Appendix A
Roots and weights

In this appendix I review some facts concerning compact Lie groups and their Lie algebras that are used in this book. I will focus in particular on the roots and weights. As we will see, these can be understood by generalizing concepts that are familiar from the simplest non-Abelian Lie group, SU(2).

A compact Lie group and its algebra may be simple, semisimple, or neither. If it is simple, the generators of the algebra cannot be divided into two mutually commuting sets and the group cannot be written, even locally, as the product of two groups. If the group can be written, at least locally, as a product of groups, but without any U(1) factors, it is semisimple, but not simple. Thus, SU(3) is simple and semisimple, $\mathrm{SO}(4) = [\mathrm{SU}(2)\times\mathrm{SU}(2)]/Z_2$ is semisimple but not simple, and $\mathrm{SU}(2)\times\mathrm{U}(1)$ is neither. Because all compact Lie groups can be built up from the simple Lie groups, I will focus on these.

A.1 Root systems

The usual choice of Hermitian generators for SU(2) is such that in the fundamental two-dimensional representation they are given by $T_a = \sigma_a/2$. They then satisfy the standard angular momentum commutation relations, and in any irreducible representation obey the relation

$$\mathrm{tr}\,(T_a T_b) = \delta_{ab} T(R)\,, \tag{A.1}$$

where $T(R)$ is a constant that depends on the representation; for the fundamental doublet representation, $T = 1/2$. From these Hermitian generators one can construct two ladder operators $T_\pm = T_1 \pm iT_2$ that raise and lower the value of the third generator, T_3, by one unit.

This construction generalizes in a natural way to larger Lie groups. If the group is compact, it is always possible to choose its generators so that they obey Eq. (A.1). From these, one can choose a maximal set of mutually commuting

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generators H_j $(j = 1, 2, \ldots, r)$. These generalize the role of T_3, in that states of an irreducible representation can be taken to be simultaneous eigenvectors of the H_j; their eigenvalues are the weights that will be discussed in the next section. The r-dimensional vector space spanned by the H_j is called the Cartan subalgebra, and r is known as the rank of the group. The remaining generators can be combined to form a set of ladder operators $E_{\boldsymbol{\alpha}}$ that obey

$$\begin{aligned} [H_j, E_{\boldsymbol{\alpha}}] &= \alpha_j E_{\boldsymbol{\alpha}} \,, \\ [E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}}] &= \alpha_j H_j \equiv \boldsymbol{\alpha} \cdot \mathbf{H} \,, \end{aligned} \tag{A.2}$$

with $E_{-\boldsymbol{\alpha}} = E^\dagger_{\boldsymbol{\alpha}}$. The r-component vectors $\boldsymbol{\alpha}$ are known as roots and are naturally viewed as living in an r-dimensional Euclidean space.

This can be illustrated with the case of SU(3). A standard choice of the generators in the fundamental three-dimensional representation is $T_a = \lambda_a/2$, where the Gell-Mann matrices λ_a generalize the Pauli matrices and are given by

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \sqrt{\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} . \end{aligned} \tag{A.3}$$

The generators of the Cartan subalgebra can be take to be $H_1 = T_3$ and $H_2 = T_8$. We can then define ladder operators

$$\begin{aligned} E_{\pm\boldsymbol{\alpha}} &= \frac{1}{\sqrt{2}} \left(T_1 \pm i T_2\right) , \\ E_{\pm\boldsymbol{\beta}} &= \frac{1}{\sqrt{2}} \left(T_4 \pm i T_5\right) , \\ E_{\pm\boldsymbol{\gamma}} &= \frac{1}{\sqrt{2}} \left(T_6 \pm i T_7\right) . \end{aligned} \tag{A.4}$$

Evaluating the commutators of these with the H_i, we find that $E_{\boldsymbol{\alpha}}$ raises H_1 by one unit and leaves H_2 unchanged, $E_{\boldsymbol{\beta}}$ raises H_1 by $1/2$ and H_2 by $\sqrt{3}/2$, and $E_{\boldsymbol{\gamma}}$ lowers H_1 by $1/2$ but raises H_2 by $\sqrt{3}/2$. These relations can be summarized by a root diagram, such as that shown in Fig. A.1a.

As a second example, consider the rotation group in N dimensions, SO(N). Its generators can be taken to be the $J_{ij} = -J_{ji}$ that generate rotations in the i-j plane. These obey

$$[J_{ij}, J_{kl}] = i \left(\delta_{ik} J_{jl} - \delta_{il} J_{jk} - \delta_{jk} J_{il} + \delta_{jl} J_{ik}\right) . \tag{A.5}$$

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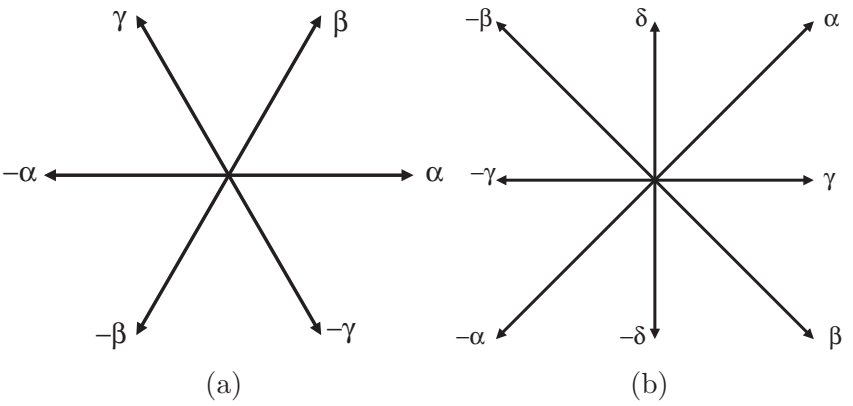


Fig. A.1. The root diagrams for SU(3) and SO(5). The horizontal and vertical directions correspond to H_1 and H_2, respectively.

If $N = 2k$ is even, the generators of the Cartan subalgebra can be taken to be J_{12}, J_{34}, …, $J_{2k-1,2k}$. The same choice suffices for $N = 2k + 1$. Thus, SO($2k$) and SO($2k + 1$) both have rank k.

Let us focus on the case of SO(5), and define

$$\begin{aligned} E_{\boldsymbol{\alpha}} &= \frac{1}{2}\left(J_{13} + iJ_{23}\right) + \frac{i}{2}\left(J_{14} + iJ_{24}\right) , \\ E_{\boldsymbol{\beta}} &= \frac{1}{2}\left(J_{13} + iJ_{23}\right) - \frac{i}{2}\left(J_{14} + iJ_{24}\right) , \\ E_{\boldsymbol{\gamma}} &= \frac{1}{\sqrt{2}}\left(J_{15} + iJ_{25}\right) , \\ E_{\boldsymbol{\delta}} &= \frac{1}{\sqrt{2}}\left(J_{35} + iJ_{45}\right) . \end{aligned} \tag{A.6}$$

Using Eq. (A.5) we find that $E_{\boldsymbol{\alpha}}$ raises both $H_1 = J_{12}$ and $H_2 = J_{34}$ by one unit, while $E_{\boldsymbol{\beta}}$ raises H_1 but lowers H_2, each by one unit. Further, $E_{\boldsymbol{\gamma}}$ raises H_1 and commutes with H_2, while $E_{\boldsymbol{\delta}}$ commutes with H_1 and raises H_2. The corresponding root diagram is shown in Fig. A.1b. In contrast with SU(3), there are two different root lengths.

Note that removing the roots $\pm\boldsymbol{\gamma}$ and $\pm\boldsymbol{\delta}$, formed from the generators J_{j5}, gives the root diagram for the semisimple SO(4).

There are strong constraints on the possible root systems. These can be obtained by noting that any root $\boldsymbol{\alpha}$ can be used to define an SU(2) subgroup with generators

$$\begin{aligned} t_1(\boldsymbol{\alpha}) &= \frac{1}{\sqrt{2\boldsymbol{\alpha}^2}}\left(E_{\boldsymbol{\alpha}} + E_{-\boldsymbol{\alpha}}\right) , \\ t_2(\boldsymbol{\alpha}) &= -\frac{i}{\sqrt{2\boldsymbol{\alpha}^2}}\left(E_{\boldsymbol{\alpha}} - E_{-\boldsymbol{\alpha}}\right) , \\ t_3(\boldsymbol{\alpha}) &= \frac{1}{\boldsymbol{\alpha}^2}\alpha_j H_j . \end{aligned} \tag{A.7}$$

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All of the other roots fall into irreducible representations of this SU(2). In the case of SU(3), for every such SU(2) there are two doublets; e.g., for the SU(2) defined by $\boldsymbol{\alpha}$ the doublets are $(\boldsymbol{\beta}, \boldsymbol{\gamma})$ and $(-\boldsymbol{\gamma}, -\boldsymbol{\beta})$. For SO(5), each short root defines an SU(2) with two triplets, while the long roots define SU(2) subgroups with two doublets and two singlets each.

Because the roots fall into SU(2) representations, the projection of any root on the root defining the SU(2) must be an integer or half-integer multiple of the length of the defining root. Thus, if $\boldsymbol{\alpha}$ defines an SU(2) and $\boldsymbol{\beta}$ is any other root,

$$\frac{\boldsymbol{\beta}\cdot\boldsymbol{\alpha}}{\boldsymbol{\alpha}^2} = \frac{p}{2}\,, \tag{A.8}$$

with p an integer. Viewing $\boldsymbol{\alpha}$ as an element of a representation of the SU(2) defined by $\boldsymbol{\beta}$ gives a dual relation,

$$\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}{\boldsymbol{\beta}^2} = \frac{q}{2}\,, \tag{A.9}$$

with q also an integer. Combining these two equations yields the two relations

$$4\cos^2\theta = \frac{4(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})^2}{\boldsymbol{\alpha}^2\,\boldsymbol{\beta}^2} = pq\,, \tag{A.10}$$

$$\frac{\boldsymbol{\beta}^2}{\boldsymbol{\alpha}^2} = \frac{p}{q}\,, \tag{A.11}$$

where θ is the angle between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

Since $\cos\theta$ must lie between -1 and 1, we see that $0 \le pq \le 3$. The only possibilities are

- $p = q = 0$: The two roots are orthogonal.
- $p = q = \pm 1$: The roots are either 60° or 120° apart and of equal length.
- $p = 2q = \pm 2$ or $q = 2p = \pm 2$: The angle between the roots is either 45° or 135° and one root is $\sqrt{2}$ times the length of the other.
- $p = 3q = \pm 3$ or $q = 3p = \pm 3$: The angle between the roots is either 30° or 150° and one root is $\sqrt{3}$ times the length of the other.

A further consequence follows from the fact that the roots form complete SU(2) representations. Reflection of a root $\boldsymbol{\beta}$ in the hyperplane through the origin orthogonal to $\boldsymbol{\alpha}$ gives a vector $\boldsymbol{\beta} - 2(\boldsymbol{\beta}\cdot\boldsymbol{\alpha})/|\boldsymbol{\alpha}|$ that is also a root; the group generated by such reflections is known as the Weyl group.

It is a straightforward geometrical exercise to determine all possible root systems satisfying these constraints; the details can be found, e.g., in [311] or [312]. There are four infinite series, A_N, B_N, C_N, and D_N (the classical Lie algebras) and five exceptional algebras, G_2, F_4, E_6, E_7, and E_8; in each case the subscript denotes the rank. The roots of these algebras (with an arbitrary overall normalization) can be given in terms of orthogonal unit vectors $\mathbf{e}_j$.

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The roots and dimensions[1] for the infinite series are

- A_N, with dimension $N(N+2)$:
  $\mathbf{e}_i - \mathbf{e}_j$ $(1 \le i \ne j \le N+1)$.
- B_N, with dimension $N(2N+1)$:
  $\pm\mathbf{e}_i \pm \mathbf{e}_j$ $(1 \le i \ne j \le N)$;
  $\pm\mathbf{e}_i$ $(1 \le i \le N)$.
- C_N, with dimension $N(2N+1)$:
  $\pm\mathbf{e}_i \pm \mathbf{e}_j$ $(1 \le i \ne j \le N)$;
  $\pm 2\mathbf{e}_i$ $(1 \le i \le N)$.
- D_N, with dimension $N(2N-1)$:
  $\pm\mathbf{e}_i \pm \mathbf{e}_j$ $(1 \le i \ne j \le N)$.

Two of these series generalize our results for SO(4) and SO(5): the B_N are the algebras for the odd orthogonal groups SO($2N+1$), and the D_N are the algebras for the even orthogonal groups SO($2N$).

A_N is the algebra of SU($N+1$). Although its roots are written in terms of an $(N+1)$-dimensional space, they all lie in the N-dimensional subspace orthogonal to $\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_{N+1}$. Writing them in this form makes it clear that the roots of SU(N) are a subset of those of SO($2N$), corresponding to the fact the former is a subgroup of the latter.

Given any root one can define a dual root

$$\boldsymbol{\alpha}^* = \frac{\boldsymbol{\alpha}}{\boldsymbol{\alpha}^2}\,. \tag{A.12}$$

It is easy to see that the duals of a root system satisfying our constraints also form an acceptable root system. A_N and D_N are self-dual (after an irrelevant rescaling), but taking the dual of B_N leads to the last of the infinite series, C_N, which is the Lie algebra of the symplectic group Sp($2N$).

The dimensions and roots for the exceptional groups are

- G_2 (dimension 14):
  $\mathbf{e}_i - \mathbf{e}_j$ $(1 \le i \ne j \le 3)$;
  $\pm(3\mathbf{e_j} - \mathbf{e_1} - \mathbf{e_2} - \mathbf{e_3})$ $(j = 1, 2, 3)$.
- F_4 (dimension 52):
  $\pm(\mathbf{e}_i \pm \mathbf{e}_j)$ $(1 \le i \ne j \le 4)$;
  $\pm\mathbf{e}_i$ $(i = 1, 2, 3, 4)$;
  $\frac{1}{2}(\pm\mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3 \pm \mathbf{e}_4)$.
- E_8 (dimension 248):
  $\pm\mathbf{e}_i \pm \mathbf{e}_j$ $(1 \le i \ne j \le 8)$;
  $\frac{1}{2}(\pm\mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3 \pm \mathbf{e}_4 \pm \mathbf{e}_5 \pm \mathbf{e}_6 \pm \mathbf{e}_7 \pm \mathbf{e}_8)$, with an even number of plus signs.

[1] Recall that the dimension of a Lie group and of its algebra is the dimension of the group as a manifold and of the algebra as a vector space. It is equal to the number of generators.

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- E_7 (dimension 133):
  $\pm\mathbf{e}_i \pm \mathbf{e}_j$ $(1 \le i \ne j \le 6)$;
  $\pm(\mathbf{e}_7 + \mathbf{e}_8)$;
  $\frac{1}{2}[(\pm\mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3 \pm \mathbf{e}_4 \pm \mathbf{e}_5 \pm \mathbf{e}_6) \pm (\mathbf{e}_7 + \mathbf{e}_8)]$, with an even number of plus signs in the first pair of parentheses.
- E_6 (dimension 78):
  $\pm\mathbf{e}_i \pm \mathbf{e}_j$ $(1 \le i \ne j \le 5)$;
  $\frac{1}{2}[\pm\mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3 \pm \mathbf{e}_4 \pm \mathbf{e}_5 \pm (\mathbf{e}_6 + \mathbf{e}_7 + \mathbf{e}_8)]$, with an even number of plus signs.

Note that the roots of E_7 are those roots of E_8 that are orthogonal to the root $\mathbf{e}_7 - \mathbf{e}_8$, and the roots of E_6 are the roots orthogonal to both $\mathbf{e}_6 - \mathbf{e}_7$ and $\mathbf{e}_7 - \mathbf{e}_8$. One could continue this pattern to obtain smaller algebras. However, the algebras obtained in this manner are not new: $E_5 = D_5 = \mathrm{SO}(10)$, $E_4 = A_4 = \mathrm{SU}(5)$, and $E_3 = A_2 \times A_1 = \mathrm{SU}(3) \times \mathrm{SU}(2)$.

In fact, the full root system of any of these algebras can be reconstructed from a special subset of r roots $\boldsymbol{\beta}_a$, known as simple roots. These form a basis for the root system and are chosen so that the inner product of any two simple roots is less than or equal to zero. Once these are known, all of the other roots can be obtained by applying successive Weyl reflections. Furthermore, all other roots are linear combinations of the simple roots with integral coefficients that are all of the same sign.

The essential properties of the simple roots can be summarized compactly by the Dynkin diagrams, which are displayed in Fig. A.2. Each node in these diagrams corresponds to a simple root. Two roots $\boldsymbol{\beta}_a$ and $\boldsymbol{\beta}_b$ are linked by

$$m_{ab} = \frac{4(\boldsymbol{\beta}_a \cdot \boldsymbol{\beta}_b)^2}{\boldsymbol{\beta}_a^2 \boldsymbol{\beta}_b^2} \tag{A.13}$$

lines. If $m_{ab} = 1$, then $\boldsymbol{\beta}_a$ and $\boldsymbol{\beta}_b$ are of equal length. If not, there is an arrow pointing toward the longer root.

Note that the Dynkin diagrams for low ranks are not all distinct. This reflects the equivalences $A_1 = B_1 = C_1$, $B_2 = C_2$, and $A_3 = D_3$. Also, note that D_2 is not simple.

The choice of the set of simple roots is not unique, although the m_{ab} are. Given any r-component vector $\mathbf{h}$, it is always possible to choose the simple roots so that their inner products with $\mathbf{h}$ are all positive. If these inner products are all nonzero, this picks out a unique set of simple roots.

With the full root systems specified as previously, the simple roots can be taken to be

- A_N: $\mathbf{e}_j - \mathbf{e}_{j+1}$ $(j = 1, 2, \ldots, N)$.
- B_N: $\mathbf{e}_j - \mathbf{e}_{j+1}$ $(j = 1, 2, \ldots, N-1)$;
  $\mathbf{e}_N$.

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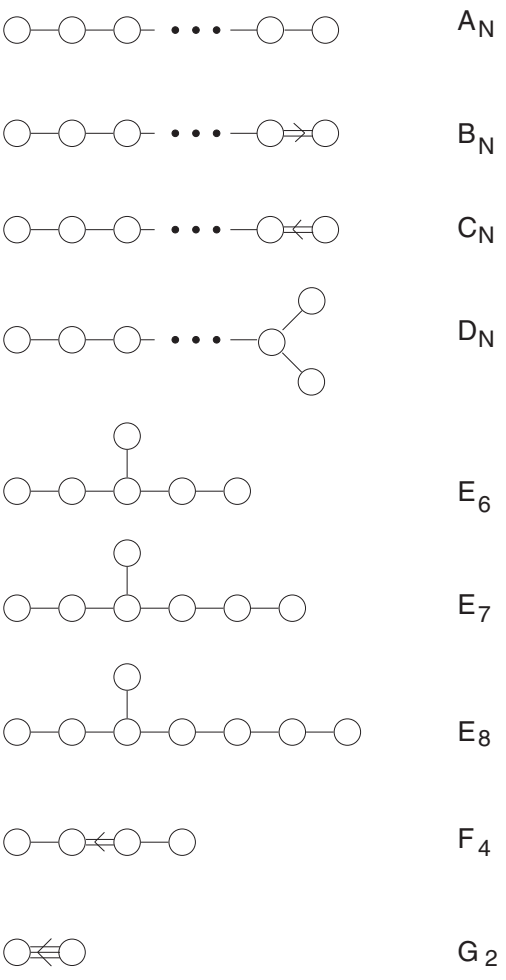


Fig. A.2. Dynkin diagrams for the simple Lie algebras.

- C_N: $\mathbf{e}_j - \mathbf{e}_{j+1}$ $(j = 1, 2, \ldots, N-1)$;
  $2\mathbf{e}_N$.
- D_N: $\mathbf{e}_j - \mathbf{e}_{j+1}$ $(j = 1, 2, \ldots, N-1)$;
  $\mathbf{e}_{N-1} + \mathbf{e}_N$.
- G_2: $\mathbf{e}_2 - \mathbf{e}_3$ and $\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3$.
- F_4: $\mathbf{e}_2 - \mathbf{e}_3$ and $\mathbf{e}_3 - \mathbf{e}_4$;
  $\mathbf{e}_4$;
  $\frac{1}{2}(\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4)$.
- E_8: $\mathbf{e}_{j+1} - \mathbf{e}_j$ $(j = 2, 3, \ldots, 7)$;
  $\mathbf{e}_2 + \mathbf{e}_3$;
  $\frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5 - \mathbf{e}_6 - \mathbf{e}_7 - \mathbf{e}_8)$.

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- E_7: $\mathbf{e}_{j+1} - \mathbf{e}_j$, $(j = 2, 3, 4, 5)$;
  $\mathbf{e}_2 + \mathbf{e}_3$ and $\mathbf{e}_7 + \mathbf{e}_8$;
  $\frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5 - \mathbf{e}_6 - \mathbf{e}_7 - \mathbf{e}_8)$.
- E_6: $\mathbf{e}_{j+1} - \mathbf{e}_j$, $(j = 2, 3, 4)$;
  $\mathbf{e}_2 + \mathbf{e}_3$;
  $\frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5 - \mathbf{e}_6 - \mathbf{e}_7 - \mathbf{e}_8)$;
  $\frac{1}{2}(\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5 + \mathbf{e}_6 + \mathbf{e}_7 + \mathbf{e}_8)$.

A.2 Weights

The states within an SU(2) multiplet can be chosen to be eigenstates of T_3, with eigenvalues that are either integers or half-integers. Within an irreducible multiplet any pair of states can be connected by successive action of the raising and lowering operators. Because these shift T_3 by one unit, the states within a given multiplet have either all integer or all half-integer eigenvalues.

For a larger group the members of an irreducible multiplet can be chosen to be simultaneous eigenstates of all of the H_j, with eigenvalues, or "weights", $w_1, w_2, \ldots, w_r$. These w_j can be viewed as forming an r-component weight vector $\mathbf{w}$. As in SU(2), the weights within a given irreducible multiplet can all be connected by repeated action of the various ladder operators, so that the difference between any two weights of an irreducible representation is an integral linear combination of root vectors.

Just as with the roots, the weights in any representation must fall into complete representations of the SU(2) defined by any root $\boldsymbol{\alpha}$, and so must obey

$$\frac{2\mathbf{w} \cdot \boldsymbol{\alpha}}{\boldsymbol{\alpha}^2} = n \tag{A.14}$$

for some integer n. We already know that the roots satisfy this constraint, so the roots themselves are weights; more specifically, they are the nonzero weights of the adjoint representation.

The set of all possible weights for a simple Lie group can be displayed as a lattice of discrete points in an r-dimensional Euclidean space. For SU(2), this is just the set of integer and half-integer points along the real line. The weight lattices for the rank 2 groups SU(3) and Spin(5), the covering group of SO(5), are shown in Fig. A.3. As indicated in the figure, these lattices can be divided into sublattices such that any two points within a sublattice can be connected by an integral sum of roots, and that two points within different sublattices cannot be. This is completely analogous to the distinction between integer and half-integer weights for SU(2).

The existence of these sublattices is closely related to the fact that several Lie groups can share the same Lie algebra. Let G be the universal covering group of a simple Lie algebra. The center K of G is defined to be the set of elements that

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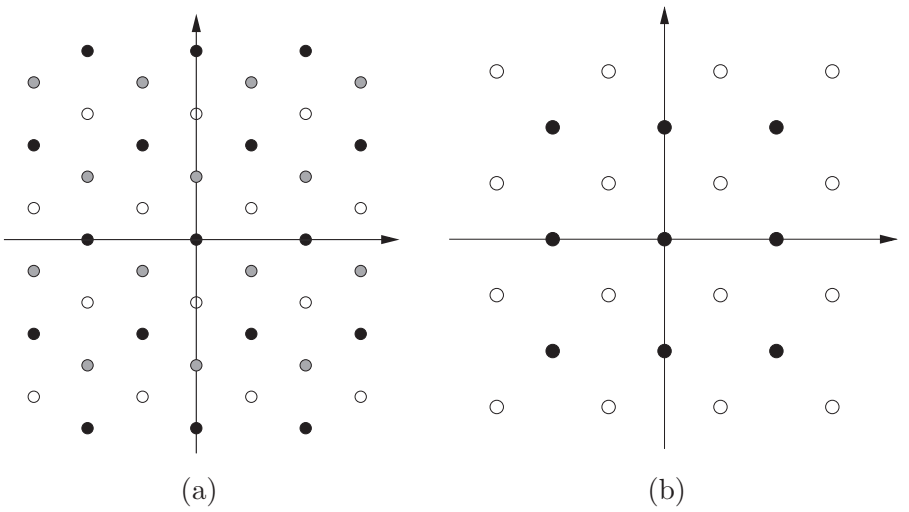


Fig. A.3. Portions of the weight lattices for (a) SU(3) and (b) Spin(5). In both cases the root lattice is indicated by solid black circles. For SU(3) the sublattices with triality 1 and triality 2 are indicated by white and gray circles, respectively. For Spin(5) the white circles indicate the weights of the spinor representations. The horizontal and vertical directions correspond to H_1 and H_2, respectively.

commute with all elements of G; these form a finite group. A new group G/K, also known as the adjoint group, can be defined by identifying the elements g and kg of G, where k is any element of the center. A similar construction can be carried out with K replaced by any of its subgroups.

For a representation of G to also be a true (i.e., single-valued) representation of G/K, the elements of K must be represented by a unit matrix. The weights of these latter representations form a sublattice of the full weight lattice of G. This sublattice contains the root vectors, and may be termed the root lattice.[2] The remaining representations of G are not true representations of G/K. Their weights fall into sublattices that can be put in a one-to-one correspondence with the nontrivial elements of K, with the properties of the direct products of these representations reflecting the group structure of K. (The groups obtained by factoring by a subgroup of the center have correspondingly fewer sublattices, and the adjoint group has only the root lattice.)

For example, SU(2) is the universal covering group of its Lie algebra, with a center Z_2 composed of the two elements I and $-I$. Identifying the elements u and $-u$ yields the group SO(3). The integer spin representations are representations of both SU(2) and SO(3), but the half-integer spin representations

[2] The root lattice is the only sublattice that is itself a lattice. The others are not, because they are not closed under addition.

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are representations only of SU(2). The direct product of two integer spin or of two half-integer spin representations is a direct sum of integer spin representations, while the product of an integer spin representation and a half-integer spin representation is a sum of half-integer spin representations. This reproduces the structure of Z_2, with the integer and half-integer sublattices corresponding to I and $-I$, respectively.

SU(N) is the simply connected universal covering group of its Lie algebra. Its center consists of the elements z, z^2, ..., $z^N = I_N$, where $z = e^{2\pi i/N} I_N$. These form the cyclic group Z_N. In any representation of SU(N) z is represented by $e^{2\pi k i/N}$ times a unit matrix, where k is a representation-dependent integer, defined modulo N, that is known as the triality for $N = 3$ and more generally as the N-ality. The weights of the representations for each value of k form a sublattice of the full weight lattice. For the fundamental representation $k = 1$. Taking the direct product of m fundamental representations gives a direct sum of representations with $k = m \pmod N$, thus reproducing the group structure of Z_N.

Spin(N) is the universal covering group of SO(N). For odd $N = 2r + 1$, its center is Z_2 and the weight lattice consists of two sublattices. One contains the weights of the vector and tensor representations that are also representations of SO($2r + 1$), while the other contains the weights of the spinor representations that are representations only of Spin($2r + 1$). With the roots normalized as in the previous section, the weights of the former are of the form $\sum p_a \mathbf{e}_a$, with the p_a all integers, while the spinor sublattice has weights $\sum (p_a + \frac{1}{2})\mathbf{e}_a$.

The structure for even N is more complicated. The center of Spin($2r$) is Z_4 if r is odd and $Z_2 \times Z_2$ if r is even. [This extra structure can be traced to the fact that SO($2r$) has a nontrivial center composed of the elements $\pm I_{2r}$.] The full weight lattice of Spin($2r$) is the same as that of Spin($2r + 1$), but the two sublattices of the latter are each divided further into two sublattices, according to whether $\sum p_a$ is even or odd. For further discussion of these and the quotient groups of Spin($2r$), see [98].

With the normalization used previously, the weights of Sp($2N$) are of the form $\sum p_a \mathbf{e}_a$, with the p_a again all integers. There are two sublattices, with $\sum p_a$ even or odd, respectively, corresponding to the fact that the center of Sp($2N$) is Z_2.

Of the exceptional groups, E_6 and E_7 have centers Z_3 and Z_2, respectively, while the other three have trivial centers.

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