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ANALYTIC CUT AND INTERPOLATION FOR BI-INTUITIONISTIC LOGIC

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Abstract. We prove that certain natural sequent systems for bi-intuitionistic logic have the analytic cut property. In the process we show that the (global) subformula property implies the (local) analytic cut property, thereby demonstrating their equivalence. Applying a version of Maehara technique modified in several ways, we prove that bi-intuitionistic logic enjoys the classical Craig interpolation property and Maximova variable separation property; its Halldén completeness follows.

§1. Introduction. Bi-intuitionistic logic is a natural extension of intuitionistic logic by a *subtraction* connective (sometimes also called *coimplication*), dual to the usual implication. This is best expressed algebraically. While implication is the residual of conjunction:

$$x \wedge y \leq z \text{ if and only if } y \leq x \to z,$$

subtraction is the co-residual of disjunction:

$$z \leq x \vee y \text{ if and only if } z - x \leq y.$$

Bi-intuitionistic logic was first investigated by Rauszer (1973/74) and Rauszer (1974), under the name *Heyting–Brouwer logic*. An algebraic semantics and a Kripke semantics were defined there, and respective completeness theorems were proved.

The algebraic semantics for bi-intuitionistic logic is the class of *double-Heyting algebras*, that is, algebras $\mathbf{A} = (A; \vee, \wedge, \to, -, 0, 1)$ such that $(A; \vee, \wedge, \to, 0, 1)$ as well as its dual $(A; \vee^\partial, \wedge^\partial, -, 0^\partial, 1^\partial)$ are Heyting algebras. In other words, both $(A; \vee, \wedge, 0, 1)$ and its dual lattice $(A; \vee^\partial, \wedge^\partial, 0^\partial, 1^\partial)$ have respective residuals: $\to$ and $-$. Here, as usual, $\vee^\partial = \wedge$, $\wedge^\partial = \vee$, $0^\partial = 1$, and $1^\partial = 0$. Put in yet another way, we have that

- $(A; \vee, \wedge, \to, 0, 1)$ is a Heyting algebra,
- $(A; \vee, \wedge, -, 0, 1)$ is a dual Heyting algebra.

Double-Heyting algebras were called semi-Boolean algebras in Rauszer (1971), and Heyting algebras with dual pseudocomplementation in Sankappanavar (1985). The class of double-Heyting algebras is a finitely based variety. As an equational base one can take

0. Some finite base for bounded lattices.
1. $x \wedge ((x \to z) \wedge y) \leq z$.

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2. $y \leq x \to ((x \wedge y) \vee z)$.
3. $z \leq (y \vee (z - x)) \vee x$.
4. $(z \wedge (y \vee x)) - x \leq y$.

We will use **BiInt** for bi-intuitionistic logic, and HB for the variety of double-Heyting algebras (on occasion, **Int** will be used for the intuitionistic logic). For some algebraic results on double-Heyting algebras, we refer the reader to Sankappanavar (1985), and also to Taylor (2016). Here we only state that HB is the *equivalent quasivariety semantics* for **BiInt**, in the sense of algebraic logic.

Kripke frames for the bi-intuitionistic logic are posets. Definition of valuation is the same as for the intuitionistic case: values are up-sets. Truth definition for the standard intuitionistic connectives also remains the same. For subtraction, we have

- $w \models_V \alpha - \beta$, if $u \models_V \alpha$ and $u \not\models_V \beta$, for some $u \leq w$,

where w and u are elements of the frame, and V is a valuation. General frames can be defined, too, and a Priestley-style duality exists. Neither will be used here; we mention them only as a matter of interest.

Since **BiInt** is a conservative extension of **Int**, and Kripke frames for the two logics are the same, it might seem that standard results about **Int** would transfer smoothly to **BiInt**. Early works, such as Rauszer (1977a, 1977b, 1977c), appeared to confirm this view. On a more careful analysis, however, the picture turned out to be more complicated.

Consider cut elimination, for example. It holds in the standard sequent system for the intuitionistic logic, and so it does in the dual system for the logic with subtraction but no implication. Yet, when the two are put together, cut elimination fails, as Uustalu showed, correcting the erroneous claim to the contrary made by Rauszer (see §2.3).

Or consider Craig interpolation. It holds for the intuitionistic logic and for its dual. Moreover, in Rauszer (1977b) a form of interpolation property for **BiInt** is shown to hold, and, unfortunately, it is *called* Craig interpolation. However, it involves the consequence relation rather than the implication connective, so it is in general (much) weaker than Craig interpolation. It is precisely the property called *deductive interpolation* in Galatos, Jipsen, Kowalski, & Ono (2007) and Kihara & Ono (2010).

We will prove in §5 that the usual Craig interpolation does hold for **BiInt**, but the proof is quite involved. First, we show (see §4) that the analytic cut property holds for a whole range of sequent systems for **BiInt**, although cut elimination fails in each. Then, applying a modification of the Maehara method to a carefully chosen sequent system, we prove Craig interpolation. Striking an appropriate balance between the modification and the choice is essential: unlike in modal logics like **S5**, Craig interpolation does not follow immediately from the analytic cut property. The observant reader will notice that our arguments in §5 never make use of negation, neither in the form $\alpha \to 0$, nor $1 - \alpha$. It is because each of these does, so to speak, only one half of the job done by the classical negation, and we would need both halves together to proceed. This obstacle has to be manoeuvred around, which causes §5 to resemble—superficially—a standard proof of cut elimination.

To sum up our short invitation to **BiInt**, an analogy may be in order. The relationship between intuitionistic and bi-intuitionistic logics is like that between modal and tense logics. That is

$$\frac{\text{Bi-intuitionistic logic}}{\text{Intuitionistic logic}} = \frac{\text{Tense logic}}{\text{Modal logic}}.$$

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In particular, an analogue of Blok-Esakia isomorphism holds. For more on this we refer the reader to Wolter (1998). It may also be profitable to compare algebraic results from Kowalski (1998) and Taylor (2016).

1.1. Methods. The methods we use, although based on quite well-known ones, have not so far been developed in this direction. Even the precise relationship between the subformula property and the analytic cut property (essential for our results: see §4.1), has not been noticed. However, as the second author has recently shown, the methods can be made to work for a wide range of modal logics (see Ono (2015)), and they also seem to be applicable in a number of other nonclassical settings. The present article presents one such application; a systematic investigation of the methods by algebraic means is planned.

1.2. Background and notation. We assume some familiarity with sequent systems, as well as algebraic and Kripke semantics (at least for intuitionistic logic). For example Ono (1998) would suffice for the sequent systems part, and a few chapters of Chagrov & Zakharyaschev (1997) for the rest.

Finally, a word on notation. In the contexts of sequents we use lowercase Greek letters for formulas, uppercase Greek letters for multisets of formulas, lowercase Roman letters for sets of formulas. Otherwise the notation is rather standard and self-explanatory.

§2. Sequent systems for bi-intuitionistic logic. An intuitively natural sequent calculus for **BiInt** arises by combining Gentzen's **LJ** with its dual. Unfortunately, it is not cut-free: a counterexample, due to Uustalu, will be briefly presented at the end of this section. At least two cut-free sequent calculi for **BiInt** were devised, one by Buisman & Goré (2007), another by Pinto & Uustalu (2009). These systems, however, are rather intricate and we were not able to obtain our results using them. We will use different systems, which are better suited for our purposes.

2.1. A minimal system. The sequents, written in the usual fashion: $\Gamma \Rightarrow \Delta$, are formally pairs (Γ, Δ) of finite multisets of formulas. We adopt all the following

$$\Rightarrow 1 \qquad \alpha \Rightarrow \alpha \qquad 0 \Rightarrow$$
$$\alpha \vee \beta \Rightarrow \alpha, \beta \qquad \alpha, \beta \Rightarrow \alpha \wedge \beta$$
$$\alpha \Rightarrow \alpha - \beta, \beta \qquad \alpha, \alpha \to \beta \Rightarrow \beta$$

as initial sequents. As structural rules we take left and right weakening:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \alpha, \Delta} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma, \alpha \Rightarrow \Delta}$$

and left and right contraction:

$$\frac{\Gamma \Rightarrow \alpha, \Delta}{\Gamma \Rightarrow \alpha, \alpha, \Delta} \qquad \frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma, \alpha, \alpha \Rightarrow \Delta}.$$

The introduction rules are the following:

$$\frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma, \alpha \wedge \beta \Rightarrow \Delta} \qquad \frac{\Gamma, \beta \Rightarrow \Delta}{\Gamma, \alpha \wedge \beta \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \alpha, \Delta}{\Gamma \Rightarrow \alpha \vee \beta, \Delta} \qquad \frac{\Gamma \Rightarrow \beta, \Delta}{\Gamma \Rightarrow \alpha \vee \beta, \Delta}$$

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta} \qquad \frac{\beta \Rightarrow \Delta, \alpha}{\beta - \alpha \Rightarrow \Delta}.$$

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Finally, the cut rule:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}.$$

We will call this sequent calculus $\mathbf{LBJ}_0$, with **L** and **J** in tribute to Gentzen's original system for intuitionism, **B** standing for Bi- (or Brouwer), and the subscript 0 suggesting minimality. The nonstandardness of $\mathbf{LBJ}_0$ consists in the fact that some initial sequents introduce connectives other than constants.

2.2. A more standard system. We will also use another system, seemingly stronger than $\mathbf{LBJ}_0$. We obtain it from $\mathbf{LBJ}_0$ by replacing the nonstandard initial sequents (those in the second and third line), by the more standard introduction rules. Namely, we adopt

$$\frac{\Gamma, \alpha \Rightarrow \Delta \quad \Gamma, \beta \Rightarrow \Delta}{\Gamma, \alpha \vee \beta \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \alpha, \Delta \quad \Gamma \Rightarrow \beta, \Delta}{\Gamma \Rightarrow \alpha \wedge \beta, \Delta}$$

$$\frac{\Gamma \Rightarrow \alpha, \Delta \quad \Sigma, \beta \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \alpha - \beta, \Delta, \Pi} \qquad \frac{\Gamma \Rightarrow \alpha, \Delta \quad \Sigma, \beta \Rightarrow \Pi}{\Gamma, \Sigma, \alpha \to \beta \Rightarrow \Delta, \Pi}.$$

We will call this system $\mathbf{LBJ}_1$. Since the sequents

$$\alpha \vee \beta \Rightarrow \alpha, \beta \qquad \alpha, \beta \Rightarrow \alpha \wedge \beta \qquad \alpha \Rightarrow \alpha - \beta, \beta \qquad \alpha, \alpha \to \beta \Rightarrow \beta$$

are provable $\mathbf{LBJ}_1$, we have that $\mathbf{LBJ}_1$ is stronger than $\mathbf{LBJ}_0$. On the other hand, in the presence of cut the four rules above are derivable in $\mathbf{LBJ}_0$. For example, the rule introducing $\to$ on the left is derivable as follows

$$\dfrac{\dfrac{\Gamma \Rightarrow \alpha, \Delta \quad \alpha, \alpha \to \beta \Rightarrow \beta}{\Gamma, \alpha \to \beta \Rightarrow \Delta, \beta} \qquad \Sigma, \beta \Rightarrow \Pi}{\Gamma, \Sigma, \alpha \to \beta \Rightarrow \Delta, \Pi}$$

and the rule introducing $\vee$ on the left is derived by

$$\dfrac{\dfrac{\alpha \vee \beta \Rightarrow \alpha, \beta \quad \Gamma, \alpha \Rightarrow \Delta}{\Gamma, \alpha \vee \beta \Rightarrow \Delta, \beta} \qquad \Gamma, \beta \Rightarrow \Delta}{\begin{array}{c}\Gamma, \Gamma, \alpha \vee \beta \Rightarrow \Delta, \Delta \\ \vdots \\ \Gamma, \alpha \vee \beta \Rightarrow \Delta\end{array}}.$$

Similar proofs derive the two other rules. Thus, in fact $\mathbf{LBJ}_0$ and $\mathbf{LBJ}_1$ are equivalent as proof systems.

LEMMA 2.1. *Let $\mathcal{S}$ be a sequent calculus in the language of bi-intuitionistic logic. If all initial sequents of $\mathbf{LBJ}_0$ are provable in $\mathcal{S}$ and all rules of $\mathcal{S}$ are instances of the rules of $\mathbf{LBJ}_1$, then $\mathcal{S}$ is an adequate proof system for bi-intuitionistic logic.*

Proof (Sketch). We use algebraic completeness of bi-intuitionistic logic with respect to the variety of double-Heyting algebras, that is, the fact that a formula φ is a theorem of **BiInt** if and only if the equation $1 \approx \varphi$ holds in HB. Since all rules of $\mathbf{LBJ}_1$ are derivable in $\mathbf{LBJ}_0$, so are all the rules of $\mathcal{S}$. Thus, to prove the lemma, it suffices to show adequacy of $\mathbf{LBJ}_0$. We will prove it by a variant of the standard Lindenbaum algebra argument. To begin with, we define a relation $\equiv$ between formulas, putting $\alpha \equiv \beta$ if the sequents $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$ are both provable in $\mathbf{LBJ}_0$. This relation is clearly reflexive and symmetric, and by cut it is also transitive, so it is an equivalence relation. We proceed to show that it is a congruence. Suppose $\alpha \equiv \beta$ and $\gamma \equiv \delta$; we wish to show that $\alpha \star \gamma \equiv \beta \star \delta$,

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for $\star \in \{\wedge, \vee, \to, -\}$. We will present the proof for $\wedge$ and $\to$, the other two being dual. The following proof

$$\dfrac{\dfrac{\dfrac{\gamma \Rightarrow \delta \qquad \dfrac{\alpha \Rightarrow \beta \quad \beta, \delta \Rightarrow \beta \wedge \delta}{\alpha, \delta \Rightarrow \beta \wedge \delta}}{\alpha, \gamma \Rightarrow \beta \wedge \delta}}{\alpha \wedge \gamma, \gamma \Rightarrow \beta \wedge \delta}}{\alpha \wedge \gamma, \alpha \wedge \gamma \Rightarrow \beta \wedge \delta}}{\alpha \wedge \gamma \Rightarrow \beta \wedge \delta}$$

shows one half of the desired property for $\wedge$; the other half follows by swapping α with β and γ with δ. The next proof

$$\dfrac{\dfrac{\alpha \Rightarrow \beta \quad \beta, \beta \to \delta \Rightarrow \delta}{\alpha, \beta \to \delta \Rightarrow \delta} \qquad \delta \Rightarrow \gamma}{\dfrac{\alpha, \beta \to \delta \Rightarrow \gamma}{\beta \to \delta \Rightarrow \alpha \to \gamma}}$$

shows one half of the desired property for $\to$; the other half follows as before by symmetry.

It follows that $\equiv$ is a congruence on the algebra $\mathbf{F}$ of formulas. To show that $\mathbf{F}/\equiv$ is a double-Heyting algebra it suffices to prove that (a) implication is the residual of meet, and (b) subtraction is the co-residual of join. To prove (a) and (b), it suffices in turn to derive the following four sequents:

1. $\alpha \wedge ((\alpha \to \gamma) \wedge \beta) \Rightarrow \gamma$,
2. $\beta \Rightarrow \alpha \to ((\alpha \wedge \beta) \vee \gamma)$,
3. $\gamma \Rightarrow (\beta \vee (\gamma - \alpha)) \vee \alpha$,
4. $(\gamma \wedge (\beta \vee \alpha)) - \alpha \Rightarrow \beta$,

which correspond to the inequalities (1)–(4) from the equational base for HB mentioned in the introduction. Here are the proofs:

$$\dfrac{\dfrac{\alpha, \alpha \to \gamma \Rightarrow \gamma}{\alpha, (\alpha \to \gamma) \wedge \beta \Rightarrow \gamma}\ (\wedge \text{ introduction on left})}{\alpha \wedge ((\alpha \to \gamma) \wedge \beta) \Rightarrow \gamma}\ (\text{two implicit contractions on left})$$

$$\dfrac{\dfrac{\alpha, \beta \Rightarrow \alpha \wedge \beta}{\alpha, \beta \Rightarrow (\alpha \wedge \beta) \vee \gamma}\ (\vee \text{ introduction on right})}{\beta \Rightarrow \alpha \to ((\alpha \wedge \beta) \vee \gamma)}\ (\to \text{ introduction on right})$$

$$\dfrac{\dfrac{\gamma \Rightarrow \gamma - \alpha, \alpha}{\gamma \Rightarrow \beta \vee (\gamma - \alpha), \alpha}\ (\vee \text{ introduction on right})}{\gamma \Rightarrow (\beta \vee (\gamma - \alpha)) \vee \alpha}\ (\text{two implicit contractions on right})$$

$$\dfrac{\dfrac{\beta \vee \alpha \Rightarrow \beta, \alpha}{\gamma \wedge (\beta \vee \alpha) \Rightarrow \beta, \alpha}\ (\wedge \text{ introduction on left})}{(\gamma \wedge (\beta \vee \alpha)) - \alpha \Rightarrow \beta}\ (- \text{ introduction on left})\ .$$

We define satisfaction of a sequent $\Gamma \Rightarrow \Delta$ in a double-Heyting algebra $\mathbf{A}$ by a valuation V in the usual way, that is, $\mathbf{A} \models_v \Gamma \Rightarrow \Delta$, if $\bigwedge\{V(\gamma) : \gamma \in \Gamma\} \leq \bigvee\{V(\delta) : \delta \in \Delta\}$ holds in $\mathbf{A}$ (with $\bigwedge \emptyset = 1$ and $\bigvee \emptyset = 0$). Again as usual, $\Gamma \Rightarrow \Delta$ is said to be true in $\mathbf{A}$ if it is satisfied in $\mathbf{A}$ by every valuation v. It is readily verified that all initial sequents are true in any double-Heyting algebra $\mathbf{A}$, and truth is preserved by all rules. It follows that for every

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sequent $\Rightarrow \varphi$ provable in $\mathbf{LBJ}_0$, the equation $1 \approx \varphi$ is true in all double-Heyting algebras. This shows soundness.

Next, let φ be a formula such that $\Rightarrow \varphi$ is not provable in $\mathbf{LBJ}_0$. Then, the equation $1 \approx \varphi$ fails in $\mathbf{F}/\equiv$, showing completeness. By algebraic completeness of **BiInt**, we have that $\Rightarrow \varphi$ is provable in $\mathbf{LBJ}_0$ if and only if φ is a theorem of **BiInt**.

A fully detailed proof based on the present sketch is not difficult to obtain. □

2.3. Counterexample to cut elimination. Observe that sequent systems satisfying the conditions of Lemma 2.1 include some calculi which, intuitively, would be deemed pathological. For example, a sequent calculus whose initial sequents are *all* sequents provable in $\mathbf{LBJ}_0$ and rules are the rules of $\mathbf{LBJ}_1$. This calculus enjoys cut elimination, but it is completely useless as a proof system. We wish to exclude such pathologies, while at the same time retaining a reasonable degree of flexibility.

DEFINITION 2.2. *Let $\mathcal{S}$ be a sequent calculus in the language of bi-intuitionistic logic. We will say that $\mathcal{S}$ lies* between $\mathbf{LBJ}_0$ and $\mathbf{LBJ}_1$*, if the following hold*:

(i) *The rules of $\mathbf{LBJ}_0$ are instances of the rules of $\mathcal{S}$, and the rules of $\mathcal{S}$ are instances of the rules of $\mathbf{LBJ}_1$.*

(ii) *The initial sequents of $\mathbf{LBJ}_0$ are cut-free provable in $\mathcal{S}$, and the initial sequents of $\mathcal{S}$ are cut-free provable in $\mathbf{LBJ}_1$.*

In §5, we will use one such system, which we dub $\mathbf{LBJ}_{0.9}$ to indicate that it is in a sense closer to $\mathbf{LBJ}_1$ than to $\mathbf{LBJ}_0$. Namely, the introduction rules for implication on the left and subtraction on the right, from $\mathbf{LBJ}_1$, are restricted as below:

$$\frac{\Gamma \Rightarrow \alpha \quad \Sigma, \beta \Rightarrow \Pi}{\Gamma, \Sigma, \alpha \to \beta \Rightarrow \Pi} \qquad \frac{\Gamma \Rightarrow \alpha, \Delta \quad \beta \Rightarrow \Pi}{\Gamma \Rightarrow \alpha - \beta, \Delta, \Pi}.$$

It is easy to see that all initial sequents of $\mathbf{LBJ}_0$ are provable in $\mathbf{LBJ}_{0.9}$, so by Lemma 2.1, it is an adequate proof system for bi-intuitionistic logic.

An example of a sequent system which is adequate for **BiInt** but does not lie between $\mathbf{LBJ}_0$ and $\mathbf{LBJ}_1$ in our sense is the calculus used by Rauszer (1974). Her calculus is a restriction of $\mathbf{LBJ}_1$ to sequents $\Delta \Rightarrow \Gamma$, where at least one of Δ, Γ is a singleton. Since not all rules of $\mathbf{LBJ}_0$ are of this form, Rauszer's calculus does not fall under Definition 2.2.

Uustalu's counterexample, mentioned at the beginning of the present section, was originally devised to show that Rauszer's calculus does not enjoy cut elimination, contrary to the claim made in Rauszer (1974). It also works for all sequent calculi between $\mathbf{LBJ}_0$ and $\mathbf{LBJ}_1$.

THEOREM 2.3. *Let $\mathcal{S}$ be a sequent calculus between $\mathbf{LBJ}_0$ and $\mathbf{LBJ}_1$. Then, the sequent $p \Rightarrow q, r \to ((p - q) \wedge r)$, where p, q, r are variables, is provable in $\mathcal{S}$, but it is not provable in $\mathcal{S}$ without cut.*

Proof (Sketch). We leave it for the reader to prove that the sequent above (we will call it *Uustalu's sequent*) is in fact provable in $\mathbf{LBJ}_1$ (or in $\mathbf{LBJ}_0$) with cut.

Now, observe that any proof P in the calculus $\mathcal{S}$ can be transformed into a proof P' in $\mathbf{LBJ}_1$ by replacing initial sequents by their proofs where necessary, and in such a way that P is a cut-free proof if and only if P' is. Thus, it suffices to show that the sequent $p \Rightarrow q, r \to ((p - q) \wedge r)$ cannot be proved in $\mathbf{LBJ}_1$ without cut. To get a contradiction assume that there is a cut-free proof P of Uustalu's sequent in $\mathbf{LBJ}_1$. Since the right-hand side of this sequent is not a singleton, the last rule of P cannot be implication introduction,

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and so it can only be weakening or contraction (possibly preceded by another weakening or contraction). We invite the reader to derive a contradiction from this setup. □

§3. Disjointness and saturation. In this section we exploit and extend some ideas of Takano, scattered across several of his publications, for example in Takano (1992) and Takano (2001). A similar approach seems to have been used by Lahav & Avron (2013), however the presentation there is rather convoluted.

Let us give a quick preview of the method. We want to construct Kripke frames for **BiInt** out of sets of formulas. To this end, as usual, we need maximally consistent sets of formulas. In this section we define them, using the analogy with prime filter-ideal pairs, well known in the theory of distributive lattices. To incorporate sets of formulas smoothly in the sequent system framework, we observe that every set is a multiset (without repetitions), so having sets of formulas in sequents is harmless. As the reader will recall, we use lowercase Roman letters for finite sets of formulas; with this device we keep them notionally separated from "genuine" multisets.

The sequent system in this and the next section can be any system between $\mathbf{LBJ}_0$ and $\mathbf{LBJ}_1$. Let $\mathbf{s}$ be an arbitrary but fixed finite set of formulas closed under subformulas (we use boldface to distinguish it from other finite sets of formulas). We say that a sequent $\Gamma \Rightarrow \Delta$ is $\mathbf{s}$-provable, if it is provable by means of a proof P such that every formula occurring in P belongs to $\mathbf{s}$. Writing $s(\Gamma)$ and $s(\Delta)$ for the sets of formulas in Γ and Δ, by obvious applications of weakening and contraction we have that $\Gamma \Rightarrow \Delta$ is $\mathbf{s}$-provable if and only if $s(\Gamma) \Rightarrow s(\Delta)$ is $\mathbf{s}$-provable.

Define a pair (a, b) of subsets of $\mathbf{s}$ to be $\mathbf{s}$-*disjoint*, if the sequent $a \Rightarrow b$ is not $\mathbf{s}$-provable. We say that a $\mathbf{s}$-disjoint pair $(a, b) \in \mathbf{s} \times \mathbf{s}$ is $\mathbf{s}$-*saturated* if it is maximally $\mathbf{s}$-disjoint, that is

1. $a, \gamma \Rightarrow b$ is $\mathbf{s}$-provable for every $\gamma \in \mathbf{s} \setminus a$,
2. $a \Rightarrow \gamma, b$ is $\mathbf{s}$-provable for every $\gamma \in \mathbf{s} \setminus b$.

REMARK 3.1. *We could also accommodate Rauszer's calculus in our argument, at the cost of certain modifications of the definitions above. Namely, we would need to define* (a, b) *to be* $\mathbf{s}$*-disjoint, if the sequent* $\bigwedge a \Rightarrow \bigvee b$ *is not* $\mathbf{s}$*-provable, and modify the definition of* $\mathbf{s}$*-saturation accordingly. We will not pursue this option, but the reader is invited to verify that all our proofs carry over with no essential complications.*

LEMMA 3.2. *For any* $a, b \subseteq \mathbf{s}$*, if* (a, b) *is a* $\mathbf{s}$*-disjoint pair, then there exists a* $\mathbf{s}$*-saturated pair* (a', b') *such that* $a \subseteq a'$ *and* $b \subseteq b'$.

Proof. Assume (a, b) is $\mathbf{s}$-disjoint. We claim that if (a, b) is not $\mathbf{s}$-saturated, then there is a formula $\gamma \in \mathbf{s}$ such that at least one of $(\{\gamma\} \cup a, b)$ and $(a, b \cup \{\gamma\})$ is $\mathbf{s}$-disjoint. Suppose towards a contradiction that neither of them is $\mathbf{s}$-disjoint. Then, the sequents $a \Rightarrow b, \gamma$ and $\gamma, a \Rightarrow b$ are both $\mathbf{s}$-provable, and so, applying cut and contraction as below

$$\dfrac{\dfrac{a \Rightarrow b, \gamma \quad \gamma, a \Rightarrow b}{a, a \Rightarrow b, b}}{a \Rightarrow b},$$

we obtain a proof of $a \Rightarrow b$ in which only formulas from $\mathbf{s}$ occur. This contradicts the assumption of $\mathbf{s}$-disjointness.

Repeating the procedure sufficiently many times, by finiteness of $\mathbf{s}$, we obtain a required $\mathbf{s}$-saturated pair. □

LEMMA 3.3. *Let* (a, b) *be* $\mathbf{s}$-saturated. *Then* $a = \mathbf{s} \setminus b$ *and* $b = \mathbf{s} \setminus a$.

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Proof. Suppose some $\gamma \in \mathbf{s}$ has $\gamma \notin a$ and $\gamma \notin b$. Since (a, b) is **s**-saturated, we have that the sequents $\gamma, a \Rightarrow b$ and $a \Rightarrow \gamma, b$ are both **s**-provable. By cut, $a \Rightarrow b$ is provable, contradicting **s**-saturation. □

We can now apply the notion of being **s**-saturated to subsets of **s**. Namely, we say $a \subseteq \mathbf{s}$ is **s**-saturated if the pair $(a, \mathbf{s} \setminus a)$ is. By Lemma 3.3, this establishes a one-to-one correspondence between **s**-saturated pairs and **s**-saturated sets. To make notation less cumbersome, we will write $\overline{a}$ for $\mathbf{s} \setminus a$.

LEMMA 3.4. *Let a be a* **s***-saturated set, and $\alpha, \beta \in \mathbf{s}$. Whenever the formula on the left belongs to* **s***, the appropriate equivalence below holds.*

1. $\alpha \wedge \beta \in a$ *if and only if* $\alpha \in a$ *and* $\beta \in a$.
2. $\alpha \vee \beta \in a$ *if and only if* $\alpha \in a$ *or* $\beta \in a$.
3. $\alpha \to \beta \in a$ *if and only if* $\alpha \in b$ *implies* $\beta \in b$ *for all* **s***-saturated* $b \supseteq a$.
4. $\alpha - \beta \in a$ *if and only if* $\alpha \in b$ *and* $\beta \notin b$ *for some* **s***-saturated* $b \subseteq a$.

Proof. For the forward direction of (1) assume that $\alpha \wedge \beta \in a$. If $\alpha \notin a$, then by saturation the sequent $\alpha, a \Rightarrow \overline{a}$ is **s**-provable. Then, $\alpha \wedge \beta, a \Rightarrow \overline{a}$ is **s**-provable as well. Since $\alpha \wedge \beta \in a$, we get, by contraction, that $a \Rightarrow \overline{a}$ is **s**-provable, contradicting the assumption.

For the backward direction of (1) assume $\alpha \in a$ and $\beta \in a$. If $\alpha \wedge \beta \notin a$, then $\alpha \wedge \beta, a \Rightarrow \overline{a}$ is **s**-provable. Since $\alpha, \beta \Rightarrow \alpha \wedge \beta$ is **s**-provable, by cut we get that $\alpha, \beta, a \Rightarrow \overline{a}$ is **s**-provable, and by contraction again, $a \Rightarrow \overline{a}$ is **s**-provable, contradicting the assumption.

The proof of (2) is similar.

For the forward direction of (3) assume $\alpha \to \beta \in a$, but for some **s**-saturated $b \supseteq a$ we have $\alpha \in b$ and $\beta \notin b$. It follows that $\alpha \in b$, $\alpha \to \beta \in b$, and $\beta \in \overline{b}$. Since $\alpha, \alpha \to \beta \Rightarrow \beta$ is **s**-provable, we get (by repeated weakening) that $b \Rightarrow \overline{b}$ is **s**-provable, contradicting the assumption.

For the backward direction of (3) assume that for all **s**-saturated $b \supseteq a$ we have $\alpha \in b$ implies $\beta \in b$, but $\alpha \to \beta \notin a$, that is $\alpha \to \beta \in \overline{a}$. Now, observe that $(a \cup \{\alpha\}, \{\beta\})$ is a **s**-disjoint pair, for otherwise $a \Rightarrow \alpha \to \beta$ would be **s**-provable, and thus so would be $a \Rightarrow \overline{a}$, contradicting saturation (the case with $\alpha \in a$ is subsumed here). By Lemma 3.2 then, there is a **s**-saturated $b \supseteq a$, with $a \cup \{\alpha\} \subseteq b$ and $\beta \in \overline{b}$. This, however, contradicts the assumption.

For the forward direction of (4) assume $\alpha - \beta \in a$, but for all **s**-saturated $b \subseteq a$ we have $\alpha \in b$ implies $\beta \in b$. It follows that $\alpha - \beta \Rightarrow \overline{a}$ is not **s**-provable, and therefore $\alpha \Rightarrow \overline{a}, \beta$ is not **s**-provable either. Thus, $(\{\alpha\}, \overline{a} \cup \{\beta\})$ is a **s**-disjoint pair, so by Lemmas 3.2 and 3.3, there is a **s**-saturated b such that $\overline{a} \cup \{\beta\} \subseteq \overline{b}$ and $\alpha \in b$. Moreover, since $\overline{a} \subseteq \overline{b}$, we have $b \subseteq a$, so applying the assumption we obtain that $\beta \in b$. But by construction $\beta \in \overline{b}$, which is a contradiction.

Finally, for the backward direction of (4) assume that for some **s**-saturated $b \subseteq a$ we have $\alpha \in b$ and $\beta \notin b$, but $\alpha - \beta \notin a$. Thus, $\alpha - \beta \notin b$. As $\alpha \Rightarrow \alpha - \beta, \beta$ is **s**-provable, we obtain that $b \Rightarrow \overline{b}$ is **s**-provable, contradicting **s**-saturation of b. □

§4. Subformula property and analytic cut. For a formula φ we will write $\mathrm{Sfm}(\varphi)$ to denote the set of all subformulas of φ. This notation naturally extends to multisets of formulas and to sequents (i.e., pairs of multisets of formulas), so $\mathrm{Sfm}(\Gamma) = \bigcup\{\mathrm{Sfm}(\gamma) : \gamma \in \Gamma\}$ and $\mathrm{Sfm}(\Gamma, \Delta) = \bigcup\{\mathrm{Sfm}(\varphi) : \varphi \in \Gamma \cup \Delta\}$, where Γ and Δ are multisets of formulas.

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Let us fix a sequent $\Gamma \Rightarrow \Delta$, and let $\mathbf{s} = \mathrm{Sfm}(\Gamma, \Delta)$. We will show that if $\Gamma \Rightarrow \Delta$ is not $\mathbf{s}$-provable, then it fails on some Kripke frame, and so it is not provable in bi-intuitionistic logic.

We define $\mathcal{F} = (W, R)$, taking W to be the set of all $\mathbf{s}$-saturated subsets of $\mathbf{s}$, and R to be the inclusion relation. Clearly, $\mathcal{F}$ is a (finite) Kripke frame for bi-intuitionistic logic. Next we define a valuation V putting $V(p) = \{a \in W : p \in a\}$. It is immediate that V satisfies the hereditary condition for variables. We will hold V fixed throughout, so abusing notation slightly we will write $a \models \alpha$ (instead of the usual $a \models_V \alpha$) for $a \in V(\alpha)$.

LEMMA 4.1. *Let $\alpha \in \mathbf{s}$, and $a \in W$. Then $a \models \alpha$ if and only if $\alpha \in a$.*

Proof. Follows from Lemma 3.4. □

Consider the sequent $\Gamma \Rightarrow \Delta$. By Lemmas 3.2 and 3.3, there is a $\mathbf{s}$-saturated set a, such that $s(\Gamma) \subseteq a$ and $s(\Delta) \subseteq \overline{a}$, where $s(\Gamma)$ and $s(\Delta)$ again stand for the sets of formulas in Γ and Δ. By Lemma 4.1, we have that $a \models \gamma$ for every $\gamma \in \Gamma$, and likewise $a \not\models \delta$ for every $\delta \in \Delta$. So, the valuation V falsifies $s(\Gamma) \Rightarrow s(\Delta)$ on a, and therefore it falsifies $\Gamma \Rightarrow \Delta$ on a. The result below follows.

LEMMA 4.2. *If a sequent $\Gamma \Rightarrow \Delta$ has a proof P in bi-intuitionistic logic, it also has a proof Q in which only the subformulas of the formulas in $\Gamma \cup \Delta$ occur.*

The property stated in Lemma 4.2 is known as the *subformula property*. Observe that Lemma 4.2 holds for any sequent system between $\mathbf{LBJ}_0$ and $\mathbf{LBJ}_1$. The next result is an immediate consequence.

THEOREM 4.3. *Let $\mathcal{S}$ be any sequent system between $\mathbf{LBJ}_0$ and $\mathbf{LBJ}_1$. Then, $\mathcal{S}$ enjoys the subformula property.*

The subformula property for $\mathbf{LBJ}_1$ is also shown in Lahav & Avron (2013, Example 5.51). Observe that the subformula property just established is a global property: it is exhibited (or not) by a proof as a whole. A local version of subformula property, which will be important in the next subsection, applies to single rules instead. Namely, we say that a rule R has subformula property if every formula occurring in the upper sequent(s) of R is a subformula of a formula occurring in the lower sequent.

Observe also that we did not assume anything about Kripke semantics for **BiInt**, except that the appropriate Kripke frames are posets. In particular, we did not assume their finiteness; instead, we obtained it as a by-product. Thus, we get a purely proof-theoretic proof of the following well-known fact.

COROLLARY 4.4. *Bi-intuitionistic logic has the finite model property and is decidable.*

4.1. Analytic cut. As a consequence of the global nature of the subformula property it is possible that a sequent $\Gamma \Rightarrow \Delta$ has a proof P with subformula property, but some subproof Q of P, with the end sequent $\Sigma \Rightarrow \Pi$, is such that not every formula occurring in Q belongs to $\mathrm{Sfm}(\Sigma, \Pi)$. One such situation occurs if the rule above $\Sigma \Rightarrow \Pi$ is cut; namely, we can have

$$\frac{\Sigma_1 \Rightarrow \Pi_1, \alpha \quad \alpha, \Sigma_2 \Rightarrow \Pi_2}{\Sigma \Rightarrow \Pi},$$

where $\Sigma = \Sigma_1, \Sigma_2$ and $\Pi = \Pi_1, \Pi_2$, and $\alpha \notin \mathrm{Sfm}(\Sigma, \Pi)$. Notice that by subformula property $\alpha \in \mathrm{Sfm}(\Gamma, \Delta)$. By inspection of the rules it is easy to see that in systems $\mathbf{LBJ}_0$, $\mathbf{LBJ}_{0.9}$ and $\mathbf{LBJ}_1$, all rules except cut have the local subformula property, so for these

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systems the example given above is the only possible example. Before we proceed further, we need a definition.

DEFINITION 4.5. *An application of cut with the* (*local*) *subformula property is called* analytic. *If every application of cut in a proof P is analytic, we say that P has* analytic cuts. *A sequent system $\mathcal{S}$ has* analytic cut property *if every sequent provable in $\mathcal{S}$ has a proof with analytic cuts.*

Let $\mathcal{S}$ be a sequent system in which all rules except cut have the (local) subformula property. Then, the analytic cut property for $\mathcal{S}$ implies the (global) subformula property for $\mathcal{S}$, of course. We will now show that the converse holds as well.

LEMMA 4.6. *Let $\mathcal{S}$ be a sequent system in which the only rule without subformula property is cut. If $\mathcal{S}$ has subformula property, then $\mathcal{S}$ has analytic cut property.*

Proof. Assume that $\mathcal{S}$ has subformula property. Let $\Gamma \Rightarrow \Delta$ be a sequent provable in $\mathcal{S}$. We will prove by induction on the cardinality of $\mathrm{Sfm}(\Gamma, \Delta)$ that $\Gamma \Rightarrow \Delta$ has a proof with analytic cuts. The base case, with $|\mathrm{Sfm}(\Gamma, \Delta)| = 0$ holds vacuously. Assume $|\mathrm{Sfm}(\Gamma, \Delta)| = k$ and the claim holds for any provable sequent $\Sigma \Rightarrow \Pi$ with $|\mathrm{Sfm}(\Sigma, \Pi)| < k$. Consider a proof P of $\Gamma \Rightarrow \Delta$. Since $\mathcal{S}$ has the subformula property, we can assume every formula occurring in P belongs to $\mathrm{Sfm}(\Gamma, \Delta)$. Now, let

$$\frac{\Sigma_1 \Rightarrow \Pi_1, \alpha \quad \alpha, \Sigma_2 \Rightarrow \Pi_2}{\Sigma \Rightarrow \Pi},$$

with $\Sigma = \Sigma_1, \Sigma_2$ and $\Pi = \Pi_1, \Pi_2$, be an occurrence of non-analytic cut in P. Without loss of generality we can assume that this occurrence is of maximal depth in P, that is, such that below it there are no occurrences of non-analytic cut. Let P' be the subproof of P above $\Sigma \Rightarrow \Pi$. Since the cut is not analytic, we have $\alpha \notin \mathrm{Sfm}(\Sigma, \Pi)$. But, by subformula property of P, we have $\alpha \in \mathrm{Sfm}(\Gamma, \Delta)$, so $\alpha \in \mathrm{Sfm}(\Gamma, \Delta) \setminus \mathrm{Sfm}(\Sigma, \Pi)$. Therefore, $k = |\mathrm{Sfm}(\Gamma, \Delta)| > |\mathrm{Sfm}(\Sigma, \Pi)|$, so the inductive hypothesis applies to the provable sequent $\Sigma \Rightarrow \Pi$. Thus, $\Sigma \Rightarrow \Pi$ has a proof Q with analytic cuts. Replace P', as a subproof of P, by Q.

Let P_r be the result of such replacements for all maximal depth occurrences of non-analytic cut in P. Then, P_r is a proof of $\Gamma \Rightarrow \Delta$ with analytic cuts. □

The main result of the section follows immediately.

THEOREM 4.7. *Let $\mathcal{S}$ be a sequent system satisfying the conditions of Lemma 2.1. Then, $\mathcal{S}$ has the analytic cut property. In particular, each of* $\mathbf{LBJ}_0$, $\mathbf{LBJ}_{0.9}$, $\mathbf{LBJ}_1$ *has the analytic cut property.*

§5. Interpolation by Maehara method. This section employs an idea due to Maehara, who in Maehara (1960) devised a syntactic method of proving Craig's interpolation theorem. The method worked very well for classical and intuitionistic logics, and it was later adapted to many other logics, mainly modal and substructural (see Ono (1998) for a survey). Here we will adapt it to **BiInt**, but, as the reader will see, this might be close to the limit of its applicability. In this section, our official system is $\mathbf{LBJ}_{0.9}$.

With each proof P, we associate its *rank*: a natural number $r(P)$ defined inductively by $r(P) = 0$ if P consists of a single initial sequent, $r(P) = c + 1$ if the last rule of P is a one-premiss rule and c is the rank of the proof above it, $r(P) = c + d + 2$ if the last rule of P is a two-premiss rule and c, d are the ranks of the respective proofs above it.

A remark on the general structure of the argument is now in order. In standard cases, the application of Maehara method proceeds by fixing a cut-free proof P (or a proof P with

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analytic cut property) and showing the existence of an interpolant for each sequent in P by using induction on the depth of that sequent in P.

Our argument proceeds by induction on the rank of proof instead. Namely, we consider all proofs with analytic cut property, and show the existence of interpolant of each sequent with a proof of rank at most m by induction on m. This allows us to access the inductive hypothesis by considering a slightly different proof with the same end sequent but with a smaller rank.

5.1. Partitions of sequents and interpolants. For a sequent $\Gamma \Rightarrow \Delta$, we define its *partition* to be a pair $\langle(\Gamma_1, \Delta_1), (\Gamma_2, \Delta_2)\rangle$, such that $\Gamma_1, \Gamma_2 = \Gamma$ and $\Delta_1, \Delta_2 = \Delta$. As usual, we will write partitions in the form $(\Gamma_1 : \Delta_1), (\Gamma_2 : \Delta_2)$.

A partition will be called *normal* if $\Gamma_1 = \Gamma$ or $\Delta_2 = \Delta$. Thus a normal partition is either $(\Gamma : \Delta_1), (\emptyset : \Delta_2)$ or $(\Gamma_1 : \emptyset), (\Gamma_2 : \Delta)$. Notice that the *degenerate* cases $(\emptyset : \emptyset), (\Gamma : \Delta)$ and $(\Gamma : \Delta), (\emptyset : \emptyset)$ are normal.

DEFINITION 5.1. *Let $\Gamma \Rightarrow \Delta$ be a provable sequent, and $(\Gamma_1 : \Delta_1), (\Gamma_2 : \Delta_2)$ be a partition of $\Gamma \Rightarrow \Delta$. A formula φ, such that*

- *the sequents $\Gamma_1 \Rightarrow \Delta_1, \varphi$ and $\varphi, \Gamma_2 \Rightarrow \Delta_2$ are provable,*
- $\mathrm{Var}(\varphi) \subseteq \mathrm{Var}(\Gamma_1, \Delta_1) \cap \mathrm{Var}(\Gamma_2, \Delta_2)$,

is called an interpolant for $(\Gamma_1 : \Delta_1), (\Gamma_2 : \Delta_2)$.

LEMMA 5.2. *Let $\Gamma \Rightarrow \Delta$ be a sequent provable in $\mathbf{LBJ}_{0.9}$ by a proof of rank at most 2. If $(\Gamma_1 : \Delta_1), (\Gamma_2 : \Delta_2)$ is a normal partition of $\Gamma \Rightarrow \Delta$, then there exist an interpolant for it.*

Proof. Observe that for degenerate partitions of any sequent, either 0 or 1 is an interpolant, so it suffices to consider nondegenerate ones. For proofs of rank 0, that is, for initial sequents of $\mathbf{LBJ}_{0.9}$, the claim is easily verified. For proofs of rank 1 we have 10 cases, corresponding to all one-premiss rules, to consider, but they are all straightforward. Proofs of rank 2 fall into two general types. The first type consists in two applications of one-premiss rules: these are straightforward, too. The second type consists in a single application of a two-premiss rule. This implies that both upper sequents must be initial. Here are all the possibilities:

$$\frac{0 \Rightarrow \quad 0 \Rightarrow}{0 \vee 0 \Rightarrow} \qquad \frac{\alpha \Rightarrow \alpha \quad \alpha \Rightarrow \alpha}{\alpha \vee \alpha \Rightarrow \alpha}$$

$$\frac{\alpha \Rightarrow \alpha \quad \alpha \Rightarrow \alpha}{\alpha \Rightarrow \alpha \wedge \alpha} \qquad \frac{\Rightarrow 1 \quad \Rightarrow 1}{\Rightarrow 1 \wedge 1}$$

$$\frac{0 \Rightarrow 0 \quad 0 \Rightarrow}{0 \Rightarrow} \qquad \frac{\alpha \Rightarrow \alpha \quad \alpha \Rightarrow \alpha}{\alpha \Rightarrow \alpha} \qquad \frac{\Rightarrow 1 \quad 1 \Rightarrow 1}{\Rightarrow 1}$$

$$\frac{\Rightarrow 1 \quad 0 \Rightarrow}{\Rightarrow 1 - 0} \qquad \frac{\alpha \Rightarrow \alpha \quad 0 \Rightarrow}{\alpha \Rightarrow \alpha - 0} \qquad \frac{\Rightarrow 1 \quad \beta \Rightarrow \beta}{\Rightarrow 1 - \beta, \beta} \qquad \frac{\alpha \Rightarrow \alpha \quad \beta \Rightarrow \beta}{\alpha \Rightarrow \alpha - \beta, \beta}$$

$$\frac{\alpha \Rightarrow \alpha \quad \beta \Rightarrow \beta}{\alpha, \alpha \to \beta \Rightarrow \beta} \qquad \frac{\alpha \Rightarrow \alpha \quad 0 \Rightarrow}{\alpha, \alpha \to 0 \Rightarrow} \qquad \frac{\Rightarrow 1 \quad \beta \Rightarrow \beta}{1 \to \beta \Rightarrow \beta} \qquad \frac{\Rightarrow 1 \quad 0 \Rightarrow}{1 \to 0 \Rightarrow}.$$

In each of these cases, it is a tedious but easy exercise to show that all normal partitions have interpolants. □

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Before we extend this result to proofs of arbitrary rank, we introduce a technicality. Let R be a rule. We will say that interpolation *transfers across* R if from the assumption that an interpolant exists for each normal partition of the upper sequent(s) of R it follows that an interpolant exists for each normal partition of the lower sequent of R.

LEMMA 5.3. *Interpolation transfers across applications of weakening and contraction.*

Proof. Straightforward. □

To present the rest of this section in a reasonably streamlined way, we now introduce a technicality, which will be familiar (in some form) to any reader acquainted with standard cut elimination proofs for classical and intuitionistic logic. Namely, we introduce a generalized version of cut that incorporates contraction. This is the *weak mix* rule below

$$\frac{\Gamma \Rightarrow \Delta, \alpha^+ \quad \alpha^+, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi},$$

where α^+ indicates one or more occurrences of α. Notice that this differs from the usual mix, where *all* occurrences of the mix formula are deleted. Here, there still may be some occurrences of α in Σ or Δ. In the proofs that follow, we will also use α^*, to indicate *zero or more* occurrences of α (our use of $+$ and $*$ is consistent with and indeed motivated by the use of these symbols in regular expressions).

Notice that there may be some occurrences of α in Γ, Σ, Δ, or Π; hence the name *weak* mix. However, we will refer to this rule simply as *mix* from now on. The *degree* of an application of mix is the complexity of the formula α, measured by the number of occurrences of connectives.

It is easy to see that the usual cut is a special form of mix, and conversely every application of mix can be replaced by an application of cut and some contractions. Therefore, the system $\mathbf{LBJ}_{0.9}$ is equivalent to its version with mix replacing cut; in particular, the analytic cut property holds for it. We will not distinguish terminologically between the two versions.

Before we state the main lemma of this section, one more technicality. Observe that if the mix formula α occurs in either Γ or Π, we can eliminate this application of mix altogether from the proof, replacing it by a number of weakenings. Therefore, we will restrict our attention to *substantial* applications of mix, that is, such that $\alpha \notin \Gamma$ and $\alpha \notin \Pi$.

We are now ready to state and prove the main lemma. The proof will proceed by double induction on rank and degree of mix, so, superficially it looks like a standard cut elimination argument. One significant difference between our proof and standard proofs of cut elimination—apart from the fact that cut *cannot* be eliminated—is that in our proof rank has higher priority than degree. In standard proofs of cut elimination it is the other way around: they use induction on the pair (degree, rank), whereas we use induction on (rank, degree).

LEMMA 5.4. *Interpolation transfers across substantial applications of analytic mix.*

Proof. Induction on the pair (rank, degree), lexicographically ordered. Let P be a proof with analytic cut property, whose last rule is a substantial instance of analytic mix:

$$\frac{\Gamma \Rightarrow \Delta, \alpha^+ \quad \alpha^+, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}. \tag{1}$$

The rank of P is $m = c + d + 2$, where c is the rank of the subproof of $\Gamma \Rightarrow \Delta, \alpha^+$ in P, and d is the rank of the subproof of $\alpha^+, \Sigma \Rightarrow \Pi$ in P. We proceed by induction on m.

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If $m \leq 2$, then Lemma 5.2 applies, so we can assume $m > 2$. The inductive hypothesis then states that for any normal partition of any sequent whose proof Q with analytic cut property has $r(Q) < m$ there exists an interpolant.

Now, consider a nondegenerate normal partition $(\Gamma_1, \Sigma_1 : \emptyset), (\Gamma_2, \Sigma_2 : \Delta, \Pi)$ of the lower sequent. Since α is a subformula of Γ, Σ, Δ, or Π, we have $\mathrm{Var}(\alpha) \subseteq \mathrm{Var}(\Gamma, \Sigma, \Delta, \Pi)$, and two general cases to deal with.

CASE A. $\alpha \in \mathrm{Sfm}(\Gamma_2, \Sigma_2, \Delta, \Pi)$. Then, consider partitions $(\Gamma_1 : \emptyset), (\Gamma_2 : \Delta, \alpha^+)$ and $(\Sigma_1 : \emptyset), (\alpha^+, \Sigma_2 : \Pi)$. By inductive hypothesis we have respective interpolants ψ and φ. Thus, the sequents

$$\Gamma_1 \Rightarrow \psi \quad \psi, \Gamma_2 \Rightarrow \Delta, \alpha^+ \quad \Sigma_1 \Rightarrow \varphi \quad \varphi, \Sigma_2, \alpha^+ \Rightarrow \Pi$$

are provable. By analytic mix, then, the sequent $\psi, \varphi, \Gamma_2, \Sigma_2 \Rightarrow \Delta, \Pi$ is provable, and therefore, by $\wedge$ introduction on left and contraction, the sequent $\psi \wedge \varphi, \Gamma_2, \Sigma_2 \Rightarrow \Delta, \Pi$ is also provable. Furthermore, by two analytic cuts applied to the sequents $\Gamma_1 \Rightarrow \psi$, $\Sigma_1 \Rightarrow \varphi$, and $\psi, \varphi \Rightarrow \psi \wedge \varphi$, we obtain that $\Gamma_1, \Sigma_1 \Rightarrow \psi \wedge \varphi$ is provable as well. Now, by inductive hypothesis we also have $\mathrm{Var}(\psi) \subseteq \mathrm{Var}(\Gamma_1) \cap \mathrm{Var}(\Gamma_2, \Delta, \alpha)$ and $\mathrm{Var}(\varphi) \subseteq \mathrm{Var}(\Sigma_1) \cap \mathrm{Var}(\alpha, \Sigma_2, \Pi)$. Therefore

$$\begin{aligned}\mathrm{Var}(\psi \wedge \varphi) &\subseteq (\mathrm{Var}(\Gamma_1) \cap \mathrm{Var}(\Gamma_2, \Delta, \alpha)) \cup \mathrm{Var}(\Sigma_1) \cap \mathrm{Var}(\alpha, \Sigma_2, \Pi) \\ &\subseteq \mathrm{Var}(\Gamma_1, \Sigma_2) \cap \mathrm{Var}(\Gamma_2, \Delta, \alpha, \Sigma_2, \Pi) \\ &= \mathrm{Var}(\Gamma_1, \Sigma_2) \cap \mathrm{Var}(\Gamma_2, \Delta, \Sigma_2, \Pi),\end{aligned}$$

where the last equality follows by $\mathrm{Var}(\alpha) \subseteq \mathrm{Var}(\Gamma_2, \Sigma_2, \Delta, \Pi)$. This shows that $\psi \wedge \varphi$ is the required interpolant in this case.

CASE B.1. $\alpha \in \mathrm{Sfm}(\Gamma_1, \Sigma_1)$, and the rank of the proof of $\alpha^+, \Sigma \Rightarrow \Pi$ in P is strictly greater than 2, that is $d > 2$.

First, for the normal partition $(\alpha^+, \Sigma_1 : \emptyset), (\Sigma_2 : \Pi)$, we obtain an interpolant ψ by inductive hypothesis. Thus, we have provable sequents $\alpha^+, \Sigma_1 \Rightarrow \psi$ and $\psi, \Sigma_2 \Rightarrow \Pi$. Next, we consider the provable sequent $\Gamma \Rightarrow \Delta, \alpha^+$. Using mix

$$\dfrac{\Gamma \Rightarrow \Delta, \alpha^+ \quad \dfrac{\alpha \Rightarrow \alpha \quad \psi \Rightarrow \psi}{\alpha, \alpha \rightarrow \psi \Rightarrow \psi}}{\alpha \rightarrow \psi, \Gamma \Rightarrow \Delta, \psi},$$

we get that the lower sequent is provable. The rank of its proof is $c + (2 + 2) = c + 4$. By assumption we have $d > 2$, so $m = c + d + 2 > c + 4$, and the inductive hypothesis applies to the sequent $\alpha \rightarrow \psi, \Gamma \Rightarrow \Delta, \psi$. Then, taking its normal partition $(\Gamma_1, \alpha \rightarrow \psi : \emptyset)$, $(\Gamma_2 : \Delta, \psi)$, we obtain an interpolant φ. Therefore, the sequents

$$\Gamma_1, \alpha \rightarrow \psi \Rightarrow \varphi \quad \varphi, \Gamma_2 \Rightarrow \Delta, \psi \quad \Sigma_1 \Rightarrow \alpha \rightarrow \psi \quad \psi, \Sigma_2 \Rightarrow \Pi$$

are provable (to see that the third one is, apply $\rightarrow$ introduction on the right to $\alpha, \Sigma_1 \Rightarrow \psi$). Now, by two applications of cut, we get that

$$\Gamma_1, \Sigma_1 \Rightarrow \varphi \qquad \varphi, \Gamma_1, \Sigma_2 \Rightarrow \Delta, \Pi$$

are provable, too. It remains to verify that $\mathrm{Var}(\varphi) \subseteq \mathrm{Var}(\Gamma_1 \Sigma_1) \cap \mathrm{Var}(\Gamma_2, \Sigma_2, \Delta, \Pi)$. By construction, we have

$$\begin{aligned}&\mathrm{Var}(\psi) \subseteq \mathrm{Var}(\alpha, \Sigma_1) \cap \mathrm{Var}(\Sigma_2, \Pi) \subseteq \mathrm{Var}(\Gamma_1, \Sigma_1) \cap \mathrm{Var}(\Sigma_2, \Pi) \\ &\mathrm{Var}(\varphi) \subseteq \mathrm{Var}(\Gamma_1, \alpha \rightarrow \psi) \cap \mathrm{Var}(\Gamma_2, \Delta, \psi) \subseteq \mathrm{Var}(\Gamma_1, \Sigma_1, \psi) \cap \mathrm{Var}(\Gamma_2, \Delta, \psi)\end{aligned}$$

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and therefore

$$\begin{aligned}\mathrm{Var}(\varphi) &\subseteq \big(\mathrm{Var}(\Gamma_1, \Sigma_1) \cap \mathrm{Var}(\Gamma_2, \Delta)\big) \cup \mathrm{Var}(\psi)\\ &\subseteq \big(\mathrm{Var}(\Gamma_1, \Sigma_1) \cap \mathrm{Var}(\Gamma_2, \Delta)\big) \cup \big(\mathrm{Var}(\Gamma_1, \Sigma_1) \cap \mathrm{Var}(\Sigma_2, \Pi)\big)\\ &= \mathrm{Var}(\Gamma_1, \Sigma_1) \cap \big(\mathrm{Var}(\Gamma_2, \Delta) \cup \mathrm{Var}(\Sigma_2, \Pi)\big)\\ &= \mathrm{Var}(\Gamma_1, \Sigma_1) \cap \mathrm{Var}(\Gamma_2, \Sigma_2, \Delta, \Pi)\end{aligned}$$

as required. Thus, φ is an interpolant in this case.

CASE B.2. $\alpha \in \mathrm{Sfm}(\Gamma_1, \Sigma_1)$, and the rank of the proof of $\alpha^+, \Sigma \Rightarrow \Pi$ in P is at most 2, that is, $d \leq 2$.

SUBCASE: INITIAL. This occurs when $\alpha^+, \Sigma \Rightarrow \Pi$ is an initial sequent. Then, it is either $\alpha \Rightarrow \alpha$ or $0 \Rightarrow$. If the former, then the application of cut is

$$\frac{\Gamma \Rightarrow \alpha^+, \Delta \quad \alpha \Rightarrow \alpha}{\Gamma \Rightarrow \Delta, \alpha}$$

in which lower sequent is identical to the left upper one, so we get an interpolant by the inductive hypothesis. If the latter, then the application of cut is

$$\frac{\Gamma \Rightarrow 0^+, \Delta \quad 0 \Rightarrow}{\Gamma \Rightarrow \Delta}$$

in which case, the partition we are considering is $(\Gamma_1 : \emptyset), (\Gamma_2, : \Delta)$. The interpolant obtained by inductive hypothesis for the partition $(\Gamma_1 : \emptyset), (\Gamma_2, : 0, \Delta)$ is clearly also an interpolant for $(\Gamma_1 : \emptyset), (\Gamma_2, : \Delta)$.

Next, will consider various subcases: one for each possible rule above $\Gamma \Rightarrow \alpha^+, \Delta$. They fall into two broad types.

TYPE B.2.1. At least one occurrence of α is a side formula in the rule above $\Gamma \Rightarrow \alpha^+, \Delta$. We will present two such cases in detail.

SUBCASE: SUBTRACTION INTRODUCTION ON TOP OF MIX. Suppose P ends as follows

$$\frac{\dfrac{\Gamma \Rightarrow \Delta', \gamma \quad \delta \Rightarrow \Delta'', \alpha^+}{\Gamma \Rightarrow \Delta, \gamma - \delta, \alpha^+} \qquad \alpha^+, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \gamma - \delta, \Pi},$$

where $\Delta = \Delta' \cup \Delta''$. Let the ranks of the proofs of the three top sequents be a, b, and c, so $m = r(P) = a + b + c + 4$. We replace the end part of P by

$$\frac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta', \gamma \qquad \dfrac{\gamma \Rightarrow \gamma \quad \delta \Rightarrow \delta}{\gamma \Rightarrow \gamma - \delta, \delta}}{\Gamma \Rightarrow \Delta', \gamma - \delta, \delta} \qquad \dfrac{\dfrac{\delta \Rightarrow \Delta'', \alpha^+}{\Gamma_0, \delta \Rightarrow \Delta'', \alpha^+} \quad \alpha^+, \Sigma \Rightarrow \Pi}{\Gamma_0, \delta, \Sigma \Rightarrow \Delta'', \Pi}}{\Gamma_0, \Gamma, \Sigma \Rightarrow \Delta, \gamma - \delta, \Pi}}{\Gamma, \Sigma \Rightarrow \Delta, \gamma - \delta, \Pi},$$

where $\Gamma_0 = \{\gamma_0\}$ if $\alpha \in \mathrm{Sfm}(\gamma_0)$ for some $\gamma_0 \in \Gamma_1$, and $\Gamma_0 = \emptyset$ otherwise. Now, we consider the proofs of $\Gamma \Rightarrow \Delta', \gamma - \delta, \delta$ and $\Gamma_0, \delta, \Sigma \Rightarrow \Delta'', \Pi$. The rank of the proof of the second one is at most $b + c + 3 < r(P)$, so the inductive hypothesis applies to it. On the other hand, the rank of the proof of the first sequent is $a + 4$, which can be equal to $r(P)$ if $b = c = 0$. However, in this application of cut, as well as in the final one, the cut formula

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is a subformula of a formula on the right-hand side of the sequent. Therefore, both of them fall under Case A, and so the interpolants can be obtained for them successively. To be explicit, we take partitions $(\Gamma_1 : \emptyset)$, $(\Gamma_2 : \Delta', \gamma - \delta, \delta)$ and $(\Sigma_1 : \emptyset)$, $(\Gamma_0, \delta, \Sigma_2 : \Delta'', \Pi)$; get interpolants φ and ψ; proceeding as in Case A, we obtain an interpolant $\varphi \wedge \psi$ for the sequent $\Gamma_0, \Gamma, \Sigma \Rightarrow \Delta, \gamma - \delta, \Pi$. The last rule is contraction, so Lemma 5.3 applies. In fact, the interpolant is the same.

SUBCASE: MIX ON TOP OF MIX. Suppose P ends as follows

$$\dfrac{\dfrac{\Gamma' \Rightarrow \Delta', \beta^+ \quad \beta^+, \Gamma'' \Rightarrow \Delta'', \alpha^+}{\Gamma \Rightarrow \Delta, \alpha^+} \qquad \alpha^+, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi},$$

where $\Gamma = \Gamma' \cup \Gamma''$ and $\Delta = \Delta' \cup \Delta''$. Let the ranks of the proofs of the three top sequents be a, b, and c. So, $r(P) = a + b + c + 4$. Further subcases follow.

MIX-ON-MIX 1. If α is a subformula of $(\Gamma'' \cap \Gamma_1) \cup \Sigma_1$, then we can replace the end of P by

$$\dfrac{\Gamma' \Rightarrow \Delta', \beta^+ \qquad \dfrac{\beta^+, \Gamma'' \Rightarrow \Delta'', \alpha^+ \quad \alpha^+, \Sigma \Rightarrow \Pi}{\beta^+, \Gamma'', \Sigma \Rightarrow \Delta'', \Pi}}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}$$

in which the mix on α is analytic. Thus, interpolants for all normal partitions of $\beta^+, \Gamma'', \Sigma \Rightarrow \Delta'', \Pi$ exist by the inductive hypothesis. Consider the lower application of cut. If the proof of $\beta^+, \Gamma'', \Sigma \Rightarrow \Delta'', \Pi$ is of rank strictly greater than 2, then it falls under Case A, or Case B.1, depending on β. In either case, interpolants can be obtained. In the remaining case, the rank of the proof of $\beta^+, \Gamma'', \Sigma \Rightarrow \Delta'', \Pi$ is at most 2. But then, the rule above $\beta^+, \Gamma'', \Sigma \Rightarrow \Delta'', \Pi$ must be one of the three types of cut listed in the proof of Lemma 5.2. It follows that $\beta^+, \Gamma'', \Sigma \Rightarrow \Delta'', \Pi$ is an initial sequent, and by discarding everything above it, we obtain a proof of $\Gamma, \Sigma \Rightarrow \Delta, \Pi$ of rank strictly smaller than $r(P)$. Then, the inductive hypothesis applies.

MIX-ON-MIX 2. If α is not a subformula of $(\Gamma'' \cap \Gamma_1) \cup \Sigma_1$, then it must be a subformula of $\Gamma_1' = \Gamma_1 \cap \Gamma'$. Then, we proceed as follows. First, we get an interpolant for the partition $(\alpha, \Sigma_1 : \emptyset), (\Sigma_2 : \Pi)$ of the provable sequent $\alpha, \Sigma \Rightarrow \Pi$, which is provable from $\alpha^+, \Sigma \Rightarrow \Pi$ by contraction (if necessary), so such an interpolant exists by Lemma 5.3. This yields

$$\alpha, \Sigma_1 \Rightarrow \psi \text{ (so, } \Sigma_1 \Rightarrow \alpha \to \psi) \qquad \psi, \Sigma_2 \Rightarrow \Pi \tag{2}$$

with $\mathrm{Var}(\psi) \subseteq \mathrm{Var}(\alpha, \Sigma_1) \cap \mathrm{Var}(\Sigma_2, \Pi)$. Next, we take the proof Q

$$\dfrac{\beta^+, \Gamma'' \Rightarrow \Delta'', \alpha^+ \quad \alpha, \alpha \to \psi \Rightarrow \psi}{\alpha \to \psi, \beta^+, \Gamma'' \Rightarrow \Delta'', \psi},$$

which has $r(Q) = b + 4$. Now, $b + 4 < a + b + c + 4$ if $0 < a + c$, so the only situation where the inductive hypothesis does not apply is when both $\Gamma' \Rightarrow \Delta', \beta^+$ and $\alpha^+, \Sigma \Rightarrow \Pi$ are initial. This case is trivial, and will be dealt with separately, under **III** below. Assume $0 < a + c$. Since the mix on β is analytic, we have $\beta \in \mathrm{Sfm}(\alpha, \Gamma, \Delta)$. But as $\alpha \in \mathrm{Sfm}(\Gamma_1') \subseteq \Gamma$, we get $\beta \in \mathrm{Sfm}(\Gamma, \Delta)$.

I. If $\beta \in \mathrm{Sfm}(\Gamma_2, \Delta)$, then the mix

$$\dfrac{\Gamma' \Rightarrow \Delta', \beta^+ \quad \beta^+, \alpha \to \psi, \Gamma'' \Rightarrow \Delta'', \psi}{\alpha \to \psi, \Gamma \Rightarrow \Delta, \psi}$$

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with the partition $(\alpha \to \psi, \Gamma_1 : \emptyset), (\Gamma_2 : \Delta, \psi)$, falls under Case A, so we get an interpolant φ for it. Thus,

$$\alpha \to \psi, \Gamma_1 \Rightarrow \varphi \qquad \varphi, \Gamma_2 \Rightarrow \Delta, \psi \tag{3}$$

with $\mathrm{Var}(\varphi) \subseteq \mathrm{Var}(\alpha, \psi, \Gamma_1) \cap \mathrm{Var}(\Gamma_2, \Delta, \psi)$. Then, by two cuts applied to the sequents in (2) and (3), we obtain

$$\Gamma_1, \Sigma_1 \Rightarrow \varphi \qquad \varphi, \Gamma_2, \Sigma_2 \Rightarrow \Delta, \Pi \tag{4}$$

and moreover

$$\begin{aligned}
\mathrm{Var}(\varphi) &\subseteq \mathrm{Var}(\alpha, \psi, \Gamma_1) \cap \mathrm{Var}(\Gamma_2, \Delta, \psi) \\
&= \big(\mathrm{Var}(\alpha, \Gamma_1) \cap \mathrm{Var}(\Gamma_2, \Delta)\big) \cup \mathrm{Var}(\psi) \\
&\subseteq \big(\mathrm{Var}(\alpha, \Gamma_1) \cap \mathrm{Var}(\Gamma_2, \Delta)\big) \cup \big(\mathrm{Var}(\alpha, \Sigma_1) \cap \mathrm{Var}(\Sigma_2, \Pi)\big) \\
&\subseteq \mathrm{Var}(\alpha, \Gamma_1, \Sigma_1) \cap \mathrm{Var}(\Gamma_2, \Sigma_2, \Delta, \Pi) \\
&= \mathrm{Var}(\Gamma_1, \Sigma_1) \cap \mathrm{Var}(\Gamma_2, \Sigma_2, \Delta, \Pi).
\end{aligned}$$

II. If $\beta \notin \mathrm{Sfm}(\Gamma_2, \Delta)$, then putting $\Gamma_i' = \Gamma_i \cap \Gamma'$ and $\Gamma_i'' = \Gamma_i \cap \Gamma''$, for $i \in \{1, 2\}$, we get $\beta \in \mathrm{Sfm}(\alpha, \Gamma_1', \Gamma_1'') = \mathrm{Sfm}(\alpha, \Gamma_1) = \mathrm{Sfm}(\Gamma_1)$, because $\alpha \in \mathrm{Sfm}(\Gamma_1)$ by assumption. Then we proceed as follows. By inductive hypothesis, we get an interpolant χ for the partition $(\alpha \to \psi, \beta, \Gamma_1'' : \emptyset), (\Gamma_2'' : \Delta'', \psi)$, of the end sequent of Q. This gives

$$\alpha \to \psi, \beta, \Gamma_1'' \Rightarrow \chi \ (\text{so}, \alpha \to \psi, \Gamma_1'' \Rightarrow \beta \to \chi) \qquad \chi, \Gamma_2'' \Rightarrow \Delta'', \psi \tag{5}$$

with $\mathrm{Var}(\chi) \subseteq \mathrm{Var}(\alpha, \psi, \beta, \Gamma_1'') \cap \mathrm{Var}(\Gamma_2'', \Delta'', \psi)$. Then, we consider the proof R

$$\frac{\Gamma' \Rightarrow \Delta', \beta^+ \quad \beta, \beta \to \chi \Rightarrow \chi}{\Gamma', \beta \to \chi \Rightarrow \Delta', \chi}$$

with $r(R) = a + 4$. Then, $r(R) < r(P)$ if $0 < b + c$, that is, it holds unless both $\beta^+, \Gamma'' \Rightarrow \Delta'', \alpha^+$ and $\alpha^+, \Sigma \Rightarrow \Pi$ are initial. If this happens, then β is identical to α, and the upper application of mix in the end part of P can be discarded, transforming it into P', which ends as follows

$$\begin{array}{c}
\dfrac{\Gamma' \Rightarrow \Delta', \alpha^+ \quad \alpha^+, \Sigma \Rightarrow \Pi}{\Gamma', \Sigma \Rightarrow \Delta', \Pi}\cdot \\
\vdots \\
\Gamma, \Sigma \Rightarrow \Delta, \Pi
\end{array}$$

Here, the rank of the proof above the mix is $a + c + 2$ and so the inductive hypothesis applies to it. Thus, we get the desired interpolant for $\Gamma, \Sigma \Rightarrow \Delta, \Pi$ by Lemma 5.3.

Now, suppose $0 < b + c$. Then, the inductive hypothesis applies to R, yielding an interpolant ρ for the partition $(\beta \to \chi, \Gamma_1' : \emptyset), (\Gamma_2' : \Delta', \chi)$. Thus,

$$\beta \to \chi, \Gamma_1' \Rightarrow \rho \qquad \rho, \Gamma_2' \Rightarrow \Delta', \chi \tag{6}$$

and $\mathrm{Var}(\rho) \subseteq \mathrm{Var}(\beta, \chi, \Gamma_1') \cap \mathrm{Var}(\Gamma_2', \Delta', \chi)$. Now, we apply two cuts

$$\dfrac{\Sigma_1 \Rightarrow \alpha \to \psi \qquad \dfrac{\alpha \to \psi, \Gamma_1'' \Rightarrow \beta \to \chi \quad \beta \to \chi, \Gamma_1' \Rightarrow \rho}{\alpha \to \psi, \Gamma_1', \Gamma_1'' \Rightarrow \rho}}{\Gamma_1, \Sigma_1 \Rightarrow \rho}$$

with the upper sequents coming from (2), (5), and (6). Then, we apply two more cuts

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$$\dfrac{\dfrac{\rho, \Gamma_2' \Rightarrow \Delta', \chi \quad \chi, \Gamma_2'' \Rightarrow \Delta'', \psi}{\rho, \Gamma_2 \Rightarrow \Delta, \psi} \qquad \psi, \Sigma_2 \Rightarrow \Pi}{\rho, \Gamma_2, \Sigma_2 \Rightarrow \Delta, \Pi}$$

with the upper sequents coming from (6), (5), and (2). Furthermore, we have

$$\begin{aligned}
\mathrm{Var}(\rho) &\subseteq \mathrm{Var}(\beta, \chi, \Gamma_1') \cap \mathrm{Var}(\Gamma_2', \Delta', \chi) \\
&= \big(\mathrm{Var}(\beta, \Gamma_1') \cap \mathrm{Var}(\Gamma_2', \Delta',)\big) \cup \mathrm{Var}(\chi) \\
&\subseteq \big(\mathrm{Var}(\beta, \Gamma_1') \cap \mathrm{Var}(\Gamma_2', \Delta')\big) \cup \big(\mathrm{Var}(\alpha, \psi, \beta, \Gamma_1'') \cap \mathrm{Var}(\Gamma_2'', \Delta'', \psi)\big) \\
&= \big(\mathrm{Var}(\beta, \Gamma_1') \cap \mathrm{Var}(\Gamma_2', \Delta')\big) \cup \big(\mathrm{Var}(\alpha, \beta, \Gamma_1'') \cap \mathrm{Var}(\Gamma_2'', \Delta'')\big) \cup \mathrm{Var}(\psi) \\
&\subseteq \big(\mathrm{Var}(\alpha, \beta, \Gamma_1) \cap \mathrm{Var}(\Gamma_2, \Delta)\big) \cup \big(\mathrm{Var}(\alpha, \Sigma_1) \cap \mathrm{Var}(\Sigma_2, \Pi)\big) \\
&\subseteq \mathrm{Var}(\alpha, \beta, \Gamma_1, \Sigma_1) \cap \mathrm{Var}(\Gamma_2, \Sigma_2, \Delta, \Pi) \\
&\subseteq \mathrm{Var}(\Gamma_1, \Sigma_1) \cap \mathrm{Var}(\Gamma_2, \Sigma_2, \Delta, \Pi).
\end{aligned}$$

III. The sequents $\Gamma' \Rightarrow \Delta', \beta^+$ and $\alpha^+, \Sigma \Rightarrow \Pi$ are initial. Since the first can be either $\beta \Rightarrow \beta$ or $\Rightarrow 1$ and the second $\alpha \Rightarrow \alpha$ or $0 \Rightarrow$, there are four subcases. In all but one case, one of the lower sequents is identical to an upper one, so we can shorten P and apply the inductive hypothesis. The remaining case is

$$\dfrac{\dfrac{\Rightarrow 1 \quad 1^+, \Gamma'' \Rightarrow \Delta'', 0^+}{\Gamma'' \Rightarrow \Delta'', 0^+} \qquad 0 \Rightarrow}{\Gamma'' \Rightarrow \Delta''},$$

where interpolants are immediately obtained, too.

SUBCASE: OTHER RULES ON TOP OF MIX. All other rules with α as a side formula can be dealt with by similar arguments. In fact, in all other cases the arguments are much simpler. In particular, contraction on top of mix presents no difficulty.

TYPE B.2.2. The formula α is a principal formula.

SUBCASE: WEAKENING AND CONTRACTION. Contraction on top of mix reduces to an application of mix of lower rank, so let us consider weakening.

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \alpha^*}{\Gamma \Rightarrow \Delta, \alpha^+} \quad \alpha^+, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}$$

If $\Gamma \Rightarrow \Delta, \alpha^*$ is $\Gamma \Rightarrow \Delta$, then $\Gamma, \Sigma \Rightarrow \Delta, \Pi$ can be obtained by a repeated use of weakening, and interpolants transfer across weakening by Lemma 5.3. If $\Gamma \Rightarrow \Delta, \alpha^*$ is $\Gamma \Rightarrow \Delta, \alpha, \alpha^*$, then the rank of mix can be reduced and the inductive hypothesis applies.

To proceed, observe that if there are multiple occurrences of α in $\Gamma \Rightarrow \Delta, \alpha^*$, then only one of them can be the principal formula, and therefore all other occurrences are side formulas, which reduces the case to Type B.2.1. Thus, without loss of generality, we assume for all further cases that $\Gamma \Rightarrow \Delta, \alpha^*$ is $\Gamma \Rightarrow \Delta, \alpha$.

SUBCASE: CONJUNCTION INTRODUCTION. α is $\alpha_1 \wedge \alpha_2$, and we have

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \alpha_1 \quad \Gamma \Rightarrow \Delta, \alpha_2}{\Gamma \Rightarrow \Delta, \alpha_1 \wedge \alpha_2} \qquad \alpha_1 \wedge \alpha_2, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}.$$

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Then, since $\alpha \notin \Pi$, and the rank of the proof of $\alpha, \Sigma \Rightarrow \Pi$ is at most 2, inspecting the possible cases we can see that there are only three:

$$\frac{\alpha_1 \Rightarrow \alpha_1}{\alpha_1 \wedge \alpha_2 \Rightarrow \alpha_1} \qquad \frac{\alpha_2 \Rightarrow \alpha_2}{\alpha_1 \wedge \alpha_2 \Rightarrow \alpha_2} \qquad \frac{\Sigma \Rightarrow \Pi}{\alpha_1 \wedge \alpha_2, \Sigma \Rightarrow \Pi},$$

where in the first two cases $\Sigma = \emptyset$, and $\Pi = \{\alpha_1\}$ or $\Pi = \{\alpha_2\}$. In either case, the sequent $\Gamma \Rightarrow \Delta, \Pi$ is provable by assumption. In the third case, the sequent $\Sigma \Rightarrow \Pi$ is provable. In any of these cases, the sequent $\Gamma, \Sigma \Rightarrow \Delta, \Pi$ is then provable by repeated weakening, and the interpolants transfer across by Lemma 5.3.

SUBCASE: DISJUNCTION INTRODUCTION. α is $\alpha_1 \vee \alpha_2$ and we have

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \alpha_1}{\Gamma \Rightarrow \Delta, \alpha_1 \vee \alpha_2} \quad \alpha_1 \vee \alpha_2, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}.$$

Here we have several possibilities for the proof above $\alpha_1 \vee \alpha_2, \Sigma \Rightarrow \Pi$, but let us only consider one, namely

$$\frac{\alpha_1 \vee \alpha_2 \Rightarrow \alpha_1 \vee \alpha_2 \quad \pi \Rightarrow \pi}{\alpha_1 \vee \alpha_2, (\alpha_1 \vee \alpha_2) \to \pi \Rightarrow \pi}.$$

Then, we use the proof

$$\dfrac{\Gamma \Rightarrow \Delta, \alpha_1 \quad \dfrac{\dfrac{\alpha_1 \Rightarrow \alpha_1}{\alpha_1 \Rightarrow \alpha_1 \vee \alpha_2} \quad \pi \Rightarrow \pi}{\alpha_1, (\alpha_1 \vee \alpha_2) \to \pi \Rightarrow \pi}}{\Gamma, (\alpha_1 \vee \alpha_2) \to \pi \Rightarrow \Delta, \pi}$$

of the same rank as P, but the cut formula is simpler, so the degree is reduced and the inductive hypothesis applies. Other cases are similar.

SUBCASE: IMPLICATION INTRODUCTION. α is $\alpha_1 \to \alpha_2$ and we have

$$\dfrac{\dfrac{\Gamma, \alpha_1 \Rightarrow \alpha_2}{\Gamma \Rightarrow \alpha_1 \to \alpha_2} \quad \alpha_1 \to \alpha_2, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Pi}.$$

Here, we exploit the fact that $\Delta = \emptyset$. Take the partitions $(\Gamma_2, : \emptyset)$, $(\Gamma_1 : \alpha_1 \to \alpha_2)$ and $(\alpha_1 \to \alpha_2, \Sigma_1, : \emptyset)$, $(\Sigma_2 : \Pi)$; get interpolants φ and ψ. The sequents

$$\Gamma_2 \Rightarrow \varphi \qquad \varphi, \Gamma_1 \Rightarrow \alpha_1 \to \alpha_2 \qquad \alpha_1 \to \alpha_2, \Sigma_1 \Rightarrow \psi \qquad \psi, \Sigma_2 \Rightarrow \Pi$$

are provable. From these we readily obtain

$$\Gamma_1, \Sigma_1 \Rightarrow \varphi \to \psi \qquad \varphi \to \psi, \Gamma_2, \Sigma_2 \Rightarrow \Pi$$

and checking the condition on variables is straightforward.

SUBCASE: SUBTRACTION INTRODUCTION. α is $\alpha_1 - \alpha_2$ and we have

$$\dfrac{\dfrac{\Gamma' \Rightarrow \alpha_1, \Delta' \quad \alpha_2 \Rightarrow \Delta''}{\Gamma \Rightarrow \Delta, \alpha_1 - \alpha_2} \quad \alpha_1 - \alpha_2, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}.$$

Since $\alpha \notin \Pi$ by assumption, inspecting the possible cases of proofs of $\alpha_1 - \alpha_2, \Sigma \Rightarrow \Pi$ of rank at most 2 shows that the only two possibilities are

$$\frac{\dfrac{\alpha_1 \Rightarrow \alpha_1}{\alpha_1 \Rightarrow \alpha_1, \alpha_2}}{\alpha_1 - \alpha_2 \Rightarrow \alpha_1} \qquad \frac{\Sigma \Rightarrow \Pi}{\alpha_1 - \alpha_2, \Sigma \Rightarrow \Pi}.$$

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In the first case, we have $\alpha_1 \in \Pi$, so the sequent $\Gamma, \Sigma \Rightarrow \Delta, \Pi$ can be obtained from $\Gamma' \Rightarrow \alpha_1, \Delta'$ by weakening; in the second case it can be obtained by weakening from $\Sigma \Rightarrow \Pi$. In either case the interpolant transfers across weakening by Lemma 5.3.

This exhausts all cases and ends the proof in the case of the normal partition $(\Gamma_1, \Sigma_2 : \emptyset), (\Gamma_2, \Sigma_2 : \Delta, \Pi)$. The proof for $(\Gamma, \Sigma : \Delta_1, \Pi_1), (\emptyset : \Delta_2, \Pi_2)$ is completely analogous: it employs the fact that the rules for subtraction are mirror images of the rules for implication. □

LEMMA 5.5. *Let* $\Gamma \Rightarrow \Delta$ *be a sequent provable in* $\mathbf{LBJ}_{0.9}$. *If the partition* $(\Gamma_1 : \Delta_1), (\Gamma_2 : \Delta_2)$ *is normal, then there exist an interpolant for it.*

Proof. As previously, degenerate partitions have either 0 or 1 as an interpolant, so it suffices to consider nondegenerate ones. Let P be a proof of $\Gamma \Rightarrow \Delta$. We proceed by induction on the rank of P. The base case, with $r(P) \leq 2$ follows by Lemma 5.2.

We consider the last rule of P. The rules of weakening and contraction have been dealt with in Lemma 5.3, and mix in Lemma 5.4. The rules introducing $\wedge$ and $\vee$ present no difficulty. We will now exhibit a detailed proof for the rules introducing implication. First, we deal with $\rightarrow$ introduction on the right

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta}$$

and a nondegenerate normal partition of the lower sequent. By inspection of possible cases, we see that it must be of the form $(\Gamma_1 : \emptyset), (\Gamma_2 : \{\alpha \rightarrow \beta\})$, with Γ_2 possibly empty. Take the partition $(\Gamma_1 : \emptyset), (\Gamma_2 \cup \{\alpha\} : \{\beta\})$ of the upper sequent. By inductive hypothesis, it has an interpolant φ. Thus, we have provable sequents $\Gamma_1 \Rightarrow \varphi$ and $\varphi, \alpha, \Gamma_2 \Rightarrow \beta$. It follows that $\varphi, \Gamma_2 \Rightarrow \alpha \rightarrow \beta$ is provable. Since $\mathrm{Var}(\Gamma_1) \cap \mathrm{Var}(\Gamma_2, \{\alpha \rightarrow \beta\}) = \mathrm{Var}(\Gamma_1) \cap \mathrm{Var}(\Gamma_2 \cup \{\alpha\}, \{\beta\})$, we have that φ is the required interpolant.

Next, we consider $\rightarrow$ introduction on the left (in the restricted $\mathbf{LBJ}_{0.9}$ form)

$$\frac{\Gamma \Rightarrow \alpha \quad \Sigma, \beta \Rightarrow \Pi}{\Gamma, \Sigma, \alpha \rightarrow \beta \Rightarrow \Pi}.$$

Now, we have three nondegenerate normal partitions to deal with.

CASE 1. $(\Gamma_1, \Sigma_1 : \emptyset), (\Gamma_2, \Sigma_2, \alpha \rightarrow \beta : \Pi)$. Take partitions $(\Gamma_1 : \emptyset), (\Gamma_2 : \alpha)$ and $(\Sigma_1 : \emptyset), (\Sigma_2, \beta : \Pi)$. By inductive hypothesis, we have the respective interpolants φ and ψ, so the following sequents are provable

$$\Gamma_1 \Rightarrow \varphi \qquad \varphi, \Gamma_2 \Rightarrow \alpha \qquad \Sigma_1 \Rightarrow \psi \qquad \psi, \Sigma_2, \beta \Rightarrow \Pi.$$

Applying $\rightarrow$ introduction on left to the second and fourth of these, we readily obtain

$$\Gamma_1, \Sigma_1 \Rightarrow \varphi \wedge \psi \qquad \varphi \wedge \psi, \Gamma_2, \Sigma_2, \alpha \rightarrow \beta \Rightarrow \Pi.$$

Now, since $\mathrm{Var}(\varphi) \subseteq \mathrm{Var}(\Gamma_1) \cap \mathrm{Var}(\Gamma_2, \alpha)$ and $\mathrm{Var}(\psi) \subseteq \mathrm{Var}(\Sigma_1) \cap \mathrm{Var}(\Sigma_2, \beta, \Pi)$ we get that

$$\begin{aligned}
\mathrm{Var}(\varphi \wedge \psi) &= \mathrm{Var}(\varphi) \cup \mathrm{Var}(\psi) \\
&\subseteq \big(\mathrm{Var}(\Gamma_1) \cup \mathrm{Var}(\Sigma_1)\big) \cap \big(\mathrm{Var}(\Gamma_2, \alpha) \cup \mathrm{Var}(\Sigma_2, \beta, \Pi)\big) \\
&= \mathrm{Var}(\Gamma_1, \Sigma_1) \cap \mathrm{Var}(\Gamma_2, \alpha, \Sigma_2, \beta, \Pi) \\
&= \mathrm{Var}(\Gamma_1, \Sigma_1) \cap \mathrm{Var}(\Gamma_2, \alpha \rightarrow \beta, \Sigma_2, \Pi),
\end{aligned}$$

so, $\varphi \wedge \psi$ is an interpolant.

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CASE 2. $(\Gamma_1, \Sigma_1, \alpha \to \beta : \emptyset), (\Gamma_2, \Sigma_2 : \Pi)$. Take partitions $(\Gamma_2 : \emptyset), (\Gamma_1 : \alpha)$ and $(\Sigma_1, \beta : \emptyset), (\Sigma_2 : \Pi)$. By inductive hypothesis, we have the respective interpolants φ and ψ, so the following sequents are provable

$$\Gamma_2 \Rightarrow \varphi \qquad \varphi, \Gamma_1 \Rightarrow \alpha \qquad \Sigma_1, \beta \Rightarrow \psi \qquad \psi, \Sigma_2 \Rightarrow \Pi.$$

Applying $\to$ introduction on left to the second and third of these, and then $\to$ introduction on the right, we obtain

$$\Gamma_1, \alpha \to \beta, \Sigma_1 \Rightarrow \varphi \to \psi.$$

Next, applying $\to$ introduction on the left to the first and the last sequent in the list, we get

$$\Gamma_2, \Sigma_2, \varphi \to \psi \Rightarrow \Pi.$$

Moreover, we have $\mathrm{Var}(\varphi) \subseteq \mathrm{Var}(\Gamma_2) \cap \mathrm{Var}(\Gamma_1, \alpha)$ and $\mathrm{Var}(\psi) \subseteq \mathrm{Var}(\Sigma_2) \cap \mathrm{Var}(\Sigma_1, \beta)$, and therefore

$$\begin{aligned}\mathrm{Var}(\varphi \to \psi) &\subseteq \big(\mathrm{Var}(\Gamma_2) \cap \mathrm{Var}(\Gamma_1, \alpha)\big) \cup \big(\mathrm{Var}(\Sigma_2, \Pi) \cap \mathrm{Var}(\Sigma_1, \beta)\big)\\ &\subseteq \mathrm{Var}(\Gamma_2, \Sigma_2, \Pi) \cap \mathrm{Var}(\Gamma_1, \Sigma_1, \alpha, \beta)\\ &= \mathrm{Var}(\Gamma_1, \Sigma_1, \alpha \to \beta) \cap \mathrm{Var}(\Gamma_2, \Sigma_2, \Pi),\end{aligned}$$

so, $\varphi \to \psi$ is an interpolant.

CASE 3. $(\Gamma, \alpha \to \beta, \Sigma : \Pi_1), (\emptyset : \Pi_2)$. Take $(\beta, \Sigma : \Pi_1), (\emptyset : \Pi_2)$ to get an interpolant φ. Thus,

$$\beta, \Sigma \Rightarrow \Pi_1, \varphi \qquad \varphi \Rightarrow \Pi_2$$

are provable. Therefore, by $\to$ introduction on left

$$\Gamma, \alpha \to \beta, \Sigma \Rightarrow \Pi_1, \varphi$$

is provable as well. Moreover,

$$\begin{aligned}\mathrm{Var}(\varphi) &\subseteq \mathrm{Var}(\beta, \Sigma, \Pi_1) \cap \mathrm{Var}(\Pi_2)\\ &\subseteq \mathrm{Var}(\Gamma, \alpha, \beta, \Sigma, \Pi_1) \cap \mathrm{Var}(\Pi_2)\\ &= \mathrm{Var}(\Gamma, \alpha \to \beta, \Sigma, \Pi_1) \cap \mathrm{Var}(\Pi_2),\end{aligned}$$

so, φ is an interpolant. This exhausts all cases.

The rules introducing subtraction are mirror images of the rules introducing implication, so an analogous proof works for them. Similarly, for the normal partition $(\Gamma, \Sigma : \Delta_1, \Pi_1)$, $(\emptyset : \Delta_2, \Pi_2)$ we can employ the mirror image symmetry between the rules. □

THEOREM 5.6. *Craig interpolation holds for bi-intuitionistic logic.*

Proof. Let $\varphi \to \psi$ be a theorem of bi-intuitionistic logic. The sequent $\Rightarrow \varphi \to \psi$ is then provable, and, by cut, so is $\varphi \Rightarrow \psi$. Take the normal partition $(\varphi : \emptyset), (\emptyset : \psi)$. By Lemma 5.5, it has an interpolant χ. Then, $\varphi \to \chi$ and $\chi \to \psi$ are theorems, and $\mathrm{Var}(\chi) \subseteq \mathrm{Var}(\varphi, \psi)$. □

§6. Halldén completeness and Maximova variable separation. In this section we present two further applications of Maehara method: the first showing Halldén completeness of **BiInt**, the second showing Maximova variable separation property for the same logic. Maximova property for **BiInt** can also be shown semantically, and indeed such a proof was produced by Badia (2015), when he got acquainted with our proof-theoretical argument. Before we begin, two remarks are in order.

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First, notice that the *disjunction property*, well known to hold in the intuitionistic logic, fails in **BiInt**. For example, the formula $p \vee (1 - p)$, with p a variable, is a theorem of **BiInt**, but neither p nor $1 - p$ is.

Secondly, as the reader will observe later on in this section, Maximova property for **BiInt** implies its Halldén completeness, so, strictly speaking, a separate proof of Halldén completeness is superfluous. We give it nevertheless, because it is an immediate corollary of our proof method from §5, whereas the proof of Maximova property requires certain modifications to the technique.

DEFINITION 6.1. *A logic* **L***, containing a connective of disjunction, is said to be* Halldén complete*, if the following holds.*

- *Whenever* $\mathrm{Var}(\alpha) \cap \mathrm{Var}(\beta) = \emptyset$ *and* $\alpha \vee \beta$ *is a theorem of* **L***, then, at least one of* α*,* β *is a theorem of* **L***.*

THEOREM 6.2. *Bi-intuitionistic logic is Halldén complete.*

Proof. Suppose $\mathrm{Var}(\alpha) \cap \mathrm{Var}(\beta) = \emptyset$ and $\alpha \vee \beta$ is a theorem of **BiInt**. Then, the sequent

$$\Rightarrow \alpha, \beta$$

is provable. Consider the normal partition $(\emptyset : \alpha), (\emptyset : \beta)$. By Lemma 5.5, there is an interpolant γ, such that

$$\Rightarrow \alpha, \gamma \qquad \gamma \Rightarrow \beta$$

are both provable and γ contains no variables. It is easy to see that such a γ is provably equivalent to either 0 of 1. Thus, without loss of generality, we can assume that $\gamma \in \{0, 1\}$. If $\gamma = 1$, then $1 \Rightarrow \beta$ is provable, so by cut

$$\frac{\Rightarrow 1 \quad 1 \Rightarrow \beta}{\Rightarrow \beta},$$

β is a theorem. If $\gamma = 0$, then, the sequent $\Rightarrow \alpha, 0$ is provable, so by cut

$$\frac{\Rightarrow \alpha, 0 \quad 0 \Rightarrow}{\Rightarrow \alpha},$$

and therefore α is a theorem. Thus, α is a theorem or β is a theorem, as needed for Halldén completeness. □

It might be worthwhile to observe that one could also define a notion dual to Halldén completeness, at least for logics with a reasonable notion of *unsatisfiability* over **L**. For extensions of **BiInt**, unsatisfiability coincides with the notion of being an *anti-theorem*. Namely, a formula φ is an anti-theorem of **L**, if $\varphi \Rightarrow$ is provable.

DEFINITION 6.3. *A logic* **L***, containing a connective of conjunction, is said to be* co-Halldén complete*, if the following holds.*

- *Whenever* $\mathrm{Var}(\alpha) \cap \mathrm{Var}(\beta) = \emptyset$ *and* $\alpha \wedge \beta$ *is unsatisfiable over* **L***, then, at least one of* α*,* β *is unsatisfiable over* **L***.*

A proof similar to the proof of Theorem 6.2 readily shows that **BiInt** is co-Halldén complete in the above sense. Observe, however, that a separate proof is needed. This could be contrasted, for example, with the case of classical logic, where using classical negation one can derive co-Halldén completeness from Halldén completeness.

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A very neat algebraic characterisation of Halldén completeness for extensions of intuitionistic logics is due to Wroński. In Wroński (1976) he shows that a superintuitionistic logic **L** is Halldén complete if and only if the corresponding variety of Heyting algebras (that is, the equivalent algebraic semantics of **L**) is generated by a single subdirectly irreducible algebra. This leads to the following natural question.

QUESTION 6.4. *Which extensions of* **BiInt** *are Halldén complete*?

A property closely related to Halldén completeness is *Maximova variable separation property*, due, as the name suggests, to Maximova (see Maksimova (1976), and, for example, Naruse, Surarso, & Ono (1998) for a syntactic approach).

DEFINITION 6.5. *Let* **L** *be a logic extending* **BiInt**. *We say that* **L** *has the* Maximova variable separation property*, if the following holds.*

- *Whenever* $\mathrm{Var}(\alpha_1, \alpha_2) \cap \mathrm{Var}(\beta_1, \beta_2) = \emptyset$ *and* $(\alpha_1 \wedge \beta_1) \to (\alpha_2 \vee \beta_2)$ *is a theorem of* **L**, *then, at least one of* $\alpha_1 \to \alpha_2$, $\beta_1 \to \beta_2$ *is a theorem of* **L**.

As we have already mentioned, Maximova property for **BiInt** implies its Halldén completeness: to see it, take $\alpha_1 = \beta_1 = 1$. One could then expect that the Maximova property might be shown to hold for **BiInt** by a syntactic proof using interpolation in a manner similar to the proof of Theorem 6.2. Unfortunately, a naive approach along these lines does not work, because the partition $(\alpha_1 : \alpha_2), (\beta_1 : \beta_2)$ of the sequent $\alpha_1, \beta_1 \Rightarrow \alpha_2, \beta_2$ is not normal. To obtain a syntactic proof by Maehara method, we will modify the notion of partition again. Namely, we say that a partition $(\Gamma_1 : \Delta_1), (\Gamma_2 : \Delta_2)$ of a sequent $\Gamma \Rightarrow \Delta$ is *variable-disjoint*, if $\mathrm{Var}(\Gamma_1, \Delta_1) \cap \mathrm{Var}(\Gamma_2, \Delta_2) = \emptyset$.

LEMMA 6.6. *Let* $\Gamma \Rightarrow \Delta$ *be a sequent provable in* $\mathbf{LBJ}_1$. *If* $(\Gamma_1 : \Delta_1), (\Gamma_2 : \Delta_2)$ *is variable-disjoint, then either* 0 *or* 1 *is an interpolant for it.*

Proof. (Sketch) Let P be a proof of $\Gamma \Rightarrow \Delta$ in $\mathbf{LBJ}_1$. By Theorem 4.7, we can assume P has analytic cuts. We proceed by induction on the rank of P. For the base case with $r(P) = 0$, observe that the only problematic subcase would arise from $\alpha \Rightarrow \alpha$, partitioned into $(\emptyset : \alpha), (\alpha : \emptyset)$, and with $\mathrm{Var}(\alpha) \neq \emptyset$. But this partition is not variable-disjoint, so the problematic case never occurs.

The inductive step then presents no difficulty. We present two cases, as examples.

CASE 1. The last rule of P is analytic cut

$$\frac{\Gamma \Rightarrow \alpha, \Delta \quad \Sigma, \alpha \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}$$

and we consider a variable-disjoint partition $(\Gamma_1, \Sigma_1 : \Delta_1, \Pi_1), (\Gamma_2, \Sigma_2 : \Delta_2, \Pi_2)$. To narrow down our example even further, we assume $\alpha \in \mathrm{Sfm}(\Gamma_2, \Sigma_2, \Delta_2, \Pi_2)$ (the other case is similar). Then, the partitions $(\Gamma_1 : \Delta_1), (\Gamma_2 : \alpha, \Delta_2)$ and $(\Sigma_1 : \Pi_1), (\alpha, \Sigma_2 : \Pi_2)$ are variable-disjoint as well, and so by inductive hypothesis there are interpolants $\varphi, \psi \in \{0, 1\}$, making the sequents

$$\Gamma_1 \Rightarrow \Delta_1, \varphi \qquad \varphi, \Gamma_2 \Rightarrow \alpha, \Delta_2 \qquad \Sigma_1 \Rightarrow \Pi_1, \psi \qquad \psi, \alpha, \Sigma_2 \Rightarrow \Pi_2$$

provable. Now, using analytic cut it is straightforward to obtain the sequents

$$\Gamma_1, \Sigma_1 \Rightarrow \Delta_1, \Pi_1, \varphi \wedge \psi \qquad \varphi \wedge \psi, \Gamma_2, \Sigma_2 \Rightarrow \Delta_2, \Pi_2$$

and so, since no variables occur in $\varphi \wedge \psi$, it is an interpolant. Then, again using analytic cut we get that 1 is an interpolant if $\varphi = \psi = 1$, and 0 is an interpolant otherwise.

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CASE 2. The last rule of P is

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}$$

and we consider a variable-disjoint partition

$$(\Gamma_1 : \alpha \to \beta), (\Gamma_2, : \emptyset).$$

Then, the partition $(\Gamma_2 : \emptyset), (\alpha, \Gamma_1 : \beta)$ is variable-disjoint as well, so by inductive hypothesis we have an interpolant $\varphi \in \{0, 1\}$, so that the sequents

$$\Gamma_2 \Rightarrow \varphi \qquad \varphi, \alpha, \Gamma_1 \Rightarrow \beta$$

are provable. Then, the sequent

$$\varphi, \Gamma_1 \Rightarrow \alpha \to \beta$$

is also provable. Now, it is easily proved using analytic cut that the sequents

$$\Gamma_1 \Rightarrow \alpha \to \beta, \varphi' \qquad \varphi', \Gamma_2 \Rightarrow$$

where $\varphi' = 0$ if $\varphi = 1$ and $\varphi' = 1$ otherwise, are also provable. □

THEOREM 6.7. *Bi-intuitionistic logic has Maximova variable separation property.*

Proof. Assume $\mathrm{Var}(\alpha_1, \alpha_2) \cap \mathrm{Var}(\beta_1, \beta_2) = \emptyset$ and $(\alpha_1 \wedge \beta_1) \to (\alpha_2 \vee \beta_2)$ is a theorem of **BiInt**. Then, the sequent $\alpha_1, \beta_1 \Rightarrow \alpha_2, \beta_2$ is provable in $\mathbf{LBJ}_1$. Consider its partition $(\alpha_1 : \alpha_2), (\beta_1 : \beta_2)$. This partition is variable-disjoint, so by Lemma 6.6 it has an interpolant γ, with $\mathrm{Var}(\gamma) = \emptyset$, and so, as before, we can assume without loss of generality that $\gamma \in \{0, 1\}$. Moreover, the sequents

$$\alpha_1 \Rightarrow \alpha_2, \gamma \qquad \gamma, \beta_1 \Rightarrow \beta_2$$

are provable. Now, if γ is 0, then $\alpha_1 \Rightarrow \alpha_2$ is provable, and if γ is 1, then $\beta_1 \Rightarrow \beta_2$ is provable. Thus, $\alpha_1 \to \alpha_2$ is a theorem or $\beta_1 \to \beta_2$, as needed for Maximova property. □

As in the case of Halldén completeness, here as well proof-theoretic methods do not readily apply to extensions of **BiInt**; semantic methods seem to be more promising in tackling our last question.

QUESTION 6.8. *Which extensions of* **BiInt** *have Maximova variable separation property*?

As this article was going to print, we learned that the forthcoming Badia (2016) provides frame theoretic and algebraic characterisations of extensions of **BiInt** with Maximova's property.

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