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Two waveguide trifurcation problems

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Abstract

We consider the diffraction of the dominant acoustic wave mode which propagates out of the mouth of a semi-infinite waveguide made of a soft and hard half plane. This semi-infinite waveguide is symmetrically located inside an infinite waveguide whose infinite plates are soft and hard. The whole system constitutes a trifurcated waveguide. Another trifurcated waveguide is obtained by interchanging the infinite plates. A closed form solution of the resulting matrix Wiener–Hopf equation is obtained for each configuration. Thus we present exact closed-form solutions to two new waveguide trifurcation problems.

Introduction

The emission of unwanted noise from the exhaust systems of various vehicle engines is an environmental nuisance. The exhausts of modern engines are designed to try to reduce the noise emitted as much as possible. One way of achieving this is novel geometrical designs and the use of various sound absorbent materials for the exhaust system, Rawlins[**13**], [**14**]. The present problem was conceived as a possible design aid that would help in the design of practical exhaust systems. In order that mathematical techniques will yield useful acoustic field distribution within such exhaust systems, various simplifications need to be made. In particular, practical exhaust systems are of finite length however, in order to apply the powerful Wiener–Hopf technique it is convenient to consider semi-infinite geometries. This simplification is not as severe as one might think if one considers the rapid attenuation of the sound field along the absorbent duct system. Also the use of absorbent materials lining the exhaust system requires a mixed boundary condition which involves an impedance parameter related to the material lining. However, a simplification that considerably reduces the complexity of the mathematical formulae is to replace these boundary conditions by a soft boundary. Although a perfectly soft surface is a somewhat ideal concept it can be used to give realistic design data for absorbent linings, Butler [**1**], Rawlins[**15**]. Also, honeycomb surfaces that behave, for a given frequency range, like soft surfaces have been designed, constructed and used successfully as noise barriers, Fujiwara and Ohkubo [**3**]. These mathematical model simplifications can be regarded as first approximations to a more sophisticated mathematical model that involves finite length structures and absorbent type boundary conditions.

Here we are mainly interested in deriving new exact closed-form solutions to two

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new waveguide trifurcation problems. These problems require the solution of 2×2 matrix Wiener–Hopf equations. The number of matrix Wiener–Hopf problems for waveguide geometries that can be solved in closed form (i.e. without having to solve approximately infinite systems of equations) is very small. Indeed, to the author's knowledge, the trifurcated waveguide problems (with the arbitrariness of spacing) dealt with here, are the only trifurcated waveguide problems, to date, that result in the factorization of a 2×2 matrix that can be explicitly solved in closed-form. They offer benchmarks for the comparison of approximate techniques applied to more practical problems that also result in matrix Wiener–Hopf problems that cannot be solved exactly. To be specific we shall couch all boundary value problems in acoustical terms.

The trifurcated waveguide problems under consideration are shown in Figs. 1 and 2. The plates which make up the waveguide are soft–hard–soft–hard and hard–hard–soft–soft, respectively, and are symmetrically positioned relative to the centre line of the system. In order to simplify the formulae that will arise in the solution, a fundamental mode is assumed to propagate out of the mouth of the semi-infinite waveguide. Higher order modes offer no significant mathematical difficulties. The trifurcated waveguide problem where the plates and the boundary conditions are symmetric with respect to the centre-line of the system can be reduced by symmetry to the solution of bifurcation problems. Such a bifurcation problem can be solved exactly by the normal scalar Wiener–Hopf technique, Pace and Mittra[**10**], Mittra and Lee[**8**] (p. 200 and 212–213), Hurd and Meister[**2**]. For the trifurcated waveguide problems we shall consider, the symmetry with respect to the boundary conditions is lost. This results in non-trivial matrix Wiener–Hopf problems. Fortunately, we are able to solve these matrix Wiener–Hopf problems explicitly. We remark that the present problems do not fall within the set of trifurcation problems solved by Lüneburg and Hurd[**6**]. In the problems considered in Lüneburg & Hurd[**6**] the spacing between the plates was equidistant and the direction of the incident wave was different.

For each trifurcated waveguide problem we shall formulate the mathematical problem that we intend to solve. We shall then solve the first problem formulated in detail; the solution of the second problem will be merely outlined, for details of this solution see [**10**]. The solution will be expressed by complex contour integrals. We shall analytically manipulate these integrals to reduce them to infinite series of modes which propagate in the waveguide system. In order not to disrupt the flow of the solution in the main text, various appendices have been included at the end of the paper. These include analytical details required in the main text.

1. *The first waveguide trifurcation problem*

1·1. *Formulation of the boundary value problem.* We shall consider the acoustic diffraction of a wave mode propagating out of the open end of a semi-infinite duct, consisting of one plate which is rigid and the other which is soft, situated symmetrically between two infinite plates one of which is soft and the other rigid. The geometry of the problem is shown in Figure 1.

The sound source field is located at $x = x_0 (x_0 < 0)$ and propagates modes along the semi-infinite duct. We shall introduce a scalar potential function $\phi(x, y, t)$ which

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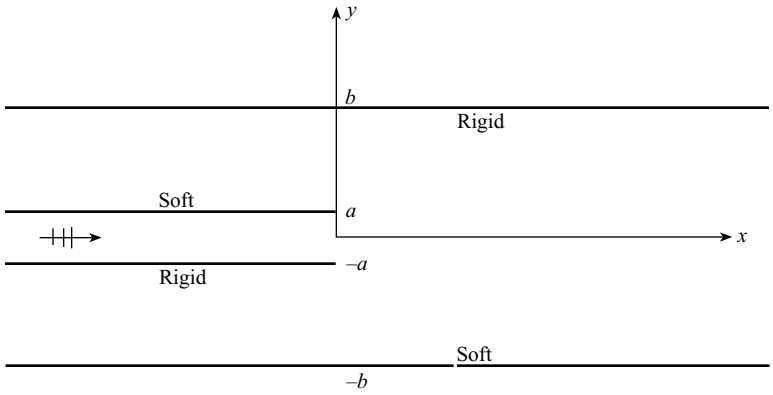


Fig. 1. Geometry of the first diffraction problem.

defines the acoustic pressure and velocity by $p = -\rho_0 \partial\phi/\partial t$, and $\mathbf{u} = \operatorname{grad}\phi$ respectively, where ρ_0 is the density of the undisturbed medium.

The incident sound field is assumed to have time harmonic variation $e^{-i\omega t}$, where the wave number $k = \omega/c$, and c is the speed of sound. In the rest of the paper we shall drop the dependence on time, it being tacitly understood that $\phi(x, y, t) = \operatorname{Re}\{e^{-i\omega t}\phi(x, y)\}$. To this end we require a representation for the solution $\phi(x, y)$ of the two-dimensional Helmholtz equation

$$\phi_{xx} + \phi_{yy} + k^2\phi = 0, \tag{1·1}$$

in the trifurcated duct system which satisfies the following boundary conditions (see Fig. 1)

$$\left.\begin{aligned} \phi_y = 0, \quad y = b \quad & -\infty < x < \infty, \\ \phi = 0, \quad y = a \quad & -\infty < x < 0, \\ \phi_y = 0, \quad y = -a \quad & -\infty < x < 0, \\ \phi = 0, \quad y = -b \quad & -\infty < x < \infty, \end{aligned}\right\} \tag{1·2a–d}$$

where it is assumed that $b > a$.

To these boundary conditions we add those conditions at infinity relevant to the nature of the lowest propagating modes which the various duct regions can sustain. We have

for $x \to -\infty$, $-a \leqslant y \leqslant a$

$$\phi(x, y) \Rightarrow e^{i\alpha_1 x}\cos\left((y+a)\frac{\pi}{4a}\right) + Re^{-i\alpha_1 x}\cos\left((y+a)\frac{\pi}{4a}\right), \tag{1·3}$$

where

$$\alpha_1 = \sqrt{(k^2 - \pi^2/16a^2)}, \quad \alpha_2 = \sqrt{(k^2 - 9\pi^2/16a^2)}, \quad \alpha_{2n-1} = \sqrt{(k^2 - ((2n-1)\pi/4a)^2)}, \quad (n = 1, 2, \ldots).$$

If we restrict $\pi/4 < ka < 3\pi/4$ then $\alpha_1 > 0$ and $\alpha_2 = i\sqrt{(9\pi^2/16a^2 - k^2)}$ so that $\operatorname{Im}\alpha_2 > 0$, $\operatorname{Re}\alpha_2 = 0$. Thus the semi-infinite duct region $-\infty < x < 0$, $-a \leqslant y \leqslant a$ can only sustain the lowest incident and reflected modes.

For $x \to +\infty$, $-b \leqslant y \leqslant b$,

$$\phi(x, y) \Rightarrow \hat{T}\, e^{i\hat{\alpha}_1 x}\sin\left((y+b)\frac{\pi}{4b}\right) + \hat{T}_1\, e^{i\hat{\alpha}_2 x}\sin\left((y+b)\frac{3\pi}{4b}\right), \tag{1·4}$$

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where

$$\hat{\alpha}_1 = \sqrt{(k^2-\pi^2/16b^2)}, \quad \hat{\alpha}_2 = \sqrt{(k^2-9\pi^2/16b^2)}, \quad \hat{\alpha}_{2n-1} = \sqrt{(k^2-((2n-1)\pi/4b)^2)}, \quad (n = 1, 2, \ldots).$$

$$\text{For } x \to -\infty, a < y \leqslant b, \quad \phi(x,y) \Rightarrow \tilde{T}_2 e^{-i\tilde{\alpha}_1 x} \sin\left((y-a)\frac{\pi}{2(b-1)}\right), \tag{1·5}$$

where

$$\tilde{\alpha}_1 = \sqrt{\left(k^2-\frac{\pi^2}{4(b-a)^2}\right)}, \quad \tilde{\alpha}_{2n-1} = \sqrt{(k^2-((2n-1)\pi/2(4b-a))^2)}, \quad (n = 1, 2, \ldots).$$

$$\text{For } x \to -\infty, -b < y \leqslant -a, \quad \phi(x,y) \Rightarrow T_1 e^{-i\tilde{\alpha}_1 x} \sin((y+b)\pi/(2(b-a))). \tag{1·6}$$

Finally, in order to ensure the uniqueness of the solution to the problem, we need to specify the 'edge condition' at the end of the semi-infinite planes, that is

$$\phi(x, \pm a) = O(1) \quad \text{and} \quad \phi_y(x, \pm a) = O(x^{-\frac{1}{2}}) \quad \text{as} \quad x \to 0. \tag{1·7}$$

1·2. *Solution of the boundary value problem.* For analytic convenience we shall assume that $k = \operatorname{Re} k + i \operatorname{Im} k$ ($\operatorname{Re} k > \operatorname{Im} k \geqslant 0$). A suitable representation for the total field $\phi(x,y)$ in all space $-\infty < x < \infty$, $|y| < b$, which satisfies (1·1) and (1·2) is given by an application of the Fourier transform approach as:

$$\phi(x,y) = \frac{1}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} e^{i\alpha x}\frac{\sin\kappa(y+b)}{\kappa\cos\kappa(b-a)}\phi_1^-(\alpha)\,d\alpha, \quad (-b \leqslant y \leqslant -a, -\infty < x < \infty); \tag{1·8}$$

$$\phi(x,y) = e^{i\alpha_1 x}\cos[(y+a)\pi/4a] + \frac{1}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau}\frac{e^{i\alpha x}}{\kappa\cos 2\kappa a}\{\sin\kappa(y-a)\,\phi_1^-(\alpha) + \kappa\cos\kappa(y+a)\,\phi_2^-(\alpha)\}\,d\alpha, \quad (-a \leqslant y \leqslant a, -\infty < x < \infty); \tag{1·9}$$

$$\phi(x,y) = \frac{1}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau}\frac{e^{i\alpha x}}{\cos\kappa(b-a)}\cos\kappa(y-b)\,\phi_2^-(\alpha)\,d\alpha, \quad (a \leqslant y \leqslant b, -\infty < x < \infty). \tag{1·10}$$

In the expressions (1·8) to (1·10), $\kappa = (k^2-\alpha^2)^{\frac{1}{2}}$ and the branch cuts are taken to be from k to $i\infty$ and from $-k$ to $-i\infty$. The cut sheet on which we shall work is defined by $0 \leqslant \arg\kappa \leqslant \pi$, see Fig. 2. The parameter τ is restricted by requiring the asymptotic behaviour (1·3)–(1·6) to be achieved. This necessitates that the contour of integration in (1·8) and (1·9) lies in the strip:

$$\operatorname{Max}\{-\operatorname{Im} k, -\operatorname{Im}\alpha_1, -\operatorname{Im}\tilde{\alpha}_1\} < \tau < \operatorname{Min}\{\operatorname{Im} k, \operatorname{Im}\hat{\alpha}_1\}.$$

It can be shown that this is achieved if $-\operatorname{Im} k < \tau < \operatorname{Im} k$. No singularities of the integrands of (1·8)–(1·10) lie within the strip $-\operatorname{Im} k < \operatorname{Im}\alpha < \operatorname{Im} k$. The unknown functions $\phi_1^-(\alpha)$ and $\phi_2^-(\alpha)$ are functions which are analytic and regular in the region $\operatorname{Im}\alpha < \operatorname{Im} k$. We must now ensure that two remaining boundary conditions are satisfied, namely that the field and its normal derivative are continuous across $y = -a$ and $y = a$, ($x > 0$) respectively, that is

$$\phi(x, -a^-) = \phi(x, -a^+), \frac{\partial\phi}{\partial y}(x, a^-) = \frac{\partial\phi}{\partial y}(x, a^+) \quad (x > 0). \tag{1·11}$$

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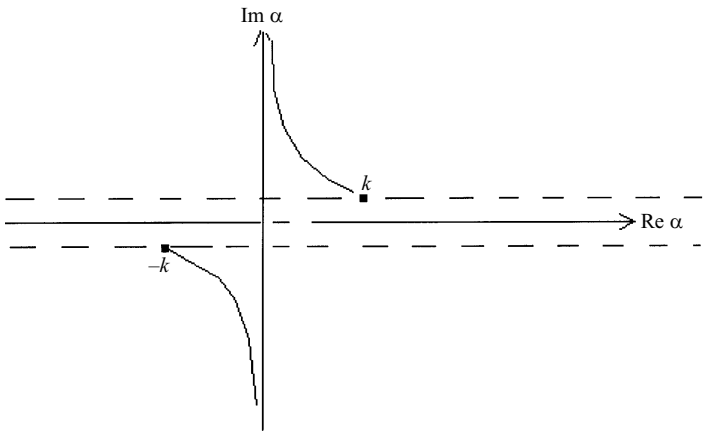


Fig. 2. Branch cuts in the complex α-plane.

Substituting (1·8)–(1·10) into (1·11) gives

$$\frac{1}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} e^{i\alpha x}\left\{\frac{-\sin\kappa(a+b)\,\phi_1^-(\alpha)}{\kappa\cos\kappa(b-a)\cos 2\kappa a}+\frac{\phi_2^-(\alpha)}{\cos 2\kappa a}+\frac{1}{\alpha-\alpha_1}\right\}d\alpha=0,\quad (x>0);\tag{1·12}$$

$$\frac{1}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} e^{i\alpha x}\left\{\frac{\phi_1^-(\alpha)}{\cos 2\kappa a}-\frac{\kappa\sin\kappa(b+a)\,\phi_2^-(\alpha)}{\cos 2\kappa a\cos\kappa(b-a)}-\frac{\kappa_1}{\alpha-\alpha_1}\right\}d\alpha=0,\quad (x>0),\tag{1·13}$$

where $\kappa_1=\pi/4a$.

A solution of (1·12) and (1·13) is given by

$$\frac{-\sin\kappa(a+b)}{\kappa\cos 2\kappa a\cos\kappa(b-a)}\phi_1^-(\alpha)+\frac{\phi_2^-(\alpha)}{\cos 2\kappa a}+\frac{1}{\alpha-\alpha_1}=\phi_2^+(\alpha),\tag{1·14}$$

$$\frac{\phi_1^-(\alpha)}{\cos 2\kappa a}-\frac{\kappa\sin\kappa(b+a)}{\cos 2\kappa a\cos\kappa(b-a)}\phi_2^-(\alpha)-\frac{\kappa_1}{\alpha-\alpha_1}=\phi_1^+(\alpha),\tag{1·15}$$

where ϕ_1^+,ϕ_2^+ are functions that are analytic and regular in the region $\operatorname{Im}\alpha>-\operatorname{Im}k$.

The equations (1·14) and (1·15) constitute a coupled system of Wiener–Hopf equations which we can write in matrix form as:

$$\boldsymbol{\psi}^+(\alpha)=\mathbf{K}(\alpha)\,\boldsymbol{\Psi}^-(\alpha)+\mathbf{D}/(\alpha-\alpha_1),\tag{1·16}$$

where

$$\boldsymbol{\psi}^\pm(\alpha)=\begin{pmatrix}\phi_1^\pm(\alpha)\\ \phi_2^\pm(\alpha)\end{pmatrix},\tag{1·17}$$

$$\mathbf{D}=\begin{pmatrix}-\kappa_1\\ 1\end{pmatrix},\tag{1·18}$$

$$\mathbf{K}(\alpha)=f(\alpha)\,\mathbf{G}(\alpha),\tag{1·19}$$

$$f(\alpha)=\sec 2\kappa a,\tag{1·20}$$

$$\mathbf{G}(\alpha)=\begin{pmatrix}1 & \dfrac{-\kappa\sin\kappa(b+a)}{\cos\kappa(b-a)}\\ \dfrac{-\sin\kappa(b+a)}{\kappa\cos\kappa(b-a)} & 1\end{pmatrix}.\tag{1·21}$$

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The standard Wiener–Hopf technique can be applied in a straightforward manner if the system (1·16) can be uncoupled into separate Wiener–Hopf equations. However, this problem requires that the matrix function $\mathbf{K}(\alpha)$ be factorized. This is a non-trivial operation and it is not always obvious how to carry it out explicitly, at least in the classical sense, Noble[**7**] (where the factors have algebraic behaviour at infinity).

The matrix $\mathbf{G}(\alpha)$ given by (1.21) is of the special form:

$$\mathbf{G}(\alpha) = \begin{pmatrix} 1 & a(\alpha) \\ b(\alpha) & 1 \end{pmatrix},$$

with $a(\alpha)/b(\alpha) =$ ratio of polynomials in α. In the present problem we have $a(\alpha)/b(\alpha) = k^2 - \alpha^2$ so that it can be factorized immediately (see Daniele[**2**], and Rawlins[**9**]). What is more, since $a(\alpha)/b(\alpha)$ is a polynomial of degree 2 the factors will have algebraic growth at infinity. Without going into the details, we obtain

$$\mathbf{K}(\alpha) = \mathbf{K}_+(\alpha)\,\mathbf{K}_-(\alpha) = \mathbf{K}_-(\alpha)\,\mathbf{K}_+(\alpha), \tag{1·22}$$

with
$$\mathbf{K}_\pm(\alpha) = \sqrt{(K_\pm(\alpha))} \begin{pmatrix} \cosh\left[\tfrac{1}{2}\kappa t_\pm(a)\right] & \kappa \sinh\left[\tfrac{1}{2}\kappa t_\pm(a)\right] \\ \dfrac{1}{k}\sinh\left[\tfrac{1}{2}\kappa t_\pm(a)\right] & \cosh\left[\tfrac{1}{2}\kappa t_\pm(a)\right] \end{pmatrix}, \tag{1·23}$$

where
$$K(\alpha) = \det\left(\mathbf{K}(\alpha)\right) = K_+(\alpha)\,K_-(\alpha) = \frac{\cos 2\kappa b}{\cos^2 \kappa(b-a)\cos 2\kappa a}, \tag{1·24}$$

$$t(\alpha) = \frac{1}{\kappa}\ln\left\{\frac{\cos\left(\dfrac{\pi}{4}+\kappa b\right)\cos\left(\dfrac{\pi}{4}+\kappa a\right)}{\cos\left(\dfrac{\pi}{4}-\kappa b\right)\cos\left(\dfrac{\pi}{4}-\kappa a\right)}\right\} = t_+(\alpha) + t_-(\alpha). \tag{1·25}$$

Explicit expressions for $K_\pm(\alpha)$ and $t_\pm(\alpha)$ are derived in Appendix A. They are:

$$K_+(\alpha) = K_-(-\alpha), \tag{1·26}$$

$$K_-(\alpha) = \frac{K_-(\alpha, b)}{2K_-(\alpha, a)\,K_-^2(\alpha, (b-a)/2)}, \tag{1·27}$$

with

$$K_-(\alpha, a) = \prod_{n=1}^{\infty}\left(\frac{a}{(2n-1)\,\pi k}\right)^2 \times (l_{4n-1}\,u - u_{4n-1}\,l)\,(l_{4n-1}\,u + u_{4n-1}\,l)\,(l_{4n-3}\,u - u_{4n-3}\,l)\,(l_{4n-3}\,u + u_{4n-3}\,l)\,e^{\frac{\alpha}{\alpha_{4n-1}}}\,e^{\frac{\alpha}{\alpha_{4n-3}}},$$

$$t_+(\alpha) = t_-(-\alpha); \tag{1·28}$$

and

$$t_-(\alpha) = \frac{1}{\kappa}\ln\prod_{n=1}^{\infty}\frac{(u_{4n-1}\,l + l_{4n-1}\,u)\,(u_{4n-3}\,l - l_{4n-3}\,u)}{(u_{4n-1}\,l - l_{4n-1}\,u)\,(u_{4n-3}\,l + l_{4n-3}\,u)} \times (\hat{u}_{4n-1}\,l + \hat{l}_{4n-1}\,u)\,(\hat{u}_{4n-3}\,l - \hat{l}_{4n-3}\,u)/\{(\bar{u}_{4n-1}\,l - \hat{l}_{4n-1}\,u)\,(\hat{u}_{4n-3}\,l + \hat{l}_{4n-3}\,u)\}, \tag{1·29}$$

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with

$$\left.\begin{aligned} &u=(k+\alpha)^{\frac{1}{2}}, \quad u_n=(k+\alpha_n)^{\frac{1}{2}}, \quad l=(k-\alpha)^{\frac{1}{2}}, \quad l_n=(k-\alpha_n)^{\frac{1}{2}}, \\ &\alpha_n=\left(k^2-\left(\frac{n\pi}{4a}\right)^2\right)^{\frac{1}{2}}, \quad \hat{u}_n=(k+\hat{\alpha}_n)^{\frac{1}{2}}, \quad \hat{l}_n=(k-\hat{\alpha}_n)^{\frac{1}{2}}, \quad \hat{\alpha}_n=\left(k^2-\left(\frac{n\pi}{4b}\right)^2\right)^{\frac{1}{2}}, \\ &\tilde{u}_n=(k+\tilde{\alpha}_n)^{\frac{1}{2}}, \quad \tilde{l}=(k-\tilde{\alpha}_n)^{\frac{1}{2}}, \quad \tilde{\alpha}_n=\left(k^2-\left(\frac{n\pi}{2(b-a)}\right)^2\right)^{\frac{1}{2}}. \end{aligned}\right\} \tag{1·30}$$

The Wiener–Hopf equation (1·16) may now be solved by using (1·22) to give

$$\mathbf{K}_-(\alpha)\,\boldsymbol{\Psi}^-(\alpha)+\frac{\mathbf{K}_+^{-1}(\alpha_1)}{\alpha-\alpha_1}\mathbf{D}=\mathbf{K}_+^{-1}(\alpha)\,\boldsymbol{\Psi}^+(\alpha)-\frac{[\mathbf{K}_+^{-1}(\alpha)-\mathbf{K}_+^{-1}(\alpha_1)]}{(\alpha-\alpha_1)}\mathbf{D}. \tag{1·31}$$

The left-hand side of (1·31) is analytic in $\operatorname{Im}\alpha<\operatorname{Im}k$; the right-hand side is analytic in $\operatorname{Im}\alpha>-\operatorname{Im}k$; consequently each side of the equation is equal to an entire function $\mathbf{E}(\alpha)$, that is a matrix with polynomial entries. Hence, setting

$$\mathbf{C}=\begin{pmatrix}c_1\\c_2\end{pmatrix}=\mathbf{K}_+^{-1}(\alpha_1)\,\mathbf{D}, \tag{1·32}$$

we have

$$\boldsymbol{\Psi}^-(\alpha)=\mathbf{K}_-^{-1}(\alpha)\,[-\mathbf{C}(\alpha-\alpha_1)^{-1}+\mathbf{E}(\alpha)]. \tag{1·33}$$

The form of $\mathbf{E}(\alpha)$ is determined from the asymptotic behaviour of various functions. From Appendix A we have, as $|\alpha|\to\infty$ in $\operatorname{Im}\alpha<\operatorname{Im}k$, that

$$K_-(\alpha)=2+O(\alpha^{-1}), \quad t_-(\alpha)=\frac{i}{\alpha}\ln\alpha+O(\alpha^{-1}). \tag{1·34}$$

Hence

$$\mathbf{K}_-(\alpha)=\frac{1}{\sqrt{2}}\begin{pmatrix}\alpha^{\frac{1}{2}}+O(1) & -i\alpha^{\frac{3}{2}}+O(\alpha)\\ i\alpha^{-\frac{1}{2}}+O(\alpha^{-1}) & \alpha^{\frac{1}{2}}+O(1)\end{pmatrix}. \tag{1·35}$$

The asymptotic behaviour of $\boldsymbol{\Psi}^-(\alpha)$ may be found from the edge conditions (1·7); it is not difficult to show that as $|\alpha|\to\infty$ in $\operatorname{Im}\alpha<\operatorname{Im}k$ then

$$\phi_1^-(\alpha)=O(\alpha^{-\frac{1}{2}}), \quad \phi_2^-(\alpha)=O(\alpha^{-1}), \tag{1·36}$$

It is now straightforward to show, by substituting (1·35) and (1·36) into (1·33), that the entire matrix $\mathbf{E}(\alpha)$ is given by

$$\mathbf{E}(\alpha)=ic_2\begin{pmatrix}1\\0\end{pmatrix}, \tag{1·37}$$

in order that the asymptotic behaviour (1·36) is obtained from (1·33).

The solution can now be written in a very succinct form. If we introduce the following functions:

$$F(u,l)=ul_1+lu_1, \tag{1·38}$$

$$\begin{aligned} P(l,u)&=\frac{e^{-i\pi/4}}{2\sqrt{k}}\cdot\frac{(l+iu)\,e^{\frac{1}{2}ikt_-(\alpha)}}{\sqrt{K_-(\alpha)}}\\ &=\frac{e^{-i\pi/4}}{\sqrt{2k}}(l+iu)\prod_{n=1}^{\infty}\left(\frac{(b-a)}{(2n-1)\,\pi}\left(\frac{a}{b}\right)^{\frac{1}{2}}\right)^2\frac{(u_{4n-1}l+l_{4n-1}u)\,(u_{4n-3}l-l_{4n-3}u)}{(\hat{u}_{4n-1}l-\hat{l}_{4n-1}u)\,(\hat{u}_{4n-3}l+\hat{l}_{4n-3}u)}\\ &\qquad\times(\tilde{u}_{4n-1}^2l^2-\tilde{l}_{4n-1}^2u^2)\,(\tilde{u}_{4n-3}^2l^2-\tilde{l}_{4n-3}^2u^2), \end{aligned} \tag{1·39}$$

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then the expression (1·33) can be written explicitly as

$$\mathbf{\Psi}^{-}(\alpha)=\frac{P(u_1,l_1)}{(\alpha-\alpha_1)}\begin{pmatrix}P(l,u)F(u,l)+P(l,-u)F(-u,l)\\ [P(l,-u)F(-u,l)-P(l,u)F(u,l)]/\kappa\end{pmatrix}. \tag{1·40}$$

It is easy to verify that $P(l,u)$, $P(l,-u)$ have neither zeros nor poles in $\operatorname{Im}\alpha<\operatorname{Im}k$. They do have branch points at $\alpha=-k$, but the particular combinations in (1·40) are invariant under the transformation $u\to-u$, so that $\mathbf{\Psi}^{-}(\alpha)$ is indeed analytic in $\operatorname{Im}\alpha<\operatorname{Im}k$. We now substitute (1·40) into the integral representations (1·8) to (1·10) and obtain the field representations for the different regions as follows:

Region A $(-b<y<-a,\ x<0)$

$$\phi(x,y)=\frac{P(u_1,l_1)}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau}e^{i\alpha x}\frac{\sin\kappa(y+b)}{\kappa\cos\kappa(b-a)}\{P(l,u)F(u,l)+P(l,-u)F(-u,l)\}\frac{d\alpha}{(\alpha-\alpha_1)}, \tag{1·41}$$

Region B $(-a<y<a, x<0)$

$$\phi(x,y)=e^{i\alpha_1 x}\cos\left[(y+a)\frac{\pi}{4a}\right]$$
$$+\frac{P(u_1,l_1)}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau}\frac{e^{i\alpha x}}{\cos 2\kappa a}\{[(\sin\kappa(y-a)-\cos\kappa(y+a))P(l,u)F(u,l)$$
$$+(\sin\kappa(y-a)+\cos\kappa(y+a))P(l,-u)F(-u,l)]/\kappa\}\frac{d\alpha}{\alpha-\alpha_1}, \tag{1·42}$$

Region C $(a<y<b,\ x<0)$

$$\phi(x,y)=\frac{P(u_1,l_1)}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau}\frac{e^{i\alpha x}\cos\kappa(y-b)}{\cos\kappa(b-a)}\{[P(l,-u)F(-u,l)-P(l,u)F(u,l)]/\kappa\}\frac{d\alpha}{(\alpha-\alpha_1)}, \tag{1·43}$$

Region D $(-b<y<b,\ x>0)$

$$\phi(x,y)=\frac{P(u_1,l_1)}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau}\frac{e^{i\alpha x}\cos\kappa(y-b)}{\cos 2\kappa b}\{[\cos(\kappa a-\pi/4)\cos(\kappa b-\pi/4)$$
$$F(u,-l)/P(u,-l)-\cos(\kappa a+\pi/4)\cos(\kappa b+\pi/4)F(u,l).P(u,l)]/\kappa\}\frac{d\alpha}{(\alpha-\alpha_1)}. \tag{1·44}$$

In the expressions (1·41)–(1·44) the pole $\alpha=\alpha_1$ lies above the contour of integration, that is $\tau<\operatorname{Im}\alpha_1$. We also note that the terms in the curly brackets of (1·41)–(1·43) have no singularities in $\operatorname{Im}\alpha<\operatorname{Im}k$, and in (1·44) no singularities in $\operatorname{Im}\alpha>-\operatorname{Im}k$. Thus the only singularities in $\operatorname{Im}\alpha<\operatorname{Im}k$ of the integrands of (1·41) and (1·43) occur at the zeros of $\cos\kappa(b-a)=0$, that is

$$\alpha=-\tilde{\alpha}_{2n-1}=-\sqrt{(k^2-((2n-1)\pi/2(b-a))^2)}\quad(n=1,2,\dots).$$

The only singularities in $\operatorname{Im}\alpha<\operatorname{Im}k$ of the integrand of (1·42) occur at the zeros of $\cos 2\kappa a=0$, that is $\alpha=-\alpha_{2n-1}=-\sqrt{(k^2-((2n-1)\pi/4a)^2)}$. The only singularities in $\operatorname{Im}\alpha>-\operatorname{Im}k$ of the integrand (1·44) occur at the zeros of $\cos 2\kappa b=0$, that is, $\alpha=\hat{\alpha}_{2n-1}=\sqrt{(k^2-((2n-1)\pi/4b)^2)}$, $n=1,2,\dots$, and also the pole $\alpha=\alpha_1$.

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1·3 *Mode field structure.* An application of Cauchy's residue theorem to the complex integrals (1·41)–(1·44) now gives the field in the various regions as a sum of waveguide modes.

Region A $(-b < y < -a,\ x < 0)$

$$\phi(x,y) = \frac{P(u_1,l_1)}{(b-a)} \sum_{n=1}^{\infty} \frac{(-)^n \sin \tilde{\kappa}_m(y+b)}{\tilde{\alpha}_m(\tilde{\alpha}_m+\alpha_1)} \{P(\tilde{u}_m,\tilde{l}_m)\,F(\tilde{l}_m,\tilde{u}_m) + P(\tilde{u}_m,-\tilde{l}_m)\,F(-\tilde{l}_m,\tilde{u}_m)\}\, e^{-i\tilde{\alpha}_m x}, \quad (m=2n-1). \tag{1·45}$$

Region B $(-a < y < a, x < 0)$

$$\phi(x,y) = e^{i\alpha_1 x}\cos[(y+a)\,\kappa_1] + \frac{P(u_1,l_1)}{a} \sum_{n=1}^{\infty} \frac{e^{-i\alpha_m x}\cos \kappa_m(y+a)}{\alpha_m(\alpha_m+\alpha_1)} \{P(u_m,(-)^{n+1}\,l_m)\,F((-)^{n+1}\,l_m,u_m)\} \quad (m=2n-1). \tag{1·46}$$

Region C $(a < y < b, x < 0)$

$$\phi(x,y) = \frac{P(u_1,l_1)}{(b-a)} \sum_{n=1}^{\infty} \frac{(-)^n\, e^{i\tilde{\alpha}_m x}\cos \tilde{\kappa}_m(y-b)}{\tilde{\alpha}_m(\tilde{\alpha}_m+\alpha_1)} \{[P(\tilde{u}_m,-\tilde{l}_m)\,F(-\tilde{l}_m,\tilde{u}_m) - P(\tilde{u}_m,\tilde{l}_m)\,F(\tilde{l}_m,\tilde{u}_m)]\}, \quad (m=2n-1). \tag{1·47}$$

Region D $(-b \leqslant y \leqslant b, x > 0)$

$$\phi(x,y) = \frac{P(u_1,l_1)}{2b} \sum_{n=1}^{\infty} \frac{(-)^{[n/2]}\cos \hat{\kappa}_m(y-b)\,F(\hat{u}_m,(-)^n\,\hat{l}_m)}{\hat{\alpha}_m(\hat{\alpha}_m-\alpha_1)\,P(\hat{u}_m,(-)^n\,\hat{l}_m)} \cos(\hat{\kappa}_m\,a+(-)^n\,\pi/4)\, e^{i\alpha_m x}, \quad (m=2n-1), \tag{1·48}$$

where $[x]$ denotes the largest integer $\leqslant x$.

Numerical results. It is of interest to know the amount of acoustic power in the incident mode which is distributed among the various waveguide regions. For example, if one assumes that only the model structure given by (1·3)–(1·6) propagates, then it is not difficult to show that a power balance gives:

$$\alpha_1(2a)\,(1-|R|^2) = \tilde{\alpha}_1(b-a)\,(|\tilde{T}_1|^2+|\tilde{T}_2|^2)+\hat{\alpha}_1(2b)\,|\hat{T}|^2+\hat{\alpha}_2(2b)\,|\hat{T}_1|^2. \tag{1·49}$$

The term $2a\alpha_1(1-|R|^2)$ represents the power that exits from the semi-infinite guide and which is distributed among the other regions. Thus $|R|$ determines the amount of energy of the incident wave which is distributed amongst the various other modes in the rest of the waveguide. By comparing (1·46) with (1·3) it can be seen that

$$R = \frac{k}{a\alpha_1^2} P^2(u_1,l_1). \tag{1·50}$$

We can obtain, after some algebra, simple closed form expressions for the modulus of the reflection coefficient $|R|$ which can be calculated numerically. Clearly as more modes are excited in the various regions the $|R|$ will change. We shall consider two situations:

(i) when the guide dimensions are such that one mode (the lowest) propagates in

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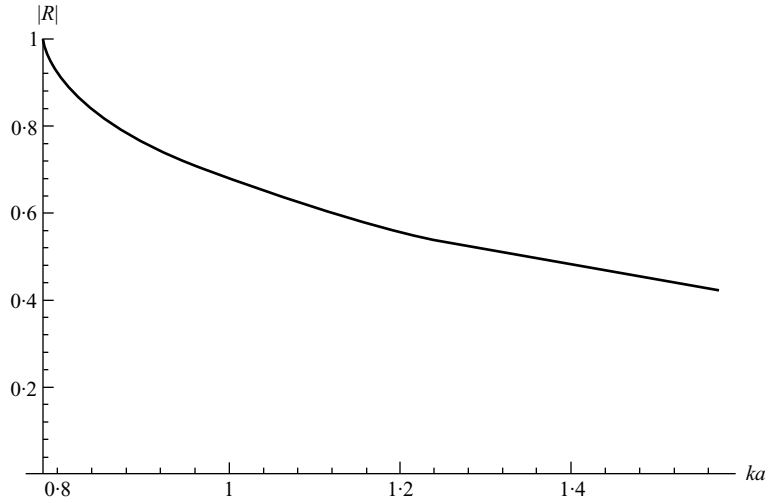


Fig. 3. The absolute value of $|R|$ when $b = 3a/2$.

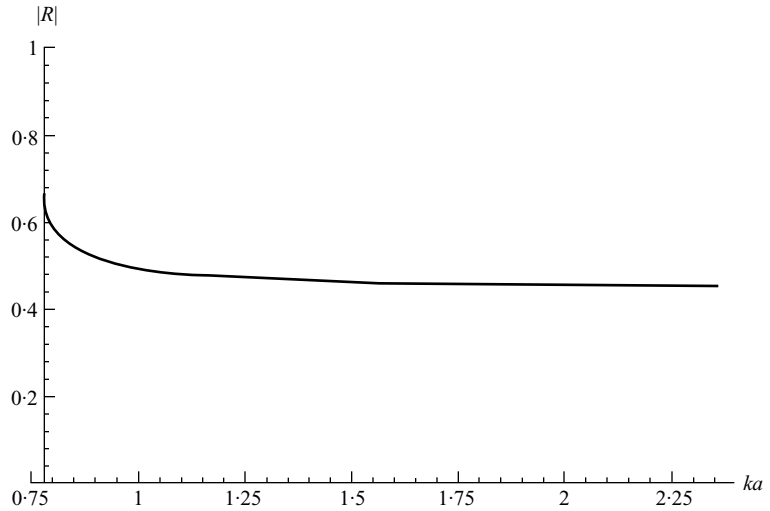


Fig. 4. The absolute value of $|R|$ when $b = 3a$.

$|y| < a, x < 0$ and one mode propagates in $|y| < b, x > 0$ and no mode propagates in $-b < y < -a, x < 0$ and $a < y < b, x < 0$. In this case it can be shown that

$$|R| = \frac{\pi k}{4ab} \frac{(a+b)}{(k^2 + \kappa_1 \hat{\kappa}_1 + \alpha_1 \hat{\alpha}_1)}. \tag{1·51}$$

The graph of $|R|$ is shown in Fig. 3 for such a situation, given by choosing $b = 3a/2$ and the frequency range $\pi/4 < ka < \pi/2$.

(ii) This is when the guide dimensions are such that only one mode propagates in $-b < y < -a, x < 0$ and one mode in $a < y < b, x < 0$ and consequently two modes in $|y| < b, x > 0$. In this situation it can be shown that

$$|R| = \frac{(b-a)^2}{b^2} \frac{(k^2 - \kappa_1 \tilde{\kappa}_1 + \alpha_1 \tilde{\alpha}_1)\,(k^2 + \kappa_1 \tilde{\kappa}_1 + \alpha_1 \tilde{\alpha}_1)}{(k^2 + \kappa_1 \hat{\kappa}_1 + \alpha_1 \hat{\alpha}_1)\,(k^2 - \kappa_1 \hat{\kappa}_3 + \alpha_1 \hat{\alpha}_3)}.$$

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The graph of $|R|$ is shown in Fig. 4 for such a situation given by choosing $b = 3a$ and the frequency range $\pi/4 < ka < 3\pi/4$.

2. *The second waveguide trifurcation problem*

2.1. *Formulation of the boundary value problem.* We shall now consider the acoustic diffraction of a wave mode propagating out of the open end of the same semi-infinite duct which is situated symmetrically between the two infinite plates which have been interchanged. The geometry of the problem is shown in Fig. 5.

To this end we require a representation for the solution $\phi(x, y)$ of the two-dimensional Helmholtz equation

$$\phi_{xx} + \phi_{yy} + k^2\phi = 0, \tag{2·1}$$

in the trifurcated duct system which satisfies the following boundary conditions (see Fig. 5)

$$\left.\begin{aligned} \phi &= 0, \quad y = b, \quad -\infty < x < \infty; \\ \phi &= 0, \quad y = a, \quad -\infty < x < 0; \\ \phi_y &= 0, \quad y = -a, \quad -\infty < x < 0; \\ \phi_y &= 0, \quad y = -b, \quad -\infty < x < \infty, \end{aligned}\right\} \tag{2·2a–d}$$

where it is assumed that $b > a$.

To these boundary conditions we add the radiation condition at infinity which, for this particular arrangement, is given by:

$$\text{for } x \to -\infty, -a \leqslant y \leqslant a, \quad \phi(x, y) \Rightarrow e^{i\alpha_1 x}\cos\left((y+a)\frac{\pi}{4a}\right) + \mathrm{Re}^{-i\alpha_1 x}\sin\left((y+a)\frac{\pi}{4a}\right), \tag{2·3}$$

where

$$\alpha_1 = \sqrt{(k^2 - (\pi^2/16a^2))}, \quad \alpha_{2n-1} = \sqrt{(k^2 - ((2n-1)\pi/4a)^2)}, \quad (n = 1, 2, \dots).$$

If we restrict $\pi/4 < ka < 3\pi/4$ then $\alpha_1 > 0$ and $\alpha_2 = i\sqrt{((3\pi/4a)^2 - k^2)}$, so that $\mathrm{Im}\,\alpha_n > 0$, $\mathrm{Re}\,\alpha_n = 0$, $n > 1$. Thus the semi-infinite duct region $-\infty < x < 0$, $-a \leqslant y \leqslant a$ can only sustain the lowest incident and reflected mode.

$$\text{For } x \to +\infty, -b \leqslant y \leqslant b, \quad \phi(x, y) \Rightarrow T e^{i\hat{\alpha}_1 x}\cos\left((y+b)\frac{\pi}{4b}\right), \tag{2·4}$$

where

$$\hat{\alpha}_1 = \sqrt{(k^2 - (\pi/4b)^2)}, \quad \hat{\alpha}_2 = \sqrt{(k^2 - (3\pi^2/4b)^2)},$$

$$\hat{\alpha}_{2n-1} = \sqrt{(k^2 - ((2n-1)\pi/4b)^2)}, \quad (n = 1, 2, \dots).$$

$$\text{For } x \to -\infty, a \leqslant y \leqslant b, \quad \phi(x, y) \Rightarrow \tilde{T} e^{-i\tilde{\alpha}_1 x}\sin\left((y-a)\frac{\pi}{(b-a)}\right), \tag{2·5}$$

where

$$\tilde{\alpha}_1 = \sqrt{((k^2 - (\pi/(b-a))^2))}, \quad \tilde{\alpha}_{2n-1} = \sqrt{((k^2 - (n\pi/(b-a))^2))}, \quad (n = 1, 2, \dots).$$

$$\text{For } x \to -\infty, -b \leqslant y \leqslant -a, \quad \phi(x, y) \Rightarrow \tilde{T}_2 e^{-ikx} + \tilde{T}_3 e^{i\tilde{\alpha}_1 x}\cos\left((y+a)\frac{\pi}{(b-a)}\right), \tag{2·6}$$

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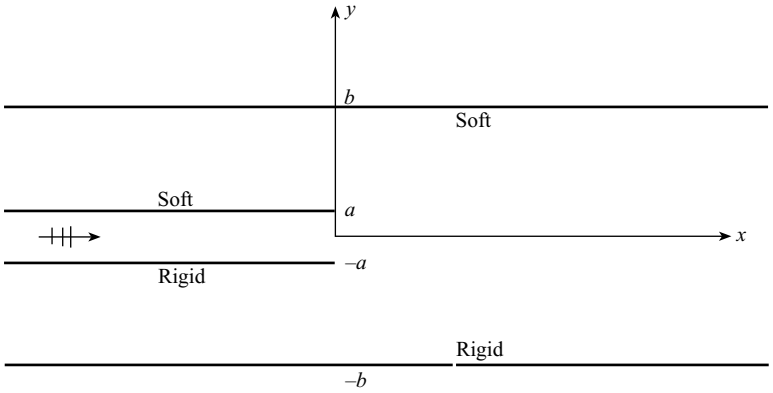


Fig. 5. Geometry of the second diffraction problem.

where

$$\tilde{\alpha}_0 = k, \quad \tilde{\alpha}_1 = \sqrt{((k^2-(\pi/(b-a))^2))}, \quad \tilde{\alpha}_n = \sqrt{((k^2-(n\pi/(b-a))^2))}, \quad (n = 0, 1, 2, \dots).$$

Finally, the 'edge condition' at the end of the semi-infinite planes requires that

$$\phi(x, \pm a) = O(1) \quad \text{and} \quad \phi_y(x, \pm a) = O(x^{-\frac{1}{2}}) \quad \text{as} \quad x \to 0. \tag{2·7}$$

2·2. *Solution of the boundary value problem.* A suitable representation for the total field $\phi(x, y)$ in all space $-\infty < x < \infty$, $|y| \leqslant b$ which satisfies (2·1) and (2·2) is given by:

$$\phi(x, y) = -\frac{1}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} e^{i\alpha x}\frac{\cos\kappa(b+y)}{\kappa\sin\kappa(b-a)}\phi_1^-(\alpha)\,d\alpha, \quad (-b \leqslant y \leqslant -a, -\infty < x < \infty), \tag{2·8}$$

$$\phi(x, y) = \frac{1}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau}\frac{e^{i\alpha x}}{\kappa\cos 2\kappa a}\{\sin\kappa(y-a)\,\phi_1^-(\alpha) + \kappa\cos\kappa(y+a)\,\phi_2^-(\alpha)\}\,d\alpha$$

$$+ e^{i\alpha_1 x}\cos\left[(y+a)\frac{\pi}{4a}\right], \quad (-a \leqslant y \leqslant a, \ -\infty < x < \infty), \tag{2·9}$$

$$\phi(x, y) = \frac{1}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau}\frac{\sin\kappa(b-y)}{\sin\kappa(b-a)}\phi_2^-(\alpha)\,d\alpha, \quad (a \leqslant y \leqslant b, \ -\infty < x < \infty). \tag{2·10}$$

In the expressions (2·8)–(2·10), $k = \operatorname{Re} k + i \operatorname{Im} k (\operatorname{Re} k > \operatorname{Im} k \geqslant 0)$, $\kappa = (k^2-\alpha^2)^{\frac{1}{2}}$ and $\operatorname{Im} k < \tau < \operatorname{Im} k$. By following the same procedure as that of the previous problem, one eventually ends up with the matrix Wiener–Hopf equation

$$\boldsymbol{\Psi}^+ = \mathbf{K}(\alpha)\,\boldsymbol{\Psi}^-(\alpha) + \mathbf{D}/(\alpha-\alpha_1), \tag{2·16}$$

where

$$\boldsymbol{\Psi}^{\pm}(\alpha) = \begin{pmatrix}\phi_1^{\pm}(\alpha)\\ \phi_2^{\pm}(\alpha)\end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix}-\kappa_1\\ 1\end{pmatrix}, \tag{2·17}$$

$$\mathbf{K}(\alpha) = \frac{1}{\cos 2\kappa a}\begin{pmatrix}1 & \dfrac{\kappa\cos\kappa(b+a)}{\sin\kappa(b-a)}\\ \dfrac{\cos\kappa(b+a)}{\kappa\sin\kappa(b-a)} & 1\end{pmatrix}. \tag{2·18}$$

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The matrix $\mathbf{K}(\alpha)$ given by (2·18) can again be factorized. Thus we obtain

$$\mathbf{K}(\alpha) = \mathbf{K}_+(\alpha)\,\mathbf{K}_-(\alpha), \tag{2·19}$$

with

$$\mathbf{K}_\pm(\alpha) = \sqrt{(K_\pm(\alpha))}\begin{pmatrix} \cosh\left[\tfrac{1}{2}\kappa t_\pm(\alpha)\right] & \kappa\sinh\left[\tfrac{1}{2}\kappa t_\pm(\alpha)\right] \\ \dfrac{1}{\kappa}\sinh\left[\tfrac{1}{2}\kappa t_\pm(\alpha)\right] & \cosh\left[\tfrac{1}{2}\kappa t_\pm(\alpha)\right] \end{pmatrix}, \tag{2·20}$$

where

$$K(\alpha) = \det\mathbf{K} = K_+(\alpha)\,K_-(\alpha) = -\frac{\cos 2\kappa b}{\cos 2\kappa a\sin^2\kappa(b-a)}, \tag{2·21}$$

$$t(\alpha) = \frac{1}{\kappa}\ln\left(\frac{e^{i\pi}\cos(\kappa a+\pi/4)\cos(\kappa b-\pi/4)}{\cos(\kappa a-\pi/4)\cos(\kappa b+\pi/4)}\right) = t_+(\alpha)+t_-(\alpha). \tag{2·22}$$

Explicit expressions for $K_\pm(\alpha)$ and $t_\pm(\alpha)$ are derived in Appendix B. They are:

$$K_-(\alpha) = \frac{i}{(b-a)\,l^2}\prod_{n=1}^{\infty}\left(\frac{2kbn\pi}{a(b-a)}\right)^2 \times\frac{[(\hat{u}_{4n-1}\,l)^2-(l_{4n-1}^2\,u)^2]\,[(\hat{u}_{4n-3}\,l)^2-(\hat{l}_{4n-3}\,u)^2]}{[(u_{4n-1}\,l)^2-(l_{4n-1}\,u)^2]\,[(u_{4n-3}\,l)^2-(l_{4n-3}\,u)^2]\,[(\tilde{u}_n\,l)^2-(\tilde{l}_n\,u)^2]^2},$$

$$K_+(\alpha) = K_-(-\alpha). \tag{2·23}$$

$$t_-(\mathrm{a}) = -\frac{1}{\kappa}\ln\left[\prod_{n=1}^{\infty}\frac{(\hat{u}_{4n-1}\,l+\hat{l}_{4n-1}\,u)\,(\hat{u}_{4n-3}\,l-\hat{l}_{4n-3}\,u)\,((u_{4n-1}\,l-l_{4n-1}\,u)\,(u_{4n-3}\,l+l_{4n-3}\,u)}{(\hat{u}_{4n-1}\,l-\hat{l}_{4n-1}\,u)\,(\hat{u}_{4n-3}\,l+\hat{l}_{4n-3}\,u)\,(u_{4n-1}\,l+l_{4n-1}\,u)\,(u_{4n-3}\,l-l_{4n-3}\,u)}\right]$$

$$-\frac{1}{\kappa}\ln\left(-\frac{\alpha-i\kappa}{k}\right), \quad t_+(\alpha) = t_-(-\alpha), \tag{2·24}$$

where $u = (k+\alpha)^{\frac{1}{2}}$, $l = (k-\alpha)^{\frac{1}{2}}$, $u_n = (k+\alpha_n)^{\frac{1}{2}}$, $l_n = (k-\alpha_n)^{\frac{1}{2}}$, $\alpha_n = (k^2-(n\pi/(4a))^2)^{\frac{1}{2}}$; $\hat{u}_n = (k+\hat{\alpha}_n)^{\frac{1}{2}}$, $\hat{l}_n = (k-\hat{\alpha}_n)^{\frac{1}{2}}$, $\hat{\alpha}_n = (k^2-(n\pi/(4b))^2)^{\frac{1}{2}}$, $\tilde{u}_n = (k+\tilde{\alpha}_n)^{\frac{1}{2}}$, $\tilde{l}_n = (k-\tilde{\alpha}_n)^{\frac{1}{2}}$, $\tilde{\alpha}_n = (k^2-(n\pi/(b-a))^2)^{\frac{1}{2}}$.

The Wiener–Hopf equation (2·16) may now be solved and the solution put in a very simple form by introducing the functions

$$F(u,l) = ul_1+lu_1, \tag{2·25}$$

$$P(l,u) = \frac{e^{i\pi/4}(l-iu)\,e^{\frac{1}{2}\kappa t_-(a)}}{2\sqrt{k}\,\sqrt{(K_-(\alpha))}},$$

$$= l\left(\frac{b-a}{2}\right)^{\frac{1}{2}}\prod_{n=1}^{\infty}\left(\frac{2n\pi kb}{a(b-a)}\right)\frac{(u_{4n-1}\,l+l_{4n-1}\,u)\,(u_{4n-3}\,l-l_{4n-3}\,u)\,((u_n\,l)^2-(l_n\,u)^2)}{(\hat{u}_{4n-1}\,l+\hat{l}_{4n-1}\,u)\,(\hat{u}_{4n-3}\,l-\hat{l}_{4n-3}\,u)}. \tag{2·26}$$

Thus one obtains:

$$\mathbf{\Psi}^-(\alpha) = \frac{P(u_1,l_1)}{(\alpha-\alpha_1)}\begin{pmatrix} P(l,u)\,F(u,l)+P(l,-u)\,F(-u,l) \\ [P(l,-u)\,F(-u,l)-P(l,u)\,F(u,l)]/\kappa \end{pmatrix}. \tag{2·27}$$

We now substitute the expression (2·27) into the integral representations (2·8) to (2·10) and obtain the field representations for the different regions as follows.

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Region A $(-b < y < -a,\ x < 0)$

$$\phi(x,y) = -\frac{P(u_1,l_1)}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} e^{i\alpha x}\frac{\cos\kappa(b+y)}{\kappa\sin\kappa(b-a)}\{P(l,u)F(u,l) + P(l,-u)F(-u,l)\}\frac{d\alpha}{(\alpha-\alpha_1)}. \quad (2{\cdot}28)$$

Region B $(-a < y < a, x < 0)$

$$\phi(x,y) = e^{i\alpha_1 x}\cos[(y+a)\pi/4a] + \frac{P(u_1,l_1)}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau}\frac{e^{i\alpha x}}{\cos 2\kappa a}\{[(\sin\kappa(y-a) - \cos\kappa(y+a)P(l,u)F(u,l) + (\sin\kappa(y-a) + \cos\kappa(y+a))P(l,u)F(-u,l)]/\kappa\}\frac{d\alpha}{(\alpha-\alpha_1)}. \quad (2{\cdot}29)$$

Region C $(a < y < b, x < 0)$

$$\phi(x,y) = \frac{P(u_1,l_1)}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} e^{i\alpha x}\frac{\sin\kappa(b-y)}{\sin\kappa(b-a)}\{[P(l,-u)F(-u,l) - P(l,u)F(u,l)]/\kappa\}\frac{d\alpha}{(\alpha-\alpha_1)}. \quad (2{\cdot}30)$$

Region D $(-b < y < b, x > 0)$

$$\phi(x,y) = \frac{P(u_1,l_1)}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} \mathrm{e}^{i\alpha x}\frac{\kappa\sin\kappa(b-y)}{\cos 2\kappa b}\{[\cos(\kappa a+\pi/4)\cos(\kappa b-\pi/4) \times F(u,-l)/P(u,-l) - \cos(\kappa a-\pi/4)\cos(\kappa b+\pi/4)F(u,l)/P(u,l)]/\kappa\}\frac{d\alpha}{(\alpha-\alpha_1)}. \quad (2{\cdot}31)$$

2·3. *Mode field representation*. An application of Cauchy's residue theorem to the complex integrals (2·28) to (2·31) then gives the field in the various regions as a sum of waveguide modes.

Region A $(-b < y < -a, x < 0)$

$$\phi(x,y) = -\frac{P(u_1,l_1)P(\surd(2k,0)\surd 2u_1 e^{-ikx}}{\surd k(b-a)(k+\alpha_1)} - \frac{P(u_1,l_1)}{(b-a)}\sum_{n=1}^{\infty}\frac{e^{-i\tilde{\alpha}_n x}(-)^n}{(\tilde{\alpha}_n+\alpha_1)\tilde{\alpha}_n}\cos\tilde{\kappa}_n(b+y)\{P(\tilde{u}_n,\tilde{l}_n)F(\tilde{l}_n,\tilde{u}_n) + P(\tilde{u}_n,-\tilde{l}_n)F(-\tilde{l}_n,\tilde{u}_n)\}. \quad (2{\cdot}32)$$

Region B $(-a < y < a, x < 0)$

$$\phi(x,y) = e^{i\alpha_1 x}\cos[(y+a)\pi/4a] + \frac{P(u_1,l_1)}{a}\sum_{n=1}^{\infty}\frac{e^{-i\alpha_m x}}{\alpha_m(\alpha_m+\alpha_1)}\cos\kappa_m(y+a)P(u_m,(-)^{n+1}l_m) \times F((-)^{n+1}l_m,u_m). \quad (m=2n-1). \quad (2{\cdot}33)$$

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Region C $(a < y < b, x < 0)$

$$\phi(x,y) = -\frac{P(u_1,l_1)}{(b-a)} \sum_{n=1}^{\infty} \frac{e^{-i\tilde{\alpha}_a x}(-)^n \sin \tilde{\kappa}_n (b-y)}{(\tilde{\alpha}_n + \alpha_1)\tilde{\alpha}_n} \{P(\tilde{u}_n, -\tilde{l}_n) F(-\tilde{l}_n, \tilde{u}_n) - P(\tilde{l}_n, \tilde{u}_n) F(\tilde{u}_n, \tilde{l}_n)\}. \quad (2{\cdot}34)$$

Region D $(-b \leqslant y \leqslant b, x > 0)$

$$\phi(x,y) = \frac{-P(u_1,l_1)}{2b} \sum_{n=1}^{\infty} \frac{(-)^{[n/2]} e^{i\hat{\alpha}_m x} \hat{\kappa}_m \sin \hat{\kappa}_m (b-y)}{\hat{\alpha}_m(\hat{\alpha}_m - \alpha_1)} \frac{F(\hat{u}_m, (-)^{n+1}\hat{l}_m)}{P(\hat{u}_m, (-)^{n+1}\hat{l}_m)} \times \cos(\hat{\kappa}_m a + (-)^{n+1}\pi/4), \quad (m = 2n-1) \quad (2{\cdot}35)$$

where $[x]$ denotes the largest integer $\leqslant x$.

3. *Conclusion*

We have solved two new problems in closed-form by using matrix factorization. These solutions can be used as a benchmark for the comparison of approximate techniques applied to more practical problems that also result in matrix Wiener–Hopf problems that cannot be solved exactly.

We also remark that if we considered the same duct geometries and boundary conditions, but with the incident wave propagating in the region $x > 0$, $|y| < b$ from $x = \infty$, then we would obtain a similar matrix Wiener–Hopf problem which can be solved exactly. As a special case of these latter problem by letting $a \to 0$ one would obtain the solution to the bifurcation problem considered by Lüneburg and Hurd [**6**].

Finally, from a consideration of the symmetry of the boundary conditions it would seem to imply that a trifurcated waveguide with a soft/hard half plane at $y = a$ and a hard/soft half plane at $y = -a$, which yields a 4×4 matrix problem should also be solvable in closed form by Daniele matrix factorization. If this proves to be true then a similar conclusion for any number of half planes that are (i) symmetrically placed about the centre line and (ii) are such that on reflecting about the centre line the hard boundary conditions lie on soft boundary conditions and vice-versa then the solution can be found in closed form by Daniele factorization.

Appendix A

Here we shall carry out the explicit factorization of (2·24) and (2·25), and give their asymptotic behaviour.

For $K(\alpha)$ given by (24) we can write it in a more convenient form

$$K(\alpha) = \frac{\cos(\kappa b + \pi/4)\cos(\kappa b - \pi/4)[\cos(\kappa a + \pi/4)\cos(\kappa\alpha - \pi/4)]^{-1}}{4\cos^2\left(\kappa\frac{(b-a)}{2} + \pi/4\right)\cos^2\left(\kappa\frac{(b-a)}{2} - \pi/4\right)}. \quad (A\ 1)$$

By using the well-known product for the cosine function

$$\cos z = \prod_{n=1}^{\infty}\left[1 - \left(\frac{2z}{(2n-1)\pi}\right)^2\right],$$

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then $$\cos\left(a\kappa\pm\frac{\pi}{4}\right)=\prod_{n=1}^{\infty}\left(\frac{2a}{(2n-1)\pi}\right)^2\left[(4n-2\mp1)\frac{\pi}{4a}-\kappa\right]\left[(4n-2\pm1)\frac{\pi}{4a}+\kappa\right].$$

If we let

$$K(\alpha,a)=\cos\left(a\kappa+\frac{\pi}{4}\right)\cos\left(a\kappa-\frac{\pi}{4}\right), \tag{A 2}$$

then we can express $K(\alpha,a)$ as the convergent infinite product

$$K(\alpha,a)=\prod_{n=1}^{\infty}\left(\frac{2a}{(2n-1)\,\pi}\right)^4\left(\alpha^2-\left(k^2-\left(\frac{\pi(4n-1)}{4a}\right)^2\right)\right)\left(\alpha^2-\left(k^2-\left(\frac{\pi(4n-3)}{4a}\right)^2\right)\right)$$

$$a\neq0,$$

so that if

$$K(\alpha,a)=K_+(\alpha,a)\,K_-(\alpha,a)$$

then $$K_+(\alpha,a)=K_-(-\alpha,a),$$

and

$$K_-(\alpha,a)=\prod_{n=1}^{\infty}\left(\frac{a}{(2n-1)\,\pi k}\right)^2(u_{4n-1}\,l-l_{4n-1}\,u)\,(u_{4n-1}\,l+l_{4n-1}\,u)\,(u_{4n-3}\,l-l_{4n-3}\,u)$$

$$\times(u_{4n-3}\,l+l_{4n-3}\,u)\exp\left[\alpha/\alpha_{4n-1}\right].\exp\left[\alpha/\alpha_{4n-3}\right], \tag{A 3}$$

where

$$l=(k-\alpha)^{\frac{1}{2}},\quad u=(k+\alpha)^{\frac{1}{2}},\quad u_n=(k+\alpha_n)^{\frac{1}{2}},\quad l_n=(k-\alpha_n)^{\frac{1}{2}},\quad \alpha_n=(k^2-(n\pi/4a)^2)^{\frac{1}{2}}.$$

We can now express the function $K(\alpha)$ given by (A 1) as

$$K(\alpha)=K_+(\alpha)\,K_-(\alpha),$$

where $$K_+(\alpha)=K_-(-\alpha)$$

and $$K_-(\alpha)=\frac{K_-(\alpha,b)}{2K_-(\alpha,a)\,\{K_-(\alpha,(b-a)/2)\}^2},$$

or more explicitly

$$K_-(\alpha)=\prod_{n=1}^{\infty}\left(\frac{2(2n-1)\,\pi k}{(b-a)}\sqrt{\frac{b}{a}}\right)^4$$

$$\times\frac{(\tilde{u}_{4n-1}^2\,l^2-\tilde{l}_{4n-1}^2\,u^2)\,(\hat{u}_{4n-3}^2\,l^2-\hat{l}_{4n-3}^2\,u^2)}{(u_{4n-1}^2\,l^2-l_{4n-1}^2\,u^2)\,(u_{4n-3}^2\,l^2-l_{4n-3}^2\,u^2)\,(\tilde{u}_{4n-1}^2\,l^2-\tilde{l}_{4n-1}^2\,u^2)^2\,(\tilde{u}_{4n-3}^2\,l^2-\tilde{l}_{4n-3}^2\,u^2)^2}, \tag{A 4}$$

where

$$u_n=(k+\alpha_n)^{\frac{1}{2}},\,l_n=(k-\alpha_n)^{\frac{1}{2}},\quad \alpha_n=\left(k^2-\left(\frac{n\pi}{4a}\right)^2\right)^{\frac{1}{2}},$$

$$\hat{u}_n=(k+\hat{\alpha}_n)^{\frac{1}{2}},\quad \hat{l}_n=(k-\hat{\alpha}_n)^{\frac{1}{2}},\quad \hat{\alpha}_n=\left(k^2-\left(\frac{n\pi}{4b}\right)^2\right)^{\frac{1}{2}},$$

$$\tilde{u}_n=(k+\tilde{\alpha}_n)^{\frac{1}{2}},\quad \tilde{l}_n=(k-\tilde{\alpha}_n)^{\frac{1}{2}},\quad \tilde{\alpha}_b=\left(k^2-\left(\frac{n\pi}{4(b-a)/2}\right)^2\right)^{\frac{1}{2}}.$$

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In order to obtain the asymptotic behaviour of the factors $K_{\pm}(\alpha)$ given by (A 4) we need to obtain the asymptotic behaviour of (A 3) as $\alpha \to \infty$ in

$$Im\,\alpha > -\operatorname{Im} k\,(\operatorname{Im}\alpha < \operatorname{Im} k).$$

We note that $K(\alpha, a) = K(-\alpha, a)$ and since $K(\alpha, a) = O(e^{2a|\alpha|})$ as $\alpha \to \pm\infty$, then by theorem 3·2 of Kranzer and Radlow [**5**]

$$K_{\pm}(\alpha, a) = O(e^{a|\alpha|}). \tag{A 5}$$

Thus
$$K_{\pm}(\alpha) = O(e^{-a|\alpha|}\, e^{b|\alpha|}\, e^{-(b-a)|\alpha|}) = O(1) \tag{A 6}$$

The factorization of

$$t(\alpha) = \frac{1}{\kappa}\ln\left[\frac{\cos(\pi/4+\kappa a)}{\cos(\pi/4-\kappa a)}.\frac{\cos(\pi/4+\kappa b)}{\cos(\pi/4-\kappa b)}\right] = t_+(\alpha)+t_-(\alpha)$$

follows directly from Lüneburg and Hurd [**6**] (when corrected), as

$$t_-(\alpha) = \frac{1}{\kappa}\ln\left[\prod_{n=1}^{\infty}\frac{(u_{4n-1}\,l+l_{4n-1}\,u)\,(u_{4n-3}\,l-l_{4n-3}\,u)\,(\hat{u}_{4n-1}\,l+\hat{l}_{4n-1}\,u)\,(\hat{u}_{4n-3}\,l-\hat{l}_{4n-3}\,u)}{(u_{4n-1}\,l-l_{4n-1}\,u)\,(u_{4n-3}\,l+l_{4n-3}\,u)\,(\hat{u}_{4n-1}\,l-\hat{l}_{4n-1}\,u)\,(\hat{u}_{4n-3}\,l+\hat{l}_{4n-3}\,u)}\right],$$

$$t_+(\alpha) = t_-(-\alpha),$$

where

$$u = (k+\alpha)^{\frac{1}{2}}, \quad u_n = (k+\alpha_n)^{\frac{1}{2}}, \quad 1 = (k-\alpha)^{\frac{1}{2}}, \quad l_n = (k-\alpha_n)^{\frac{1}{2}},$$

$$\alpha_n = \left(k^2-\left(\frac{n\pi}{4a}\right)^2\right)^{\frac{1}{2}}, \quad \hat{u}_n = (k+\hat{\alpha}_n)^{\frac{1}{2}}, \quad \hat{l}_n = (k+\hat{\alpha}_n)^{\frac{1}{2}}, \quad \hat{\alpha}_n = \left(k^2-\left(\frac{n\pi}{4b}\right)^2\right)^{\frac{1}{2}}.$$

The behaviour of $t_{\pm}(\alpha)$ as $\alpha \to \infty$ is easily shown, by using the method of Lüneburg and Hurd [**6**], to be

$$t_-(\alpha) = \frac{i}{\alpha}\ln\alpha + O(\alpha^{-1}).$$

Appendix B

Here we shall carry out the explicit factorization of (2·23) and (2·24), and give their asymptotic behaviour as $|\alpha| \to \infty$. For $K(\alpha)$ given by (2·23) we can write it in a more convenient form

$$K(\alpha) = -\frac{\cos(\kappa b+\pi/4)\cos(\kappa b-\pi/4)}{\cos(\kappa a+\pi/4)\cos(\kappa a-\pi/4)\sin^2\kappa(b-a)}$$

and if we let

$$K(\alpha, a) = \cos(\kappa a+\pi/4)\cos(\kappa a-\pi/4) = K_+(\alpha, a)\,K_-(\alpha, a),$$
$$S(\alpha, (b-a)) = \sin^2\kappa(b-a) = S_+(\alpha, (b-a))\,S_-(\alpha, (b-a)),$$

then
$$K_{\pm}(\alpha) = i\frac{K_{\pm}(\alpha, b)}{K_{\pm}(\alpha, a)\,S_{\pm}(\alpha, b-a)}.$$

Now from Appendix A it has been shown that

$$K_-(\alpha, a) = K_+(-\alpha, a) = \prod_{n=1}^{\infty}\left(\frac{a}{(2n-1)\,\pi k}\right)^2 (u_{4n-1}\,l-l_{4n-1}\,u)\,(u_{4n-1}\,l+l_{4n-1}\,u)$$
$$\times\,(u_{4n-3}\,l-l_{4n-3}\,u)\,(u_{4n-3}\,l-l_{4n-3}\,u)\exp[\alpha/\alpha_{4n-1}]\exp[\alpha/\alpha_{4n-3}],$$

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and that $K_{\pm}(\alpha, a) = O(e^{a|\alpha|})$ as $|\alpha| \to \infty$ in $\operatorname{Im}\alpha > -\operatorname{Im} k$ and $\operatorname{Im}\alpha < \operatorname{Im} k$ respectively. From the well-known product

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2 \pi^2}\right)$$

it is not difficult to show that

$$S_-(\alpha, (b-a)) = S_+(-\alpha, (b-a)) = (b-a)\, l^2 \prod_{n=1}^{\infty} \left(\frac{(b-a)}{2kn\pi}\right)^2 ((\tilde{u}_n\, l)^2 - (\tilde{l}_n\, u)^2)^2.$$

Thus

$$K_-(\alpha) = \frac{i}{(b-a)\, l^2} \prod_{n=1}^{\infty} \left(\frac{2kn\pi b}{a(b-a)}\right)^2 \times \frac{((\hat{u}_{4n-1}\, l)^2 - (\hat{l}_{4n-1}\, u)^2)\, ((\hat{u}_{4n-3}\, l)^2 - (\hat{l}_{4n-3}\, u)^2)}{((u_{4n-1}\, l)^2 - (l_{4n-1}\, u)^2)\, ((u_{4n-3}\, l)^2 - (l_{4n-3}\, u)^2)\, ((\tilde{u}_n\, l)^2 - (\tilde{l}_n\, u)^2)^2}.$$

Now since $K(-\alpha) = K(\alpha)$ then by Kranzer and Radlow[**5**] we have since $K(a) = 4 + O(1)$ that

$$K_{\pm}(\alpha) = 2 + o(1).$$

The factorization of

$$t(\alpha) = \frac{1}{\kappa} \ln \left[\frac{e^{i\pi} \cos(\pi/4 - \kappa b) \cos(\pi/4 + \kappa a)}{\cos(\pi/4 + \kappa b) \cos(\pi/4 - \kappa a)}\right]$$

follows directly from Appendix A, since

$$t_-(\alpha) = \frac{-1}{\kappa} \ln\left(\frac{-\alpha - i\kappa}{k}\right) + \frac{1}{\kappa} \ln\left[\prod_{n=1}^{\infty} \frac{(\hat{u}_{4n-1}\, l - \hat{l}_{4n-1}\, u)\, (\hat{u}_{4n-3}\, l + \hat{l}_{4n-3}\, u)\, (u_{4n-1}\, l + l_{4n-1}\, u)\, (u_{4n-3}\, l - l_{4n-3}\, u)}{(\hat{u}_{4n-1}\, l + \hat{l}_{4n-1}\, u)\, (\hat{u}_{4n-3}\, l - \hat{l}_{4n-3}\, u)\, (u_{4n-1}\, l - l_{4n-1}\, u)\, (u_{4n-3}\, l + l_{4n-3}\, u)}\right]$$

where

$$t_+(\alpha) = t_-(-\alpha)$$

and

$$u = (k+\alpha)^{\frac{1}{2}}, \quad u_n = (k+\alpha_n)^{\frac{1}{2}}, \quad l = (k-\alpha)^{\frac{1}{2}}, \quad l_n = (k-\alpha_n)^{\frac{1}{2}},$$
$$\alpha_n = (k^2 - (n\pi/(4a))^2)^{\frac{1}{2}}, \quad \hat{u}_n = (k+\hat{\alpha}_n), \quad \hat{l}_n = (k-\hat{\alpha}_n), \quad \hat{\alpha}_n = (k^2 - (n\pi/(4b))^2)^{\frac{1}{2}}.$$

The behaviour of $t_-(\alpha)$ as $\alpha \to \infty$ in $\operatorname{Im}\alpha < \operatorname{Im} k$ can be shown to be

$$t_-(\alpha) = -\frac{i}{\alpha} \ln(\alpha) + O(\alpha^{-1}).$$

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