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The derivative of a constructible function is constructible

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Abstract. The notion of constructible functions in the setting of tame real geometry has been introduced by Cluckers and Miller in their work on parametric integration of globally subanalytic functions. A function on a globally subanalytic set is called constructible if it is a finite sum of finite products of globally subanalytic functions and the logarithm of positive globally subanalytic functions. We show that the class of constructible functions is stable under taking derivatives.

1 Introduction

Log-analytic functions have been defined by Lion and Rolin in their seminal paper [8]. They are iterated compositions from either side of globally subanalytic functions and the global logarithm and allow a preparation theorem.

An important subclass is given by the so-called constructible functions. This notion has been coined by Cluckers and Miller in their work on parametric integration of globally subanalytic functions [2–4], succeeding the work of Lion and Rolin [9] and Comte et al. [5]. A function on a globally subanalytic set is called constructible if it is a finite sum of finite products of globally subanalytic functions and the logarithm of positive globally subanalytic functions. The key property of the class of constructible functions is its closedness under parametric integration. For establishing this, Cluckers and Miller have also proved a preparation theorem and a decay result in this setting.

In [7], it was shown that the derivative of a differentiable log-analytic function is again log-analytic. In this note, we show that the subclass of constructible functions is also stable under taking derivatives. Here is the main result.

Theorem 1 *Let $U \subset \mathbb{R}^n$ be an open and globally subanalytic set and let $f : U \to \mathbb{R}$ be constructible. Let $i \in \{1, \ldots, n\}$ be such that f is differentiable with respect to the variable x_i on U. Then $\partial f/\partial x_i$ is constructible.*

This result is a necessary ingredient in the recent work of Huber et al. [6], establishing a constructible de Rham theorem in the globally subanalytic setting. The proof of the theorem here adapts the strategy of [7, Theorem A], establishing a suitable preparation theorem for constructible functions on so-called simple

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cells. The preparation theorem is an appropriate variant of the one formulated in [3, Corollary 3.5]. Aizenbud et al. follow in their preprint [1, Propositions 2.7 and 4.12] a similar approach to show that a much larger class called $\mathcal{C}^{\exp}$ is stable under taking limits and derivatives.

2 Notations and preliminaries

The empty sum is by definition 0 and the empty product is by definition 1.

By $\mathbb{N} = \{1, 2, \ldots\}$ we denote the set of natural numbers and by $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ the set of nonnegative integers.

We set $\mathbb{R}_{>0} := \{x \in \mathbb{R} \mid x > 0\}$. For $a, b \in \mathbb{R}$ with $a \leq b$, we denote by $[a, b]$ the closed interval and by (a, b) the open interval with endpoints a, b, respectively. By $|\ |$ we denote the Euclidean norm on $\mathbb{R}^n$.

We assume familiarity with the notion of o-minimal structures and of globally subanalytic sets and functions (see, e.g., van den Dries [10] or van den Dries and Miller [11]; see also the preliminaries of [7]).

We give the definition of a constructible function established by Cluckers and Miller in [2].

Definition Let $X \subset \mathbb{R}^n$ be a globally subanalytic set. A function $f : X \to \mathbb{R}$ is called *constructible* (on X) if there are $k, l \in \mathbb{N}_0$ and globally subanalytic functions $(f_i : X \to \mathbb{R})_{1 \leq i \leq k}$ and $(g_{ij} : X \to \mathbb{R}_{>0})_{1 \leq i \leq k, 1 \leq j \leq l}$ such that

$$f = \sum_{i=1}^{k} f_i \prod_{j=1}^{l} \log(g_{ij}).$$

Note that a constructible function is definable in the structure $\mathbb{R}_{\mathrm{an,exp}}$ and that the set of constructible functions on a given globally subanalytic set is a ring with respect to pointwise addition and multiplication which contains the globally subanalytic functions on that set. Note also that the class of constructible functions is closed under plugging in globally subanalytic maps.

3 Results

We let $x = (x_1, \ldots, x_n)$ range over $\mathbb{R}^n$ and y over $\mathbb{R}$. We set $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $(x, y) \mapsto x$.

For the following definition compare with [2, 8] and also with van den Dries and Speissegger [12]. Let $C \subset \mathbb{R}^n \times \mathbb{R}$ be globally subanalytic.

Definition 1 A function $\widetilde{y} : C \to \mathbb{R}$ is called a globally subanalytic y-coordinate on C with *center* Θ if the following holds:

(a) We have $\widetilde{y} > 0$ or $\widetilde{y} < 0$ on C.
(b) The function Θ is a globally subanalytic function on $\pi(C)$.
(c) We have $\widetilde{y}(x, y) = y - \Theta(x)$.
(d) There is $\varepsilon \in (0, 1)$ such that $0 < |\widetilde{y}(x, y)| < \varepsilon |y|$ for all $(x, y) \in C$ or $\Theta = 0$.

Note that Θ is uniquely determined by $\widetilde{y}$.

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Remark 2 Note that Cluckers and Miller omit condition (d) in their results [3, Theorem 3.4 and Corollary 3.5]. But we can add it without problems by the original formulation of the globally subanalytic preparation theorem in [8, p. 862] or [12, Theorem 2.4].

We give the notion of so-called strong functions (cf. [3, Definition 3.3]).

Definition 3 A function $u : C \to \mathbb{R}$ is called *strong* on C with globally subanalytic y-coordinate $\widetilde{y}$ on C if $u = v \circ \varphi$ where the following holds:

(a) The function φ is given by

$$\varphi : C \to [-1,1]^s,$$
$$\varphi(x,y) = \Big(b_1(x), \ldots, b_{s-2}(x), b_{s-1}(x)|\widetilde{y}|^p, b_s(x)|\widetilde{y}|^{-p}\Big),$$

where $s \in \mathbb{N}_0$, $p \in \mathbb{Q}$, and $b_1, \ldots, b_s$ are globally subanalytic functions on $\pi(C)$.

(b) The function v is a power series which converges on a neighborhood of $[-1,1]^s$.

Note that a strong function is globally subanalytic and bounded.

Given $x \in \mathbb{R}^n$, we set $C_x := \{y \in \mathbb{R} \mid (x,y) \in C\}$. We introduce the notion of a simple cell (see [7, Definition 2.15]).

Definition 4 We call C *simple* if, for every $x \in \pi(C)$, we have $C_x = (0, d_x)$ for some $0 < d_x < +\infty$.

Remark 5 Let $X \subset \mathbb{R}^n$ be a globally subanalytic set and let $\mathcal{C}$ be a globally subanalytic cell decomposition of $X \times (0,1)$. Then

$$X = \bigcup\{\pi(C) \mid C \in \mathcal{C} \text{ simple}\}.$$

Remark 6 Let C be simple. Then y is the only globally subanalytic y-coordinate on C.

Proof That y is a globally subanalytic y-coordinate on C is clear. Let $\widetilde{y}$ be a globally subanalytic y-coordinate on C with center Θ. Assume that $\Theta \neq 0$. Then, by Definition 3, there is $\varepsilon \in (0,1)$ such that

$$|y - \Theta(x)| < \varepsilon |y|$$

for all $(x,y) \in C$. Let $x \in \pi(C)$ be such that $\Theta(x) \neq 0$. Then we obtain

$$+\infty = \lim_{y \searrow 0} \left|1 - \frac{\Theta(x)}{y}\right| \leq \varepsilon,$$

which is a contradiction. ∎

Theorem 7 *Let $X \subset \mathbb{R}^n$ be a globally subanalytic set and let $F : X \times (0,1)$ be a constructible function. Then there exists a cell decomposition $\mathcal{C}$ of $X \times (0,1)$ such that, for every $C \in \mathcal{C}$ which is simple, the following holds:*

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(1) *We have*

$$f|_C(x,y) = \sum_{i=1}^{M} a_i(x)u_i(x,y)y^{p_i}|\log y|^{l_i},$$

where $M \in \mathbb{N}_0$ *and for each* $i \in \{1,\dots,M\}$, $a_i : \pi(C) \to \mathbb{R}$ *is constructible,* $u_i : C \to \mathbb{R}$ *is strong,* $p_i \in \mathbb{Q}$, *and* $l_i \in \mathbb{N}_0$.

(2) *For each* $i \in \{1,\dots,M\}$, *we have* $u_i = 1$ *or* $p_i > 0$.

(3) *The pairs* (p_i, l_i) *where* $p_i \le 0$ *are pairwise distinct.*

Proof By [3, Corollary 3.5] in connection with Remarks 2 and 6, we get such a cell decomposition such that Property (1) holds. Applying the additional statement of [3, Corollary 3.5], we immediately obtain a presentation such that, for each i, we have $u_i = 1$ or $p_i > -1$. But the proof thereof can be exactly used to obtain also Property (2). Property (3) is then in view of Property (2) easily obtained by collecting summands. ∎

Theorem 8 *Let* $X \subset \mathbb{R}^n$ *be globally subanalytic and let* $F : X \times (0,1) \to \mathbb{R}$ *be constructible. Assume that, for every* $x \in X$, *we have that* $\lim_{y \searrow 0} F(x,y)$ *exists and is finite. Then the function* $h : X \to \mathbb{R}, x \mapsto \lim_{y \searrow 0} F(x,y)$, *is constructible.*

Proof By Theorem 7, we find a cell decomposition $\mathcal{C}$ of $X \times (0,1)$ such that, for every simple $C \in \mathcal{C}$, we have

$$F|_C(x,y) = \sum_{i=1}^{M} a_i(x)u_i(x,y)y^{p_i}|\log y|^{l_i}$$

with the properties mentioned therein. We show that $h|_{\pi(C)}$ is constructible for such C, and we are done by Remark 5. For $x \in \pi(C)$ and $i \in \{1,\dots,M\}$ with $p_i > 0$, we have

$$\lim_{y \searrow 0} a_i(x)u_i(x,y)|y|^{p_i}|\log y|^{l_i} = 0.$$

Hence, we can replace $F|_C$ by

$$\sum_{p_i \le 0} a_i(x)u_i(x,y)y^{p_i}|\log y|^{l_i} = \sum_{p_i \le 0} a_i(x)y^{p_i}|\log y|^{l_i}.$$

We can therefore assume that a priori $p_i \le 0$ for all $i \in \{1,\dots,M\}$. We may also assume that $M \ge 1$ and that a_i does not vanish identically for each $i \in \{1,\dots,M\}$. We may further assume that p_1 is minimal among the p_i and that l_1 is maximal among those l_i for which $p_i = p_1$. Then

$$\lim_{y \searrow 0} \frac{y^{p_i}|\log y|^{l_i}}{y^{p_1}|\log y|^{l_1}} = 0$$

for every $i > 1$. Since a_1 does not vanish identically, we obtain by the assumption $\lim_{y \searrow 0} F(x,y) \in \mathbb{R}$ for every $x \in \pi(C)$ that $(p_1, l_1) = (0,0)$ (and that $M = 1$). We conclude that $h|_{\pi(C)} = a_1$ and are done.

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With the above theorem, we are able to establish our main result (compare the proof of [7, Theorem A]). ∎

Theorem 9 *Let $U \subset \mathbb{R}^n$ be an open and globally subanalytic set and let $f : U \to \mathbb{R}$ be constructible. Let $i \in \{1, \ldots, n\}$ be such that f is differentiable with respect to the variable x_i on U. Then $\partial f / \partial x_i$ is constructible.*

Proof We may assume that f is differentiable with respect to the last variable x_n. We have to show that $\partial f / \partial x_n$ is constructible. Let $e_n := (0, \ldots, 0, 1) \in \mathbb{R}^n$ be the n^{th} unit vector. We define

$$F : U \times (0,1) \to \mathbb{R}, (x, y) \mapsto \begin{cases} \frac{f(x+ye_n)-f(x)}{y}, & x + ye_n \in U, \\ \text{if} & \\ 0, & x + ye_n \notin U. \end{cases}$$

Then F is constructible. Since

$$\frac{\partial f}{\partial x_n}(x) = \lim_{y \searrow 0} F(x, y)$$

for $x \in U$, we are done by Theorem 8. ∎

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