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A note on symmetric basic sequences in $L_p(L_q)$

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Introduction

Subspaces of L_p spanned by symmetric independent identically distributed random variables were identified as Orlicz spaces by Bretagnolle and Dacunha-Castelle [1], who showed that, conversely, in the case $p \leqslant 2$, every p-convex, 2-concave Orlicz space is isomorphic to a subspace of L_p. This was extended by Dacunha-Castelle [3] to subspaces of L_p with symmetric basis, which appear as 'p-means' of Orlicz spaces (see [9] for the corresponding finite-dimensional result, and [12] for the case of rearrangement invariant function spaces). On the contrary the only subspaces with symmetric basis of L_p for $p \geqslant 2$ are l_p and l_2 (if one does not care about isomorphy constants).

A study of the spaces $L_p(L_q)$ was initiated in [11], where a dichotomy was pointed out between the case $p > q$ and the case $p \leqslant q$: for instance the space $L_p(L_q), p \leqslant q$ contains l_r, $p \leqslant r \leqslant q$, as well as the p-convex, q-concave Orlicz spaces, while $L_p(L_q)$, $p > q$ does not contain l_r, unless l_r belongs to L_p or L_q.

Here we are interested in symmetric basis subspaces of $L_p(L_q)$, for which the same distinction between the cases $p \geqslant q$ and $p < q$ is relevant. In the case $p \geqslant q$, these spaces are simply p-means of subspaces of L_q with symmetric basis. In the case $p < q$ the situation cannot be so simple (as the example of l_r shows): one could expect that these spaces are p-means of p-convex, q-concave Orlicz spaces (and this is indeed so when $p \leqslant q \leqslant 2$ or $2 \leqslant p \leqslant q$), but for the case when $p < 2 < q$ we give an example of a subspace spanned by i.i.d. L_q-valued random variables, which does not have this structure. This example is close (but not isomorphic) to a Lorentz space.

1. *The case* $p \geqslant q$

THEOREM 1. *A subspace with symmetric basis of* $L_p(L_q)$, $p \geqslant q$, *is isomorphic either to* l_p *or to a* p*-mean of subspaces with symmetric basis of* L_q (*with uniformly bounded symmetric basis constants*).

By a p-mean of subspaces of L_q with symmetric bases, we mean a space E generated by a sequence $(x_i)_{i=1}^{\infty}$ in $L_p(L_q)$ such that, for a.e. ω, $(x_i(\omega, \cdot))_{i=1}^{\infty}$ is a K-symmetric basic sequence in L_q. If $\mathbf{a} \in \mathbb{R}^{(\mathbb{N})}$ and if we denote by $\|\mathbf{a}\|_\omega$ the norm $\|\sum a_i x_i(\omega, \cdot)\|_q$, we have

$$\|\mathbf{a}\|_E = \left(\int \|\mathbf{a}\|_\omega^p \, d\mu(\omega) \right)^{1/p}$$

We have the following corollary (the first statement of which could be simply obtained by a Kadec–Pelczynski type argument):

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COROLLARY 2. (1) *A subspace of* $L_p(L_q)$, $p \geqslant q \geqslant 2$ *with symmetric basis is isomorphic to one of* l_p, l_q *or* l_2.

(2) *The subspaces of* $L_p(L_q)$, $p \wedge 2 \geqslant q$, *with symmetric basis are the* $L_p(L_q)$*-means of (sequential) Orlicz spaces associated to q-convex, 2-concave Orlicz functions.*

In the second case, the norm is given by

$$\|\mathbf{a}\| = \| \|\mathbf{a}\|_{\phi_{\cdot,\cdot}} \|_{L_p(\Omega)(L_q(\Omega'))}$$

where $(\phi_{\omega,\omega'})_{\omega\in\Omega,\,\omega'\in\Omega'}$ is a measurable family of Orlicz functions. Note that an interesting example of the second case is given by the Lorentz spaces $l_{r,p}$ $(q < r < 2)$, as we have

$$\|\mathbf{a}\|_{l_{r,p}} \sim \left(\int_0^\infty (\|\mathbf{a}\|_{\phi_t} t^{-\theta})^p \frac{dt}{t} \right)^{1/p}$$

where $(1-\theta)/q + \theta/2 = 1/r$ and $u \mapsto \phi_t(u)$ is an Orlicz function equivalent to $u^q \wedge t^2u^2$.

The proof of the scalar analogue of Theorem 1 was performed by Dacunha-Castelle by reducing to the case where the symmetric basic sequence is split into a disjoint sequence and a sequence of symmetrically exchangeable (in the probabilistic sense) random variables, the measure space being a probability space (see [3]). In the vectorial frame, the exchangeability cannot be obtained (when the measure space underlying L_q is not atomic), but we can obtain the following substitute:

PROPOSITION 3. *Let* $(\Omega, \mathscr{A}, \mu)$ *be a probability space, and* $L_p = L_p(\Omega, \mathscr{A}, \mu)$. *Let* $(x_i)_{i=1}^\infty$ *be a K-symmetric basic sequence in the space* $L_p(L_q)$ $(1 \leqslant p, q < \infty)$. *There exist two sequences* $(X_i')_{i=1}^\infty$, $(X_i'')_{i=1}^\infty$ *such that*

(i) *the sequence* (X_i'), *considered as a sequence of* L_q*-valued random variables, is symmetric and, up to a measure-preserving transformation of* L_q, *exchangeable (indeed conditionally independent and identically distributed);*

(ii) *the sequence* (X_i'') *is symmetric, disjoint (in the lattice* $L_p(L_q)$*), and disjoint from the* X_i', $i \geqslant 1$;

(iii) *the sequence* (x_i) *is K-equivalent to* $(X_i' + X_i'')$.

Proof. We can suppose that the Lebesgue spaces here are separable. Taking a subsequence if necessary, we can suppose that the sequence (x_i) defines a type Θ on the space $L_p(L_q)$ (see [8] for the definition and basic properties of types on Banach spaces), i.e. for all $x \in L_p(L_q)$, the limit

$$\Theta(x) = \lim_{i\to\infty} \|x + x_i\|_{L_p(L_q)}$$

exists. The type Θ can be described, following [5], by

$$\Theta(x)^p = \mathbf{E}_\omega \int_{\mathscr{T}(L_q)} \tau(x(\omega, \cdot))^p \, d\nu_\omega(\tau) + a^p \quad (x \in L_p(L_q))$$

where a is a non-negative real, $\mathscr{T}(L_q)$ is the space of types on the space L_q and ν_ω is a random probability measure on $\mathscr{T}(L_q)$ (such that $\mathbf{E}_\omega \int \tau(0)^p \, d\nu_\omega(\tau) < \infty$).

A type on $L_q(\Omega')$ can in turn be represented with a positive real and a random probability distribution on $\mathbb{R}$; or, equivalently, by a random variable X belonging to

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$L_q(\Omega' \times S)$ (where S is another fixed measure space, say $[0,1]$ with Lebesgue measure) and a positive real A by the formula

$$\tau(y) = (\|y+X\|_q^q + A^q)^{1/q}.$$

Conversely this formula associates to a couple $(X,A) \in L_q(\Omega' \times S) \times \mathbb{R}_+$ a type $\tau = F(X,A)$, and the map F is continuous (here the space $\mathscr{T}(L_q)$ is equipped with the topology of simple convergence on L_q and is locally compact; see [**8**]). Hence by the Jankov–Von Neumann Selection Theorem F admits a universally measurable section $\tau \mapsto (X^\tau, A^\tau)$. Let μ_ω be the image of ν_ω under this map. Then $(\mu_\omega)_{\omega \in \Omega}$ is a measurable family of probability distributions on $L_q(\Omega' \times S) \times \mathbb{R}_+$. Use again the Jankov–Von Neumann theorem to associate to each probability distribution μ on $L_q(\Omega' \times S) \times \mathbb{R}_+$ a couple $(X,A) = s(\mu)$ of respectively $L_q(\Omega' \times S)$- and $\mathbb{R}_+$-valued random variables (defined, say, on the probability space $(\bar{\Omega}, \bar{\mu})$) such that μ is the distribution of (X,A). Then $s(\mu_\omega) =: (X_\omega, A_\omega)$ depends measurably on ω, or equivalently $(X(\omega,\bar{\omega}), A(\omega,\bar{\omega})) := (X_\omega(\bar{\omega}), A_\omega(\bar{\omega}))$ is a $L_q(\Omega' \times S) \times \mathbb{R}_+$-valued random variable defined on $\Omega \times \bar{\Omega}$, and finally

$$\Theta(x)^p = \mathbf{E}_{\omega\bar{\omega}}[\|x+X\|^q_{L_q(\Omega' \times S)} + A^q]^{p/q} + a^p.$$

Now to evaluate $\|\sum_i a_i x_i\|$ we remark that

$$\left\| \sum_{i=1}^n a_i x_i \right\| \underset{K}{\sim} \lim_{i_n \to \infty} \lim_{i_{n-1} \to \infty} \dots \lim_{i_1 \to \infty} \left\| \sum_{j=1}^n a_j x_{i_j} \right\|$$

(the existence of this iterated limit is a consequence of the 'stability' of the space $L_p(L_q)$; see [**8**]). Defining now the type $a_1\Theta * a_2\Theta * \dots * a_n\Theta$ as usual by

$$a_1\Theta * a_2\Theta * \dots * a_n\Theta(x) = \lim_{i_n \to \infty} \lim_{i_{n-1} \to \infty} \dots \lim_{i_1 \to \infty} \left\| x + \sum_{j=1}^n a_j x_{i_j} \right\|$$

we have thus

$$\left\| \sum_{i=1}^n a_i x_i \right\| \underset{K}{\sim} a_1\Theta * a_2\Theta * \dots * a_n\Theta(0).$$

But we have (see [**5**])

$$a_1\Theta * a_2\Theta * \dots * a_n\Theta(x)^p$$

$$= \mathbf{E}_\omega \int_{\mathscr{T}(L_q)^n} [(a_1\tau_1) * (a_2\tau_2) * \dots * (a_n\tau_n)(x)]^p \, d\nu_\omega(\tau_1)\, d\nu_\omega(\tau_2) \dots d\nu_\omega(\tau_n) + a^p \sum_{j=1}^n |a_j|^p.$$

Now if τ_1, τ_2 are two types on L_q, in order to represent $\tau_1 * \tau_2$ we take two copies S_1, S_2 of the preceding measure space S, and couples $(X^{\tau_1}, A^{\tau_1}) \in L_q(\Omega' \times S_1) \times \mathbb{R}_+$ and $(X^{\tau_2}, A^{\tau_2}) \in L_q(\Omega' \times S_2) \times \mathbb{R}_+$ representing respectively τ_1 and τ_2. Then the couple $(X^{\tau_1} + X^{\tau_2}, ((A^{\tau_1})^q + (A^{\tau_2})^q)^{1/q})$ represents $\tau_1 * \tau_2$. Thus we take copies $\bar{\Omega}_1, \bar{\Omega}_2, \dots, \bar{\Omega}_n, \dots$ of $\bar{\Omega}$; $S_1, S_2, \dots, S_n, \dots$ of S and copies $X^1 \in L_p(\Omega \times \bar{\Omega}_1; L_q(\Omega' \times S_1))$, $A_1 \in L_p(\Omega \times \bar{\Omega}_1)$, $X^2 \in L_p(\Omega \times \bar{\Omega}_2; L_q(\Omega' \times S_2))$, $A_2 \in L_p(\Omega \times \bar{\Omega}_2)$, $\dots, X^n \in L_p(\Omega \times \bar{\Omega}_n; L_q(\Omega' \times S_n))$, $A_n \in L_p(\Omega \times \bar{\Omega}_n) \dots$ of the same $X \in L_p(\Omega \times \bar{\Omega}; L_q(\Omega' \times S))$, $A \in L_p(\Omega \times \bar{\Omega})$ and obtain

$$|a_1\Theta * a_2\Theta * \dots * a_n\Theta(x)|^p = \mathbf{E}_{\omega, \bar{\omega}_1, \dots, \bar{\omega}_n} \left(\|x(\omega) + a_1 X^1(\omega, \bar{\omega}_1) + \dots + a_n X^n(\omega, \bar{\omega}_n)\|_q^q + \sum_{j=1}^n |a_j|^q A_j^q(\omega, \bar{\omega}_j) \right)^{p/q} + a^p \sum_{j=1}^n |a_j|^p.$$

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Hence $$\left\|\sum_{i=1}^n a_i x_i\right\| \sim \left\|\sum_{i=1}^n a_i X^i\right\|_{L_p(L_q)} + \left\|\left(\sum_{i=1}^n |a_i|^q A_i^q\right)^{1/q}\right\|_p + a\left(\sum_{i=1}^n |a_i|^p\right)^{1/p},$$

where the X^i belong to $L_p(\Omega \times \prod_i \bar{\Omega}_i; L_q(\Omega' \times \prod_i S_i))$. Note that for a.e. $\omega \in \Omega$ the X_ω^i are independent $L_q(\Omega' \times \prod_i S_i)$-valued random variables, equidistributed modulo the isomorphisms of the measure spaces S_i. By unconditionality of the sequence (x_i), we can replace the (X^i) by $(\epsilon_i \otimes X^i)$, where (ϵ_i) is a sequence of ± 1 valued independent Bernouilli variables (independent from (X_i)). Finally we set $X_i' = \epsilon_i \otimes X^i$ and $X_i'' = A^i \otimes f_i + aB^i \otimes f_i$, where (f_i), resp. (B^i) is a fixed normalized disjoint sequence in L_q, resp. L_p. ∎

We are led by Proposition 3 and its proof to investigate the span in $L_p(L_q)$ of a sequence of L_q-valued, symmetric, independent and, up to a measure-preserving transformation in L_q, identically distributed random variables.

PROPOSITION 4. *Let $1 \leqslant q \leqslant p < \infty$ and (X_i) be a sequence of p-integrable L_q-valued, mean zero, independent random variables, which are identically distributed (up to a measure preserving transformation in L_q). Then (X_i) is equivalent to a symmetric basic sequence in $L_q \oplus l_p$ in the case $p \geqslant 2$, resp. in L_q in the case $p \leqslant 2$, the equivalence constant depending only on p, q.*

Proof. By the usual symmetrization argument, we can suppose the X_i to be symmetric. By the Hoffmann-Jorgensen inequality ([6], theorem 3·1), we have

$$\|\sum a_n X_n\|_{L_p(L_q)} \leqslant K(p)[\|\sum a_n X_n\|_{L_q(L_q)} + \|\bigvee |a_n| \, \|X_n\|_q\|_p].$$

Let us suppose first $p \geqslant 2$. We have

$$\begin{aligned}\|\bigvee |a_n| \, \|X_n\|_q\|_p &\leqslant \|(\sum |a_n|^p \, \|X_n\|_q^p)^{1/p}\|_p = [\sum |a_n|^p \, \|X_n\|_{L_p(L_q)}^p]^{1/p} \\ &\leqslant \|X_0\|_{L_p(L_q)} (\sum |a_n|^p)^{1/p}.\end{aligned}$$

Conversely $\|\sum a_n X_n\|_{L_p(L_q)} \geqslant C_{p,q}(\sum |a_n|^p \, \|X_n\|_{L_p(L_q)}^p)^{1/p}$, where $C_{p,q}$ is the cotype p-constant of $L_p(L_q)$.

If now $p \leqslant 2$, we have

$$\|\bigvee |a_n| \, \|X_n\|_q\|_p \leqslant \|(\sum |a_n|^2 \, \|X_n\|^2)^{\frac{1}{2}}\|_p \leqslant C' \|\sum a_n \epsilon_n \otimes \|X_n\|_q\|_p,$$

where (ϵ_n) is a sequence of real Bernouilli variables. Conversely

$$\|\sum a_n X_n\|_q \geqslant C_q[\sum |a_n|^2 \, \|X_n\|_q^2]^{\frac{1}{2}},$$

where C_q is the cotype 2 constant of L_q, hence

$$\|\sum a_n X_n\|_{L_p(L_q)} \geqslant C_q \|\sum a_n \epsilon_n \otimes \|X_n\|_q\|_p.$$

Finally $$\|\sum a_n X_n\|_{L_p(L_q)} \sim \|\sum a_n X_n\|_{L_q(L_q)} + \|\sum a_n Y_n\|_p$$

where (Y_n) is a symmetric sequence in L_p, hence isometrically equivalent (as $q \leqslant p \leqslant 2$) to a symmetric sequence in L_q. ∎

It is now straightforward to deduce Theorem 1 from Propositions 3 and 4. (In the disjoint case, a $(2/q)$-convexification or concavification reduces to a symmetric sequence in $L_{2p/q}$ $(L_2) \approx L_{2p/q}$, with $2p/q \geqslant 2$.)

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2. *The case $p \leqslant q$*

We may suppose that $q > 2$: otherwise $L_p(L_q)$ isometrically embeds in L_p.

When $p \geqslant 2$, and all the measure spaces are probability spaces, then if (X_n) is a symmetric basic sequence in $L_p(L_q)$, we may suppose by a standard subsequence splitting lemma that $X_n = X'_n + X''_n$, where (X'_n) is $L_2(L_2)$-bounded and (X''_n) is a disjoint sequence in the lattice $L_p(L_q)$, and disjoint from (X'_n). Then $(X'_n) \sim l_2$-basis (by the type 2 property of $L_p(L_q)$ and the fact that the norm in $L_p(L_q)$ dominates that of $L_2(L_2)$), unless $X'_n \to 0$, while the norm in $\overline{\text{span}}\,[X''_n]$ is dominated by that of l_2 (again by type 2). If $X'_n \to 0$, we may suppose $X_n = X''_n$; then $\overline{\text{span}}\,[X_n]$ is the $\frac{1}{2}q$-convexification of a subspace of $L_{2p/q}$ with symmetric basis, i.e. of a $(2p/q)$-mean of $(2p/q)$-convex, 2-concave Orlicz spaces: thus $\overline{\text{span}}\,[X_n]$ is a p-mean of p-convex, q-concave Orlicz sequence spaces.

In the case $p < 2 < q$, we shall exhibit an example of a symmetric basic sequence which spans a subspace not isomorphic to a p-mean of p-convex, q-concave Orlicz spaces (or, equivalently, to a sublattice of $L_p(L_q)$). This sequence consists in fact of i.i.d. symmetric L_q-valued random variables.

PROPOSITION 5. *Let $(X_k)_k$ be a sequence of p-integrable symmetric independent $L_q(S)$-valued random variables $(p < 2 < q)$. Then*

$$\left\| \sum_{k=1}^n X_k \right\|_{L_p(L_q)} = 1 \quad \textit{implies} \quad c \leqslant H(X_1, \ldots, X_n) \leqslant C,$$

where

$$H(X_1, \ldots, X_n) = \sum_{k=1}^n \mathbf{E}(\|X_k\|_q^q \wedge \|X_k\|_q^p) + \int_S \left[\mathbf{E}_\omega \left(\sum_{k=1}^n \frac{|X_k(\cdot, s)|^2}{\|X_k\|_q^2} (\|X_k\|_q^2 \wedge \|X_k\|_q^p) \right) \right]^{q/2} ds$$

and the constants c, C depend only on p, q.

The proof of Proposition 5 makes use of the following remark, inspired by Dacunha-Castelle [**4**] (see also lemma 3·1 of [**7**]).

LEMMA 6. *Let E be a Banach space, $0 < p \leqslant q < \infty$, and $(X_k)_k$ a sequence of p-integrable symmetric independent E-valued random variables. There exist constants a_1, c_1, c_2 (depending on p, q) such that*

$$\left\| \sum_{k=1}^n X_k \right\|_{L_p(E)} = a_1 \quad \textit{implies} \quad c_1 \leqslant G(X_1, \ldots, X_n) \leqslant c_2,$$

where

$$G(X_1, \ldots, X_n) = \sum_{k=1}^n \mathbf{E}(\|X_k\|_E^p \cdot \mathbb{1}_{\{\|X_k\|_E > 1\}}) + \left\| \sum_{k=1}^n X_k \mathbb{1}_{\{\|X_k\|_E \leqslant 1\}} \right\|_{L_q(E)}^q. \tag{2·1}$$

Proof. We give the proof for the sake of completeness. Set $X'_k = X_k \cdot \mathbb{1}_{\{\|X_k\|_E \leqslant 1\}}$ and $X''_k = X_k \cdot \mathbb{1}_{\{\|X_k\|_E > 1\}}$. We have first

$$\begin{aligned} \left\| \sum_{k=1}^n X_k \right\|_{L_p(E)} &\leqslant \left\| \sum_{k=1}^n X'_k \right\|_{L_p(E)} + \left\| \sum_{k=1}^n X''_k \right\|_{L_p(E)} \\ &\leqslant \left\| \sum_{k=1}^n X'_k \right\|_{L_q(E)} + C_p \left(\left\| \sum_{k=1}^n X''_k \right\|_{L_1(E)} + \left\| \bigvee_{k=1}^n \|X''_k\|_E \right\|_p \right) \end{aligned}$$

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by the Hoffmann-Joergensen inequality. But (if $p \geqslant 1$)

$$\left\| \sum_{k=1}^{n} X_k'' \right\|_{L_1(E)} \leqslant \mathbf{E}\left(\sum_{k=1}^{n} \|X_k''\|_E \right) \leqslant \mathbf{E}\left(\sum_{k=1}^{n} \|X_k''\|_E^p \right)$$

(if $p < 1$ use simply the fact that E is a p-Banach space) and

$$\left\| \bigvee_{k=1}^{n} \|X_k''\|_E \right\|_p^p \leqslant \mathbf{E}\left(\sum_{k=1}^{n} \|X_k''\|_E^p \right).$$

Thus $$\left\| \sum_{k=1}^{n} X_k \right\|_{L_p(E)} \leqslant C.(G(X_1, \ldots, X_n) + G(X_1, \ldots, X_n)^{1/p}).$$

Conversely we majorize the first term in G using the Hoffmann-Joergensen inequality:

$$\left\| \sum_{k=1}^{n} X_k' \right\|_{L_q(E)} \leqslant C_{p,q}\left(\left\| \sum_{k=1}^{n} X_k' \right\|_{L_p(E)} + \left\| \bigvee_{k=1}^{n} \|X_k'\|_E \right\|_q \right) \leqslant C_{p,q}\left(\left\| \sum_{k=1}^{n} X_k' \right\|_{L_p(E)} + 1 \right)$$

and by the symmetry and the independence of the variables $X_1, \ldots, X_n$, we have

$$\left\| \sum_{k=1}^{n} X_k' \right\|_{L_p(L_q)} \lesssim \left\| \sum_{k=1}^{n} X_k \right\|_{L_p(L_q)}$$

(if $p < 1$ a 2^{1-p} factor is needed in the right-hand side). The second term in G will now be estimated. We have

$$\begin{aligned} \mathbf{P}\left(\bigvee_{k=1}^{n} \|X_k''\|_E > u \right) &= 1 - \prod_{k=1}^{n} (1 - \mathbf{P}(\|X_k''\|_E > u)) \\ &\geqslant 1 - \exp\left(- \sum_{k=1}^{n} \mathbf{P}(\|X_k''\|_E > u) \right) \\ &\geqslant \frac{1}{e}\left(1 \wedge \sum_{k=1}^{n} \mathbf{P}(\|X_k''\|_E > u) \right). \end{aligned}$$

Hence if $$\mathbf{E}\left(\bigvee_{k=1}^{n} \|X_k''\|_E^p \right) < e^{-1}$$

we have $$\sum_{k=1}^{n} \mathbf{P}(\|X_k''\|_E \neq 0) = \sum_{k=1}^{n} \mathbf{P}(\|X_k''\|_E > 1) < 1,$$

and so $$\begin{aligned} \frac{1}{e} \geqslant \mathbf{E}\left(\bigvee_{k=1}^{n} \|X_k''\|_E^p \right) &= \int_0^\infty \mathbf{P}\left(\bigvee_{k=1}^{n} \|X_k''\|_E > u \right) d(u^p) \\ &\geqslant \frac{1}{e} \int_0^\infty \sum_{k=1}^{n} \mathbf{P}(\|X_k''\|_E > u)\, d(u^p) \\ &= \sum_{k=1}^{n} \|X_k''\|_{L_p(E)}^p. \end{aligned}$$

But as a consequence of P. Lévy's inequality we have

$$\mathbf{E}\left(\bigvee_{k=1}^{n} \|X_k''\|_E^p \right) \leqslant \mathbf{E}\left(\bigvee_{k=1}^{n} \|X_k\|_E^p \right) \leqslant C_p \mathbf{E} \left\| \sum_{k=1}^{n} X_k \right\|_E^p. \quad \blacksquare$$

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Proof of Proposition 5. Set $X'_k = X_k \cdot \mathbb{1}_{\{\|X_k\|_q \leqslant 1\}}$ and $X''_k = X_k \cdot \mathbb{1}_{\{\|X_k\|_q > 1\}}$. We can choose a version of the process $(X'_k(\cdot, s))_{k=1}^n$ such that, for every s, the random variables $X'_k(\cdot, s)$, $k = 1, \ldots, n$, are symmetric i.i.d. Applying to them Rosenthal's inequality ([**13**]), we obtain

$$\int_\Omega \left| \sum_{k=1}^n X'_k(\omega, s) \right|^q d\mathbf{P}(\omega) \sim \left(\sum_{k=1}^n \int_\Omega |X'_k(\omega, s)|^2 \, d\mathbf{P}(\omega) \right)^{q/2} + \sum_{k=1}^n \int_\Omega |X'_k(\omega, s)|^q \, d\mathbf{P}(\omega)$$

(with constants depending only on q); thus

$$\left\| \sum_{k=1}^n X'_k \right\|_{L_p(L_q)}^q \sim \int_S \left(\sum_{k=1}^n \mathbf{E}|X'_k(\cdot, s)|^2 \right)^{q/2} ds + \sum_{k=1}^n \mathbf{E}\|X'_k\|_q^q. \tag{2·2}$$

On the other hand, as the identity map is bounded from $L_q(L_2)$ into $L_2(L_q)$, with norm 1, when $q \geqslant 2$, we have

$$\int_S \left(\mathbf{E}\left(\sum_{k=1}^n \frac{|X''_k(\cdot, s)|^2}{\|X_k\|_q^{2-p}} \right) \right)^{q/2} ds \leqslant \left(\sum_{k=1}^n \mathbf{E}(\|X''_k\|_q^p) \right)^{q/2}. \tag{2·3}$$

Putting (2·1), (2·2) and (2·3) together, we obtain

$$\begin{aligned} H(X_1, \ldots, X_n) &\sim \sum_{k=1}^n \mathbf{E}(\|X'_k\|_q^q + \|X''_k\|_q^p) + \int_S \left(\mathbf{E}\left(\sum_{k=1}^n \left(|X'_k(\cdot, s)|^2 + \frac{|X''_k(\cdot, s)|^2}{\|X_k\|_q^{2-p}} \right) \right) \right)^{q/2} ds \\ &\lesssim G(X_1, \ldots, X_n) + G(X_1, \ldots, X_n)^{q/2}. \\ G(X_1, \ldots, X_n) &\lesssim \sum_{k=1}^n \mathbf{E}(\|X'_k\|_q^q + \|X''_k\|_q^p) + \int_S \left(\mathbf{E}\left(\sum_{k=1}^n \left(|X'_k(\cdot, s)|^2 + \frac{|X''_k(\cdot, s)|^2}{\|X_k\|_q^{2-p}} \right) \right) \right)^{q/2} ds \\ &\leqslant H(X_1, \ldots, X_n) \end{aligned}$$

from which Proposition 5 is easily deduced, noticing that

$$(\lambda^p \wedge \lambda^q) H(X_1, \ldots, X_n) \leqslant H(\lambda X_1, \ldots, \lambda X_n) \leqslant (\lambda^p \vee \lambda^q) H(X_1, \ldots, X_n)$$

for all $0 < \lambda < \infty$. ∎

COROLLARY 7. *Let $p \leqslant 2 \leqslant q$ and $(X_k)_{k=1}^\infty$ a sequence of symmetric identically distributed $L_q(S)$-valued p-integrable variables. Set*

$$\psi(t) = \mathbf{E}(t^q \|X_1\|_q^q \wedge t^p \|X_1\|_q^p) \quad \text{and} \quad \phi(t, s) = \mathbf{E}_\omega\left(\frac{|X_1(\omega, s)|^2}{\|X_1\|_q^2} (t^2 \|X_1\|_q^2 \wedge t^p \|X_1\|_q^p) \right).$$

Denote by ϕ the map $t \mapsto \phi(\cdot, t)$, $\mathbb{R}_+ \to L_{q/2}(S)$. Define a modular M on $\mathbb{R}^{(\mathbb{N})}$ by setting, for any $\mathbf{a} = (a_k)_k \in \mathbb{R}^{(\mathbb{N})}$

$$M(\mathbf{a}) = (\sum_k \psi(a_k))^{q/2} + \|\sum_k \phi(a_k)\|_{q/2}^{q/2}. \tag{2·4}$$

Then $\|\sum_k a_k X_k\|_{L_p(L_q)} \sim \|\mathbf{a}\|_M$, where $\|\mathbf{a}\|_M := \inf\{\lambda > 0 / M(\lambda^{-1}\mathbf{a}) \leqslant 1\}$ is the norm associated to M.

Note that M defined as above is not convex, but one can define an equivalent convex modular $\bar{M}$ with convex Orlicz functions $\bar{\psi}, \bar{\phi}$ equivalent to ψ, ϕ. The space spanned by (X_k) is thus isomorphic to the 'Orlicz sequence space' associated to a vectorial Orlicz function with values in $\mathbb{R}_+ \times L_{q/2}^+$. We shall denote this space by E_M.

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A particular case. We shall now give explicitly an example where E_M is not a (usual) Orlicz sequence space, nor a p-mean of Orlicz spaces, but is close to a Lorentz sequence space.

We set $S=(0,\infty)$, define a positive random variable Y by its probability distribution $\mathbf{P}(Y>u)=u^{-2}$ for $u\geqslant 1$, $\mathbf{P}(Y>u)=1$ for $u\leqslant 1$, and set $X=\epsilon Y^{1-1/q}\mathbb{1}_{[0,Y]}$, where ϵ is a Bernouilli random variable independent of Y. Then $\|X\|_q=Y$, and the space spanned by independent copies of X is associated to the functions ϕ,ψ of Corollary 7 which are given by

$$\psi(t)=2\int_1^\infty t^q u^q\wedge t^p u^p\frac{du}{u^3}\sim t^2\wedge t^p,$$

$$\begin{aligned}\phi(t)(s)&=\mathbf{E}(Y^{-2/q}.\mathbb{1}_{[0,Y]}(s).t^2Y^2\wedge t^pY^p)\\&=2\int_{s\vee 1}^\infty u^{-2/q}(t^2u^2\wedge t^pu^p)\frac{du}{u^3}\sim\frac{t^2s^2\wedge t^ps^p}{s^{2+2/q}}\quad(\text{if } s\geqslant 1)\end{aligned}$$

(resp. $\phi(t)(s)\sim t^2\wedge t^p$ if $s\leqslant 1$). The modular M defined in Corollary 7 is thus (up to equivalence) given by

$$M(\mathbf{a})=\int_1^\infty\left(\sum_k\frac{(|a_k|s)^p\wedge(|a_k|s)^2}{s^2}\right)^{q/2}\frac{ds}{s}\tag{2·5}$$

(as the first term in (2·4) is absorbed by the second one). In fact the space E_M does not depend (from the isomorphic point of view) on p, as an equivalent norm is given by the modular

$$M_1(\mathbf{a})=(\sum_k|a_k|^2)^{q/2}+\int_1^\infty(\sum_k|a_k|\mathbb{1}_{\{|a_k|s\leqslant 1\}})^{q/2}\frac{ds}{s}.$$

For we have

$$\left(\int_1^\infty(\sum_k|a_k|^p s^{p-2}\mathbb{1}_{\{|a_k|s\geqslant 1\}})^{q/2}\frac{ds}{s}\right)^{2/q}\leqslant\sum_k|a_k|^p\left(\int_1^\infty\mathbb{1}_{\{|a_k|s\geqslant 1\}}\frac{ds}{s^{1+q(1-p/2)}}\right)^{2/q}\sim\sum_k|a_k|^2,$$

and on the other hand, if $\|\mathbf{a}\|_\infty\leqslant 1$, we have

$$M(\mathbf{a})\geqslant\int_1^2(\sum_k(|a_k|s)^p\wedge(|a_k|s)^2)^{q/2}\frac{ds}{s^{q+1}}\sim\sum_k|a_k|^p\wedge|a_k|^2=\sum_k|a_k|^2.$$

We shall now give some elements of the geometry of the space E_M (in this particular case).

We shall denote by $\bar\phi$ an Orlicz function equivalent to $u\mapsto u^p\wedge u^2$, such that $u\mapsto u^{-1}\bar\phi(u)$ is non-increasing and $\bar\phi(u)=u^2$ for $u\leqslant 1$, and we denote by $\bar M$ the corresponding modular.

LEMMA 8. *The fundamental function of E_M is given (up to equivalence) by*

$$\lambda(N)\sim N^{\frac{1}{2}}(\log N)^{1/q}.$$

Proof. Recall that $\lambda(N)=\|\mathbb{1}_{[0,N]}\|$. An elementary calculation shows that

$$M(\lambda^{-1}\mathbb{1}_{[0,N]})\sim\lambda^{-q}N^{q/2}(\log\lambda+1).$$ ∎

LEMMA 9. *The upper Boyd index q_{E_M} of E_M equals 2.*

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Proof. If $a \in E_M$ and $0 \neq l \in \mathbb{N}$, we define $D_l a$ by $(D_l a)_k = a_j$ for $l(j-1) < k \leqslant lj$. It is immediate that $\|D_l a\|_M \geqslant l^{\frac{1}{2}} \|a\|_M$, hence $q_{E_M} \leqslant 2$. Lemma 8 implies that $q_{E_M} \geqslant 2$. ∎

As a consequence of Lemma 9 and theorem 2·b·6 of [**10**], we have

LEMMA 10. *E_M contains uniformly spaces l_2^n isomorphically as sublattices.*

LEMMA 11. *If $(y_n)_{n=1}^{\infty}$ is a bounded sequence in E_M which converges to zero in c_0, it converges to zero in l_2.*

Proof. We have, for all $y \in E_M$,

$$M(y) \geqslant \int_1^{1/\|y\|_\infty} (\sum_k (|y_k| s)^p \wedge (|y_k| s)^2)^{q/2} \frac{ds}{s^{q+1}} = \|y\|_2^q \log \frac{1}{\|y\|_\infty}. \quad \blacksquare$$

LEMMA 12. *If $(y^{(n)})_{n=1}^{\infty}$ is a bounded sequence in E_M which converges to zero in c_0, then, for all $x \in E_M$,*

$$|\bar{M}(x+y^{(n)}) - (\bar{M}(x) + \bar{M}(y^{(n)}))| \underset{n\to\infty}{\longrightarrow} 0$$

(and this convergence is uniform for x belonging to an order interval of E_M).

Proof. We note first that the modified convex modular $\bar{M}$ is uniformly continuous on bounded subsets of E_M (at the difference of M). For we have

$$|\bar{M}(x+y)^{2/q} - \bar{M}(x)^{2/q}| \leqslant \left(\int_1^{\infty} (\sum_k |\bar{\phi}((x_k+y_k)s) - \bar{\phi}(x_k s)|)^{q/2} \frac{ds}{s^{q+1}} \right)^{2/q}.$$

Setting $y = \tau z$ with $\bar{M}(z) = 1$, and noticing that, by convexity of $\bar{\phi}$

$$0 \leqslant \tau \leqslant 1 \quad \text{implies} \quad |\bar{\phi}(u+\tau h) - \bar{\phi}(u)| \leqslant \tau |\bar{\phi}(u+h) - \bar{\phi}(u)|$$

we obtain $$|\bar{M}(x+y)^{2/q} - \bar{M}(x)^{2/q}| \leqslant \|y\|_{\bar{M}} (\bar{M}(x+z)^{2/q} + \bar{M}(x)^{2/q}).$$

Thus in the setting of Lemma 12 we can suppose that x has finite support $[0, N]$, and that the supports of the elements $y^{(n)}$ are disjoint from $[0, N]$. Then

$$\bar{M}(x+y^{(n)}) = \int_1^{\infty} (\sum_{k \leqslant N} \bar{\phi}(x_k s) + \sum_{k > N} \bar{\phi}(y_k^{(n)} s))^{q/2} \frac{ds}{s^{q+1}}$$

and as $\bar{\phi}(s) \sim s^2$ at 0, we have

$$\int_1^A \left(\sum_{k>N} \frac{\bar{\phi}(y_k^{(n)} s)}{s^2} \right)^{q/2} \frac{ds}{s} \leqslant C(A) \|y^{(n)}\|_2^{q/2}$$

which goes to zero by Lemma 11. On the other hand, as $\bar{M}(x) < \infty$,

$$\int_A^{\infty} (\sum_k \bar{\phi}(x_k s))^{q/2} \frac{ds}{s^{q+1}} \underset{A\to\infty}{\longrightarrow} 0.$$

Finally

$$\begin{aligned} \bar{M}(x+y^{(n)}) &\simeq \int_1^A (\sum_{k \leqslant N} \bar{\phi}(x_k s))^{q/2} \frac{ds}{s^{q+1}} + \int_A^{\infty} (\sum_{k>N} \bar{\phi}(y_k^{(n)} s))^{q/2} \frac{ds}{s^{q+1}} \\ &\simeq \bar{M}(x) + \bar{M}(y^n), \end{aligned}$$

which ends the proof of Lemma 12.

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LEMMA 13. *For all $x \in E_M$ and $\lambda \leqslant 1$, we have*

$$|\bar{M}(\lambda x)^{2/q} - \lambda^2 \bar{M}(x)^{2/q}| \leqslant C\|x\|_2^2 \lambda^2((\log \lambda^{-1})^{2/q} + 1),$$

where C depends only on $\bar{\phi}$.

Proof. We have

$$|\bar{M}(\lambda x)^{2/q} - \lambda^2 \bar{M}(x)^{2/q}| \leqslant \left(\int_1^\infty (\sum_k |\bar{\phi}(\lambda |x_k| s) - \lambda^2 \bar{\phi}(|x_k| s)|)^{q/2} \frac{ds}{s^{q+1}} \right)^{2/q}$$

$$\leqslant \sum_k \left(\int_1^\infty |\bar{\phi}(\lambda |x_k| s) - \lambda^2 \bar{\phi}(|x_k| s)|^{q/2} \frac{ds}{s^{q+1}} \right)^{2/q}.$$

As we suppose $\bar{\phi}(u) = u^2$ for $u \leqslant 1$, the contribution of the segment $(0, |x_k|^{-1})$ in the preceding integrals is zero; noticing also that $u^{-2}\bar{\phi}(u)$ is non-increasing, we obtain

$$|\bar{M}(\lambda x)^{2/q} - \lambda^2 \bar{M}(x)^{2/q}| \leqslant \sum_k \left(\int_{(1/|x_k|)}^{(1/\lambda|x_k|)} (\lambda^2 |x_k|^2)^{q/2} \frac{ds}{s} \right)^{2/q} + \sum_k \left(\int_{(1/\lambda|x_k|)}^\infty (\bar{\phi}(\lambda |x_k| s))^{q/2} \frac{ds}{s^{q+1}} \right)^{2/q}$$

$$= \sum_k \lambda^2 |x_k|^2 (\log \lambda^{-1})^{2/q} + \sum_k \lambda^2 |x_k|^2 \left(\int_1^\infty \bar{\phi}(s)^{q/2} \frac{ds}{s^{q+1}} \right)^{2/q}.$$

LEMMA 14. *Every infinite-dimensional subspace of E_M contains isomorphically the space l_q.*

Proof. Let V be an infinite-dimensional subspace of E_M. Using Lemmas 11 and 12, we can choose recursively a sequence $(y^{(n)})$ in V such that

$$\|y^{(n)}\|_{\bar{M}} = 1, \quad \sum_n \|y^{(n)}\|_2^q \leqslant \delta/2C$$

(where C is the constant of Lemma 13) and such that for all λ_n with $0 \leqslant \lambda_n \leqslant 1$, we have

$$|\bar{M}(\sum_n \lambda_n y^{(n)}) - \sum_n \bar{M}(\lambda_n y^{(n)})| \leqslant \epsilon.$$

Then an equivalent norm on $W = \overline{\text{span}}\,[y^{(n)}]_{n=1}^\infty$ is given by the modular

$$\Phi(\lambda) = \sum_n \bar{M}(\lambda_n y^{(n)}).$$

We have by Lemma 13

$$|\Phi(\lambda)^{2/q} - (\sum_n |\lambda_n|^q)^{2/q}| \leqslant (\sum_n |\bar{M}(\lambda_n y^{(n)})^{2/q} - |\lambda_n|^2|^{q/2})^{2/q}$$

$$\leqslant C \left\{ \sum_n \lambda_n^q \|y^{(n)}\|_2^q \left[\left(\log \frac{1}{|\lambda_n|} \right)^{2/q} + 1 \right]^{q/2} \right\}^{2/q}$$

$$\leqslant C \max_{\lambda \leqslant 1} \lambda^2 \left[\left(\log \frac{1}{\lambda} \right)^{2/q} + 1 \right] (\sum_n \|y^{(n)}\|_2^q)^{2/q} \leqslant \delta.$$

Hence $\sum_n |\lambda_n|^q \leqslant 1$ implies that $\Phi(\lambda) \leqslant (1+\delta)^{q/2}$ and conversely $\Phi(\lambda) \leqslant 1$ implies that $\sum_n |\lambda_n|^q \leqslant (1+\delta)^{q/2}$. ∎

PROPOSITION 15. *The space E_M contains $(\oplus l_2^n)_{l_q}$ as a sublattice.*

As the space E_M is isomorphic to a stable space (a subspace of $L_p(L_q)$), and contains

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uniformly the spaces l_2^n (as sublattices) it contains isomorphically, as a sublattice, a space $(\oplus l_2^n)_{l_r}$ for some r (this is the lattice version of a result of Krivine, see [11]), and by Lemma 14 we have $r = q$. This can also be checked directly, noticing that an l_2^n basis in E_M can be obtained with c_0-norm as small as desired and applying then Lemmas 12 and 13 as in the proof of Lemma 14. ∎

COROLLARY 16. *The space E_M is not isomorphic to a p-mean of (p-convex, q-concave) Orlicz spaces, or, equivalently, to a sublattice of $L_p(L_q)$ generated by disjoint elements.*

Proof. If E_M is isomorphic to a sublattice of $L_p(L_q)$ (generated by disjoint elements), it is lattice-isomorphic to (another) sublattice of $L_p(L_q)$. But a sublattice of $L_p(L_q)$ cannot contain $(\oplus l_2^n)_{l_q}$ as a sublattice (see [11]).

Remark 17. In view of formula (2·4), we could interpret the space E_M as the interpolation space $[A_0^p, A_1^2]_{0,q/2}$, where $A_0 = l_p$, $A_1 = l_2$ (see [2] for interpolation of normed groups and in particular for the space $[A_0^{p_0}, A_1^{p_1}]_{\theta,q}$, $0 < \theta < 1$). In the case $0 < \theta < 1$, the 'Power Theorem' ([2], theorem 3·11·6) states that $[A_0^{p_0}, A_1^{p_1}]_{\theta,q} = [A_0, A_1]^p_{\eta, qp}$, with $p = \theta p_0 + (1-\theta)\, p_1$ and $\eta = \theta p_1/p$. The corresponding space would be here the space $[l_p, l_2]_{1,q}$ (which we shall denote by $\lambda_{2,q}$) equipped with the norm

$$\|x\|_{\lambda_{2,q}} = \left(\int_1^\infty (t^{-1}K(t, x; l_p, l_2))^q \frac{dt}{t}\right)^{1/q}$$

where $K(t, x)$ is the usual K-functional of interpolation.

This last norm is, up to equivalence, linked to the usual Lorentz $l_{2,q}$-norm by the formula

$$\|x\|_{\lambda_{2,q}} \sim \|N_2(x)\|_{2,q},$$

where $N_2(x) \in c_0$ is defined by setting

$$N_2(x)_k = \left(\frac{1}{k}\sum_{l \geqslant k} |x_l^*|^2\right)^{\frac{1}{2}},$$

where x^* is the non-increasing rearrangement of x.

This space $\lambda_{2,q}$ is close to the space E_M; in particular it has the same fundamental function and gives the same norm (up to equivalence) to 'sufficiently regular' vectors (i.e. $x^*_{2k} \neq 0$ implies $(1-\epsilon)\, x_k^* \geqslant x_{2k}^* \geqslant c\,.\,x_k^*$). However these spaces are not equal, as is shown by the example of vectors $x^{(N)}$, defined by $x_k^{(N)} = (\log(k+1))^{-1}$ if $1 \leqslant k \leqslant N$ and $x_k^{(N)} = 0$ if $k > N$. Elementary calculation shows that

$$\|x^{(N)}\|_{\lambda_{2,q}} \sim \frac{N^{\frac{1}{2}}}{(\log N)^{1-1/q}}, \quad \text{but} \quad \|x^{(N)}\|_M \sim \frac{N^{\frac{1}{2}}}{\log N}(\log\log N)^{1/q}.$$

To end this paper we mention the following two questions which remain open.

Question 1. Is the space E_M isomorphic to a $L_p(L_q)$ mean of Orlicz spaces?

Question 2. Is the Lorentz sequence space $l_{2,q}$ isomorphically embeddable in $L_p(L_q)$?

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