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FINITE HORIZON RUIN PROBABILITIES FOR RANDOM WALKS WITH HEAVY TAILED INCREMENTS

YINGDONG LU
IBM T.J. Watson Research Center
Yorktown Heights, NY 10598
E-mail: yingdong@us.ibm.com

We study the asymptotic behavior of finite horizon ruin probabilities for random walks with heavy tailed increment via corrected diffusion approximation. We follow the main idea in [4] of inverting Fourier transformation, and the Fourier transformation is calculated through optimal stopping and a central limit theorem for renewal process.

1. INTRODUCTION

Consider a random walk $S_n = \sum_{i=1}^{n} x_i, n = 1, 2, \ldots,$ of independent and identically distributed (i.i.d.) random variables x_n, $n = 1, 2, \ldots,$ defines on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with $\mathsf{E}[x_1] = 0$, $\mathsf{Var}[x_1]^2 = 1$. In this paper, we attempt to obtain approximations to the quantity,

$$\psi(a, T) := \mathsf{P}[S_n > a, n \leq T], \quad \forall a \in \mathbb{R}_+, T \in \mathbb{Z}_+,$$

otherwise known as the finite horizon ruin probability due to its applications in actuarial science.

Ruin probability calculation is an important yet challenging problem in the probability theory. It is also critical in many applications of applied probability, such as engineering, management, and insurance industry; hence, draws considerable attentions, from researchers in various fields. Several classes of methods have been developed for efficient calculation and approximation, for a comprehensive study of the problem, see, for example Asmussen's book [1]. Many of the methods developed, although accurate, require special distributional properties for the increment, such as, being phase type distributions. For problems with increments following a general probability distribution, diffusion approximation methods, that is, the application of variants of (functional) central limit theorems, are proven to be quite efficient when the mean of the increment falls into certain region, for details, again, see [1]. The approximation fails to be satisfactory when the parameters are outside this region. To overcome this difficulty, Sigmund [6] proposed the idea of corrected diffusion approximation. Later, Asmussen and his colleagues generalized the frame to very general case with light tailed increments. In [4], Hogan generalized the corrected diffusion approximation method to the case of infinite horizon ruin problems using Fourier analysis techniques,

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the purpose of this paper is to extend his result to the finite horizon case. In [5], the author extends the corrected diffusion approximation to the finite horizon problem with the time horizon approximated by a random horizon.

The key idea of diffusion approximation is to use a Brownian motion, whose first passage time is easy to compute, to replace the random walk. This Brownian motion is obtained when we scale the time and space properly. Namely, for n very large, due to the functional central limit theorem, $(1/\sqrt{n})S_{\lfloor nt \rfloor}$ should behave like a Brownian motion, for any time t. Directly, applying the formula of the first passage time for Brownian motion, we get,

$$\psi(a/\epsilon, \infty) \approx e^{-2a}.$$

When x_1 has moment generating function, the corrected diffusion approximation scheme to further expand the terms of expansion hence get more accurate approximation as follows:

$$\psi(a/\epsilon, T/\epsilon^2) \approx G(c_0 T + \epsilon\beta, -\frac{\gamma}{2\epsilon}, 1 + \beta'\epsilon),$$

where $G(t, \xi, c)$ denote the inverse Gaussian distribution function, and constants β, γ and β' are parameters given in [3]. When the increments follow a heavy tailed distribution, Hogan derived a similar approximation for the infinite horizon ruin probability through Fourier analysis,

$$\psi(a/\epsilon, \infty) \approx \exp[-2\xi] \left(1 - \frac{4\xi\gamma\epsilon}{3} - 2\epsilon\beta\right) + o(\epsilon).$$

The purpose of this paper is to extend these techniques to obtain an estimation of $\psi(a/\epsilon, T/\epsilon^2)$ when the increments follow a heavy tailed distribution. We will take an approach of inverting Fourier transforms as those in [4]. During the process, we have developed many estimation to quantities related to the ladder process of a random walk, which we believe are of independent interests.

The rest of the paper will be organized as follows. In Section 2, we collect basic definitions and some preliminary results mostly related to random walk and its ladder hight process. Then we collect estimations of key quantities in Section 3. Finally, in Section 4, we apply all the results to the inverting Fourier transforms and obtain asymptotic estimation of the ruin probability.

2. PRELIMINARY RESULTS AND NOTATIONS

For the random walk S_n, the ladder process is defined as the following class of stopping times:

$$\begin{aligned}
\tau^{(0)} &= 0, \\
\tau^{(1)} &= \tau_+, \\
\tau^{(n+1)} &= \begin{cases} \inf\{n > \tau : S_n > S_{\tau^{(n)}}\}, & \tau^{(n)} < \infty \\ \infty, & \tau^{(n)} = \infty, \end{cases}
\end{aligned}$$

where

$$\tau_- = \inf\{n : S_n \le 0\}, \quad \tau_b = \inf\{n : S_n > b\},$$

and $\tau_+ = \tau_0$. Furthermore, let $G_+(x)$ denote the distribution of the the ladder height process, that is $G_+(x) = \mathsf{P}[S_{\tau_+} \le x]$. It is known that, $\mathsf{E}[\tau_+] = (1 - \|G_-\|)^{-1}$, where

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$\|G_+\| = \mathsf{P}[\tau_- < \infty]$. For more of its properties, see for example [2]. Furthermore, define.

$$\nu = \inf\{n : \tau^{(n)} > T/\epsilon^2, \text{ or, } \tau^{(n)} = \infty\}.$$

For any small positive number $\epsilon > 0$, define the following probability measure, $\mathsf{P}_\epsilon[x_1 \in A] = \mathsf{P}[x_1 - \epsilon \in A]$, hence $\mathsf{E}_\epsilon[x_1] = -\epsilon$, $\mathsf{E}_\epsilon[x_1^2] = 1 + \epsilon^2$ and $\mathsf{E}_\epsilon[x_1^3] = \mathsf{E}_0[x_1^3] + 3\epsilon + \epsilon^3$. Therefore, the variance of x_1 under P_ϵ, $\sigma_\epsilon^2 = 1$. Our goal is to estimate the following quantity

$$1 - \mathsf{P}_\epsilon[S_n < a/\epsilon, n \le T/\epsilon^2]$$

to the order of ϵ for fixed positive numbers $a, T > 0$. Thus,

$$\mathsf{P}_\epsilon[S_n \le a/\epsilon, \forall n \le T/\epsilon^2] = \mathsf{P}_\epsilon[S_{\tau^{(\nu-1)}} \le a/\epsilon].$$

It can also be seen easily that $\tau^{(n)}$, $\tau^{(n)} < \infty$, $n = 0, 1, \ldots$, forms a renewal process with increment $\{S_{\tau+}, \tau < \infty\}$. Therefore, ν can then be treated as the counting process of this renewal process at the time T/ϵ^2. Furthermore, we have,

$$\begin{aligned}\mathsf{P}_\epsilon[S_{\tau^{(\nu-1)}} \le a/\epsilon] = \mathsf{P}_\epsilon[S_{\tau_\nu} \le a/\epsilon, \tau^{(\nu)} < \infty] - \mathsf{P}_\epsilon[S_{\tau_{\nu-1}} \le a/\epsilon, S_{\tau_\nu} > a/\epsilon, \tau^{(\nu)} < \infty] \\ + \mathsf{P}_\epsilon[S_{\tau^{(\nu-1)}} \le a/\epsilon, \tau^{(\nu)} = \infty]. \qquad \textbf{(1)}\end{aligned}$$

However, a careful look at the term

$$\mathsf{P}_\epsilon[S_{\tau_{\nu-1}} \le a/\epsilon, S_{\tau_\nu} > a/\epsilon]$$

reveals that it is a term that is at least of order $O(\epsilon^2)$.

LEMMA 1:

$$\mathsf{P}_\epsilon[S_{\tau_{\nu-1}} \le a/\epsilon, S_{\tau_\nu} > a/\epsilon] = o(\epsilon) \qquad \textbf{(2)}$$

PROOF: We know that there exist an integer $M = o(1/\epsilon^{1+\delta})$ for some small $\delta > 0$, such that, $\mathsf{P}[S_{\tau_1} > M] < \epsilon^2$, therefore,

$$\mathsf{P}[S_{\tau_{\nu-1}} \le a/\epsilon, S_{\tau_\nu} > a/\epsilon] \le \mathsf{P}[S_{T/\epsilon^2} \in [a/\epsilon - M, a/\epsilon + M]] + O(\epsilon^2).$$

Meanwhile,

$$\mathsf{P}[S_{T/\epsilon^2} \in [a/\epsilon - M, a/\epsilon + M]] = \mathsf{P}[\epsilon S_{T/\epsilon^2} \in [a - M\epsilon, a + M\epsilon]].$$

From the classic Central Limit Theorem, we know that

$$\mathsf{P}[\epsilon S_{T/\epsilon^2} \in [a - M\epsilon, a + M\epsilon]] = \mathsf{P}[N(\mu, \sigma) \in [a - M\epsilon, a + M\epsilon]] + o(\epsilon).$$

Therefore, (2) holds. ■

In conjunction with the independence between the increments of the ladder heights process, we have, from (1),

$$\mathsf{P}_\epsilon[S_{\tau^{(\nu-1)}} \le a/\epsilon] = \frac{1}{1-\epsilon}\mathsf{P}_\epsilon[S_{\tau_\nu} \le a/\epsilon | \tau^{(\nu)} < \infty]. \qquad \textbf{(3)}$$

The quantity we need to estimate, hence, becomes,

$$\mathsf{P}_\epsilon[S_{\tau_\nu} \le a/\epsilon | \tau^{(\nu)} < \infty].$$

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3. ESTIMATION OF KEY QUANTITIES

For the calculation of the ruin probability, we need to estimate several important quantities. Not all the quantities will be needed in the final result in its most precise form. For some of them, a rough estimation of their order will be sufficient. However, we believe that these are fundamental quantities for a general random walk, might be of independent interest, thus present the estimations in the most precise form we can obtain.

First, let us define the following notations:

$$p = \mathsf{P}_\epsilon[\tau_+ < \infty],$$
$$\mu_i = \mathsf{E}_\epsilon[S^i_{\tau_+} | \tau_+ < \infty],$$
$$f(t) = \mathsf{E}_\epsilon[\exp(iS_{\tau_+}t) | \tau_+ < \infty].$$

$1 - \mathsf{P}_\epsilon[\tau_+ = \infty]$ is the probability that the random walk will be above zero in finite time. Theorem 2 in [4] gives an estimate to the order of ϵ^2 of the infinite horizon ruin probability. Therefore, to get an estimation of $\mathsf{P}_\epsilon[\tau_+ = \infty]$, just set the level to be zero in the theorem, the following conclusion can be reached.

LEMMA 2: $1 - p = \mathsf{P}_\epsilon[\tau_+ = \infty] = \epsilon(\mathsf{E}_0[S^2_{\tau_+}]/\mathsf{E}_0[S_{\tau_+}]) + o(\epsilon^2)$.

PROOF: Theorem 2 in [4] provides the expression up to the $O(\epsilon^2)$ term, but by a quick look at the derivation we can conclude that the $O(\epsilon^2)$ varnishes when the level is set to be zero. ■

Next, we estimate $\mathsf{E}_\epsilon[\tau_+, \tau_+ < \infty]$. Wald's first identity implies that, $\mathsf{E}_\epsilon[\tau_+, \tau_+ < \infty] = \mathsf{E}_\epsilon[S_{\tau_+}, \tau_+ < \infty]/\epsilon$. Therefore, it is suffice to estimate, $\mathsf{E}_\epsilon[S_{\tau_+}, \tau_+ < \infty]$. Given $\mathsf{E}_0[x_1^3] < \infty$, Lemma 1 from [4] gives

$$\mathsf{E}_\epsilon[S_{\tau_+}, \tau_+ < \infty]\mathsf{E}_\epsilon[S_{\tau_-}] = -\frac{1}{2} + \frac{\epsilon \mathsf{E}_0[S^2_{\tau_-}]}{2\mathsf{E}_0[S_{\tau_-}]} + o(\epsilon).$$

Again by Wald's identity, we have, $\mathsf{E}_\epsilon[S_{\tau_-}] = \epsilon \mathsf{E}_\epsilon[\tau_-]$. Meanwhile, from the classic results on ladder height process, $\mathsf{E}_\epsilon[\tau_-] = [1/(1-p)]$. Therefore,

LEMMA 3:

$$\begin{aligned}\mathsf{E}_\epsilon[S_{\tau_+}, \tau_+ < \infty] &= -\frac{1-p}{2\epsilon} + \frac{(1-p)\mathsf{E}_0[S^2_{\tau_-}]}{2\mathsf{E}_0[S_{\tau_-}]} + o(\epsilon) \\ &= -\frac{\mathsf{E}_0[S^2_{\tau_+}]}{2\mathsf{E}_0[S_{\tau_+}]} + \epsilon \frac{\mathsf{E}_0[S^2_{\tau_+}]\mathsf{E}_0[S^2_{\tau_-}]}{2\mathsf{E}_0[S_{\tau_+}]\mathsf{E}_0[S_{\tau_-}]} + o(\epsilon). \end{aligned} \tag{4}$$

Next, we will estimate some second-order quantities. First, let us examine $\mathsf{E}_\epsilon[S^2_{\tau_+}, \tau_+ < \infty]$. Using the similar approach as in [4], we can get,

LEMMA 4:

$$-\mathsf{E}_\epsilon[S_{\tau_-}]\mathsf{E}[S^2_{\tau_+}, \tau_+ < \infty] - \mathsf{E}_\epsilon[S^2_{\tau_-}]\mathsf{E}[S_{\tau_+}, \tau_+ < \infty] = \frac{\mathsf{E}_\epsilon[x^3]}{3} + \frac{\epsilon \mathsf{E}_0[S^3_{\tau_-}]}{\mathsf{E}_0[S_{\tau_-}]} + O(\epsilon^2). \tag{5}$$

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PROOF: The key equation is still the following identity due to Wiener–Hopf decomposition:

$$1-\phi(\lambda)-(1-X_-(\lambda))(1-p)=(1-X_-(\lambda))(X_+(0)-X_+(\lambda)), \tag{6}$$

where

$$\begin{aligned} X_-(\lambda) &= \mathsf{E}_\epsilon[\exp[i\lambda S_{\tau_-}]], \\ X_+(\lambda) &= \mathsf{E}_\epsilon[\exp[i\lambda S_{\tau_+}], \tau_+<\infty], \\ \phi(\lambda) &= \mathsf{E}_\epsilon \exp(i\lambda x). \end{aligned}$$

By rewriting (6), we have

$$\begin{aligned} &1-\phi(\lambda)+\lambda\phi'(0)+\frac{\lambda^2\phi''(0)}{2}-\left[1-X_-(\lambda)+\lambda X'_-(0)+\frac{\lambda^2 X''_-(0)}{2}\right](1-p) \\ &\quad=(1-X_-(\lambda))(X_-(0)-X_-(\lambda))-\lambda^2 X'_-(0)X'_+(0). \end{aligned}$$

Divide both sides by λ^3, let λ go to zero, then we obtain (5). ■

Using the Eq. (1) and (2) in [4], we can obtain,

$$\frac{\mathsf{E}_\epsilon[S^2_{\tau_-}]}{\mathsf{E}_\epsilon[S_{\tau_-}]}=\frac{\mathsf{E}_\epsilon[x_1^2]}{\mathsf{E}_\epsilon[x_1]}-\frac{1}{\epsilon}+\frac{\mathsf{E}_0[S^2_{\tau_-}]}{\mathsf{E}_0[S_{\tau_-}]}+o(1).$$

Hence,

$$\frac{\mathsf{E}_\epsilon[S^2_{\tau_-}]}{\mathsf{E}_\epsilon[S_{\tau_-}]}=-\frac{1+\sigma^2}{\epsilon}+\left(1+\frac{\mathsf{E}_0[S^2_{\tau_-}]}{\mathsf{E}_0[S_{\tau_-}]}\right)+o(1).$$

In conjunction with (5), we have

$$\begin{aligned} \mathsf{E}[S^2_{\tau_+},\tau_+<\infty] &= \left[\frac{1+\sigma^2}{\epsilon}-\left(1+\frac{\mathsf{E}_0[S^2_{\tau_-}]}{\mathsf{E}_0[S_{\tau_-}]}\right)\right]\left[-\frac{\mathsf{E}_0[S^2_{\tau_+}]}{2\mathsf{E}_0[S_{\tau_+}]}+\epsilon\frac{\mathsf{E}_0[S^2_{\tau_+}]\mathsf{E}_0[S^2_{\tau_-}]}{2\mathsf{E}_0[S_{\tau_+}]\mathsf{E}_0[S_{\tau_-}]}+o(\epsilon)\right] \\ &\quad+\frac{\mathsf{E}_0[S^2_{\tau_+}]}{\mathsf{E}_0[S_{\tau_+}]}\left[\frac{\mathsf{E}_\epsilon[x^3]}{3}+\frac{\epsilon\mathsf{E}_0[S^3_{\tau_-}]}{\mathsf{E}_0[S_{\tau_-}]}+O(\epsilon^2)\right]. \end{aligned} \tag{7}$$

The second moment of $[\tau_+,\tau_+<\infty]$ has the following expression through Wald's second identity:

$$\mathsf{E}_\epsilon\left\{\left[\sum_{i=1}^{\tau_+}(x_i-\epsilon)\right]^2,\tau_+<\infty\right\}=\mathsf{E}_\epsilon[x_i-\epsilon,\tau_+<\infty]^2\mathsf{E}_\epsilon[\tau_+,\tau_+<\infty].$$

Meanwhile,

$$\mathsf{E}_\epsilon\left\{\left[\sum_{i=1}^{\tau_+}(x_i-\epsilon)\right]^2,\tau_+<\infty\right\}=\mathsf{E}_\epsilon[S^2_{\tau_+}-2\epsilon\tau_+S_{\tau_+}+\epsilon^2\tau_+^2,\tau_+<\infty].$$

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Hence, the only term we need to estimate is $\mathsf{E}_\epsilon[\tau_+ S_{\tau_+},,\tau_+<\infty]$. To do this, we will adapt Siegmund's approach in proving Lemma 4 in [6].

$$\begin{aligned}
&\mathsf{E}_\epsilon[x_1]\mathsf{E}_\epsilon[\tau_+ S_+^k, \tau_+<\infty] \\
&\quad = \mathsf{E}_\epsilon[S_{\tau_+}, \tau_+\infty] \sum_{n=0}^{\infty} \int_{\{\tau_+>n,,\tau_+<\infty\}} S^k_{\tau_+,\tau_+<\infty} d\mathsf{P}_\epsilon \\
&\quad = \mathsf{E}_\epsilon[S_{\tau_+}, \tau_+\infty] \sum_{n=0}^{\infty} \int_0^{\infty} \mathsf{E}_\epsilon[S_{\tau(x)}-x]^k \mathsf{P}_\epsilon[\tau_+>n, -S_n\in[x,x+dx), \tau_+<\infty].
\end{aligned}$$

meanwhile, we have the distribution version,

$$\sum_{n=0}^{\infty} \mathsf{P}_\epsilon[\tau_+>n, S_n\le -x] = \mathsf{E}_\epsilon \tau_+ \mathsf{P}_\epsilon[M\le -x].$$

Hence, we need to estimate

$$\int_0^{\infty} \mathsf{E}_\epsilon[S_{\tau(x)}-x]^k \mathsf{P}_\epsilon[-M\in[x,x+dx)].$$

Furthermore, we know that,

$$\mathsf{E}_\epsilon[S_{\tau(x)}-x] = \mathsf{E}[S_{\tau(x/\epsilon)}-\epsilon\tau - x/\epsilon]$$

tend to the equilibrium distribution $\mathsf{E}_e[S_{\tau_+}]$, which denote the mean of a random variable that has density $(1-G_+(x))/\mathsf{E}_0[S_{\tau_+}]$. Let $\nu_k = \int_0^\infty [1-G_+(x)]^k e^{-2x}dx$, then, the above plus Kingman's formula, $\mathsf{P}_\epsilon[-M\in[x,x+dx)]\to e^{-2x}$, with guarantees on the convergence via the dominated convergence theorem, we have,

$$\int_0^{\infty} \mathsf{E}_\epsilon[S_{\tau(x)}-x]^k \mathsf{P}_\epsilon[-M\in[x,x+dx)] = \frac{\nu_k}{\mathsf{E}_0[S_{\tau_+},\tau_+<\infty]} + o(\epsilon).$$

Hence,

Lemma 5:

$$\epsilon\mathsf{E}_\epsilon[\tau_+ S_+^k, \tau_+<\infty] = \left(-\frac{\mathsf{E}_0[S^2_{\tau_+}]}{2\mathsf{E}_0[S_{\tau_+}]} + \frac{\mathsf{E}_0[S^2_{\tau_-}]\mathsf{E}_0[S^2_{\tau_+}]}{2\mathsf{E}_0[S_{\tau_+}]\mathsf{E}_0[S_{\tau_-}]}\epsilon\right)\frac{\nu_1}{\mathsf{E}_0[S_{\tau_+},\tau_+<\infty]} + O(\epsilon^2).$$

4. MAIN RESULTS

Recall that our goal is to estimate the following quantity.

$$\mathsf{P}_\epsilon[S_{\tau_\nu}\le a/\epsilon|\tau_\nu<\infty].$$

To do so, let us consider the Fourier transform of S_{τ_ν}, The key is to estimate

$$\mathsf{E}[\exp[i\lambda S_\nu|\tau^{(\nu)}<\infty]] = \mathsf{E}[p\phi(\lambda)^\nu].$$

For this we need the following lemma.

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LEMMA 6: *For a renew process $N(t)$, we have, as $t \to \infty$,*

$$\mathsf{E}[\exp(\hat{\lambda}N(t)] = \exp\left[-\frac{t\sigma^2\hat{\lambda}^2}{2\mu^3} + \frac{\hat{\lambda}t}{\mu} + o(1)\right]. \tag{8}$$

PROOF: Central limit theorem gives us

$$\frac{N(t) - (t/\mu)}{\sqrt{t(\sigma^2/\mu^3)}} \Rightarrow N(0,1),$$

or in the form of characteristic function, uniformly,

$$\mathsf{E}\left\{\exp\left[\lambda\left(\frac{N(t) - (t/\mu)}{\sqrt{t(\sigma^2/\mu^3)}}\right)\right]\right\} = e^{-(\lambda^2/2)+o(1)}, \quad \text{as } t \to \infty.$$

Therefore,

$$\mathsf{E}\left\{\exp\left[\lambda\left(\frac{N(t)}{\sqrt{t(\sigma^2/\mu^3)}}\right)\right]\right\} = \exp\left[-\frac{\lambda^2}{2} + \lambda\frac{\sqrt{t}}{\sqrt{\sigma^2/\mu}} + o(1)\right], \quad \text{as } t \to \infty.$$

Let $\hat{\lambda} = \frac{\lambda}{\sqrt{t\sigma^2/\mu^3}}$, so $\lambda = \hat{\lambda}\sqrt{t\sigma^2/\mu^3}$, hence,

$$\mathsf{E}[\exp(\hat{\lambda}N(t)] = \exp\left[-\frac{t\sigma^2\hat{\lambda}^2}{2\mu^3} + \frac{\hat{\lambda}t}{\mu} + o(1)\right],$$

which is desired. ■

THEOREM 1: *Assuming $\mathsf{E}_0[x_1^5] < \infty$, the ruin probability has the following estimation:*

$$\psi\left(\frac{a}{\epsilon}, \frac{T}{\epsilon^2}\right) = A_1 + A_2\epsilon + O(\epsilon^2) \tag{9}$$

where

$$A_1 = 1 - \frac{1}{\pi}\int_{-\infty}^{\infty} \exp[I_1]\frac{\sin az}{z}\frac{dz}{\sqrt{1 + (I_3 z)^2}},$$

$$A_2 = \frac{1}{\pi}\int_{-\infty}^{\infty} \exp[I_2]\frac{\sin az}{z}\frac{dz}{\sqrt{1 + (I_4 z)^2}},$$

$$I_1 = 2T\left(1 + \beta z\sqrt{\frac{\mathsf{E}_0[S_{\tau_-}]}{\beta\mathsf{E}_0[S_{\tau_-}^2]} - \frac{1}{4\mathsf{E}_0[x^3]}}\right),$$

$$I_2 = T\left(\frac{\mathsf{E}_0[S_{\tau_+}^2]}{\mathsf{E}_0[S_{\tau_+}]}\right)\left(\frac{\beta}{2}z^2 - \frac{\mathsf{E}_0[S_{\tau_-}^2]}{\mathsf{E}_0[S_{\tau_-}]}\left(1 + \beta z\sqrt{1 - \frac{\mathsf{E}_0[S_{\tau_-}^2]}{\mathsf{E}_0[S_{\tau_-}]}\left(1 - \frac{\mathsf{E}_0[x^3]}{6}\right)^2}\right)\right).$$

and $I_3 = T\beta/2\mathsf{E}_0[x^3]$ and $I_4 = T\beta/2\mathsf{E}_0[x^3]\,\mathsf{E}_0[S_{\tau_-}^2]/\mathsf{E}_0[S_{\tau_-}]$.

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PROOF: Now, recall that, applying the inverse of the Fourier transformation, $1-\psi(a/\epsilon, T/\epsilon^2)$ is represented by

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin at/\epsilon}{t}\mathsf{E}[\exp(itS_{\tau_\nu})]/\mathsf{P}[\tau_\nu<\infty]dt,$$

or equivalently, with variable transform, $t=\epsilon z$,

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin az}{z}\mathsf{E}[\exp\left(i\epsilon z S_{\tau_\nu}\right),\tau_\nu<\infty]/\mathsf{P}[\tau_\nu<\infty]dz.$$

Apparently, we need only to look at the real part of the integrand, we have,

$$\begin{aligned}Re\{\mathsf{E}[\exp[itS_{\tau_\nu},\tau_\nu<\infty]]\}&=Re\{\mathsf{E}[\phi(t)^\nu]\}\\&=Re\left\{[(\exp[\log|p\phi(\lambda)|+i\alpha])^{\frac{1}{\epsilon^2}}]^{(T/\mu)+\sqrt{(T\sigma^2\epsilon^2/\mu)}N(0,1)+O(\epsilon^2)}\right\},\end{aligned}$$

where $\alpha=\arctan(f_1/f_2)$. We know that $\phi(t)=f_1+if_2$, where,

$$f_1=1-\frac{\mu_2}{2}z^2+O(z^4),\ f_2=\mu_1 z+O(z^2).$$

The fact that the ladder process is a renewal process, see, for example [2] allow us to apply Lemma 6, therefore, we have, uniformly in t,

$$\begin{aligned}&\mathsf{E}[\exp[itS_{\tau_\nu},\tau_\nu<\infty]]\\&\quad=\mathsf{E}[(p\phi(t))^\nu] &\textbf{(10)}\\&\quad=\mathsf{E}\left[\left(p\left[1-\frac{\mu_2}{2}z^2+i\mu_1 z+O(z^3)\right]\right)^{(T/\mu_\tau\epsilon^2)+\sqrt{(T\sigma^2/\epsilon^2\mu_\tau^3)}N(0,1)+O(1)}\right] &\textbf{(11)}\\&\quad=\exp\left\{[\log|p\phi(z\epsilon)|+i\alpha]\frac{T}{\mu_\tau\epsilon^2}+\frac{T\sigma^2}{2\epsilon^2\mu_\tau^3}[(\log|\phi(z\epsilon)|)^2+2i\alpha\log|\phi(z\epsilon)|-\alpha^2]\right\}\\&\quad=\exp\left\{\left[\frac{T}{\mu_\tau\epsilon^2}\log|\phi(z\epsilon)|+\frac{T\sigma^2}{2\epsilon^2\mu_\tau^3}[(\log|\phi(z\epsilon)|)^2-\alpha^2]\right]\right.\\&\qquad\left.+\,i\left[2\alpha\log|\phi(z\epsilon)|\frac{T\sigma^2}{2\epsilon^2\mu_\tau^3}+\alpha\frac{T}{\mu_\tau\epsilon^2}\right]\right\}. &\textbf{(12)}\end{aligned}$$

From Lemma 3, we have

$$|\phi(z\epsilon)|=1-\sigma_\tau z\epsilon+\frac{\mu_2}{2}z^2\epsilon^2+O(\epsilon^3).$$

From Lemmas 4 and 5, we can conclude that,

$$\frac{T\sigma^2}{2\epsilon^2\mu_\tau^3}=O(\epsilon),\log|p\phi(z\epsilon)|=O(\epsilon).$$

Hence,

$$\begin{aligned}&\frac{T\sigma^2}{2\epsilon^2\mu_\tau^3}(\log|p\phi(z\epsilon)|)^2\\&\quad=\frac{T[\mathsf{E}_\epsilon[\tau_+]+2\epsilon\mathsf{E}_\epsilon[\tau_+S_+^k,\tau_+<\infty]-\mathsf{E}_\epsilon[S_{\tau_+}^2,\tau_+<\infty]]}{2\epsilon\left[-\frac{1-p}{2}+((1-p)\epsilon\mathsf{E}_0[S_{\tau_-}^2]/2\mathsf{E}_0[S_{\tau_-}])+o(\epsilon^2)\right]^3}(\log|p\phi(r)|)^2=O(\epsilon^3).\end{aligned}$$

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Similarly, we have,

$$\alpha\frac{T\sigma^2}{2\epsilon^2\mu_\tau^3}(\log|p\phi(z\epsilon)|) = O(\epsilon^3).$$

Hence, we can write (12) as,

$$\exp\left\{\left[\frac{T}{\mu_\tau\epsilon^2}\log|\phi(z\epsilon)| + O(\epsilon^3)\right] + i\left[\alpha\frac{T}{\mu_\tau\epsilon^2} + O(\epsilon^3)\right]\right\}. \tag{13}$$

Now let us take a detailed look at each term, first, we know that $\epsilon\mu_\tau = \mathsf{E}[S_{\tau_+}, \tau_+ < \infty]$

$$\begin{aligned}\frac{T}{\mu_\tau\epsilon^2}\log|p\phi(z\epsilon)| &= \frac{T}{-\frac{1-p}{2} + ((1-p)\epsilon\mathsf{E}_0[S_{\tau_-}^2]/2\mathsf{E}_0[S_{\tau_-}]) + o(\epsilon^2)}\log|p\phi(z\epsilon)| \\ &= \frac{T}{-\frac{1-p}{2} + ((1-p)\epsilon\mathsf{E}_0[S_{\tau_-}^2]/2\mathsf{E}_0[S_{\tau_-}]) + o(\epsilon^2)} \\ &\quad [(1-p) + \sigma_\tau z\epsilon + \frac{\mu_2}{2}z^2\epsilon^2 + O(\epsilon^3)].\end{aligned}$$

Furthermore,

$$\frac{\epsilon\sigma_\tau}{1-p} = \sqrt{\frac{(\mu_2-\mu_1^2)\epsilon^2}{(1-p)^2}} = \sqrt{\left(\frac{\mu_2\epsilon}{1-p}\right)\left(\frac{\epsilon}{1-p}\right) - \left(\frac{\mu_2\epsilon}{1-p}\right)^2\left(\frac{\mu_1}{\mu_2}\right)^2}$$

and

$$\frac{\epsilon\mu_1}{1-p} = \frac{\epsilon\mu_2}{1-p}\times\frac{\mu_1}{\mu_2} \to \beta\left(-\frac{1}{2}/\mathsf{E}_0[x^3]\right).$$

Hence,

$$\frac{\epsilon\sigma_\tau}{1-p} \to \beta\sqrt{\frac{\mathsf{E}_0[S_{\tau_-}]}{\beta\mathsf{E}_0[S_{\tau_-}^2]} - \frac{1}{4\mathsf{E}_0[x^3]}}.$$

So the $O(1)$ term of the ruin problem,

$$I_1 = 2T\left(1 + \beta z\sqrt{\frac{\mathsf{E}_0[S_{\tau_-}]}{\beta\mathsf{E}_0[S_{\tau_-}^2]} - \frac{1}{4\mathsf{E}_0[x^3]}}\right).$$

Similarly, the $O(\epsilon)$ term,

$$I_2 = T\left(\frac{\mathsf{E}_0[S_{\tau_+}^2]}{\mathsf{E}_0[S_{\tau_+}]}\right)\left(\frac{\beta}{2}z^2 - \frac{\mathsf{E}_0[S_{\tau_-}^2]}{\mathsf{E}_0[S_{\tau_-}]}\left(1 + \beta z\sqrt{1 - \frac{\mathsf{E}_0[S_{\tau_-}^2]}{\mathsf{E}_0[S_{\tau_-}]}\left(1 - \frac{\mathsf{E}_0[x^3]}{6}\right)^2}\right)\right).$$

Moreover,

$$\alpha = \arctan\left(\frac{\mu_1\epsilon z + O((\epsilon z)^3)}{1 - \frac{\mu_2}{2}(\epsilon z)^2 + O(z^4)}\right).$$

It is an $O(\epsilon)$ term. Therefore, the quantity (13) becomes,

$$\exp\left\{\left[I_1 + I_2\epsilon + O(\epsilon^2)\right] + i\left[I_3 + I_4\epsilon + O(\epsilon^2)\right]\right\}$$

with $I_3 = T\beta(1/2\mathsf{E}_0[x^3])$, and $I_4 = T\beta(1/2\mathsf{E}_0[x^3][\mathsf{E}_0[S_{\tau_-}^2]/\mathsf{E}_0[S_{\tau_-}])$. Thus, (9) follows the same arguments as those in [4], under the assumption that $\mathsf{E}_0[x_1^5] < \infty$, which guarantees the boundness of the integral operators involved as well as the uniformality of the infinitesimal orders. ■

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