the second half, it is straightforward to see that it is satisfied for leximin, leximax and decision rules based on strictly increasing R, because for these rules, being null corresponds to non-zero probability.

If we require E to be non-null in a comoncone containing both f and g, then surprisingly, the second half is also satisfied by classic maximin and maximax, because the only non-null sets in a comoncone include lower or upper intervals. But this is a kind of trivial satisfaction.

To eliminate trivial satisfaction, it is sufficient to allow E to be non-null in *any* comoncone, since every event of non-zero probability is non-null in some comoncone, even if it is null in a comoncone containing f and g.

This version of the axiom effectively states that if an event *can* make a difference for some comparison, then it *always* makes a difference when everything else is equal. This is effectively stochastic dominance.

1.3 Preference Richness

Her axiom of Preference Richness (A3) has two parts. The first part says there exist outcomes x, y with $x > y$. The second part says, for any event E, gambles f, g, and outcomes x, y, with $x_E f > g > y_E f$, there is z with $z_E f \sim g$.

In the context of her other axioms, this axiom is meant to ensure that there are outcomes with each real number utility, but it doesn't do that work when some outcomes (such as the best or worst) can be weighted infinitely more than others.

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The first part is trivially satisfied by all the decision rules I consider. But the second part has the opposite pattern from (A2) – it is satisfied as stated by classic maximin and maximax, but violated by the preferences given by locally strictly increasing R with an unbounded range, and by leximin and leximax.

It is straightforward to see that classic maximin satisfies the axiom as stated. If $x_E f > g > y_E f$, then it must be that E has non-zero probability, and the worst possible outcome of g is worse than x, and worse than the worst outcome of f outside of E, but better than y. Setting z equal to this worst possible outcome of f outside E verifies the conclusion. (A similar argument works for classic maximax.) However, this form of satisfaction prevents the axiom from doing the work it is meant to do. In Buchak's *REU*, the precise value of z depends not only on x, y, f, g, but also on the probability of E.

To see that none of the other decision rules satisfy this axiom even in this trivial sense, note that it is impossible to satisfy this axiom, together with both stochastic dominance, and the maximin property. If the worst outcome of g is z, and it yields this outcome at every state in E, and f only yields outcomes better than z, but yields strictly worse outcomes than g at all states outside of E, then we can see the conflict. If $x > z$, then $x_E f > g$, and if $z > y$ then $g > y_E f$, but because of stochastic dominance, we have $g > z_E f$, rather than $g \sim z_E f$. (A similar argument shows that it is impossible to satisfy this axiom together with both stochastic dominance and the maximax property. I believe this is also true for the maximedian property, but there are extra complexities in the argument.)

1.4 Small Event Continuity

Her axiom Small Event Continuity (A4) is violated by all of the preferences considered here. This axiom says, for any gambles $f > g$, and outcome x, there is a finite partition of events $E_1, \ldots, E_n$ such that $x_{E_i} f > g$ and $f > x_{E_i} g$.

In the context of her representation theorem, it does the work of ensuring that every event has finite weight, so it is unsurprising that these preferences violate it whenever x occurs at a point of the scale that gets infinite weight. More subtly, even ignoring the E_i that occur at points of infinite weight (which is all points for leximin and leximax), if R goes to infinity, then with increasingly fine partitions, there is no upper bound to how large the weights of the E_i can get.

1.5 Comonotonic Archimedean Axiom

The Comonotonic Archimedean Axiom (Buchak's A5) and Comonotonic Tradeoff Consistency (A6), in the context of her other axioms, are used to ensure that the utilities of outcomes are structured cardinally (that is, unique up to affine transformation). Since leximin, leximax and classic maximin and maximax depend only on the ordinal structure of outcomes (that is, any order-preserving permutation of utilities preserves preferences, not just affine transformations), one expects that these axioms should fail.

However, in Buchak's formal statement of these axioms, it turns out that all of these decision rules trivially satisfy them, while my decision rules with strictly increasing functions R non-trivially satisfy them. I will give very slight strengthenings of these axioms that prevent trivial satisfaction.

To state the Comonotonic Archimedean Axiom (A5), we need to define a "standard sequence". A sequence $x^1, x^2, \ldots$ of outcomes is said to be a *standard sequence for a comoncone F* iff there are disjoint non-null events E_1, E_2, outcomes y, z, and act f such that for all k,

$$(x^{k+1})_{E_1}(y)_{E_2} f \sim (x^k)_{E_1}(z)_{E_2} f,$$

and both events are in F.

The Comonotonic Archimedean Axiom states that, in every comoncone, every bounded standard sequence is finite.

For classic maximin and maximax, there are no disjoint non-null events in any comoncone, so there are no standard sequences, so the axiom trivially holds. And for leximin and leximax, the relation $\sim$ never holds between gambles in a comoncone that differ on a non-null event, so again there are no standard sequences, so again the axiom trivially holds.

But for strictly increasing R, it is not hard to show that the axiom non-trivially holds. The x^k form a standard sequence iff $x^k - x^{k+1}$ multiplied by the risk weighting of E_1 in the comoncone equals $-z$ multiplied by the risk weighting of E_2 in the comoncone. Thus, the sequence must have utilities that form an

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arithmetic sequence, and if it is bounded, then the Archimedean principle of the reals non-trivially establishes that the sequence must be finite.

A slightly stronger statement of the axiom would say that in every comoncone, there exist bounded standard sequences of every finite size, but there are no infinite bounded standard sequences. This version of the axiom is satisfied by locally strictly increasing R (including ones that put infinite weight at finitely many positions in the scale – we just have to choose E_1 and E_2 that aren't at these positions) but not by classic maximin or maximax, leximin or leximax, since they have no standard sequences.

1.6 Comonotonic Tradeoff Consistency

Comonotonic Tradeoff Consistency states that for any comoncone F, and event E that is non-null for F, if there are gambles f, g, and outcomes x, y, z, w, with $x_E f \sim y_E g$ and $z_E f \sim w_E g$, with all four gambles in the comoncone, then there is no $y' \neq y$ with $x_E f \sim y'_E g$.

For classic maximin, a non-null event must include a lower interval of positive probability, and thus if there is a single outcome that a gamble has throughout that lower interval, the evaluation of the gamble must equal the utility of that outcome. Thus, if $x_E f \sim y_E g$ and $x_E f \sim y'_E g$ we must have $x = y = y'$, so the axiom is trivially satisfied. (Similarly for classic maximax.)

For leximin and leximax, the relation $\sim$ only holds between gambles with the same probabilities of the same outcomes, so again if $x_E f \sim y_E g$ and $x_E f \sim y'_E g$, we must have $x = y = y'$, so the axiom is trivially satisfied.

However, if we use a strictly increasing R, satisfaction is non-trivial if E does not contain any infinitely weighted event. If $z_E f \sim w_E g$ and E doesn't include any infinitely weighted event, then f and g must agree on all the infinitely weighted events, and their differences on finitely weighted events must precisely balance the difference between z and w multiplied by the weight of E in this comoncone. Similarly, since $x_E f \sim y_E g$, we can see that $x - y$ multiplied by this weight must also precisely balance this same difference, and if $x_E f \sim y'_E g$, then $x - y'$ must as well. Thus, we must have $y = y'$. This result is non-trivial.

To strengthen the axiom in a way that leaves it satisfied by any locally strictly increasing R (including ones that give infinite weight to finitely many positions in the scale) while ruling out the others, we can add the statement that for any comoncone F, there exists an event E that is non-null for F, and there exist distinct gambles f, g and distinct outcomes x, y, z, w, such that $x_E f \sim y_E g$ and $z_E f \sim w_E g$.

1.7 Strong Comparative Probability

Her axiom of Strong Comparative Probability (A7) states that if E_1 and E_2 are disjoint, and $x' > x$ and $y' > y$, then for any acts g, h, we have $x'_{E_1} x_{E_2} g \geq x_{E_1} x'_{E_2} g$ iff $y'_{E_1} y_{E_2} h \geq y_{E_1} y'_{E_2} h$.

In the context of Buchak's other axioms, this axiom ensures that the probability scale is unique, separate from the risk-weighting. To see this, note that if E_1 and E_2 do both have meaningful probabilities, then $x'_{E_1} x_{E_2} g$ stochastically dominates $x_{E_1} x'_{E_2} g$ iff E_1 has greater probability than E_2 (and similarly for $y'_{E_1} y_{E_2} h$ and $y_{E_1} y'_{E_2} h$). (Incidentally, this observation allows us to see that this axiom holds for locally strictly increasing R, and for leximin and leximax.)

Slightly surprisingly, this axiom also trivially holds for classic maximin and maximax. This is for the same reason as the trivial satisfaction for Statewise Dominance – within the comoncone for $x'_{E_1} x_{E_2} g$, exactly one of E_1, E_2 and their complement is non-null, and the same is true within the comoncone for $x_{E_1} x'_{E_2} g$, except that it switches if it is E_1 or E_2, so the two gambles are evaluated the same. Requiring that there be some events E_1 and E_2, and some distinct values x, y, x', y' making the inequalities strict, eliminates trivial satisfaction of this axiom.

1.8 Summary

Classic maximin and maximax only satisfy A1 non-trivially, and satisfy A2, A3, A5, A6 and A7 in trivial ways that are removed with natural strengthenings of the axioms.

Leximin and leximax satisfy A2 and A7 non-trivially, and satisfy A5 and A6 only in trivial ways that are removed with natural strengthenings of the axioms. For some continuous gambles, A1 is falsified, but these are hard to construct.

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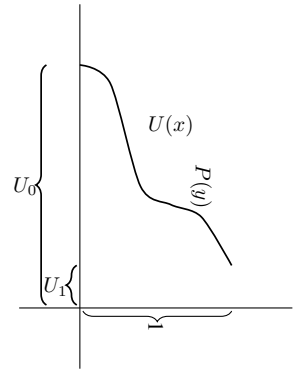


Figure 17. The graphs of $U(x)$ and $P(y)$.

Locally strictly increasing R satisfy A2, A5, A6 and A7 in non-trivial ways. They also satisfy A1 if there is only one point in the scale with infinite weight. But the failures when there are multiple points with infinite weight seem analogous to the failures of A1 that arise even for standard *EU*, with continuous gambles whose probabilities of positive and negative outcomes don't die off quickly enough.

Thus, my strictly increasing R do a better job of satisfying the axioms for Buchak's *REU* than these other ways of satisfying the maximin or maximax property, in addition to staying closer to her motivations.

Appendix 2. Continuous Gambles

The main text considers gambles G with only finitely many possible outcomes. This appendix shows how all the results can be modified to accommodate certain infinite gambles. Say that G is *continuous over an interval* iff there is a minimum utility U_1 (possibly $-\infty$) and maximum utility U_0 (possibly $+\infty$), and for any $U_1 \leq u \leq u' \leq U_0$, the probability of an outcome with utility in $[u, u']$ is 0 iff $u = u'$. Many of the formulas considered here can likely be generalized to an even wider class of gambles, but in some cases modifications may be needed – I do not consider them here.

For $x \in [0, 1]$ define $U(x)$ to be the unique utility value in $[U_1, U_0]$ such that the probability of an outcome at least that good is x. For y in $[U_1, U_0]$, define $P(y)$ to be the probability of receiving an outcome with utility at least y. Continuity states that for any positive width interval in $[U_1, U_0]$, there is strictly positive probability of an outcome with utility in that interval, while any particular utility value has probability 0 of occurring, so that U and P are both continuous and strictly decreasing. (It will eventually be useful to define P on *all* values, so that $P(y) = 1$ if $y \leq U_1$ and $P(y) = 0$ if $y \geq U_0$. P will be continuous at all real values, but will only be strictly decreasing for $U_1 \leq y \leq U_0$.)

Figure 17 shows both of these functions – U in the standard orientation, and P when rotated counterclockwise. Note that when rotating the graph to look at $P(y)$, the y-axis has higher values to the left and lower values to the right, so an upwards-sloping curve in that rotation is actually *decreasing*. By virtue of the relevant definitions, $U(0) = U_0$, $U(1) = U_1$, $P(U_1) = 1$, and $P(U_0) = 0$. In general, we have $U(P(y)) = y$ whenever $y \in [U_0, U_1]$ and $P(U(x)) = x$ whenever $x \in [0, 1]$, so that these functions are inverses of each other.

2.1 Standard Expected Utility: Vertical or Horizontal Integrals

Standard expected utility can be defined by integration with vertical rectangles by

$$EU(G) = \int_{x=0}^{1} U(x)dx.$$

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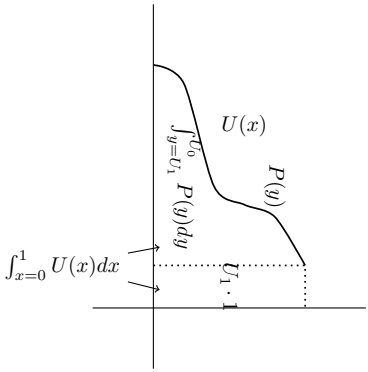


Figure 18. Calculating expected utility with horizontal or vertical integration.

If U_1 is finite, it can also be defined by integration with horizontal rectangles by

$$EU(G) = U_1 + \int_{y=U_1}^{U_0} P(y)dy.$$

This correction by adding U_1 corresponds to the correction of the convention declaring $U_{n+1} = 0$ in the finite case – it fills in the last block visible in Figure 18. Note that there is a slight convenience to doing the vertical calculation rather than the horizontal one, in that no correction term is needed. I omit the mathematical details of the definition of integration, in favour of using Figure 18 to make clear why this should work. However, it is sufficient to derive the above formulas that we require that a continuous gamble should get a higher *EU* than any continuous or finite gamble that it stochastically dominates, and a lower *EU* than any continuous or finite gamble that stochastically dominates it. (Again, note that if you rotate your head to look at this diagram, then the newly horizontal *y*-axis is reversed, which is why the integral has its limits in the opposite order from what you expect.)

2.2 Comparative Expectation

One detail of integration that is relevant though is that if either U_1 or U_0 is infinite, it is possible that the integral be infinite, and if both are infinite, then it is possible that the integral be undefined, depending on how U approaches infinity. Standard expected utility theory doesn't say how to compare two gambles if one of them has undefined *EU*, or if both have the same infinite value. However, we can adapt an idea of Colyvan (2008) to make comparisons in at least some of these cases.[12]

Let G and G' be two continuous gambles over finite intervals of possible utilities. When $EU(G)$ and $EU(G')$ are not both infinite with the same sign, $EU(G) > EU(G')$ iff $EU(G) - EU(G') > 0$. Letting U and U' be the respective cumulative utility functions, the integrals over vertical rectangles show that this happens iff

$$\int_{x=0}^{1} U(x)dx - \int_{x=0}^{1} U'(x)dx > 0.$$

[12]Strictly speaking, this is the revision proposed by Colyvan and Hájek (2016). Colyvan's original proposal assigns probabilities and utilities to states individuated independently of utility, and evaluates a pair of gambles based on the difference of their outcomes in these states. The idea here ignores the identity of the states, and uses the cumulative distribution to identify states based on the probability of getting an outcome at least that good. Colyvan makes this proposal in response to examples from Nover and Hájek (2004) of gambles where *EU* is undefined because there are infinite contributions of both positive utilities and negative utilities.

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Since integrals are linear, and these integrals are over the same interval, this happens iff

$$\int_{x=0}^{1} [U(x) - U'(x)]dx > 0.$$

The idea of comparative expected utility is to apply this same comparison to all gambles, regardless of whether $EU(G)$ and $EU(G')$ are defined or finite.

We define the comparative expected utility of G and G' to be

$$CEU(G, G') = \int_{x=0}^{1} [U(x) - U'(x)]dx,$$

and say that G is preferred to G' if $CEU(G, G') > 0$. This naturally extends standard EU comparison to many cases of two gambles with infinite EU of the same sign. ($CEU(G, G')$ may still be undefined if $U(x) - U'(x)$ is both unbounded above and below, and sufficiently ill-behaved. CEU defines a merely partial ordering on gambles that are continuous over intervals, though this ordering does extend to many pairs of gambles where one or both lacks a standard EU.)

This extension naturally satisfies stochastic dominance. For G to stochastically dominate G', it must be that $U(x) \geq U'(x)$ for all x, with at least one value for which the inequality is strict. Because U and U' are continuous, this means there must be some interval of positive width such that $U(x) > U'(x)$ on this entire interval, so $CEU(G, G') > 0$.

A similar comparison is possible using integrals over horizontal rectangles, though some of the calculations are a bit trickier. Letting P and P' be the respective cumulative probability functions, and U_1 and U'_1 be the respective minimum utilities, and U_0 and U'_0 be the respective maximum utilities, we see that

$$EU(G) - EU(G') = \left[U_1 + \int_{y=U_1}^{U_0} P(y)dy\right] - \left[U'_1 + \int_{y=U'_1}^{U'_0} P'(y)dy\right].$$

It is natural to define $P(y) = 0$ when $y > U_0$, and similarly $P'(y) = 0$ when $y > U'_0$. This quantity is then

$$\left[U_1 + \int_{y=U_1}^{+\infty} P(y)dy\right] - \left[U'_1 + \int_{y=U'_1}^{+\infty} P'(y)dy\right].$$

If we temporarily assume that $U_1 < U'_1$, then this is

$$[U_1 - U'_1] + \left[\int_{y=U_1}^{U'_1} P(y)dy\right] + \left[\int_{y=U'_1}^{+\infty} [P(y) - P'()]y\right].$$

If we notice that $U_1 - U'_1 = -\int_{y=U_1}^{y=U'_1} 1dy$, then this is

$$\left[\int_{y=-\infty}^{U_1} [1 - 1]dy\right] + \left[\int_{y=U_1}^{U'_1} [P(y) - 1]dy\right] + \left[\int_{y=U'_1}^{+\infty} [P(y) - P'(y)]dy\right].$$

If we again conventionally define $P(y) = 1$ when $y < U_1$ and $P'(y) = 1$ when $y < U'_1$ then this simplifies to

$$CEU(G, G') = \int_{y=-\infty}^{+\infty} [P(y) - P'(y)]dy.$$

These conventional definitions of P outside the interval of utilities the gamble takes on allow us to eliminate the correction term U_1 from the calculation of $EU(G)$ by integrals over horizontal rectangles. This formula for $CEU(G, G')$ is equivalent to the calculation yielded by integrals over vertical rectangles, though this is not obvious.

$P(y) - P'(y)$ must go to 0 as y goes to $-\infty$ or $+\infty$, but if it does not go to 0 quickly enough at both ends then $CEU(G, G')$ may be infinite or undefined. If G stochastically dominates G', then $P(y) - P'(y) \geq 0$ for all y, and is strictly positive at some point, and thus by continuity is strictly positive over some interval, yielding another way to see that CEU agrees with stochastic dominance.

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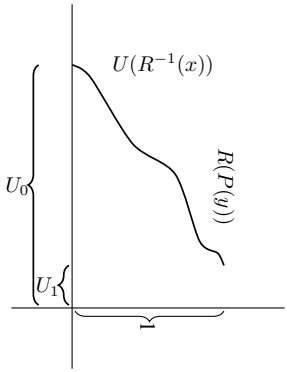


Figure 19. The graphs of $U\big(R^{-1}(x)\big)$ and $R\big(P(y)\big)$.

2.3 Risk-weighted Expected Utility: Horizontal or Vertical Integrals, and Comparative Risk-weighted Expectation

To add risk weightings for the continuous case, we again start with R some continuous strictly increasing function from $[0, 1]$ to $[0, 1]$ with $R(0) = 0$ and $R(1) = 1$, as illustrated in Figure 19. However, there are a few complications compared with the finite case. Note that the x-axis here now represents the cumulative risk-weighted probabilities, and not the cumulative probabilities themselves. Thus, we must first undo the risk-weighting in order to get the right input for the U function.

To calculate the risk-weighted expected utility, we can do the integration by horizontal rectangles straightforwardly to get

$$REU(G) = U_1 + \int_{y=U_0}^{U_1} R\big(P\big(y\big)\big)dy.$$

Again, the U_1 is added to account for the missing rectangle as seen in Figure 20.

The integration using vertical rectangles has to deal with the fact that U is composed with R^{-1}. Because U is a function of the cumulative probability itself rather than its risk-weighting, it is natural in some sense to calculate this integral with respect to cumulative probability, rather than the x-axis of this diagram. So we define an auxiliary variable w that doesn't appear in this diagram, representing the cumulative probability itself. Thus, $x = R(w)$, and $y = U(w)$. Equal tiny steps in w no longer correspond to equal-width tiny rectangles. With a suitably general form of integration, we can define $REU(G) = \int_{w=0}^{1} U(w)dR(w)$. But when R is differentiable,[13] we can think in the familiar Riemann way of an integral composed of tiny rectangles – but the width of the rectangle is proportional both to the step in w *and* to the derivative of $R(w)$ with respect to w. That is, instead of just integrating $U(w)$ with respect to w, we have to multiply by a correction factor. This version of the formula is already demonstrated by (Quiggin, 1982, 330).

If we let $r(w) = \frac{dR(w)}{dw}$, then

$$REU(G) = \int_{w=0}^{1} U(w)r(w)dw.$$

This $r(w)$ gives the marginal risk-weighting function corresponding to cumulative risk-weighting function $R(w)$. Roughly, $r(w)$ measures the significance of the contribution of the wth best outcome to REU. Because $R(w)$ is strictly increasing, $r(w)$ is everywhere non-negative and there is no interval over which it is zero. This tells us that REU agres with stochastic dominance, as desired.

[13]The fact that R is continuous, increasing, and bounded does not guarantee that it is *everywhere* differentiable. The Cantor staircase function, whose value at any w is the fraction of the Cantor set that is below w, is non-differentiable at the continuum-many points of the Cantor set. Although this function is not *strictly* increasing, mixing it with a linear function can make it so, while preserving its infinitely many points of non-differentiability. The move from R to r thus does result in a slight loss of generality, in that we miss out on non-differentiable Rs, but as we will see in the next few sections, it increases generality in other ways.

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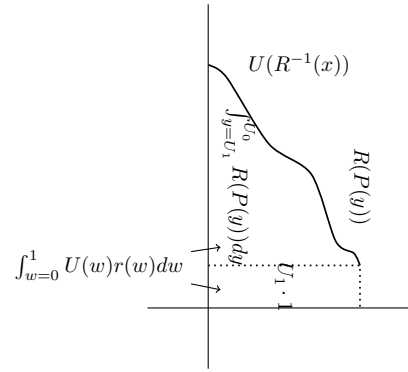


Figure 20. Calculating risk-weighted expected utility with horizontal or vertical integration.

Because $R(1) = 1$ and $R(0) = 0$, the area under the curve contains the rectangle of width 1 and height U_1, and is contained within the rectangle of width 1 and height U_0. Thus, in the continuous case as well, $U_n \leq REU(G) \leq U_0$.

Standard *EU* corresponds to the case where $R(w) = w$, which means that $r(w) = 1$, so that

$$\int_{w=0}^{1} U(w)r(w)dw = \int_{x=0}^{1} U(x)dx.$$

As before, we can deal with cases where $REU(G)$ and/or $REU(G')$ are undefined, or infinite with the same sign, by moving to

$$CREU(G, G') = \int_{w=0}^{1} [U(w) - U'(w)]r(w)dw = \int_{y=-\infty}^{+\infty} \left[R(P(y)) - R(P'(y))\right]dy,$$

again with the conventions that $P(y) = 1$ when $y < U_1$ and $P(y) = 0$ when $y > U_0$, and similarly for P'. Again, these conventions for the *CREU* allow us to eliminate the correction term of U_1 that appears in the calculation of the absolute *REU* by horizontal integrals.

Since r is everywhere non-negative, and not zero over an entire interval, and since R is strictly increasing, we again see that this rule satisfies stochastic dominance. Furthermore, we can see that r in a sense measures how much weight the decision-maker is giving to utilities at the w probability. If r is strictly increasing (so that R is concave up), then the agent gives strictly more weight to the worse outcomes of the gamble (i.e. the ones for which there is a higher probability of exceeding them) validating Buchak's thought that this is a way to characterize risk-aversion.

2.4 Rescaling R beyond $[0, 1]$

For the continuous case, we get a diagram like the one in Figure 21. Under the vertical method, we again have

$$REU(G) = \int_{w=0}^{1} U(w)r(w)dw,$$

where w represents cumulative probability in $[0, 1]$, and where $r(w)$ is the factor for the change of variables from probabilities to risk-weightings, and is given by $r(w) = dR(w)/dw$. When we calculate *REU* horizontally instead, we need to subtract off one correction term for the rectangle on the left, and add back in the correction term for the rectangle on the bottom. These two methods of integration are both illustrated in Figure 22.

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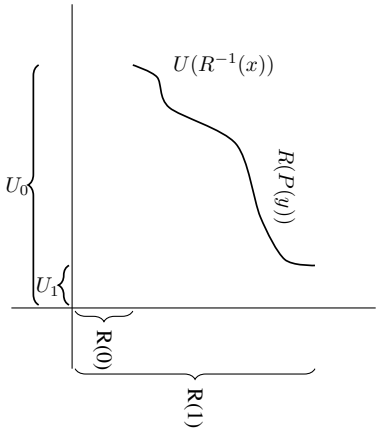


Figure 21. The graphs of $U\big(R^{-1}(x)\big)$ and $R\big(P(y)\big)$ with non-[0, 1] risk function.

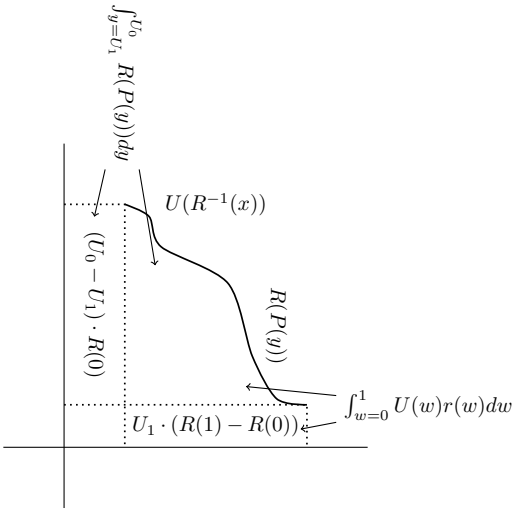


Figure 22. Calculating risk-weighted expected utility by horizontal or vertical integrals, with non-[0, 1] risk function.

That is,

$$\begin{aligned} REU(G) &= \int_{y=U_1}^{U_0} R\big(P\big(y\big)\big)dy - (U_0 - U_1) \cdot R(0) + (R(1) - R(0))U_1 \\ &= \int_{y=U_1}^{U_0} R\big(P\big(y\big)\big)dy - U_0R(0) + U_1R(1). \end{aligned}$$

Again, when R sends the [0, 1] interval to a different interval, we end up needing correction terms when calculating *REU* for continuous gambles by the horizontal method, but not the vertical method.

Again, when we move to *CREU*, the correction terms disappear. We again get:

$$CREU(G, G') = \int_{w=0}^{1} [U(w) - U'(w)]r(w)dw = \int_{y=-\infty}^{+\infty} [R\big(P\big(y\big)\big) - R\big(P'\big(y\big)\big]dy,$$

again with the conventions that $P\big(y\big) = 1$ when $y < U_1$ and $P\big(y\big) = 0$ when $y > U_0$, and similarly for P'.

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2.5 Affine Transformations

Just as in the finite case, let f be any affine transformation, and let $f(G)$ be the gamble that replaces U by $f(U)$. Again, from the formulations of EU in terms of vertical rectangles, it is straightforward to see that $EU(f(G)) = f(EU(G))$ and $REU(f(G)) = f(REU(G))$. Furthermore, if $f(x) = ax + b$, then it is also straightforward to see that $CEU(f(G), f(G')) = a \cdot CEU(G, G')$ and that $CREU(f(G), f(G')) = a \cdot CREU(G, G')$. Thus, changing the utility scale by an affine transformation again doesn't change any pairwise evaluations of gambles. Similarly, when applying an affine transformation $f(x) = ax + b$ to the risk-weighting, to use $R'(x) = f(R(x))$, we see that $r'(x) = a \cdot r(x)$. Thus, using the vertical calculations in terms of r, it is easy to see that $R'EU(G) = a \cdot REU(G)$ and $CR'EU(G, G') = a \cdot CREU(G, G')$. Changing the scale of risk-weighting by an affine transformation again doesn't change any pairwise evaluations of gambles, and there is a unique affine transformation of the risk-weighting that sends it to the $[0, 1]$ interval.

2.6 Infinite Risk-weighting

As we saw in the case of finite gambles, when the risk-weighting goes to infinity at one or more points, *REU* will only be finite if the limiting utility of the gamble is 0 at the relevant points of the scale. But in the continuous case, even if the limiting utility is 0, the integral will only be finite if the utility goes to 0 *quickly enough* to make up for the growth of r, just as Colyvan noted that EU is undefined if the probability doesn't go to 0 quickly enough as utility goes to infinity. Thus, a large number of evaluations are better done comparatively than by assigning a numerical *REU* to each individual gamble.

This is much like the potential problem that arose when utility could go to infinity at one or both ends of the scale, but the same formulas we defined for *CREU* there work here as well:

$$CREU(G, G') = \int_{w=0}^{1} [U(w) - U'(w)]r(w)dw = \int_{y=-\infty}^{+\infty} [R(P(y)) - R(P'(y)]dy,$$

where $r(w)$ is the marginal risk-weighting function $r(w) = dR(w)/dw$, $P(y)$ is the cumulative probability function, such that the probability of getting more than y utility is $P(y)$, and $U(w)$ is the cumulative utility function, such that the probability of getting more than that utility is w. (P and U are continuous functions, and P is strictly decreasing for $U_1 < y < U_0$, while U is strictly decreasing for $0 < w < 1$. Recall that $P(y)$ was defined to be constantly 1 if $y < U_1$ and 0 if $y > U_0$.) The fact that R is locally strictly increasing means that r is always non-negative, and there is no interval over which r is almost always 0. But now we allow r to have any positive integral, including infinity, instead of requiring it to integrate to 1.

The formula $\int_{=0}^{1}[U(w) - U'(w)]r(w)dw$ has the advantage that it is well-defined even in cases where r has infinite integral over multiple intervals (as in section 6.4), as long as $U - U'$ either has the same sign over all these intervals, or goes to 0 fast enough to make the relevant segment of the interval finite. Even when r has finite integral, this version of the formula makes clear precisely what matters – it is just the ratios of r at various points of the probability scale. It doesn't matter whether the integral of r over this whole scale is equal to 1, and it doesn't matter what the absolute values of R are, just their differences, which are entirely reflected in r.

Kenny Easwaran is Professor of Logic and Philosophy of Science at the University of California, Irvine. His work spans a variety of topics in formal epistemology, decision theory, philosophy of mathematics, and related areas.

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