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On computably locally compact Hausdorff spaces

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Locally compact Hausdorff spaces generalise Euclidean spaces and metric spaces from 'metric' to 'topology'. But does the effectivity on the latter (Brattka and Weihrauch 1999; Weihrauch 2000) still hold for the former? In fact, some results will be totally changed. This paper provides a complete investigation of a specific kind of space – computably locally compact Hausdorff spaces. First we characterise this type of effective space, and then study computability on closed and compact subsets of them. We use the framework of the representation approach, TTE, where continuity and computability on finite and infinite sequences of symbols are defined canonically and transferred to abstract sets by means of notations and representations.

1. Introduction

Computable analysis is a branch of computability theory studying computable functions and subsets of Euclidean space and other topological spaces. It connects computability/computational complexity with analysis/numerical computation by combining concepts of approximation and computation. Beginning with Turing's famous paper in 1936 (Turing 1936), various similar but not exactly equivalent models of real number computation have been proposed for researching computable analysis (Grzegorczyk 1955; Ko 1991; Kusner 1984; Mazur 1963; Pour-El and Richards 1989; Scott 1970). Among these models, Type-2 Theory of Effectivity (TTE), which is based on representations, seems to be particularly realistic, flexible and expressive. Its basic idea is to transfer computability on continuous objects such as real numbers to computability on infinite sequences of symbols, using sequences as 'names' of objects. A Type-2 machine, as a generalised Turing machine, transforms infinite sequences to infinite sequences.

Computable analysis was first proposed by Grzegorczyk (Grzegorczyk 1955) and Lacombe (Lacombe 1955), and further developed by Hauck (Hauck 1973) and Kreitz and Weihrauch (Kreitz and Weihrauch 1984). Later, Zhou (Zhou 1996), Ziegler (Ziegler 2002) and Brattka and Weihrauch (Brattka and Weihrauch 1999) used the framework of TTE to study computability on real-valued functions and subsets of Euclidean spaces. Weihrauch (Weihrauch 1993), Yasugi (Yasugi *et al.* 1999) and Brattka and Presser (Brattka and

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Presser 2003) generalised the corresponding work to computable metric spaces. In addition, locally compact Hausdorff spaces (Collins 2005), quasi-metric spaces (Brattka 2003) and non-metrisable (Zhong and Weihrauch 2003) spaces have also been introduced. We note that the study of computability on topological spaces has, in the main, been developed top-down, that is, from special spaces to more general ones.

In this paper we mainly investigate computability on locally compact Hausdorff spaces. Our motivation for this was, on the one hand, locally compact Hausdorff spaces are more general than metric spaces – they are not based on the concept of a metric – and, on the other hand, locally compact Hausdorff spaces are quite 'close' to metric spaces and inherit a good many important properties of metric spaces. Roughly speaking, a Hausdorff space with a countable base is quasi-metrisable, and with the addition of local compactness it becomes metrisable. Hence, locally compact Hausdorff spaces actually form a 'bridge' connecting metric spaces and general topological spaces. In addition, they have practical applications. For example, Collins (2005) uses locally compact Hausdorff spaces to study computability of reachable sets for non-linear dynamic and control systems. For these reasons we believe that this research aim is both meaningful and achievable.

This paper is organised as follows. In the next section, we sketch some basic notions on TTE and provide some fundamental definitions and properties of representations of points and sets in computable T_0-spaces. In Section 3 we introduce and characterise computably locally compact Hausdorff spaces. Section 4 defines and compares various representations of closed subsets of them. Section 5 is specially discussing representations of compact subsets of them. Section 6 presents our conclusions.

2. Preliminaries

This section consists of two parts: in Section 2.1 we sketch the concept of TTE; and in Section 2.2 we introduce computable T_0-spaces and the underlying representations of points and sets.

2.1. *Type-2 Theory of Effectivity (TTE)*

In this paper we will use the framework of TTE (Weihrauch 2000) to explore several aspects of computability in locally compact Hausdorff spaces. The Type2 theory of effectivity defines computability on Σ^* and Σ^ω using Type-2 machines and transfers a computability concept to 'abstract' sets by means of naming systems.

We assume that Σ is a fixed, finite alphabet containing the symbols 0 and 1, and consider computable functions on finite and infinite sequences of symbols Σ^* and Σ^ω, respectively, which can be defined, for example, by Type-2 machines, that is, Turing machines reading from and writing on finite or infinite tapes. A Type-2 machine may have several one-way read-only input tapes, several two-way work tapes and a unique one-way write-only output tape. It permits infinite input or output, and has a finiteness property, that is, each tuple of prefixes of the inputs determines a unique prefix of the

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output. We use $f :\subseteq X \to Y$ to denote a partial function from X to Y. For $\alpha_i \in \{*, \omega\}$, a function $f :\subseteq \Sigma^{\alpha_1} \times \ldots \times \Sigma^{\alpha_k} \to \Sigma^{\alpha_0}$ is said to be computable if $f = f_M$ for some Type-2 machine M.

We define the 'wrapping function' $\iota : \Sigma^* \to \Sigma^*$ by $\iota(a_1 a_2 \ldots a_k) := 110 a_1 0 a_2 0 \ldots a_k 011$ for all $k \in \mathbb{N}$. We consider standard functions for finite or countable tupling on Σ^* and Σ^ω denoted by $\langle \cdot \rangle$, and projections of the inverse π_i for $i \in \mathbb{N}$. We use '$\lhd$' to denote the subword relation. A sequence $p \in \Sigma^* \cup \Sigma^\omega$ is said to be a list of M if $M = \{u \mid \iota(u) \lhd p\}$. $u \sqsubseteq p$ means that u is a prefix of p.

Given $Y := Y_1 \times \ldots \times Y_k (Y_1, \ldots, Y_k \in \{\Sigma^*, \Sigma^\omega\})$, we define $(y_1, \ldots, y_k) \circ Y := U_1 \times \ldots \times U_k$, where $U_i := \{y_i\}$ if $Y_i = \Sigma^*$ and $U_i := y_i \Sigma^\omega$ if $Y_i = \Sigma^\omega$.

$A \subseteq_{\text{fin}} B$ if $A \subseteq B$ and A is a finite set.

Notations $\nu :\subseteq \Sigma^* \to M$ and representations $\delta :\subseteq \Sigma^\omega \to M$ are used to introduce relative continuity and computability on 'abstract' sets M.

For a representation $\delta :\subseteq \Sigma^\omega \to M$, if $\delta(p) = x$, the point $x \in M$ can be identified by the 'name' $p \in \Sigma^\omega$. For representations $\delta :\subseteq \Sigma^\omega \to M$ and $\delta' :\subseteq \Sigma^\omega \to M'$, we define $[\delta, \delta'] :\subseteq \Sigma^\omega \to M \times M'$ by $[\delta, \delta']\langle p, p' \rangle := (\delta(p), \delta(p'))$ and $[\delta]^\omega :\subseteq \Sigma^\omega \to M^\omega$ by $[\delta]^\omega \langle p_0, p_1, \ldots \rangle := (\delta(p_0), \delta(p_1), \ldots)$.

For a naming systems $\gamma :\subseteq Y_i \to M_i$, $Y_i = \Sigma^*$ or Σ^ω, the set X is said to be γ-open (γ-clopen, γ-r.e., γ-decidable) if and only if $\gamma^{-1}[X]$ is open (clopen, r.e. open, decidable) in $\text{dom}(\gamma)$.

For naming systems $\gamma_i :\subseteq Y_i \to M_i$ $(i = 0, \ldots, k)$, a function $h :\subseteq Y_1 \times \ldots \times Y_k \to Y_0$ is a $(\gamma_1, \ldots, \gamma_k, \gamma_0)$-realisation of $f :\subseteq M_1 \times \ldots \times M_k \rightrightarrows M_0$ if $\gamma_0 \circ h(p_1, \ldots, p_k) \in f(\gamma_1(p_1), \ldots, \gamma_k(p_k))$ whenever $f(\gamma_1(p_1), \ldots, \gamma_k(p_k))$ exists. The function h is said to be a strong realisation of f if $h(p_1, \ldots, p_k) = \uparrow$ for all $\langle p_1, \ldots, p_k \rangle \in \text{dom}([\gamma_1, \ldots, \gamma_k])$ with $[\gamma_1, \ldots, \gamma_k](p_1, \ldots, p_k) \notin \text{dom}(f)$. The multi-function f is $(\gamma_1, \ldots, \gamma_k, \gamma_0)$-continuous (-computable) if it has a continuous (computable) $(\gamma_1, \ldots, \gamma_k, \gamma_0)$-realisation.

For naming systems $\gamma :\subseteq Y \to M$ and $\gamma' :\subseteq Y' \to M'$ $(Y, Y' \in \{\Sigma^*, \Sigma^\omega\})$, $M \subseteq M'$, we let $\gamma \leqslant_t \gamma'$ (t-reducible) and $\gamma \leqslant \gamma'$ (reducible) if and only if there is some continuous or computable function $f :\subseteq Y \to Y'$ such that $\gamma(y) = \gamma' f(y)$ for all $y \in \text{dom}(\gamma)$, respectively. We define t-equivalence and equivalence by $\gamma \equiv_t \gamma' \iff (\gamma \leqslant_t \gamma'$ and $\gamma' \leqslant_t \gamma)$ and $\gamma \equiv \gamma' \iff (\gamma \leqslant \gamma'$ and $\gamma' \leqslant \gamma)$, respectively.

Let $\nu_{\mathbb{N}} :\subseteq \Sigma^* \to \mathbb{N}$ be some standard notation of the natural numbers, and ρ be the standard representation of $\mathbb{R}$.

$\rho^<(p) = x$ if p is a list of all rational numbers $a < x$.

Let η^{ab} be a standard representation of F^{ab}, the partial continuous functions $f :\subseteq \Sigma^a \to \Sigma^b$ with open or G_δ domain if $b = *$ or $b = \omega$, respectively, with properties utm(ζ) and smn(ζ) (defined below).

Let $a, b, c \in \{*, \omega\}$, G^{ab} be a set of functions $g :\subseteq \Sigma^a \to \Sigma^b$ and let $\zeta :\subseteq \Sigma^c \to G^{ab}$ be a naming system of G^{ab}:

— utm(ζ) property:

There is a computable universal function $u :\subseteq \Sigma^c \times \Sigma^a \to \Sigma^b$ with $\zeta_x(y) = u(x, y)$ for all $x \in \text{dom}(\zeta)$ and $y \in \Sigma^a$.

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— smn(ζ) property:

For every computable function $f :\subseteq \Sigma^c \times \Sigma^a \to \Sigma^b$ there is a total computable function $s : \Sigma^c \to \Sigma^c$ such that $s(x) \in \mathrm{dom}(\zeta)$ for all $x \in \Sigma^c$ and $f(x, y) = \zeta_{s(x)}(y)$ for all $x \in \Sigma^c$ and $y \in \Sigma^a$.

2.2. *Representations of points and sets in computable T_0-spaces*

In this section we introduce computable T_0-spaces together with some fundamental representations of points and sets.

A topological space $\mathbf{X} = (X, \tau)$ is a T_0-space if for all $x, y \in X$ such that $x \neq y$, there is an open set $O \in \tau$ such that $x \in O$ if and only if $y \notin O$. In a T_0-space, every point can be identified by the set of its neighbourhoods $O \in \tau$. $\mathbf{X}$ is said to be second countable if it has a countable base (Engelking 1989).

In the following we will only consider second-countable T_0-spaces. In order to introduce concepts of effectivity, we assume that some notation ν of a base β with recursive domain is given.

Definition 2.1 (computable T_0-space). A computable T_0-space is a tuple $\mathbf{X} = (X, \tau, \beta, \nu)$ such that (X, τ) is a second-countable T_0-space and $\nu :\subseteq \Sigma^* \to \beta$ is a notation of a base β of τ with recursive domain, $U \neq \varnothing$ for $U \in \beta$ and $\mathbf{X}$ has computable intersection: there is a computable function $h :\subseteq \Sigma^* \times \Sigma^* \to \Sigma^\omega$ such that for all $u, v \in \mathrm{dom}(\nu)$,

$$\nu(u) \cap \nu(v) = \bigcup \{\nu(w) \mid w \in \mathrm{dom}(\nu) \ \text{ and } \ \iota(w) \lhd h(u, v)\}. \tag{1}$$

We say two computable T_0-spaces $\mathbf{X}_1 = (X, \tau, \beta_1, \nu_1)$ and $\mathbf{X}_2 = (X, \tau, \beta_2, \nu_2)$ are recursively related if and only if there are computable functions $g, g' :\subseteq \Sigma^* \to \Sigma^\omega$ such that

$$\nu_1(u) = \bigcup_{\iota(w) \lhd g(u)} \nu_2(w) \quad \text{and} \quad \nu_2(v) = \bigcup_{\iota(w) \lhd g'(v)} \nu_1(w). \tag{2}$$

In the following, we let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable T_0-space. Let $\mathscr{A}$ be the set of closed, $\mathscr{O}$ the set of open and $\mathscr{K}$ the set of compact subsets of X.

We will now introduce the standard representation of X.

Definition 2.2 (standard representation δ of X). We define the standard representation $\delta :\subseteq \Sigma^\omega \to X$ by $\delta(p) = x$ if and only if:

— $u \in \mathrm{dom}(\nu)$ if $\iota(u) \lhd p$

— $\{u \in \mathrm{dom}(\nu) \mid x \in \nu(u)\} = \{u \mid \iota(u) \lhd p\}$.

A δ-name p of an element $x \in X$ is a list of all words u such that $x \in \nu(u)$. The definition of δ corresponds to the definition of δ'_S in Lemma 3.2.3 of Weihrauch (2000).

Definition 2.3 (union representation of open sets). We define the union representation $\theta^{\mathrm{un}} :\subseteq \Sigma^\omega \to \tau$ by

$$\mathrm{dom}(\theta^{\mathrm{un}}) := \{q \in \Sigma^\omega \mid u \in \mathrm{dom}(\nu) \text{ if } \iota(u) \lhd q\} \ \text{ and } \ \theta^{\mathrm{un}}(p) := \bigcup_{\iota(u) \lhd p} \nu(u).$$

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Thus, $\theta^{\mathrm{un}}(p)$ is the union of all $v(u)$ such that u is listed by p. Similarly, we will define the union representation ψ^{un} of the set of closed subsets in Definition 4.1.

For technical reasons we define a notation $v^* :\subseteq \Sigma^* \to \{M \subseteq \beta \mid M \text{ is finite}\}$ of all finite sets of base elements by $\mathrm{dom}(v^*) := \{w \in \Sigma^* \mid u \in \mathrm{dom}(v) \text{ if } \iota(u) \lhd w\}$ and $v^*(w) := \{v(u) \mid \iota(u) \lhd w\}$. We define the notation $\theta^* :\subseteq \Sigma^* \to \tau^{\mathrm{fin}}$ of all open sets that can be written as the union of finitely many base elements by $\theta^*(w) := \bigcup v^*(w)$.

The representations δ and θ^{un} are not only very natural but can also be characterised up to equivalence as maximal elements among representations for which the element relation is open or r.e., respectively. Furthermore, the following properties hold.

Lemma 2.1. For computable T_0-spaces $\mathbf{X} = (X, \tau, \beta, v)$:

(1) '$O \neq \varnothing$' is θ^{un}-r.e..
(2) Countable union on τ is $([\theta^{\mathrm{un}}]^\omega, \theta^{\mathrm{un}})$-computable.
(3) Intersection is $(\theta^{\mathrm{un}}, \theta^{\mathrm{un}}, \theta^{\mathrm{un}})$-computable.
(4) Finite intersection is $(v^*, \theta^{\mathrm{un}})$-computable.

Proof.

(1) There is a Type-2 machine that on input $p \in \mathrm{dom}(\theta^{\mathrm{un}})$ halts if

$$(\exists u \in \Sigma^*)\ (\iota(u) \lhd p \wedge u \in \mathrm{dom}(v))\,,$$

since $\mathrm{dom}(v)$ is recursive and $\varnothing \notin \beta$.

(2) Let M be a Type-2 machine that on input $\langle p_0, p_1, \ldots\rangle$ $(p_i \in \Sigma^\omega)$ computes a list of all $\iota(w)$ such that $\iota(w) \lhd p_i$ for some i. The function f_M realises countable union on τ.

(3) Let h be the computable function from Definition 2.1. Let M be a Type-2 machine that on input $(p, q) \in (\Sigma^\omega)^2$ computes a sequence r such that $\iota(w) \lhd r$ if and only if

$$(\exists u, v \in \Sigma^*)\ (\iota(u) \lhd p \wedge \iota(v) \lhd q \wedge \iota(w) \lhd h(u, v))\,.$$

The function f_M realises intersection on τ.

(4) Let h be the computable function from Definition 2.1. Let M be a Type-2 machine that on input w listing $\{u_0, \ldots, u_k\}$ computes a sequence $p \in \Sigma^\omega$ such that $\iota(v) \lhd p$ if and only if

$$\begin{aligned}(\exists v_1, \ldots, v_{k-1})\ \ & \iota(v_1) \in h(u_0, u_1)\\ & \iota(v_2) \in h(v_1, u_2)\\ & \quad\vdots\\ & \iota(v_{k-1}) \in h(v_{k-2}, u_{k-1})\\ & \iota(v) \in h(v_{k-1}, u_k).\end{aligned}$$

The function f_M realises finite intersection on β. □

Equivalent to the computability of finite intersection on the base is the existence of an r.e. set $I \subseteq \Sigma^* \times \mathrm{dom}(v)$ such that for all $w \in \Sigma^*$

$$\bigcap_{\iota(v) \lhd w} v(v) = \bigcup_{(w,u) \in I} v(u)\,. \tag{3}$$

Furthermore, the set $P := \{w \in \Sigma^* \mid (\exists u \in \mathrm{dom}(v))\,(w, u) \in I\}$ of all finite prefixes of δ-names is r.e..

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A topological space is a T_2-space (also called a Hausdorff space) if for all $x, y \in X$ such that $x \neq y$, there are disjoint open sets $O, O' \in \tau$ such that $x \in O$ and $y \in O'$. A subset $K \subseteq X$ of a second-countable Hausdorff space (X, τ) with a fixed base is compact if every open cover of K by elements of the base has a finite subcover. Let

$$\mathscr{K}(\mathbf{X}) := \{K \subseteq X \mid K \text{ compact}\}$$

denote the set of all compact subsets of a Hausdorff space (X, τ). We write $\mathscr{K}$ instead of $\mathscr{K}(\mathbf{X})$ if there is no need to specify the space or if it is obvious which space we refer to.

In the following we generalise the representation κ_c of the compact subsets of the Euclidean space (Weihrauch 2000) and δ_{cover} of the compact subsets of a computable metric space (Brattka and Presser 2003) defined by listing all finite basic covers from a countable base.

Definition 2.4 (cover representation κ^c of compact sets). Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable T_0-space and let (X, τ) be a Hausdorff space. We define a representation $\kappa^c :\subseteq \Sigma^\omega \to \mathscr{K}$ by $K = \kappa^c(p)$ if and only if:

— $w \in \text{dom}(\theta^*)$ if $\iota(w) \lhd p$.
— $\{w \in \Sigma^* \mid \iota(w) \lhd p\} = \{w \in \Sigma^* \mid K \subseteq \theta^*(w)\}$.

Roughly speaking, p is a name of K if it is a list of all (!) names of all finite basic subcovers of K with base elements.

For technical reasons we define a representation of all finite sets of compact sets $\kappa^* :\subseteq \Sigma^\omega \to \{M \subseteq \mathscr{K} \mid M \text{ is finite }\}$ by

$$\kappa^*(p) = \{K_1, \ldots, K_k\} :\iff \begin{array}{l} p = 1^k 0 \langle p_1, \ldots, p_k \rangle \text{ and} \\ \kappa^c(p_i) = K_i \text{ for all } i \in \{1, \ldots, k\}. \end{array}$$

Lemma 2.2. Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable T_0-space and (X, τ) be a Hausdorff space. Then:

(1) '$K \subseteq O$' is $(\kappa^c, \theta^{\text{un}})$-r.e..
(2) Finite union on $\mathscr{K}$ is (κ^*, κ^c)-computable.

Proof.

(1) The set $X := \{(p, q) \in \Sigma^\omega \times \Sigma^\omega \mid \kappa^c(p) \subseteq \theta^{\text{un}}(q)\}$ is r.e. open in $\text{dom}(\kappa^c) \times \text{dom}(\theta^{\text{un}})$ as

$$X = A \circ (\Sigma^\omega \times \Sigma^\omega) \cap \text{dom}(\kappa^c) \times \text{dom}(\theta^{\text{un}})$$

where $A := \{(w_p, w_q) \in \Sigma^* \times \Sigma^* \mid (\exists \iota(w) \lhd w_p)(\forall \iota(u) \lhd w)\, \iota(u) \lhd w_q\}$ is r.e. (see Weihrauch (2000, Theorem 2.4.7)).

(2) A finite set of base elements $\zeta \subseteq_{\text{fin}} \beta$ covers $K_1 \cup \ldots \cup K_k$ if and only if $\zeta = \zeta_1 \cup \ldots \cup \zeta_k$ and ζ_i covers K_i for $i \in \{1, \ldots, k\}$. Let M be a machine that on input p listing $\{q_1, \ldots, q_k\}$ computes a sequence $r \in \Sigma^\omega$ such that $\iota(w) \lhd r$ if and only if

$$\begin{array}{l} (\exists w_1, \ldots, w_k) \\ (\forall i \in \{1, \ldots, k\})\, \iota(w_i) \lhd q_i \text{ and } \{u \mid \iota(u) \lhd w\} = \{u \mid (\exists i \in \{1, \ldots, k\})\, \iota(u) \lhd w_i\}. \end{array}$$

The function f_M realises finite union on $\mathscr{K}$. □

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Every closed subset of a compact set is compact. The next lemma gives an effective version.

Lemma 2.3. Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable T_0-space and let (X, τ) be a Hausdorff space. The mapping $F : \mathscr{A} \times \mathscr{K} \to \mathscr{K}$ defined by

$$F(A, K) := A \cap K$$

is $(\psi^{\mathrm{un}}, \kappa^{\mathrm{c}}, \kappa^{\mathrm{c}})$-computable.

Proof. The set $A \cap K$ is covered by finitely many base elements $U := \{U_i \mid i \in I\}$ if and only if there are finitely many base elements $V := \{V_j \mid j \in J\}$ in A^{c} such that $U \cup V$ covers K. Let $\psi^{\mathrm{un}}(p) = A$ and $K = \kappa^{\mathrm{c}}(q)$. A sequence r can be computed from p and q such that $\iota(w) \lhd r$ if and only if

$$(\exists w' \in \Sigma^*)\,(\exists V \subseteq_{\mathrm{fin}} \Sigma^*)(\forall v \in V)$$
$$(\iota(v) \lhd p \wedge \iota(w') \lhd q \wedge \{u \mid \iota(u) \lhd w'\} = \{u \mid \iota(u) \lhd w\} \cup V)\,. \qquad \square$$

3. Computably locally compact Hausdorff spaces

In this section we introduce an effective version of a Hausdorff space first, and then we give the definition of computably locally compact Hausdorff spaces.

Definition 3.1 (computably Hausdorff space). We say that a computable T_0-space $\mathbf{X} = (X, \tau, \beta, \nu)$ is a *computably Hausdorff space* if there exists an r.e. set $H \subseteq \mathrm{dom}(\nu) \times \mathrm{dom}(\nu)$ such that:

$$(\forall (u, v) \in H) \quad \nu(u) \cap \nu(v) = \varnothing \tag{4}$$

$$(\forall x, y \in X \text{ with } x \neq y)\,(\exists (u, v) \in H)\ \ x \in \nu(u) \wedge y \in \nu(v)\,. \tag{5}$$

Lemma 3.1. For computable T_0-spaces,

$$\mathbf{X} \text{ computably Hausdorff} \iff \{(x, y) \in X \times X \mid x \neq y\} \text{ is } (\delta, \delta)\text{-r.e.}.$$

Proof.

'$\Rightarrow$' Let $\mathbf{X}$ be computably Hausdorff. Then $X := \{(p, q) \in \Sigma^\omega \times \Sigma^\omega \mid \delta(p) \neq \delta(q)\}$ is r.e. open in $\mathrm{dom}(\delta) \times \mathrm{dom}(\delta)$ as

$$X = A \circ (\Sigma^\omega \times \Sigma^\omega) \cap \mathrm{dom}(\delta) \times \mathrm{dom}(\delta)$$

where $A := \{(w_p, w_q) \in \Sigma^* \times \Sigma^* \mid (\exists (u, v) \in H)\, \iota(u) \lhd w_p \text{ and } \iota(v) \lhd w_q\}$ is r.e. – see Weihrauch (2000, Theorem 2.4.7).

'$\Leftarrow$' Suppose $\{(x, y) \in X \times X \mid x \neq y\}$ is (δ, δ)-r.e.. There exists a Type-2 machine M that halts on input (p, q) if $p, q \in \mathrm{dom}(\delta)$ and $\delta(p) \neq \delta(q)$. Until such time as M halts, only finite prefixes $v \sqsubseteq p$ and $w \sqsubseteq q$ have been read. The set

$$A := \{(v, w) \in \Sigma^* \times \Sigma^* \mid M \text{ halts after reading at most } v \text{ and } w\}$$

is r.e.. If $(v, w) \in A$, then v or w must not be a prefix of some δ-name. Such a $v \in \Sigma^*$ is a prefix of a δ-name if there exists some $u_v \in \mathrm{dom}(\nu)$ such that $(v, u_v) \in I$, where

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the r.e. set I is defined in (3). The set

$$B := \{(\langle v, w\rangle, u_v, u_w) \mid (v, w) \in A \ \text{ and } \ (v, u_v) \in I \ \text{ and } \ (w, u_w) \in I\}$$

is obviously r.e.. By the Projection Theorem for r.e. sets, we get the required r.e. set

$$H := \{(u_v, u_w) \in (\mathrm{dom}(v))^2 \mid (\exists \langle v, w\rangle)(\langle v, w\rangle, u_v, u_w) \in B\}. \qquad \square$$

Lemma 3.2. Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computably Hausdorff space.

(1) The mapping $F_1 :\subseteq X \times X \rightrightarrows \beta \times \beta$ defined by $\mathrm{dom}(F_1) = \{(x, y) \mid x \neq y\}$ and

$$(U, V) \in F_1(x, y) :\Longleftrightarrow \ x \in U \ \text{ and } \ y \in V \ \text{ and } \ U \cap V = \varnothing$$

is $(\delta, \delta, \nu, \nu)$-computable.

(2) The mapping $F_2 :\subseteq X \times \mathscr{K} \rightrightarrows \beta \times \tau$ defined by $\mathrm{dom}(F_2) = \{(x, K) \mid x \notin K\}$ and

$$(U, O) \in F_2(x, K) :\Longleftrightarrow \ x \in U \ \text{ and } \ K \subseteq O \ \text{ and } \ U \cap O = \varnothing$$

is $(\delta, \kappa^{\mathrm{c}}, \nu, \theta^*)$-computable.

(3) The mapping $F_3 :\subseteq \mathscr{K} \times \mathscr{K} \rightrightarrows \tau \times \tau$ defined by $\mathrm{dom}(F_3) = \{(K, K') \mid K \cap K' = \varnothing\}$ and

$$(O, O') \in F_3(K, K') :\Longleftrightarrow \ K \subseteq O \ \text{ and } \ K' \subseteq O' \ \text{ and } \ O \cap O' = \varnothing$$

is $(\kappa^{\mathrm{c}}, \kappa^{\mathrm{c}}, \theta^*, \theta^*)$-computable.

Proof.

(1) Let M be a Type-2 machine that on input $\langle p, q\rangle$ with $p, q \in \Sigma^\omega$ computes some $\langle u, v\rangle$ with $u, v \in \Sigma^*$ such that

$$\iota(u) \lhd p \ \text{ and } \ \iota(v) \lhd q \ \text{ and } \ (u, v) \in H\,.$$

The function f_M realises F_1.

(2) Let M be a Type-2 machine that on input $\langle p, q\rangle$ with $p, q \in \Sigma^\omega$ computes some $\langle u, w\rangle$ with $u, w \in \Sigma^*$ such that $\iota(u) \lhd p$ and $\iota(w) \lhd q$. There exists a finite $N \subseteq H$ such that for all $(u', v') \in N$

$$\iota(u') \lhd p \ \text{ and } \ \{v \mid \iota(v) \lhd w\} = \mathrm{pr}_2(N) \ \text{ and } \ \nu(u) \subseteq \bigcap_{(u', v') \in N} \nu(u')\,.$$

By Lemma 2.1, we know that the finite intersection on β is computable, so the function f_M realises F_2.

(3) Let M be a Type-2 machine that on input $\langle p, q\rangle$ with $p, q \in \Sigma^\omega$ computes some $\langle w, w'\rangle$ with $w, w' \in \Sigma^*$ such that $\iota(w) \lhd p$ and $\iota(w') \lhd q$ and

$$\begin{aligned}&(\forall \iota(u) \lhd w, \iota(w'_u) \lhd q)\,(\forall \iota(v_u) \lhd w'_u)\\ &\big((u, v_u) \in H \ \text{ and } \ \nu^*(w') \subseteq \bigcap_{\iota(u) \lhd w} \nu^*(w'_u)\big)\,.\end{aligned}$$

The function f_M realises F_3. $\square$

Every compact subspace of a Hausdorff space is closed. The next theorem is an effective version.

Theorem 3.1. For computably Hausdorff spaces, $\kappa^{\mathrm{c}} \leqslant \psi^{\mathrm{un}}$.

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Proof. Let $\kappa^c(p) = K, I$ be the r.e. set for finite intersection defined in (3) and H be the r.e. Hausdorff set. A sequence r can be computed from p such that $\iota(w) \lhd r$ if and only if

$$(\exists \iota(u) \lhd p \text{ listing } u_0 \cdots u_k)(\exists v \in \Sigma^* \text{ listing } v_0 \cdots v_k)$$
$$((u_0, v_0), \ldots, (u_k, v_k) \in H \quad \text{and} \quad (v, w) \in I).$$

So $v(w) \subseteq K^c$ if $\iota(w) \lhd r$. Next we show that $K^c \subseteq \bigcup_{\iota(w) \lhd r} v(w)$ holds. Let $y \in K^c$. For all $x \in K$ there is some $(u, v) \in H$ such that $x \in v(u)$ and $y \in v(v)$. The family of all these $v(u)$ covers K. As K is compact, it has a finite subcover $\{v(u_i) \mid j \in J\}$ covering K. Let v list $\{v_i \mid j \in J\}$. As $y \in \bigcap_{j \in J} v(v_j) = \bigcap_{v_j \lhd v} v(v_j)$, there is some $w \in \Sigma^*$ such that

$$(v, w) \in I \quad \text{and} \quad y \in v(w).$$

Therefore r is a ψ^{un}-name of K. □

A topological space (X, τ) is said to be locally compact if for every point $x \in X$ there exists a neighbourhood O of x such that the closure $\bar{O}$ is compact. Next we introduce an effective version of a locally compact Hausdorff space by means of the representation κ^c of the compact subsets of a Hausdorff space.

Definition 3.2 (computably locally compact Hausdorff space). A computable T_0-space $\mathbf{X}' = (X, \tau, \beta', \nu')$ is said to be a *computably locally compact Haudorff space* if:

— $\mathbf{X}'$ is a computably Hausdorff space.
— There is some computably Hausdorff space $\mathbf{X} = (X, \tau, \beta, \nu)$ such that $\mathrm{CLS} : \beta \to \mathscr{K}(\mathbf{X})$ defined by $\mathrm{CLS}(U) := \bar{U}$ is (ν, κ^c)-computable.
— $\mathbf{X}'$ and $\mathbf{X}$ are recursively related.

In the following, if $\mathbf{X} = (X, \tau, \beta, \nu)$ is a computably locally compact Hausdorff space, we suppose CLS to be (ν, κ^c)-computable (without changing the base or its notation).

If $\mathbf{X}$ is a computably locally compact Hausdorff space, it is locally compact since the closure of each base element is compact, so $\mathbf{X}$ is Tychonoff, and thus regular (and a Hausdorff space) and even metrisable since $\mathbf{X}$ is second countable (Engelking 1989).

Lemma 3.3. For computably locally compact Hausdorff spaces:

(1) '$\bar{U} \subseteq O$' is $(\nu, \theta^{\text{un}})$-r.e..
(2) '$\bar{O} \subseteq O'$' is $(\theta^*, \theta^{\text{un}})$-r.e..

Proof.

(1) There is a Type-2 machine that on input $(u, p) \in \mathrm{dom}(\nu) \times \mathrm{dom}(\theta^{\text{un}})$ halts if and only if

$$(\exists w \in \Sigma^*)\ (\iota(w) \lhd h(u) \quad \text{and} \quad \{u \mid \iota(u) \lhd w\} \subseteq \{u \mid \iota(u) \lhd p\})$$

where h is a computable realisation of CLS (Definition 3.2).

(2) There is a Type-2 machine that on input $(w, p) \in \mathrm{dom}(\nu^*) \times \mathrm{dom}(\theta^{\text{un}})$ halts if and only if

$$(\forall \iota(u) \lhd w)(\exists \iota(w_u) \lhd h(u))\ (\{v \mid \iota(v) \lhd w_u)\} \subseteq \{v \mid \iota(v) \lhd p\})$$

where h is a computable realisation of CLS (Definition 3.2). □

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The property given by part (1) of Lemma 3.3 is defined in Brattka and Presser (2003) and Collins (2005) as the 'effective covering property' of a computable metric space. As this property holds true for each computably locally compact Hausdorfff space, we do not need an additional axiom.

For every compact subspace K of a locally compact Hausdorff space X and every open set $V \subseteq X$ that contains K, there exists an open set $U \subseteq X$ such that $K \subseteq U \subseteq \bar{U} \subseteq V$ and $\bar{U}$ is compact. The next lemma gives an effective version.

Lemma 3.4. Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computably locally compact Hausdorff space. The mapping $\mathrm{F} :\subseteq \mathscr{K} \times \tau \rightrightarrows \tau$ defined by $\mathrm{dom}(F) = \{(K, O) \in \mathscr{K} \times \tau \mid K \subseteq O\}$ and

$$U \in \mathrm{F}(K, O) :\Longleftrightarrow K \subseteq U \subseteq \bar{U} \subseteq O$$

is $(\kappa^{\mathrm{c}}, \theta^{\mathrm{un}}, \theta^{*})$-computable.

Proof. By Lemma 3.3, '$\bar{O} \subseteq O'$' is $(\nu^{*}, \theta^{\mathrm{un}})$-r.e.. Furthermore, the mapping $O \mapsto \bar{O}$ is $(\nu^{*}, \kappa^{\mathrm{c}})$-computable as $\mathbf{X}$ is computably locally compact. Therefore, there exists a Type-2 machine that on input $(p, q) \in \mathrm{dom}(\kappa^{\mathrm{c}}) \times \mathrm{dom}(\theta^{\mathrm{un}})$ computes some $w \in \Sigma^{*}$ with

$$\imath(w) \lhd p \quad \text{and} \quad \overline{\nu^{*}(w)} \subseteq \theta^{\mathrm{un}}(q)\,. \qquad \square$$

Computably regular spaces have been introduced in Schröder (1998) and Grubba and Weihrauch (2006).

Definition 3.3 (computably regular space). We say that a computable T_0-space $\mathbf{X} = (X, \tau, \beta, \nu)$ is *computably regular* if there exists a computable function $t_3 :\subseteq \Sigma^{*} \times \Sigma^{*} \to \Sigma^{\omega}$ such that $R := \mathrm{dom}(t_3)$ is r.e., and

$$(\forall v \in \mathrm{dom}(\nu))\left(\nu(v) = \bigcup_{(u,v) \in R} \nu(u)\right) \tag{6}$$

$$(\forall (u, v) \in R)\left(\nu(u) \subseteq \psi^{\mathrm{un}}(t_3\,(u, v)) \subseteq \nu(v)\right). \tag{7}$$

The following theorem gives an effective version of the classical hierarchy: every locally compact Hausdorff space is regular and every regular space is a Hausdorff space.

Theorem 3.2.

(1) A computable T_0-space is computably regular if it is a computably locally compact Hausdorff space.
(2) A computable T_0-space is computably Hausdorff if it is computably regular.

Proof.

(1) Since any locally compact Hausdorff space is regular, for computably locally compact Hausdorff spaces the set $R := \{(u, v) \in \mathrm{dom}(\nu) \times \mathrm{dom}(\nu) \mid \overline{\nu(u)} \subseteq \nu(v)\}$ is r.e. (this follows from Lemma 3.3). For computably Hausdorff spaces, $\kappa^{\mathrm{c}} \leqslant \psi^{\mathrm{un}}$ holds (Theorem 3.1). Let f be a computable reduction of κ^{c} to ψ^{un} and g be a computable realisation of CLS. Define $t_3 :\subseteq \Sigma^{*} \times \Sigma^{*} \to \Sigma^{\omega}$ by $\mathrm{dom}(t_3) := R$ and $t_3(u, v) := f(g(u))$.

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(2) Suppose $\mathbf{X}$ is computably regular. There is a Type-2 machine that on input $(p,q) \in \mathrm{dom}(\delta) \times \mathrm{dom}(\delta)$ halts if and only if

$$(\exists (u,v) \in R)\,(\exists v' \in \Sigma^*)\ \big(\iota(u) \lhd p \ \text{ and } \ \iota(v') \lhd q \ \text{ and } \ \iota(v') \lhd t_3(u,v)\big).$$

Therefore $\{(x,y) \in X \times X \mid x \neq y\}$ is (δ,δ)-r.e., so $\mathbf{X}$ is computably Hausdorff by Lemma 3.1. □

4. Computability on closed subsets

In this section we study several representations of the set of closed subsets of a computably locally compact Hausdorff space.

Definition 4.1 (representations of closed sets). Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computably locally compact Hausdorff space, and $\eta_p^{\omega *}, \eta_p^{\omega\omega}$ be standard representation of the set of continuous functions of $F :\subseteq \Sigma^\omega \to \Sigma^*$ and $F :\subseteq \Sigma^\omega \to \Sigma^\omega$, respectively. $D_\nu := \{p \mid w \lhd p, \nu(w) \in \beta\}$.

1 Define the domain representation $\psi^{\mathrm{dom}} :\subseteq \Sigma^\omega \to \mathscr{A}$ by

$$\psi^{\mathrm{dom}}(p) = A :\iff \eta_p^{\omega *} \text{ is a strong } (\delta, \nu_{\mathbb{N}})\text{-realisation of } f :\subseteq X \to \mathbb{N} \text{ such that } \mathrm{dom}(f) = A^{\mathrm{c}}.$$

2 Define the Sierpinski representation $\psi^{\mathrm{sie}} :\subseteq \Sigma^\omega \to \mathscr{A}$ by

$$\psi^{\mathrm{sie}}(p) = A \ :\iff \ \eta_p^{\omega\omega} \text{ is a } (\delta, \rho^<)\text{-realisation of } \mathrm{cf}_A : X \to \mathbb{R}.$$

Here $\mathrm{cf}_A : X \to \mathbb{R}$ means that

$$x \mapsto \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{otherwise}. \end{cases}$$

3 Define the fibre representation $\psi^{\mathrm{fibre}} :\subseteq \Sigma^\omega \to \mathscr{A}$ by

$$\psi^{\mathrm{fibre}}(p) = A :\iff \eta_p^{\omega\omega} \text{ is a } (\delta,\rho)\text{-realisation of some } f : X \to \mathbb{R} \text{ such that } f^{-1}\{0\} = A.$$

4 Define the union representation $\psi^{\mathrm{un}} :\subseteq \Sigma^\omega \to \tau$ by $\mathrm{dom}(\psi^{\mathrm{un}}) := D_\nu$ and

$$\psi^{\mathrm{un}}(p) = A :\iff A^{\mathrm{c}} = \bigcup_{\iota(w) \lhd p} \nu(w). \tag{8}$$

5 Define the outer representation $\psi^> :\subseteq \Sigma^\omega \to \mathscr{A}$ by:
$\psi^>(p) = A :\iff$
— $u \in \mathrm{dom}(\nu)$ if $\iota(u) \lhd p$
— $\{w \mid \iota(w) \lhd p\} = \{w \mid \overline{\nu(w)} \cap A = \varnothing\}$.

Remark 4.1. All of these representations are well defined even for computable T_0-spaces (for ψ^{fibre}, surjective is required), except for $\psi^>$.

Proposition 4.1. $\psi^>$ is well defined for computably locally compact Hausdorff spaces.

Proof. For closed sets A and B, if

$$\{w \in \mathrm{dom}(\nu) \mid \overline{\nu(w)} \cap A = \varnothing\} = \{w \in \mathrm{dom}(\nu) \mid \overline{\nu(w)} \cap B = \varnothing\},$$

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then

$$\{w \in \mathrm{dom}(v) \mid \overline{v(w)} \subseteq A^{\mathrm{c}}\} = \{w \in \mathrm{dom}(v) \mid \overline{v(w)} \subseteq B^{c}\}.$$

If $\mathbf{X} = (X, \tau, \beta, v)$ is a computably locally compact Hausdorff space, then X is regular. If $x \in A^{\mathrm{c}}$, there exists some $V \in \beta$ such that $x \in V \subseteq \bar{V} \subseteq A^{\mathrm{c}}$ since X is regular. So $x \in V \subseteq B^{c}$, and thus $A^{\mathrm{c}} \subseteq B^{c}$. By symmetry, we can conclude that $A^{\mathrm{c}} = B^{c}$, and thus $A = B$. □

The 'well defined' property of $\psi^{>}$, however, does not always hold. Now we give some typical examples to show that, $\psi^{>}$ is not, in general, well defined for more general topological spaces.

Example 4.1. $\psi^{>}$ is not well defined for T_0-spaces in general: for example, $\mathbb{R}_{<}$, whose base elements are all $U := (q, \infty), q \in \mathbb{Q}$ and $cls(U) = (-\infty, \infty)$. If A closed, $cls(U) \subseteq A$ and U is non-empty, we have $A = (-\infty, \infty)$.

Example 4.2. $\psi^{>}$ is not well defined for T_1-spaces in general: for example, the co-finite topology of $\mathbb{Q}$, that is, $\tau = \{O | \mathbb{Q} \setminus O \text{ is finite}\} \cup \{\varnothing\}$. We will now show that $(\mathbb{Q}, \tau)$ is a T_1-space but not a T_2-space.

Let $r_1, r_2 \in \mathbb{Q}$ and $r_1 \neq r_2$. Take $\mathbb{Q} \setminus \{r_1\}$ as an open set. Then we see that it contains r_2 but not r_1. By symmetry, we can conclude that $(\mathbb{Q}, \tau)$ is a T_1-space. Suppose there exist open sets O_1, O_2 such that $r_1 \in O_1, r_2 \in O_2$ and $O_1 \cap O_2 = \varnothing$. On the one hand, $\mathbb{Q} \setminus (O_1 \cap O_2) = (\mathbb{Q} \setminus O_1) \cup (\mathbb{Q} \setminus O_2)$ is finite, since both $\mathbb{Q} \setminus O_1$ and $\mathbb{Q} \setminus O_2$ are finite. On the other hand, $\mathbb{Q} \setminus (O_1 \cap O_2) = \mathbb{Q} \setminus \varnothing = \mathbb{Q}$ is infinite, which gives a contradiction. Hence, $(\mathbb{Q}, \tau)$ is not a T_2-space.

For any non-empty open set $O \in \tau$, $cls(O) = \mathbb{Q}$. Therefore, given a closed non-empty subset A, there is no base element U such that $cls(U) \cap A = \varnothing$.

Example 4.3. $\psi^{>}$ is not well defined for Hausdorff spaces in general: for example, $\tau = \{G \setminus E | G \in \tau_{\mathbb{R}}, E \subseteq \mathbb{Q}\}$, where $\tau_{\mathbb{R}}$ is the common topology of $\mathbb{R}$.

$(\mathbb{R}, \tau)$ is a T_2-space, since for any two different $x_1, x_2 \in \mathbb{R}$, by the density of real numbers, there exists x between them, we can find two open sets $(-\infty, x) \setminus E_1$ and $(x, \infty) \setminus E_2$ that contain x_1 and x_2, respectively, such that $((-\infty, x) \setminus E_1) \cap ((x, \infty) \setminus E_2) = \varnothing$.

We claim that for any $G \in \tau_{\mathbb{R}}$, we have that $G \setminus E_i$ have the same closure as G, $i \in \mathbb{N}$. To prove this claim, we just need to show $cls(G) \subseteq cls(G \setminus E)$ for any $E \subseteq \mathbb{Q}$.

Suppose $x \in cls(G)$. By the definition of closure, for any neighbourhood $G' \setminus E'$ of x, we have $(G' \setminus E') \cap G \neq \varnothing$. Since G and G' are open sets of $\mathbb{R}$ with common topology, and by the density of irrational numbers, we conclude that there exists some irrational number $y \in (G' \setminus E') \cap G$. As $E \subseteq \mathbb{Q}$, we have $y \in (G' \setminus E') \cap (G \setminus E)$, that is, $(G' \setminus E') \cap (G \setminus E) \neq \varnothing$. Hence, $x \in cls(G \setminus E)$, as required.

Given the closed subset $\mathbb{Q}$, there is no open set $G \setminus E$ such that $cls(G \setminus E) \cap \mathbb{Q} = G \cap \mathbb{Q} = \varnothing$, where G is the closure of $G \setminus E$.

The following theorem states quite an important fact: although defined from different viewpoints, all of the above representations are in fact equivalent.

Theorem 4.1. For computably locally compact Hausdorff spaces,

$$\psi^{\mathrm{fibre}} \equiv \psi^{\mathrm{dom}} \equiv \psi^{\mathrm{sie}} \equiv \psi^{\mathrm{un}} \equiv \psi^{>}.$$

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Proof.

$\psi^{\text{fibre}} \leqslant \psi^{\text{dom}}$: (As in Brattka and Weihrauch (1999).)

Let A be a closed subset of X and $\psi^{\text{fibre}}(p) = A$, that is, $\eta_p^{\omega\omega}$ is a (δ, ρ)-realisation of a function $f_p : X \to \mathbb{R}$ such that $f_p^{-1}\{0\} = A$. Define $g_p :\subseteq X \to \mathbb{N}$ as follows:

$$g_p(x) = \begin{cases} 1 & \text{if } f_p(x) \neq 0 \\ \uparrow & \text{otherwise} \end{cases}$$

for all $x \in X$. Then $\text{dom}(g_p) = A^{\text{c}}$. As $\eta_p^{\omega\omega}$ is a realisation of f_p, we have that if $\rho\eta_p^{\omega\omega}(q) \neq 0$, then $g_p\delta(q) = 1$, otherwise $g_p\delta(q)$ diverges . By the utm-Theorem, there is a computable function $G :\subseteq \Sigma^\omega \times \Sigma^\omega \to \Sigma^*$ such that $g_p\delta(q) = \nu_{\mathbb{N}} G(p, q)$. And by the smn-Theorem for η^{ω^*}, there is a computable function $H : \Sigma^\omega \to \Sigma^\omega$ such that $g_p\delta(q) = \nu_{\mathbb{N}}\eta^{\omega^*}_{H(p)}(q)$. Furthermore, for any $x \notin \text{dom}(g_p)$, we have $f_p(x) = 0$, so $G(p, q)$ diverges. This shows that $\eta^{\omega^*}_{H(p)}$ is a strong realisation. Therefore, $H(p)$ is a ψ^{dom}-name of A, as required.

$\psi^{\text{dom}} \leqslant \psi^{\text{sie}}$: (As in Brattka and Presser (2003).)

Let $\psi^{\text{dom}}(p) = A$, that is, η^{ω^*} is a strong $(\delta, \nu_{\mathbb{N}})$-realisation of function $f_p :\subseteq X \to \mathbb{N}$ such that $\text{dom}(f_p) = A^{\text{c}}$. Let M be a Type-2 machine computing the universal function of η^{ω^*}. Define

$$H(p, q) = \begin{cases} 0^\omega & \text{if } M \text{ does not halt on input } (p, q) \\ 0^k 1^\omega & \text{if } M \text{ halts on input } (p, q) \text{ after } k \text{ steps}. \end{cases}$$

H is computable, and by the utm- and smn-Theorems for $\eta^{\omega\omega}$, there exists a computable function F such that $\eta^{\omega\omega}_{F(p)}(q) = H(p, q)$. Now we have

$$\rho^< \eta^{\omega\omega}_{F(p)}(q) = \rho^< H(p, q) = \text{cf}_A\delta(q),$$

that is, $\eta^{\omega\omega}_{F(p)}$ is a $(\delta, \rho^<)$-realisation of cf_A. Therefore, $F(p)$ is a ψ^{sie}-name of A, as required.

$\psi^{\text{sie}} \leqslant \psi^{\text{un}}$: Let $\psi^{\text{sie}}(p) = A$ and M be a

Type-2 machine computing the universal function of $\eta^{\omega\omega}$. There is a Type-2 machine that on input p computes a sequence r such that $\iota(u) \lhd r$ if and only if $(\exists w \in \text{dom}(\nu^*))$ (M on input (p, q) writes some $\iota(a)$ such that $\nu_{\mathbb{Q}}(a) > 0$ and M has at most read the prefix w of q) and $(w, u) \in I$, where I is the r.e. set for finite intersection defined in expression (3).

$\psi^{\text{un}} \leqslant \psi^>$:

Note that a ψ^{un}-name of A is just a θ^{un}-name of A^{c}. Since $\bar{U} \subseteq A^{\text{c}}$ is $(\nu, \theta^{\text{un}})$-r.e., by Lemma 3.3, we can now construct a Type-2 machine M that upon input of a θ^{un}-name p outputs a list of u such that $\overline{\nu(u)} \subseteq A^{\text{c}}$, that is, $\overline{\nu(u)} \cap A = \varnothing$. So f_M translates ψ^{un} to $\psi^>$.

$\psi^> \leqslant \psi^{\text{fibre}}$:

We show this through $\psi^> \leqslant \psi^{\text{un}} \leqslant \psi^{\text{fibre}}$. $\psi^> \leqslant \psi^{\text{un}}$ (the identity function can realise the reduction). Let $\psi^>(p) = A$. If $x \in A^{\text{c}}$, there exists some $V \in \beta$ such that

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$x \in V \subseteq cls(V) \subseteq A^c$ as X is regular. But the V is already listed by p. $\psi^{un} \leqslant \psi^{fibre}$ (by Theorem 3.2, we know that **X** is computably regular if it is computably locally compact Hausdorff space). So the multivalued Urysohn operator UR mappping every pair (A, B) of disjoint closed sets to all continuous functions $f : X \to [0;1]$ such that $f[A] = 0$ and $f[B] = 1$ is $(\psi^{un}, \psi^{un}, [\delta \to \rho])$-computable (Grubba and Weihrauch 2006). Let $\psi^{un}(p) = A$ and let p be a list of $\{u_i \in \mathrm{dom}(v) \mid i \in \mathbb{N}\}$. There is a Type-2 machine M that on input p:

— computes a ψ^{un}-name p_i of $\overline{v(u_i)}$ for all $i \in \mathbb{N}$ ($U \to \bar{U}$ is (v, κ^c)-computable and $\kappa^c \leqslant \psi^{un}$);

— computes a $[\delta \to \rho]$-name q_i of some $f_i \in \mathrm{UR}(\psi^{un}(p), \psi^{un}(p_i))$ for all $i \in \mathbb{N}$;

— computes a $[\delta \to \rho]$-name q of $f : X \to [0;1]$ defined by

$$f(x) := \sum_{i=0}^{\infty} 2^{-i} f_i(x).$$

So $A = \psi^{fibre}(q)$. □

These equivalent representations induce the same computability. Besides the above five representations, we will now introduce two more, which induce different computabilities.

Definition 4.2 (continued).

6 Define the enumeration representation $\psi^{range} :\subseteq \Sigma^\omega \to \mathscr{A}$ by

$$\begin{aligned} \psi^{range}(0^\omega) &:= \varnothing \\ \psi^{range}(0^k 1 p) &:= cls \circ range([v_{\mathbb{N}} \to \delta]_{\mathbb{N}}(p)). \end{aligned}$$

7 Define the inner representation $\psi^< :\subseteq \Sigma^\omega \to \mathscr{A}$ by $\psi^<(p) = A$ if and only if:

— $u \in \mathrm{dom}(v)$ if $\iota(u) \lhd p$

— $\{w \mid \iota(w) \lhd p\} = \{w \mid v(w) \cap A \neq \varnothing\}$.

Theorem 4.2. For computably locally compact Hausdorff spaces,

$$\psi^{range} \leqslant \psi^<.$$

Proof. Let $\psi^{range}(p) = A$. If $A = \varnothing$, then $p = 0^\omega$ and, furthermore, $\psi^<(0^\omega) = \varnothing$. If $A \neq \varnothing$, then

$$\begin{aligned} v(w) \cap A \neq \varnothing \iff & \text{ there exists some } x \in \mathrm{range}([v_{\mathbb{N}} \to \delta](p)) \text{ such that } x \in v(w) \\ \iff & \text{ there exists some } u \in \mathrm{dom}(v_{\mathbb{N}}) \text{ such that } w \text{ is listed by } \eta_p(u). \end{aligned}$$

The following Typ-2 machine M realises the reduction: on input p the machine M copies all zeros on the input tape to the output tape until it reads a symbol $a \neq 0$, and then M writes $\iota(w)$ if and only if

$$(\exists u \in \mathrm{dom}(v_{\mathbb{N}}))\, \iota(w) \lhd u_\eta(p, u),$$

where $u_\eta(p, u)$ is the output of $\eta_p(u)$. □

Remark 4.2. In general, $\psi^< \leqslant \psi^{range}$ does not hold. See Brattka and Presser (2003) for a counterexample for computable metric spaces.

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5. Computability on compact subsets

In this section we study several representations of the set $\mathscr{K}$ of the compact subsets of a computably locally compact Hausdorff space.

Definition 5.1 (representations of compact sets). Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computably locally compact Hausdorff space.

(1) Define the union representation $\kappa^{\mathrm{un}} :\subseteq \Sigma^\omega \to \mathscr{K}$ by $\kappa^{\mathrm{un}}\langle p, w\rangle = K$ if and only if:
   — $p \in \mathrm{dom}(\psi^{\mathrm{un}})$ and $w \in \mathrm{dom}(\nu^*)$
   — $K = \psi^{\mathrm{un}}(p)$ and $K \subseteq \theta^*(w)$.

(2) Define the net representation $\kappa^{\mathrm{net}} :\subseteq \Sigma^\omega \to \mathscr{K}$ by $K = \kappa^{\mathrm{net}}(p)$ if and only if:
   — $p = \langle w_1, w_2, \ldots\rangle$ and $w_i \in \mathrm{dom}(\nu^*)$ for all $i \in \mathbb{N}$
   — $\overline{\theta^*(w_{i+1})} \subseteq \theta^*(w_i)$ for all $i \in \mathbb{N}$
   — $K = \bigcap_{i=1}^{\infty} \theta^*(w_i)$.

For any compact subset of a second countable locally compact Hausdorff space, there exists a 'strictly' decreasing cover sequence converging to it. Since every locally compact Hausdorff space is a Hausdorff space, the limit of a cover sequence is unique. Therefore, κ^{net} is well defined. Compared with the cover representation, the advantage of the net representation is that it does not require all finite basic covers.

Theorem 5.1. For computably locally compact Hausdorff spaces,

$$\kappa^{\mathrm{c}} \equiv \kappa^{\mathrm{un}} \equiv \kappa^{\mathrm{net}}.$$

Proof.

$\kappa^{\mathrm{c}} \leqslant \kappa^{\mathrm{un}}$:

By Theorem 3.1, $\kappa^{\mathrm{c}} \leqslant \psi^{\mathrm{un}}$ for computably Hausdorff spaces. Furthermore, if p is a κ^{c}-name of K, then $K \subseteq \theta^*(w)$ for any $\iota(w) \lhd p$.

$\kappa^{\mathrm{un}} \leqslant \kappa^{\mathrm{c}}$:

Let $\kappa^{\mathrm{un}}\langle p, w\rangle = K$, where w lists $\{v_1, \ldots, v_k\}$. Then $K = \psi^{\mathrm{un}}(p)$ and $K \subseteq B := \bigcup\{\overline{\nu(v_i)} \mid i = 1, \ldots, k\}$. As $\mathbf{X}$ is computably locally compact, there is a Type-2 machine M that on input $(v_1, \ldots, v_k)$ computes a κ^*-name $\langle q_1, \ldots, q_k\rangle \in \Sigma^\omega$ such that $\kappa^{\mathrm{c}}(q_i) = \overline{\nu(v_i)}$ for all $i \in \{1, \ldots, k\}$. By Lemma 2.2, a sequence $q \in \Sigma^\omega$ can be computed from $\langle q_1, \ldots, q_k\rangle$ with $\kappa^{\mathrm{c}}(q) = B$. By Lemma 2.3, the mapping $(K, B) \to K \cap B$ is $(\psi^{\mathrm{un}}, \kappa^{\mathrm{c}}, \kappa^{\mathrm{c}})$-computable. As $K = K \cap B$, we have shown $\kappa^{\mathrm{un}} \leqslant \kappa^{\mathrm{c}}$.

$\kappa^{\mathrm{net}} \leqslant \kappa^{\mathrm{c}}$:

Let $\kappa^{\mathrm{net}}(p) = K$, where p is a ι-combination (see Definition 5.1). There is a Type-2 machine that on input p computes a sequence r such that $\iota(w) \lhd r$ if and only if

$$(\exists \iota(w') \lhd p)\, \overline{\theta^*(w')} \subseteq \theta^*(w),$$

where $\theta^*(w)$ can be written as the union of finitely many base elements. Combining this with Lemma 3.3, we know that the relation above is r.e..

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$\kappa^{\mathrm{c}} \leqslant \kappa^{\mathrm{net}}$:

Let $\kappa^{\mathrm{c}}(p) = K$, where p is a list of $\{w_i \in \mathrm{dom}(\theta^*) \mid i \in \mathbb{N}\}$, and F be the $(\kappa^{\mathrm{c}}, \theta^{\mathrm{un}}, \theta^*)$-computable mapping from Lemma 3.4 with

$$U \in F(K, O) : \Longleftrightarrow\ K \subseteq U \subseteq \bar{U} \subseteq O\,.$$

There is a Type-2 machine M such that on input p:

— M starts with output $\iota(w_0)$.

— If $\iota(w_0')\iota(w_1')\ldots\iota(w_i')$ has been written on the output tape, then M writes some $\iota(w_{i+1}')$ such that

$$\theta^*(w_{i+1}') \in F(\kappa^{\mathrm{c}}(p), \theta^*(w_i') \cap \theta^*(w_{i+1}))\,.$$

Lemma 2.1 says finite intersection is computable based on specific representations, which means that we can effectively check the relation $\kappa^{\mathrm{c}}(p) \subseteq \theta^*(w_{i+1}') \subseteq \overline{\theta^*(w_{i+1}')} \subseteq \theta^*(w_i') \cap \theta^*(w_{i+1})$. □

Definition 5.1 (continued).

(3) Define the minimal cover representation $\kappa^{\mathrm{mc}} :\subseteq \Sigma^\omega \to \mathscr{K}$ by $K = \kappa^{\mathrm{mc}}(p)$ if and only if:

— $w \in \mathrm{dom}(\theta^*)$ if $\iota(w) \lhd p$

— $\{w \in \Sigma^* \mid \iota(w) \lhd p\} = \{w \in \Sigma^* \mid K \subseteq \theta^*(w)$ and $(\forall \iota(u) \lhd w)\, v(u) \cap K \neq \varnothing\}$.

Theorem 5.2. For computably locally compact Hausdorff spaces,

$$\kappa^{\mathrm{mc}} \leqslant \kappa^{\mathrm{c}}\,.$$

Proof.

$\kappa^{\mathrm{mc}} \leqslant \kappa^{\mathrm{c}}$:

Let $\kappa^{\mathrm{mc}}(p) = K$. As $\mathrm{dom}(\nu^*)$ is r.e., there is a Type-2 machine that on input p computes a sequence $r \in \Sigma^\omega$ such that $\iota(w) \lhd r$ if and only if

$$(\exists \iota(w') \lhd p)(\exists w'' \in \mathrm{dom}(\nu^*))\, \{u \mid \iota(u) \lhd w\} = \{u \mid \iota(u) \lhd w'\} \cup \{u \mid \iota(u) \lhd w''\}.$$ □

Remark 5.1. Note that $\kappa^{\mathrm{c}} \leqslant \kappa^{\mathrm{mc}}$ is not true in general (Weihrauch 2000).

6. Conclusion and future work

In this paper, we have characterised and completely investigated computably locally compact Hausdorff spaces, including computability on their closed and compact subsets.

In future we expect to generalise this research to computable T_0-spaces. Moreover, we will apply these basic results to examine the effectivity of certain theorems in general topology.

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