Math. Proc. Camb. Phil. Soc. (1976), **80**, 437
MPCPS 80–43
Printed in Great Britain

A Fitting class construction

BY TREVOR HAWKES
University of Warwick

(*Received* 5 *January* 1976)

This note is devoted to the construction of a $\langle Q, E_\Phi \rangle$-closed Fischer class which is not a formation. It provides a negative answer to the Questions 3–7 raised by Cossey in (1). All groups considered are finite and soluble.

1. *Preliminaries.* Classes of groups (denoted by Gothic capitals) are closed under isomorphisms and contain groups of order 1. If S is a collection of groups, (S) will denote the smallest class containing S. A *closure operation* is a map from classes to classes satisfying

$$\mathfrak{X} \subseteq C\mathfrak{X} = C^2\mathfrak{X} \subseteq C\mathfrak{Y},$$

whenever $\mathfrak{X} \subseteq \mathfrak{Y}$. As usual $S\mathfrak{X}$ ($S_n\mathfrak{X}$) denotes the class of subgroups (subnormal) subgroups of $\mathfrak{X}$-groups, $Q\mathfrak{X}$ the class of epimorphic images of $\mathfrak{X}$-groups; S, S_n and Q are well known to be closure operations, as are R_0, N_0 and E_Φ defined thus:

$$R_0\mathfrak{X} = (G: \exists N_1, \ldots, N_r \trianglelefteq G \text{ with each } G/N_i \in \mathfrak{X} \text{ and } \bigcap_{i=1}^{r} N_i = 1);$$

$$N_0\mathfrak{X} = (G: G = \langle N_1, N_2, \ldots, N_r \rangle \text{ with each } N_i \operatorname{sn} G \text{ and } N_i \in \mathfrak{X});$$

$$E_\Phi\mathfrak{X} = (G: \exists K \trianglelefteq G, K \leq \phi(G), \text{ such that } G/K \in \mathfrak{X}).$$

If A and B are closure operations, a class $\mathfrak{X}$ is A-closed if $\mathfrak{X} = A\mathfrak{X}$, and is $\langle A, B \rangle$-closed if it is both A-closed and B-closed. A *formation* is a $\langle Q, R_0 \rangle$-closed class; associated with each formation $\mathfrak{X}$ is the $\mathfrak{X}$-*residual* $G^{\mathfrak{X}}$, the smallest normal subgroup of G with quotient in $\mathfrak{X}$. If a formation is E_Φ-closed, it is called *saturated*. $\mathfrak{F}$ is a *Fitting class* if $\mathfrak{F} = \langle S_n, N_0 \rangle \mathfrak{F}$. A *Fischer class* is a Fitting class $\mathfrak{F}$ which satisfies:

$$H \leq G \in \mathfrak{F} \quad \text{and} \quad H/\operatorname{Core}_G(H) \in \mathfrak{N} \Rightarrow H \in \mathfrak{F};$$

here $\mathfrak{N}$ denotes the class of nilpotent groups. Fischer classes may also be described by means of closure operations: define

$$S_F\mathfrak{X} = (H: H \leq G \in \mathfrak{X} \quad \text{and} \quad H^{\mathfrak{N}} \operatorname{sn} G).$$

LEMMA 1. (*a*) S_F *is a closure operation*;
(*b*) $S_n \leq S_F \leq S$;
(*c*) $\mathfrak{F}$ *is a Fischer class if and only if* $\mathfrak{F} = \langle S_F, N_0 \rangle \mathfrak{F}$.

Proof. (*a*) It is obvious that S_F is expanding and monotonic. We show it is also idempotent. When $H \leq G$ and $H^{\mathfrak{N}} \operatorname{sn} G$, write $H \operatorname{sf} G$. Suppose that $L \operatorname{sf} H \operatorname{sf} G$; we must show that $L \operatorname{sf} G$, i.e. that $L^{\mathfrak{N}} \operatorname{sn} G$. But $L/L \cap H^{\mathfrak{N}} \cong LH^{\mathfrak{N}}/H^{\mathfrak{N}} \leq H/H^{\mathfrak{N}} \in \mathfrak{N}$ and therefore $L^{\mathfrak{N}} \leq H^{\mathfrak{N}}$. Since $L^{\mathfrak{N}} \operatorname{sn} H$, we have $L^{\mathfrak{N}} \operatorname{sn} H^{\mathfrak{N}} \operatorname{sn} G$, and the result follows.

https://doi.org/10.1017/S030500410005307X Published online by Cambridge University Press

(*b*) It is clear from the definitions that an S-closed class is S_F-closed, and that an S_F-closed class is S_n-closed.

(*c*) Let $\mathfrak{F} = S_F\mathfrak{F}$, and let $H \leq G$ and $H/\mathrm{Core}_G(H) \in \mathfrak{N}$. Then $H^{\mathfrak{N}} \trianglelefteq \mathrm{Core}_G(H) \trianglelefteq G$ and so $H^{\mathfrak{N}} \,\mathrm{sn}\, G$. Therefore $H \in S_F\mathfrak{F} = \mathfrak{F}$. The sufficiency of the condition is now clear.

Conversely, suppose that $\mathfrak{F}$ is a Fischer class; and let $H \,\mathrm{sf}\, G \in \mathfrak{F}$. Then

$$H^{\mathfrak{N}} = L_t \trianglelefteq L_{t-1} \trianglelefteq \ldots \trianglelefteq L_1 = G.$$

Moreover, the subgroups L_i in this subnormal chain may be chosen H-invariant. For suppose inductively that $L_i\,(\neq H^{\mathfrak{N}})$ has been so chosen; since $H^{\mathfrak{N}} \,\mathrm{sn}\, L_i$ and therefore $H^{\mathfrak{N}} \,\mathrm{sn}\, K \lhd L_i$ for some K, we may then take $L_{i+1} = \bigcap_{h \in H} K^h$. We now show by induction on i that $L_iH \in \mathfrak{F}$ for $i = 1, 2, \ldots$. This is certainly true for $i = 1$, so assume that $L_kH \in \mathfrak{F}$ for some $k \geqslant 1$. Then $L_{k+1}H/L_{k+1} \cong H/H \cap L_{k+1} \in \mathrm{Q}(H/H^{\mathfrak{N}}) \subseteq \mathfrak{N}$. Since $L_{k+1} \trianglelefteq L_kH$, the quotient $L_{k+1}H/\mathrm{Core}_{L_kH}(L_{k+1}H)$ is therefore nilpotent. Since $\mathfrak{F}$ is a Fischer class, $L_{k+1}H \in \mathfrak{F}$. Therefore by induction $H = L_tH \in \mathfrak{F}$, and thus $\mathfrak{F} = S_F\mathfrak{F}$.

Use '$M \lessdot G$' to denote the statement 'M is a maximal subgroup of G'. The next result provides a useful criterion for checking that a given class is S_F-closed.

LEMMA 2. *A class $\mathfrak{X}$ is S_F-closed if and only if*:

(†) *whenever $1 \neq G \in \mathfrak{X}$ and $G^{\mathfrak{N}^2} \leq M \lessdot G$, then $M \in \mathfrak{X}$.*

Proof. We prove the necessity first. Let $1 \neq G \in \mathfrak{X} = S_F\mathfrak{X}$ and $G^{\mathfrak{N}^2} \leq M \lessdot G$. If $M \lhd G$, certainly $M^{\mathfrak{N}} \,\mathrm{sn}\, G$. If $M \not\lhd G$, we have $MG^{\mathfrak{N}} = G$. Next we observe that if K is a nilpotent normal subgroup of a group X and if $M \lessdot X = MK$, then $M \cap K \lhd G$, because $M \cap K \lhd M$ and $N_K(M \cap K) > M \cap K$. By setting $X = G/G^{\mathfrak{N}^2}$, we may conclude that $M \cap G^{\mathfrak{N}} \lhd G$. Since $M^{\mathfrak{N}} \lhd M \cap G^{\mathfrak{N}}$, we again have $M^{\mathfrak{N}} \,\mathrm{sn}\, G$ and hence $M \in S_F\mathfrak{X} = \mathfrak{X}$. Thus (†) holds.

Next suppose that $\mathfrak{X}$ satisfies condition (†). Let $H \,\mathrm{sf}\, G \in \mathfrak{X}$, and assume inductively that for all $L \in \mathfrak{X}$ with $|L| < |G|$ we have $S_F(L) \subseteq \mathfrak{X}$. If $HG^{\mathfrak{N}^2} = G$, we have $H^{\mathfrak{N}}G^{\mathfrak{N}^2} = G^{\mathfrak{N}}$. But $H^{\mathfrak{N}} \,\mathrm{sn}\, G^{\mathfrak{N}}$, and therefore $H^{\mathfrak{N}} = G^{\mathfrak{N}}$; this is because $H^{\mathfrak{N}}$ is a subnormal supplement of $[G^{\mathfrak{N}}, G^{\mathfrak{N}}]$ in $G^{\mathfrak{N}}$ and $G^{\mathfrak{N}}$ is soluble. Thus $H = HH^{\mathfrak{N}} = HG^{\mathfrak{N}} \geq HG^{\mathfrak{N}^2} = G \in \mathfrak{X}$. On the other hand, if $HG^{\mathfrak{N}^2} \neq G$, let $HG^{\mathfrak{N}^2} \leq M \lessdot G$. Clearly $H \,\mathrm{sf}\, M$, by (†) $M \in \mathfrak{X}$, and therefore $H \in \mathfrak{X}$ by the inductive hypothesis. This completes the proof.

The next result is well known, and is easily deduced from Huppert (3), I.12.6 (p. 70).

LEMMA 3. *Let B be a directly indecomposable group such that $(|B:B'|, |Z(B)|) = 1$. Let*

$$D = B_1 \times \ldots \times B_t \quad \textit{with} \quad B_i \cong B.$$

If $\alpha \in \mathrm{Aut}\,(D)$, α induces a permutation on the subgroups $\{B_1, \ldots, B_t\}$, and

$$\mathrm{Aut}\,(D) \cong \mathrm{Aut}\,(B) \wr \mathrm{Sym}\,(t).$$

(Here the wreath product is with respect to the natural representation of $\mathrm{Sym}\,(t)$, the symmetric group of degree t.)

LEMMA 4. *Let G be a π-group, and let $\mathfrak{X}$ be a Fitting formation containing $\mathfrak{N}_\pi$, the class of nilpotent π-groups. Let $G = N_1N_2$, where $N_i \trianglelefteq G$ and $G/N_i \in \mathfrak{N}$, $i = 1, 2$; then*

$$G^{\mathfrak{X}} = N_1^{\mathfrak{X}}N_2^{\mathfrak{X}}.$$

https://doi.org/10.1017/S030500410005307X Published online by Cambridge University Press

Proof. Set $K = G^{\mathfrak{X}}$ and $L = N_1^{\mathfrak{X}} N_2^{\mathfrak{X}}$. Since $\mathfrak{N}_\pi \subseteq \mathfrak{X}$, we have $K \leq G^{\mathfrak{N}_\pi} = G^{\mathfrak{N}} \leq N_i$ by hypothesis. Therefore $N_i/K \in \mathrm{S}_n\mathfrak{X} = \mathfrak{X}$, and hence $N_i^{\mathfrak{X}} \leq K$ for $i = 1, 2$. Thus $L \leq K$.

Next let $\{i, j\} = \{1, 2\}$. Then $N_iL/L \cong N_i/(N_i \cap N_i^{\mathfrak{X}} N_j^{\mathfrak{X}}) \in \mathrm{Q}(N_i/N_i^{\mathfrak{X}}) \subseteq \mathrm{Q}\mathfrak{X} = \mathfrak{X}$. Therefore $G/L = (N_1L/L)(N_2L/L) \in \mathrm{N}_0\mathfrak{X} = \mathfrak{X}$. Hence $K \leq L$ and equality holds.

We conclude this section by stating explicitly the following elementary fact.

REMARK. *Let $\mathfrak{X}$ be an* S*-closed formation. If $H \leq G$, then $H^{\mathfrak{X}} \leq G^{\mathfrak{X}}$.*

2. *The class* $\mathfrak{F}$. Let E denote an extraspecial group of order 5^3 and exponent 5. Let N denote an irreducible $\mathbb{Z}_{11}E$-module, faithful for E, and B the semi-direct product of N by E. Since $5 \mid (11-1)$, N has degree 5 and is absolutely irreducible (see Huppert (3), V.16.14 on p. 562). Although E has four isomorphism classes of such modules N, the group B is unique (up to isomorphism); this is because E admits an automorphism of order 4 which permutes transitively the elements of $Z(E)\backslash 1$. When, in the sequel, B_i denotes an isomorphic copy of B, the symbols E_i and N_i will denote the images of E and N respectively.

If $\mathfrak{A}$ and $\mathfrak{B}$ are classes of groups, as usual let $\mathfrak{A}\mathfrak{B} = (G: \exists N \trianglelefteq G, N \in \mathfrak{A}$ such that $G/N \in \mathfrak{B})$. Recall that if $\mathfrak{A} = \mathrm{N}_0\mathfrak{A}$ and $\mathfrak{B}$ is a Q-closed class, then $(\mathfrak{A}\mathfrak{B})\mathfrak{C} = \mathfrak{A}(\mathfrak{B}\mathfrak{C})$, and we can leave the brackets out. Also take note that if $\mathfrak{A}$ and $\mathfrak{B}$ are formations with $\mathfrak{A} = \mathrm{S}_n\mathfrak{A}$, then $\mathfrak{A}\mathfrak{B}$ is a formation and $G^{\mathfrak{A}\mathfrak{B}} = (G^{\mathfrak{B}})^{\mathfrak{A}}$. If π is a set of primes, let $\mathfrak{S}_\pi$ denote the class of soluble π groups, and for notational convenience define classes

$$\mathfrak{W} = \mathfrak{S}_{11}\mathfrak{S}_5\mathfrak{S}_3;$$
$$\mathfrak{X} = \mathfrak{S}_{11}\mathfrak{S}_5\mathfrak{S}_3\mathfrak{S}_2 = \mathfrak{W}\mathfrak{S}_2;$$
$$\mathfrak{Y} = \mathfrak{S}_{11}\mathfrak{S}_3\mathfrak{S}_{11}\mathfrak{S}_5\mathfrak{S}_2\mathfrak{S}_{11}\mathfrak{S}_5\mathfrak{S}_3 = \mathfrak{S}_{11}\mathfrak{S}_3\mathfrak{Z};$$
$$\mathfrak{Z} = \mathfrak{S}_{11}\mathfrak{S}_5\mathfrak{S}_2\mathfrak{S}_{11}\mathfrak{S}_5\mathfrak{S}_3 = \mathfrak{S}_{11}\mathfrak{S}_5\mathfrak{S}_2\mathfrak{W}.$$

Now we come to the class $\mathfrak{F}$, which we aim to show is a Q-closed Fischer class but not a formation:

$$\mathfrak{F} = (G \in \mathfrak{X}: \text{if } L = G^{\mathfrak{Y}}, \text{ then } L/0_5(L) = \bar{B}_1 \times \ldots \times \bar{B}_t,\ t \geq 0,$$
$$\text{where } \bar{B}_i \cong B \text{ for each } i, \text{ and if } \bar{B}_i = B_i/0_5(L), \text{ then } B_i \trianglelefteq G^{\mathfrak{Z}}).$$

Here we use the convention that $\bar{B}_1 \times \ldots \times \bar{B}_t = 1$, when $t = 0$. For $G \in \mathfrak{X}$ and $L = G^{\mathfrak{Y}}$, call $L/0_5(L)$ the *key section* of G. Thus the key section of a group in $\mathfrak{F}$ is either trivial or a direct product of copies of B. It is perhaps worth remarking that $\mathfrak{N}^3 \cap \mathfrak{X} \subseteq \mathfrak{F}$. For, if $G \in \mathfrak{X}$, then $L \leq G^{\mathfrak{N}^2}$, and therefore if $G \in \mathfrak{X} \cap \mathfrak{N}^3$, we have

$$L = 0^{11}(L) \in \mathfrak{N} \cap \mathfrak{S}_{\{5,11\}},$$

whence $L \in \mathfrak{S}_5$ and G has trivial key section.

LEMMA 5. *Let $G \in \mathfrak{X}$. Then $G \in \mathfrak{F}$ if and only if $G^{\mathfrak{Y}} \in \mathfrak{F}$.*

Proof. Let $K = G^{\mathfrak{Y}}$ and $K_0 = K^{\mathfrak{Y}}$. Then $G/K_0 \in \mathfrak{X} \cap \mathfrak{S}_{\{3,5,11\}} = \mathfrak{W}$, and therefore $K = K_0$. Hence $K^{\mathfrak{Y}} = G^{\mathfrak{Y}}$ and $K^{\mathfrak{Z}} = G^{\mathfrak{Z}}$, and the result is now clear.

LEMMA 6. *Let $G \in \mathfrak{X}$ and set $L = G^{\mathfrak{Y}}$. Let $K \trianglelefteq G$ and assume that $L \cap K \leq 0_5(L)$. Then $G \in \mathfrak{F}$ if and only if $G/K \in \mathfrak{F}$.*

https://doi.org/10.1017/S030500410005307X Published online by Cambridge University Press

Proof. If $L \cap K \leq 0_5(L)$, then $0_5(L/L \cap K) = 0_5(L)/L \cap K$, and therefore

$$0_5(LK/K) = 0_5(L)\,K/K$$

because of the isomorphism $LK/K \cong L/L \cap K$. Since

$$L/0_5(L) \underset{G}{\cong} LK/0_5(L)\,K \underset{G}{\cong} (LK/K)/(0_5(L)\,K/K),$$

there exists a G-isomorphism θ from $L/0_5(L)$ onto $(LK/K)/0_5(LK/K)$. Because $(G/K)^{\mathfrak{V}} = LK/K$, it follows that G and G/K have isomorphic key sections. Furthermore, since $(G/K)^{\mathfrak{F}} = G^{\mathfrak{F}}K/K$, the normality requirements for the direct factors translate intact from one key section to the other under θ and θ^{-1}. The result now follows.

COROLLARY. *Let $G \in \mathfrak{X}$, and assume that G has a normal $\{2,3,5\}$-subgroup K such that $G/K \in \mathfrak{F}$. Then $G \in \mathfrak{F}$.*

Proof. Let $L = G^{\mathfrak{V}}$. Because $G \in \mathfrak{X}$, we have $L \leq G^{\mathfrak{N}_2} \in \mathfrak{S}_{\{5,11\}}$ and therefore

$$L \cap K \in \mathfrak{S}_{\{5,11\}} \cap \mathfrak{S}_{\{2,3,5\}} = \mathfrak{S}_5.$$

Hence $L \cap K \leq 0_5(L)$, and $G \in \mathfrak{F}$ by Lemma 6.

Next we discuss the automorphism groups of E and B. First observe that $\mathrm{Inn}\,(E)$ is elementary abelian of order 25 and that $\mathrm{Aut}\,(E)/\mathrm{Inn}\,(E) \cong GL(2,5)$. If

$$S = C_{\mathrm{Aut}(E)}(Z(E)),$$

then $S/\mathrm{Inn}\,(E) \cong SL(2,5)$, which contains a non-abelian subgroup T^* of order 12 with cyclic Sylow subgroups. Hence, by the Schur–Zassenhaus theorem, S contains a subgroup $T \cong T^*$. Since $[Z(E), T] = 1$ and $(|E|, |T|) = 1$, by (2·6) of Hawkes (2), the absolutely irreducible $\mathbb{Z}_{11}E$-module N can be considered as a module for the semi-direct product ET; in fact, there are two inequivalent extensions of N from E to ET, and so, to be specific, let N denote the (unique) $\mathbb{Z}_{11}ET$-module (extending N) for which the elements of ET induce on N linear transformations of determinant 1 (again compare (2·6) of (2)). Now denote the semi-direct product of N with ET by H. Thus H has order $2^2 \cdot 3 \cdot 5^3 \cdot 11^5$, has a normal subgroup $B = NE$ such that $C_H(B) = 1$, and has nilpotent length 4, with lower nilpotent series

$$1 < N < B < B[T,T] < H.$$

Moreover, we have $H^{\mathfrak{V}} = B$ and $H^{\mathfrak{F}} = B[T,T]$, and therefore $H \in \mathfrak{F}$; in particular, $\mathfrak{F}$ contains groups of length 4.

LEMMA 7. (*a*) *Let* $\mathrm{Inn}\,(B) \leq A \leq \mathrm{Aut}\,(B)$ *and assume that* $A \in \mathfrak{W}$. *Then*

$$|A : \mathrm{Inn}\,(B)| = 1, 3 \text{ or } 5.$$

(*b*) *If* $U \leq \mathrm{Aut}\,(B)$ *and* $3 \mid |U|$, *then* $[B, U] = B$.

Proof. For $b \in B$, let ρ_b denote the inner automorphism $x \to b^{-1}xb$ $(x \in B)$ of B. Since $Z(B) = 1$, the map $b \to \rho_b$ is an $\mathrm{Aut}\,(B)$-monomorphism from B into $\mathrm{Aut}\,(B)$; therefore identify B with its image $\mathrm{Inn}\,(B)$. Let $K \trianglelefteq A$. Since N char $B \trianglelefteq A$ and $N \cdot \lhd B$, we have $N \cdot \lhd A$ (using '$\cdot \trianglelefteq$' to mean 'is a minimal normal subgroup of'). Therefore if

https://doi.org/10.1017/S030500410005307X Published online by Cambridge University Press

$N \not\leq K$, then $[B, K] \leq B \cap K = 1$ and hence $K = 1$ because $K \leq \mathrm{Aut}(B)$. Hence the assumption $K \neq 1$ implies that $N \leq K$. In particular, $F = F(A)$ is an 11-group containing N. Since $F \cap EN = N$, we have $[F, EN] \leq N$, i.e. F centralizes the action of E on N. But N is absolutely irreducible as $\mathbb{Z}_{11}E$-module, and therefore the centralizer in $GL(5, 11)$ of the group of linear transformations induced by E on N is $Z(GL(5, 11))$, which is cyclic of order 10. Hence $F = N = C_A(N)$. Since by the Frattini argument $N_A(E)$ complements N in A, if $A > B$, we may choose an element $a \in N_A(E) \backslash B$. First suppose that a generates a $5'$-group, whence $\langle a \rangle \in \mathfrak{S}_{5'} \cap \mathfrak{W} = \mathfrak{S}_{\{3,11\}}$. If a centralizes $E/Z(E)\,(= E/\phi(E))$, it centralizes E, and as above $|\langle a \rangle N/N|$ divides 10, which is not the case. Hence $\langle a \rangle$ induces on $E/Z(E)$ a non-trivial group of automorphisms. Since the only non-trivial $\{3, 11\}$-subgroup of $GL(2, 5)$ has order 3, it follows that $a^3 \in B$. Next suppose that $\langle a \rangle \in \mathfrak{S}_5$. If a centralizes $E/Z(E)$, a induces an inner automorphism on E. Hence, for some $x \in E$, ax centralizes E, and as above $ax \in Z(E)$, which is not the case because $a \notin E$. Hence $\langle a \rangle / \langle a \rangle \cap E \cong C_5 \in \mathrm{Syl}_5(GL(2, 5))$ (C_n denotes a cyclic group of order n). Finally we remark that $\langle a \rangle B/B$ cannot have 'mixed' order because $GL(2, 5)$ contains no subgroup of order 15. This completes the proof of part (a). Part (b) is an easy consequence of this analysis. For, if $W \in \mathrm{Syl}_3(U)$, we may take $A = BW$ in the above. W acts non-trivially (and irreducibly) on $EN/Z(E)N$ and so $[B, W]$ is a normal subgroup of $B = EN$ covering this section. Hence $[B, W] = B$ and the result is clear.

3. $\mathfrak{F} = \langle \mathrm{Q}, \mathrm{S}_F \rangle \mathfrak{F}$

LEMMA 8. *Let $L = B_1 \times \ldots \times B_t$, with $B_i \cong B$, and let $K \trianglelefteq L$. Let $R/K = 0_5(L/K)$. Then $R = B_{s+1} \times \ldots \times B_t$ after suitable relabelling of the B_i's, and*

$$(L/K)/0_5(L/K) = (B_1R/K)/(R/K) \times \ldots \times (B_sR/K)/(R/K),$$

where each direct factor is isomorphic with B.

Proof. Set $N_i = 0_{11}(B_i)$ and $S = N_1 \times \ldots \times N_t$, the socle of L. Since the minimal normal subgroups $\{N_i\}$ are pairwise non-isomorphic as L-modules, the normal subgroup $S^\sharp := S \cap K$ of L is the direct product of a suitable subset of them. Say

$$S^\sharp = N_{s+1} \times \ldots \times N_t$$

after suitable relabelling. Note that $0_{11}(K) = 0_{11}(L) \cap K = S \cap K = S^\sharp$. Set

$$L^\flat = B_1 \times \ldots \times B_s, \quad L^\sharp = B_{s+1} \times \ldots \times B_t;$$

then $L^\sharp K/K \cong L^\sharp/L^\sharp \cap K \in \mathrm{Q}(L^\sharp/S^\sharp) \subseteq \mathfrak{S}_5$. Therefore $L^\sharp K \leq R$. Since $R/0_{11}(K) \in \mathfrak{S}_5$, we have $0_{11}(R) = S^\sharp$. With $S^\flat = N_1 \times \ldots \times N_s$, we have $S^\flat \cap R \leq 0_{11}(R) = S^\sharp$, whence $S^\flat \cap R \leq S^\flat \cap S^\sharp = 1$. Hence $L^\flat \cap R = 1$ and $R = R \cap L^\flat L^\sharp = L^\sharp$. It follows that $L/R \cong B_1R/R \times \ldots \times B_tR/R$, and the last statement of the lemma is now clear.

PROPOSITION A. $\mathfrak{F} = \mathrm{Q}\mathfrak{F}$.

Proof. Let $W \trianglelefteq G \in \mathfrak{F}$ and let $L = G^{\mathfrak{B}}$. In showing that $G/W \in \mathfrak{F}$, we may suppose without loss of generality that $0_5(L) = 1$. Then $L = B_1 \times \ldots \times B_t$, where $B_i \cong B$ and $B_i \trianglelefteq G^{\mathfrak{B}}$. Let $K = L \cap W \trianglelefteq L$ and $R/K = 0_5(L/K)$. We have $(G/W)^{\mathfrak{B}} = LW/W \underset{G}{\cong} L/K$.

https://doi.org/10.1017/S030500410005307X Published online by Cambridge University Press

Hence the key section of G/W is G-isomorphic with $(L/K)/0_5(L/K)$, which by Lemma 8 is $(B_1R/K)/(R/K) \times \ldots \times (B_sR/K)/(R/K)$ for an appropriate ordering of the B_i's. Since $(G/W)^{\mathfrak{F}} = G^{\mathfrak{F}}W/W$, it is now easy to see that G/W fulfils all the requirements of an $\mathfrak{F}$-group.

PROPOSITION B. $\mathfrak{F} = S_F\mathfrak{F}$.

Proof. We suppose that $\mathfrak{F} \neq S_F\mathfrak{F}$ and derive a contradiction. By Lemma 2, there exists a group $G \in \mathfrak{F}$ which possesses a maximal subgroup M such that $G^{\mathfrak{N}^2} \leq M \notin \mathfrak{F}$. Suppose that G has minimal order among such groups.

We first assert that the subgroup $0_{\{2,3,5\}}(G)$, $= R$ say, is trivial. By Proposition A, $G/R \in \mathfrak{F}$. Suppose that $R \neq 1$. Then either (i) $MR/R = G/R \in \mathfrak{F}$, or (ii) MR/R is a maximal subgroup of G/R containing $G^{\mathfrak{N}^2}R/R = (G/R)^{\mathfrak{N}^2}$. In case (ii) we have $MR/R \in \mathfrak{F}$ by minimal choice of G, and therefore in any case $M/M \cap R \in \mathfrak{F}$. But then by the Corollary to Lemma 6 we have $M \in \mathfrak{F}$, which is not the case. Therefore $R = 1$.

Now let $L = G^{\mathfrak{V}}$. Then $0_5(L) (\leq R) = 1$, and by hypothesis $L = B_1 \times \ldots \times B_t$, where $B_i \cong B$ and $B_i \trianglelefteq G^{\mathfrak{F}}$. Since $\mathfrak{N}^2 \cap \mathfrak{X} \subseteq \mathfrak{V}$, we have $L \leq G^{\mathfrak{N}^2} \leq M$. Let $K = M^{\mathfrak{F}}$ and $C_i = C_K(B_i)$; then $C_i \trianglelefteq K$ because $K \leq G^{\mathfrak{F}} \leq N_G(B_i)$ by the Remark of section 1. We distinguish two cases:

(1) $C_i < K$. Because $K^{\mathfrak{S}_{11}\mathfrak{S}_5} = K \in \mathfrak{V}$, we have $K/K^{\mathfrak{N}} \in \mathfrak{S}_3$, and therefore $3 \mid\mid K/C_i\mid$. Hence by Lemma 7(b), we have $B_i = [B_i, K] \leqslant B_i \cap K$ because $B_i \leq M \leq N_G(K)$; in particular, $B_i \leq K$. Let t be a 3-element in $K \backslash C_i$. It is clear from Lemma 7(b) that $(B_i\langle t\rangle)^{\mathfrak{S}_3} = B_i$, and using the Remark of section 1 we have

$$B_i = B_i^{\mathfrak{S}_{11}} = (B_i\langle t\rangle)^{\mathfrak{S}_{11}\mathfrak{S}_3} \leq K^{\mathfrak{S}_{11}\mathfrak{S}_3} = (M^{\mathfrak{F}})^{\mathfrak{S}_{11}\mathfrak{S}_3} = M^{\mathfrak{S}_{11}\mathfrak{S}_3\mathfrak{F}} = M^{\mathfrak{V}}.$$

(2) $C_i = K$. In this case $B_i \cap M^{\mathfrak{V}} \leq B_i \cap K = B_i \cap C_i = 1$, because

$$Z(B_i) = 1.$$

If we label the B_i so that $C_i = K$ for precisely the suffices $i = 1, \ldots, r$ (with the usual interpretation for $r = 0$), we have

$$B_1 \times \ldots \times B_r \cap M^{\mathfrak{V}} = 1; \qquad (\P)$$

this is because $\{N_i (= 0_{11}(B_i)) : i = 1, \ldots, r\}$ are the only minimal normal subgroups of the direct product $B_1 \times \ldots \times B_r$, and so its non-trivial normal subgroups have non-trivial intersection with at least one director factor. Since

$$B_{r+1} \times \ldots \times B_t \leq M^{\mathfrak{V}} \leq L = B_1 \times \ldots \times B_t,$$

it follows from ($\P$) that $M^{\mathfrak{V}} = B_{r+1} \times \ldots \times B_t$. Since $M \in S\mathfrak{X} = \mathfrak{X}$, we therefore have $M \in \mathfrak{F}$. This contradiction completes the proof.

4. *$\mathfrak{F}$ is N_0-closed.*

PROPOSITION C. $\mathfrak{F} = N_0\mathfrak{F}$.

Proof. Suppose, for a contradiction, that $\mathfrak{F} \neq N_0\mathfrak{F}$, and let G be a group of minimal order in $N_0\mathfrak{F} \backslash \mathfrak{F}$. Then by a standard argument $G = NN^*$, where N and N^* are maximal normal subgroups of G and belong to $\mathfrak{F}$. Let $p = |G:N|$ and $p^* = |G:N^*|$.

https://doi.org/10.1017/S030500410005307X Published online by Cambridge University Press

Let $R = 0_5(G)$. Then $G/R = (NR/R)(N^*R/R) \in \mathrm{N}_0\mathfrak{F}$ by Proposition A. If $R \neq 1$, by choice of G the class $\mathfrak{F}$ contains G/R and hence contains G by the Corollary to Lemma 6. Therefore $R = 1$. Let $L = N^{\mathfrak{V}}$, $K = N^{\mathfrak{I}}$, $L^* = (N^*)^{\mathfrak{V}}$, $K^* = (M^*)^{\mathfrak{I}}$. Then

$$L = B_1 \times \ldots \times B_t \quad \text{with} \quad B \cong B_i \trianglelefteq K,$$

and

$$L^* = B_1^* \times \ldots \times B_t^* \quad \text{with} \quad B \cong B_i^* \trianglelefteq K^*.$$

By Lemma 4 $G^{\mathfrak{V}} = LL^*$ and $G^{\mathfrak{I}} = KK^*$.

Now let $W = G^{\mathfrak{V}}$. If $W \leq N$, we have $W \in \mathrm{S}_n\mathfrak{F} = \mathfrak{F}$ by Proposition B and therefore $G \in \mathfrak{F}$ by Lemma 5. (Note that $G \in \mathrm{N}_0\mathfrak{X} = \mathfrak{X}$.) Hence $W \not\leq N$ and $p = 2$; similarly $p^* = 2$. Thus W and $N \cap N^*$ have coprime index in G, whence $G = W(N \cap N^*)$. In consequence $W/(W \cap N \cap N^*)$ is a 'four group', and therefore $W = (W \cap N)(W \cap N^*)$. Hence $W \in \mathrm{N}_0\mathfrak{F}$. If $W < G$, the class $\mathfrak{F}$ contains W by the minimality of G and therefore contains G by Lemma 5. Since this is not the case, $W = G$.

The next step is to show that $B_i \trianglelefteq G^{\mathfrak{I}}$ for $i = 1, \ldots, t$. First note that $L \lhd G$ because L char $N \lhd G$. Let $g \in 0_{11}(G)$, the Sylow 11-subgroup of G. If $g \notin N_G(B_j)$ for some j, by Lemma 3 a power x of g permutes a subset of $\{B_i : i = 1, \ldots, t\}$ in a cycle of length 11. But then $L\langle x\rangle$ contains a section isomorphic with $C_5 \wr C_{11}$, which is impossible since every section of G has a normal Sylow 11-subgroup. Therefore $0_{11}(G) \leq N_G(B_i)$ for $i = 1, \ldots, t$. Now let $L_0 = LL^*$, and choose $S_0 \in \mathrm{Syl}_5(L_0)$. Then $S_0 = SS^*$, where $S\,(= S_0 \cap L)$ and $S^*\,(= S_0 \cap L^*)$ are normal in S_0. Moreover, S and S^*, as Sylow 5-subgroups of L and L^*, are direct products of copies of E, an extraspecial group of order 5^3. Let $S_1 = S \cap S^*$. With $\gamma_1(S_0) = S_0$ and $\gamma_{i+1}(S_0) = [\gamma_i(S_0), S_0]$, $i = 1, 2, \ldots$, we have

$$\gamma_2(S_0) = [S, S]\,[S^*, S^*]\,[S, S^*] \leq Z(S)\,Z(S^*)\,S_1.$$

Hence

$$\begin{aligned}\gamma_3(S_0) &\leq [Z(S), S^*]\,[(Z(S^*), S]\,[S_1, S]\,[S_1, S^*] \\ &\leq (Z(S) \cap S_1)\,(Z(S^*) \cap S_1),\end{aligned}$$

and so

$$\begin{aligned}\gamma_4(S_0) &\leq [Z(S) \cap S_1, S^*]\,[Z(S^*) \cap S_1, S] \\ &\leq (Z(S) \cap (S^*)')\,(Z(S^*) \cap S') \\ &= Z(S) \cap Z(S^*) \leq Z(S_0).\end{aligned}$$

Therefore $\gamma_5(S_0) = 1$, whence S_0 and every 5-section of L_0 has class at most 4. Let $g \in S_0$. If g does not normalize some B_j, some power x of g permutes a subset of $\{B_i\}$, say $\{B_1, B_2, \ldots, B_5\}$, in a cycle of length 5. Choose a Sylow 5-subgroup $E^\sharp$ of $\langle B_1, x\rangle$ containing x. Since x^5 normalizes $E_i = E^\sharp \cap B_i$, $i = 1, \ldots, 5$, it centralizes $Z(E_i)$, and therefore $\langle Z(E_1), x\rangle/\langle x^5\rangle \cong C_5 \wr C_5$, which has class 5. From this contradiction, we conclude that $S_0 \leq N_G(B_i)$ for $i = 1, \ldots, t$. Now let $g \in J^* \in \mathrm{Syl}_3(K^*)$, and suppose, for a contradiction, that $g \notin N_G(B_j)$ for some j. Then by a now familiar argument some power x of g permutes a subset of $\{B_i\}$, say $\{B_1, B_2, B_3\}$, in a cycle of length 3. For $i = 1, 2, 3$, x^3 normalizes B_i, and by Lemma 7 (*b*) $[B_i, \langle x^3\rangle] = 1$ or B_i; in either case, if U denotes $\langle B_1, x\rangle$, $U^{\mathfrak{S}_{11}\mathfrak{S}_3\mathfrak{S}_{11}\mathfrak{S}_5}$ contains a subgroup of $B_1 \times B_2 \times B_3$ of index 5^2, and

https://doi.org/10.1017/S030500410005307X Published online by Cambridge University Press

in particular contains $0_{11}(B_1 \times B_2 \times B_3) = N_1 \times N_2 \times N_3$. Since $U \leq LJ^* \leq LK^*$ and $LK^*/K^* \cong L/L \cap K^* \in \mathfrak{S}_{11}\mathfrak{S}_5$, we have $U^{\mathfrak{S}_{11}\mathfrak{S}_5} \leq K^*$ and therefore

$$U^{\mathfrak{S}_{11}\mathfrak{S}_3\mathfrak{S}_{11}\mathfrak{S}_5} \leq (K^*)^{\mathfrak{S}_{11}\mathfrak{S}_3} = L^*.$$

Hence $N_i \leq L^*, i = 1, 2, 3$. N_i is a characteristic subgroup of B_i which we have already shown to be normalized by $0_{11}(G)S_0$. Since $L^* \leq 0_{11}(G)S_0$, we have $N_i \trianglelefteq L^*$. Now $N_1^*, \ldots, N_{t^*}^*$ are the only minimal normal subgroups of L^*, and therefore $N_1 = N_j^*$ for some j. But $N_j^* \operatorname{char} B_j^* \trianglelefteq K^*$; hence J^* normalizes B_j^*, contradicting the supposition that $x \in J^*$ permutes N_1, N_2, N_3 cyclically. Therefore $J^* \leq N_G(B_i), i = 1, \ldots, t$. Since $K^* \leq L^* 0_{11}(G) J^*$, we have proved that $B_i \trianglelefteq KK^* = G^{\mathfrak{F}}$, $i = 1, \ldots, t$. An identical argument shows that $B_i^* \trianglelefteq G^{\mathfrak{F}}$, $i = 1, \ldots, t^*$.

Finally we must show that LL^* is the direct product of a subset of $\{B_i, B_j^*\}$. Label the components of L so that $B_i \not\leq L^*$ for $i = 1, \ldots, r$ and $B_i \leq L^*$ for $i = r+1, \ldots, t$. We assert that $LL^* = (B_1 \times \ldots \times B_r) \times L^*$, and for this, it is sufficient to prove that $(B_1 \times \ldots \times B_r) \cap L^* = 1$. If not, there is a minimal normal subgroup of $B_1 \times \ldots \times B_r$, without loss of generality say N_1, contained in L^*; again without loss, we can take $N_1 = N_1^*$. Hence $B_1 B_1^* = N_1(E_1 E_1^*)$, with E_1 and E_1^* ($\trianglelefteq E_1 E_1^*$) Sylow 5-subgroups of B_1 and B_1^*. Since $B_1 \cap B_1^*$ is a normal subgroup of K properly contained in both B_1 and B_1^* and since $B_1/N_1 Z(E_1)$ and $B_1^*/N_1 Z(E_1^*)$ are chief factors of K (a fact implicit in the proof of Proposition B, cases (1) and (2)), it follows that $E_1 \cap E_1^* = 1$ or $Z(E_1)$. In either case, N_1 is an irreducible $\mathbb{Z}_{11}$-module for the central product $E_1 E_1^*$ and is faithful for $Z(E_1)$ and $Z(E_1^*)$. But then $E_1 E_1^*/C_{E_1 E_1^*}(N_1)$ is an extraspecial group of order 5^5 and its irreducible representations have dimension 25 over $\mathbb{Z}_{11}$ (cf. Huppert (3), V.16.14, p. 562), a contradiction. Hence $G^{\mathfrak{B}} = B_1 \times \ldots \times B_r \times B_1^* \times \ldots \times B_{t^*}^*$, and, since $G \in \mathfrak{X}$, it is clear that $G \in \mathfrak{F}$. This final contradiction completes the proof.

Thus far we have shown that $\mathfrak{F}$ is a Q-closed Fischer class. To see that it is not a formation, let N_1 and N_2 denote two copies of the $\mathbb{Z}_{11}(ET)$-module N, described in section 2; then the semi-direct product $(N_1 \times N_2)(ET)$ is a group in $\mathrm{R}_0\mathfrak{F} \setminus \mathfrak{F}$.

5. The main theorem.

THEOREM. *There exists a $\langle \mathrm{Q}, \mathrm{E}_\Phi \rangle$-closed Fischer class which is not a formation.*

The candidate is $\mathfrak{N}\mathfrak{F}$, and that it is a $\langle \mathrm{Q}, \mathrm{E}_\Phi \rangle$-closed Fischer class is an immediate consequence of Propositions A, B and C, and the following result.

LEMMA 9. *If $\mathfrak{X}$ is a Q-closed Fischer class, then $\mathfrak{N}\mathfrak{X}$ is a $\langle \mathrm{Q}, \mathrm{E}_\Phi \rangle$-closed Fischer class.*

Proof. Q-closure. Let $G \in \mathfrak{N}\mathfrak{X}$ and $K \trianglelefteq G$. G has a normal nilpotent subgroup N such that $G/N \in \mathfrak{X}$. Since $NK/K \cong N/N \cap K \in \mathrm{Q}\mathfrak{N} = \mathfrak{N}$, G/K has a normal nilpotent subgroup NK/K, and $(G/K)/(NK/K) \cong G/NK \in \mathrm{Q}(G/K) \subseteq \mathrm{Q}\mathfrak{X} = \mathfrak{X}$, and therefore $G/K \in \mathfrak{N}\mathfrak{X}$.

S_F-closure. Let $G \in \mathfrak{N}\mathfrak{X}$ and $N \trianglelefteq G$ with $N \in \mathfrak{N}$ and $G/N \in \mathfrak{X}$. Let $G^{\mathfrak{N}^2} \leq M < G$. If $MN = G$, then $M/M \cap N \cong G/N \in \mathfrak{X}$; since $M \cap N$ is a normal nilpotent subgroup of M, we have $M \in \mathfrak{N}\mathfrak{X}$. If $MN \neq G$, then $N \leq M$, and M/N is a maximal subgroup of G/N containing $G^{\mathfrak{N}^2}N/N = (G/N)^{\mathfrak{N}^2}$. Since $\mathfrak{X} = \mathrm{S}_F\mathfrak{X}$, by Lemma 2 $M/N \in \mathfrak{X}$. Hence $M \in \mathfrak{N}\mathfrak{X}$, and, again by Lemma 2, $\mathfrak{N}\mathfrak{X}$ is S_F-closed.

https://doi.org/10.1017/S030500410005307X Published online by Cambridge University Press

N_0-*closure.* Let $G = N_1N_2$ with $N_i \trianglelefteq G$ and $N_i \in \mathfrak{N}\mathfrak{X}$. If $F(G) \not\leq N_i$ for $i = 1$ or 2, we have $G = F(G)N_i$ and therefore $G/F(G) \cong N_i/F(G) \cap N_i = N_i/F(N_i)$. Since $N_i \in \mathfrak{N}\mathfrak{X}$, $N_i/F(N_i) \in \mathrm{Q}\mathfrak{X} = \mathfrak{X}$, and therefore $G/F(G) \in \mathfrak{X}$ and $G \in \mathfrak{N}\mathfrak{X}$. Therefore suppose that $F(G) \leq N_1 \cap N_2$. Then $G/F(G) = (N_1/F(G))(N_2/F(G)) \in \mathrm{N}_0\mathrm{Q}\mathfrak{X} = \mathfrak{X}$, and again $G \in \mathfrak{N}\mathfrak{X}$.

E_Φ-*closure.* Let $K \trianglelefteq G$, $K \leq \phi(G)$ and $G/K \in \mathfrak{N}\mathfrak{X}$. By Huppert (3), III.4.2 (d), p. 277 $F(G/K) = F(G)/K$, and so $G/F(G) \cong (G/K)/F(G/K) \in \mathrm{Q}\mathfrak{X} = \mathfrak{X}$. Hence $G \in \mathfrak{N}\mathfrak{X}$. This completes the proof of Lemma 9.

To complete the proof of the Theorem, we must show that $\mathfrak{N}\mathfrak{F}$ is not a formation. Let $H_i = N_iE_iT_i$ be a copy of the group H described in section 2, $i = 1, 2$. Let $p > 11$ be a prime, and let V_i be an irreducible $\mathbb{Z}_pH_i$-module, faithful for H_i, $i = 1, 2$; such a module exists because H_i is primitive and $0_p(H_i) = 1$. Let G_i be the semi-direct product of V_i by H_i, and let D denote the direct product $G_1 \times G_2$. Let $\theta: E_1T_1 \to E_2T_2$ be an isomorphism, let $W = \{w\theta(w): w \in E_1T_1\} \leq D$, and let $G = (V_1N_1 \times V_2N_2)W$. Then

$$F(G/V_1) = (V_1V_2N_1)/V_1$$

and therefore $(G/V_1)/F(G/V_1) \cong G/(V_1V_2N_1) \cong N_2W \cong H \in \mathfrak{F}$. Hence $G/V_1 \in \mathfrak{N}\mathfrak{F}$. Similarly $G/V_2 \in \mathfrak{N}\mathfrak{F}$. But $F(G) = V_1 \times V_2$ and $G/F(G) \cong (N_1 \times N_2)W \notin \mathfrak{F}$. Therefore $G \in \mathrm{R}_0(\mathfrak{N}\mathfrak{F}) \setminus \mathfrak{N}\mathfrak{F}$.

6. *Concluding remarks.* The construction of the Fitting class $\mathfrak{F}$ is capable of abstraction. The underlying idea is to specify a certain characteristic section of G below $G^{\mathfrak{N}_1}$, to force this to be a given direct product, and to exploit the resulting control over its automorphism group. In the case of $\mathfrak{F}$, the key section L^* is a direct product of copies of a single group B; we needed to know that the automorphisms of L^* permute the direct factors as well as the minimal normal subgroups of L^*, and that $\mathrm{Aut}(B)$ has a sufficiently simple structure to prove S_F-closure while being large enough to ensure that $\mathfrak{F}$ contains groups with a non-trivial key section.

Another example constructed along these lines may be described as follows:

Let $\mathfrak{X}_0 = \mathfrak{S}_5\mathfrak{S}_3\mathfrak{S}_2$, $\mathfrak{Y}_0 = \mathfrak{S}_5\mathfrak{S}_2\mathfrak{S}_5\mathfrak{S}_3$, and let B_0 denote a non-abelian group of order 75. Define

$$\mathfrak{F}_0 = (G \in \mathfrak{X}_0: \text{if } L = G^{\mathfrak{Y}_0},\ L/0_3(L) = B_1^* \times \ldots \times B_t^*,$$
$$\text{where } B_i^* \cong B_0 \text{ and } B_i^* = B_i/0_3(L) \text{ with } B_i \trianglelefteq G^{\mathfrak{S}_5\mathfrak{S}_3}).$$

It is not hard to adapt the arguments of sections 3 and 4 to prove that $\mathfrak{F}_0$ is a Q-closed Fitting class which is not a formation. For $p \neq 3$, the symmetric group S_3 has a 2-dimensional faithful irreducible $\mathbb{Z}_p$-module, V say; let H_p denote the semi-direct product VS_3. The above group B_0 is the derived subgroup of H_5, and the properties of $\mathfrak{F}_0$ hang on the existence of H_5. If p is taken as 7 instead of 5, the analogous construction (with $B_0 = [H_7, H_7]$) breaks down. Because B_0 is no longer primitive, the class corresponding to $\mathfrak{F}_0$ is not even N_0-closed.

Note that $\mathfrak{F}_0 \subset \mathfrak{N}^3$ but $\mathfrak{F}_0 \neq \mathrm{S}_F\mathfrak{F}_0$. Thus suggests the following question:

Question. Is a Q-closed Fischer class contained in $\mathfrak{N}^3$ necessarily a formation?

Two final comments. First, a similar construction seems to work for suitable groups B with non-trivial centre provided that 'central product' is substituted for 'direct

https://doi.org/10.1017/S030500410005307X Published online by Cambridge University Press

product'. Secondly, the foregoing ideas appear to be of little help in constructing metanilpotent Fitting classes. A description of $\langle S_n, N_0 \rangle (S_3)$ is as elusive as ever.

This work was completed at the University of Oregon with support from NSF Grant MPS73-08619 A02.

REFERENCES

(1) COSSEY, JOHN. Classes of finite soluble groups. *Proc. Second Internat. Conf. Theory of Groups, Canberra* 1973, pp. 226–237, Lecture Notes in Mathematics, No. 372 (Berlin, Heidelberg, New York; Springer-Verlag, 1974).

(2) HAWKES, TREVOR. Bounding the nilpotent length of a finite group. *Proc. London Math. Soc.* (to appear).

(3) HUPPERT, B. *Endliche Gruppen I* (Berlin, Heidelberg, New York; Springer-Verlag, 1967).

https://doi.org/10.1017/S030500410005307X Published online by Cambridge University Press