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A POLYHEDRAL STUDY OF A TWO LEVEL FACILITY LOCATION MODEL *

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Abstract. We study an uncapacitated facility location model where customers are served by facilities of level one, then each level one facility that is opened must be assigned to an opened facility of level two. We identify a polynomially solvable case, and study some valid inequalities and facets of the associated polytope.

Keywords. Uncapacitated facility location problem, two level facility location.

Mathematics Subject Classification. 05C85, 90C27.

1. Introduction

We study a two level uncapacitated facility location model. Each customer has to be assigned to an opened facility of the first level, then each opened facility of the first level must be assigned to an opened facility of the second level. There are fixed costs for each assignment, and fixed costs for opening each facility. The objective is to minimize the total cost. This represents a hierarchical structure with minor regional depots and major central warehouses. Here we study a polytope associated with this model, and identify a polynomially solvable case. Surprisingly the only reference to this model that we found is [10] where an approximation algorithm has been given for the extension to k levels.

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A different related model arises when each client must be serviced by a sequence of k different facilities, each of these sequences has a transportation cost, and each facility has a fixed cost. An approximation algorithm for this case has been given in [1]. A polytope associated with the two level version of this last model has been studied in [2].

Now we give a precise definition of the problem. Let $G = (V, A)$ be a tripartite directed graph, with $V = V_0 \cup V_1 \cup V_2$, $V_i \cap V_j = \emptyset$, for $i \neq j$; $A = A_1 \cup A_2$, $A_1 \subseteq V_0 \times V_1$ and $A_2 \subseteq V_1 \times V_2$. Here V_0 corresponds to a set of customers, V_1 corresponds to a set of potential facilities of level one, and V_2 corresponds to a set of potential facilities of level two. An integer programming formulation of the *two level facility location problem* is

$$\min \sum_{(u,v)\in A_1\cup A_2} c(u,v)x(u,v) + \sum_{v\in V_1\cup V_2} f(v)y(v)$$

$$\sum_{(u,v)\in A_1} x(u,v) = 1, \quad \forall u \in V_0 \tag{1.1}$$

$$x(u,v) \leq y(v), \quad \forall (u,v) \in A_1 \tag{1.2}$$

$$\sum_{(u,v)\in A_2} x(u,v) = y(u), \quad \forall u \in V_1 \tag{1.3}$$

$$x(u,v) \leq y(v), \quad \forall (u,v) \in A_2 \tag{1.4}$$

$$x(u,v) \in \{0,1\}, \quad \forall (u,v) \in A_1 \cup A_2 \tag{1.5}$$

$$y(v) \in \{0,1\}, \quad \forall v \in V_1 \cup V_2. \tag{1.6}$$

The variable $x(u,v)$ represents the assignment from u to v, the cost of this assignment is $c(u,v)$. The cost of opening a facility v is $f(v)$. Equation (1.1) imply that each customer should be assigned to one facility of level one. Equation (1.3) imply that if a facility of level one is opened, then it should be assigned to a facility of level two. Inequalities (1.2) and (1.4) imply that if a facility is not opened then no assignment can be made to that facility.

We call *two level facility location polytope of* G ($TLFLP(G)$) the convex hull of all vectors satisfying (1.1)−(1.6). Let $P(G)$ be the polytope defined by (1.1)−(1.4) and

$$0 \leq x(u,v) \leq 1, \quad \forall (u,v) \in A_1 \cup A_2, \tag{1.7}$$

$$0 \leq y(v) \leq 1, \quad \forall v \in V_1 \cup V_2. \tag{1.8}$$

Clearly $TLFLP(G) \subseteq P(G)$, in this paper we characterize the graphs for which $TLFLP(G) = P(G)$. For this class of graphs the problem is polynomially solvable. We also study the facial structure of $TLFLP(G)$.

This paper is organized as follows. In Section 2 we give some definitions. In Section 3 we give some inequalities that are valid for $TLFLP(G)$. In Section 4 we characterize the graphs for which $P(G) = TLFLP(G)$. Section 5 is devoted to study the facial structure of $TLFLP(G)$.

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2. Preliminaries

In this section we give some definitions and notations. Given a graph $G = (V, A)$, a simple *cycle* C is an ordered sequence $v_0, a_0, v_1, a_1, \ldots, a_{t-1}, v_t = v_0$, where all v_i's, for $i = 0, \ldots, t-1$, are distinct and such that for $0 \leq i \leq t-1$: either v_i is the tail of a_i and v_{i+1} is the head of a_i, or v_i is the head of a_i and v_{i+1} is the tail of a_i. Let $V(C)$ and $A(C)$ denote the nodes and the arcs of C, respectively. By setting $a_t = a_0$, we associate with C three more sets as below.

- We denote by $\hat{C}$ the set of nodes v_i, such that v_i is the head of a_{i-1} and also the head of a_i, $1 \leq i \leq t$.
- We denote by $\dot{C}$ the set of nodes v_i, such that v_i is the tail of a_{i-1} and also the tail of a_i, $1 \leq i \leq t$.
- We denote by $\tilde{C}$ the set of nodes v_i, such that either v_i is the head of a_{i-1} and also the tail of a_i, or v_i is the tail of a_{i-1} and also the head of a_i, $1 \leq i \leq t$.

Notice that $|\hat{C}| = |\dot{C}|$. A cycle will be called *g-odd* if $|\tilde{C}| + |\hat{C}|$ is odd, otherwise it will be called *g-even*. A cycle C with $V(C) = \tilde{C}$ is a *directed* cycle. The notion of g-odd (g-even) cycles generalizes the notion of odd (even) directed cycles. Also a cycle C is called d-odd (d-even) if $|\dot{C}|$ is odd (even).

A polyhedron is called integral if all its extreme points are integral. A polytope is a bounded polyhedron. An inequality $ax \leq \alpha$ is called *valid* with respect to a polyhedron P, if $P \subseteq \{x \,|\, ax \leq \alpha\}$. Valid inequalities are useful if they cut off factional extreme points of a linear programming relaxation of an integer programming problem. If $ax \leq \alpha$ is valid for a polyhedron P, then $F = \{x \in P \,|\, ax = \alpha\}$ is called a *face of* P. A *facet* of P is a maximal face of P. If an inequality defines a facet, it can be viewed as a "strongest" valid inequality.

Given a vector $x \in \mathbb{R}^E$, we denote by $x(S)$ the sum $\sum_{e \in S} x(e)$. We denote by G_1 the subgraph of G induced by $V_0 \cup V_1$. We also denote by G_2 the subgraph of G induced by $V_1 \cup V_2$. For a directed graph $G = (V, A)$ and a set $W \subset V$, we denote by $\delta^+(W)$ the set of arcs $(u, v) \in A$, with $u \in W$ and $v \in V \setminus W$. We write $\delta^+(v)$ instead of $\delta^+(\{v\})$.

For a directed graph $H = (W, B)$, we call the *(one level) facility location polytope of* H $(FLP(H))$ the convex hull of all vectors satisfying

$$\begin{aligned}
&\sum_{(u,v) \in A} x(u, v) + y(u) \leq 1 \quad \forall u \in W,\\
&x(u, v) \leq y(v) \quad \forall (u, v) \in B,\\
&y(v) \in \{0, 1\} \quad \forall v \in W,\\
&x(u, v) \in \{0, 1\} \quad \forall (u, v) \in B.
\end{aligned}$$

This polytope was studied in [4], and a class of valid inequalities was given as follows. If C is a g-odd cycle of H, then the inequality below is valid for $FLP(H)$.

$$\sum_{a \in A(C)} x(a) - \sum_{v \in \hat{C}} y(v) \leq \frac{|\tilde{C}| + |\hat{C}| - 1}{2}. \tag{2.1}$$

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For a family of inequalities, the *separation problem* consists of given a vector $(\bar{x}, \bar{y})$, finding an inequality in this family that is violated by $(\bar{x}, \bar{y})$, if there is any. An algorithm for the separation of inequality (2.1) was given in [4], its complexity is $O(|W|^2|B|^2)$. An extension of these inequalities to the two level case will be studied in the next section.

For an undirected graph $U = (W, E)$, the *stable set polytope* is the convex hull of all vectors satisfying

$$x(i) + x(j) \leq 1 \text{ for every edge } \{i, j\} \in E,$$
$$x(i) \in \{0, 1\} \text{ for every node } i \in V.$$

3. Valid inequalities

In this section we present three families of valid inequalities for $TLFLP(G)$ and discuss their separation problem.

3.1. d-odd cycle inequalities

Lemma 3.1. *If C is a d-odd cycle, then the following inequality is valid for $TLFLP(G)$.*

$$\sum_{u \in \dot{C}, (u,v) \in A(C)} x(u, v) - \sum_{v \in \hat{C}} y(v) - \sum_{v \in \tilde{C}} x(\delta^+(v) \setminus A(C)) \leq \frac{|\dot{C}| - 1}{2}. \tag{3.1}$$

Proof. From inequalities (1.1)–(1.4) and (1.7)–(1.8) we obtain

$$x(u, v) - y(v) \leq 0, \text{ for every arc } (u, v) \in A(C), \ v \in \tilde{C} \cup \hat{C},$$
$$y(v) - x(\delta^+(v)) = 0, \ v \in \tilde{C},$$
$$x(\delta^+(v)) \leq 1, \text{for } v \in \dot{C},$$
$$-x(u, v) \leq 0, \text{ for } (u, v) \in \delta^+(u) \setminus A(C), \text{ for } u \in \dot{C} \cup \tilde{C}.$$

Their sum gives

$$2 \sum_{u \in \dot{C}, (u,v) \in A(C)} x(u, v) - 2 \sum_{v \in \hat{C}} y(v) - 2 \sum_{v \in \tilde{C}} x(\delta^+(v) \setminus A(C)) \leq |\dot{C}|.$$

Dividing by 2 and rounding down the right hand side, as in Chvátal's procedure [8], we obtain inequality (3.1). □

In the next section we show fractional vectors in $P(G)$ that are separated from $TLFLP(G)$ by inequality (3.1). Now we discuss how to solve the separation problem for inequality (3.1). Assume that $(\bar{x}, \bar{y})$ is a vector of $P(G)$. We first build a graph $H = (W, B)$, where $W = V_0 \cup V_1' \cup V_2$, and

$$V_1' = \{v' \mid v \in V_1\} \cup \{v'' \mid v \in V_1\},$$
$$B = \{(u, v') \mid (u, v) \in A_1\} \cup \{(v'', w) \mid (v, w) \in A_2\} \cup \{(v', v'') \mid v \in V_1\}.$$

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So the graph H is obtained by splitting every node in V_1 into two copies, and adding an arc from the first copy to the second copy. Now we build a vector (x', y') as follows,

$$\begin{aligned} x'(u, v') &= \bar{x}(u, v), \quad \forall (u, v) \in A_1, \\ x'(v'', w) &= \bar{x}(v, w), \quad \forall (v, w) \in A_2, \\ x'(v', v'') &= 1 - \bar{y}(v), \quad \forall v \in V_1, \\ y'(w) &= \bar{y}(w), \quad \forall w \in V_2, \\ y'(v') &= \bar{y}(v), \quad \forall v \in V_1, \\ y'(v'') &= 1 - \bar{y}(v), \quad \forall v \in V_1. \end{aligned}$$

We need the following two lemmas.

Lemma 3.2. *There is a one to one correspondence between d-odd cycles in G and g-odd cycles in H.*

Proof. Let C be a d-odd cycle of G, we build a cycle C' in H as below. For every arc $(u, v) \in A(C) \cap A_1$ we add (u, v') to C', for every arc $(v, w) \in A(C) \cap A_2$ we add (v'', w) to C'. Finally for every node $v \in \tilde{C}$ we add the arc (v', v'') to C'. Notice that $|\tilde{C'}|$ is even, thus C' is g-odd.

On the other hand if D' is a g-odd cycle of H we obtain a cycle D in G by shrinking every arc (v', v'') in D into a single node. Clearly D is d-odd. □

Lemma 3.3. *The vector $(\bar{x}, \bar{y})$ violates an inequality (3.1) if and only if (x', y') violates an inequality (2.1).*

Proof. Assume that there is a violated inequality (3.1), and let C be the corresponding cycle of G. Let C' be the associated cycle in H. We have that $|\dot{C'}| = |\dot{C}|$, $|\tilde{C'}| = 2|\tilde{C}|$, and $\sum_{v \in \hat{C}} \bar{y}(v) = \sum_{v \in \hat{C'}} y'(v)$.

For every node $v \in \tilde{C}$, we have $x'(v', v'') = 1 - \bar{x}(\delta^+(v))$. Thus

$$\sum_{u \in \dot{C}, (u,v) \in A(C)} \bar{x}(u, v) - \sum_{v \in \tilde{C}} \bar{x}(\delta^+(v) \setminus A(C)) + |\tilde{C}| = \sum_{(u,v) \in A(C')} x'(u, v),$$

and the lemma follows. □

Therefore the separation problem for inequalities (3.1) reduces to the separation problem for inequalities (2.1) in the graph H. This can be done with the algorithm of [4].

3.2. Cut inequalities

Let $S \subseteq V$ such that $S \cap V_0 \neq \emptyset$. The following inequality is valid for $TLFLP(G)$.

$$x(\delta^+(S)) + y(V_2 \cap S) \geq 1. \tag{3.2}$$

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To see the validity, consider a 0−1 vector $(\bar{x}, \bar{y})$ of $TLFLP(G)$. If $\bar{y}(V_2 \cap S) = 0$, then at least for one arc $(u, v) \in \delta^+(S)$ we should have $\bar{x}(u, v) = 1$.

Consider now the separation problem, and suppose that we are given a nonnegative vector (x', y'). We add an extra vertex t and the arcs (w, t) for all $w \in V_2$. Then we define capacities c for each arc as follows.

$$c(u, v) = x'(u, v) \quad \forall (u, v) \in A_1 \cup A_2,$$
$$c(w, t) = y'(w) \quad \forall w \in V_2.$$

Next we fix a node $u \in V_0$ and find a set of nodes S, with $u \in S$, $t \notin S$ that minimizes $c(\delta^+(S))$. This can be done with a minimum cut algorithm in $O(|V|^3)$ time, see [3]. There is a violated inequality if and only if there is a set $\bar{S}$ with $c(\delta^+(\bar{S})) < 1$. This procedure should be repeated for every node $u \in V_0$.

3.3. Matching inequalities

The following is an adaptation of a family of inequalities given in [2]. Suppose that G_1 admits a matching M where for every node in V_1 there is an arc in M incident to it, then the following inequality is valid for $TLFLP(G)$.

$$\sum_{(u,v)\in M} x(u, v) + \sum_{v \in V_1} y(v) \geq 2. \tag{3.3}$$

To see the validity, consider the case when from all the nodes in V_1 there is exactly one, $\bar{v}$, whose variable $y(\bar{v})$ takes the value one. Then we should have $x(\bar{u}, \bar{v}) = 1$, for $(\bar{u}, \bar{v}) \in M$.

The separation problem for a vector $(\bar{x}, \bar{y})$ can be solved by assigning the weight $\bar{x}(u, v) + \bar{y}(v)$ to each arc $(u, v) \in A_1$, then one should find a minimum weight matching that saturates every node in V_1.

4. On the integrality of $P(G)$

In this section we characterize the graphs for which $P(G) = TLFLP(G)$. Let $\bar{V}_0$ be the set of nodes $v \in V_0$ with $|\delta^+(v)| = 1$. Let $\bar{V}_1$ be the set of nodes in V_1 that are adjacent to a node in $\bar{V}_0$. Clearly the variables associated with the nodes in $\bar{V}_1$ should be fixed, *i.e.*, $y(v) = 1$ for all $v \in \bar{V}_1$. Let $\bar{V}_2$ be the set of nodes in V_2 that are adjacent to a node $v \in \bar{V}_1$ with $|\delta^+(v)| = 1$, we should have $y(w) = 1$ for all $w \in \bar{V}_2$. Let us denote by $\bar{G}$ the subgraph induced by $V \setminus \bar{V}_2$ after removing all the arcs $(u, v) \in A$ with $v \in \bar{V}_1$. We are going to prove the following.

Theorem 4.1. *The polytope $P(G)$ is integral if and only if each cycle of $\bar{G}$ is d-even.*

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First suppose that $\bar{G}$ contains a cycle C that is d-odd. Let us define a fractional vector as follows:

- $\bar{y}(v) = 1/2, \quad \forall v \in (\tilde{C} \cup \hat{C})$.
- $\bar{y}(v) = 1, \quad \forall v \in \dot{C} \cap V_1$.
- $\bar{x}(a) = 1/2, \quad \forall a \in A(C)$.
- $\bar{y}(v) = 1, \quad \forall v \in V_2 \setminus V(C)$.
- For every node $u \in V_1 \setminus V(C)$, we look for an arc $(u, v) \in \delta^+(u)$, we set $\bar{x}(u, v) = \bar{y}(v)$. If $|\delta^+(u)| \geq 2$ and $\bar{y}(v) = 1/2$, we pick another arc (u, w) and set $\bar{x}(u, w) = 1/2$. Then we set $\bar{y}(u) = \bar{x}(\delta^+(u))$.
- For every node $u \in V_0 \setminus V(C)$, we look for an arc $(u, v) \in \delta^+(u)$, if $\bar{y}(v) = 1$ we set $\bar{x}(u, v) = 1$. If $\bar{y}(v) = 1/2$ then there is another arc $(u, w) \in \delta^+(u)$ with $\bar{y}(w) = 1$ or $\bar{y}(w) = 1/2$, then we set $\bar{x}(u, v) = \bar{x}(u, w) = 1/2$.
- For every node $u \in V_1 \cap \hat{C}$, we look for an arc $(u, v) \in \delta^+(u)$ and set $\bar{x}(u, v) = 1/2$.
- For every other arc a, we set $\bar{x}(a) = 0$.

It is easy to see that $(\bar{x}, \bar{y}) \in P(G)$. Since

$$\sum_{u \in \dot{C}, (u,v) \in A(C)} \bar{x}(u, v) - \sum_{v \in \dot{C}} \bar{y}(v) - \sum_{v \in \bar{C}} \bar{x}(\delta^+(v) \setminus A(C)) = \frac{|\dot{C}|}{2},$$

we have that $(\bar{x}, \bar{y})$ violates an inequality (3.1) that is valid for $TLFLP(G)$. So $P(G) \neq TLFLP(G)$.

Now we assume that $\bar{G}$ has no d-odd cycle. First we need to introduce a related polytope studied in [4]. Let $H = (W, B)$ be a directed graph, and let $Q(H)$ be the polytope defined below.

$$\sum_{(u,v) \in A} x(u, v) + y(u) = 1 \quad \forall u \in W, \tag{4.1}$$

$$x(u, v) \leq y(v) \quad \forall (u, v) \in B, \tag{4.2}$$

$$0 \leq y(v) \leq 1 \quad \forall v \in W, \tag{4.3}$$

$$x(u, v) \geq 0 \quad \forall (u, v) \in B. \tag{4.4}$$

In [4] we proved the following.

Theorem 4.2 [4]. *If H does not contain a g-odd cycle, then $Q(H)$ is integral.*

In order to use Theorem 4.2 we have to transform G to a directed graph H such that $\bar{G}$ has no d-odd cycle if and only if H has no g-odd cycle. This will be done in the following three steps.

- For every arc $a = (u, v)$ with $v \in \bar{V}_1 \cup \bar{V}_2$, replace it by (u, v_a), where v_a is a new node.
- Split every node $v \in V_1$, replacing it by v' and v'', and adding an arc (v', v''). Also every arc (u, v) is replaced by (u, v'), and every arc (v, w) is replaced by (v'', w).

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- For every node $w \in (V_2 \setminus \bar{V}_2)$, add a node w' and an arc (w, w').

Let H be this new graph. The goal here is to make a correspondence between some extreme point of $Q(H)$ and those of $P(G)$. If we consider any solution in $Q(H)$, then its restriction on the variables associated with G may not belong to $P(G)$. To have the desired correspondence we study a face of $Q(H)$ defined as follows.

$$
\begin{aligned}
&y(u) = 0, \text{ for each node } u \in V_0,\\
&x(v', v'') = y(v''), \text{ for each arc } (v', v'') \text{ with } v \in V_1,\\
&y(v') = 1, \text{ for each node } v' \text{ such that } v \in \bar{V}_1,
\end{aligned}
$$

First notice that since $\bar{G}$ has no d-odd cycle, then H has no g-odd cycle. Therefore, from Theorem 4.2, $Q(H)$ is an integral polytope. The polytope $P(G)$ is obtained from the face of $Q(H)$ defined above, by projecting the variables associated with extra nodes and extra arcs. Thus the proof of Theorem 4.1 is complete.

We can also state the following.

Theorem 4.3. *The two level facility location problem is polynomially solvable for graphs G such that $\bar{G}$ has no d-odd cycle.*

In order to decide if $\bar{G}$ has a d-odd cycle, we start with $\bar{G}$ and split the nodes in $V_1 \setminus \bar{V}_1$ as in Section 3.1. Let $\bar{H}$ be the new graph. We have seen that $\bar{G}$ has a d-odd cycle if and only if $\bar{H}$ has a g-odd cycle. In [4] we gave a polynomial algorithm to find a g-odd cycle in a graph, if there is any; this algorithm can be applied to $\bar{H}$.

5. On the facial structure of $TLFLP(G)$

Let H be a bipartite graph $H = (U_1 \cup U_2, B = U_1 \times U_2)$. The *uncapacitated facility location polytope of H* ($UFLP(H)$) is the convex hull of all vectors satisfying

$$\sum_{(u,v)\in B} x(u,v) = 1, \quad \forall u \in U_1 \tag{5.1}$$

$$x(u,v) \leq y(v), \quad \forall (u,v) \in B \tag{5.2}$$

$$x(u,v) \in \{0,1\}, \quad \forall (u,v) \in B \tag{5.3}$$

$$y(v) \in \{0,1\}, \quad \forall v \in U_2. \tag{5.4}$$

The facial structure of $UFLP(H)$ has been studied in [5–7, 9] and others. In this section we extend some of these results to the two level case. Here we assume that $A_1 = V_0 \times V_1$, $A_2 = V_1 \times V_2$. First we study the dimension of $TLFLP(G)$. Let $m = |V_0|$, $n = |V_1|$ and $p = |V_2|$. In what follows, we suppose that $n > 1$ and $p > 1$.

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Theorem 5.1. *The dimension of* $TLFLP(G)$ *is* $mn + p - m + np$.

Proof. Since equations (1.1) and (1.3) are linearly independent, the dimension of $TLFLP(G)$ is at most $mn + p - m + np$.

Now suppose that $TLFLP(G) \subseteq \{(x, y) \,|\, ax + by = \alpha\}$. Adding multiples of equations (1.3), we may assume that $b(v) = 0$ for all $v \in V_1$. Also adding multiples of equations (1.1) we may assume that $a(u, v_0) = 0$ for a fixed node $v_0 \in V_1$ and all $u \in V_0$. Let (x^1, y^1) be defined as follows.

$$
\begin{aligned}
&x^1(u, v_0) = 1, \quad \forall u \in V_0,\\
&x^1(v_0, w_0) = 1, \text{ for a fixed node } w_0 \in V_2,\\
&y^1(v_0) = y^1(w_0) = 1,\\
&x^1(u, v) = 0 \text{ otherwise},\\
&y^1(v) = 0 \text{ otherwise}.
\end{aligned}
$$

We have that $ax^1 + by^1 = \alpha$. Now for $w \in V_2$, $w \neq w_0$, let (x^2, y^2) be the vector obtained from (x^1, y^1) by setting $y(w) = 1$. Since $ax^2 + by^2 = \alpha$, we have $b(w) = 0$, for $w \in V_2$, $w \neq w_0$. Since (x^1, y^1) can also be built with a node different from w_0, we also have $b(w_0) = 0$.

Now starting from (x^2, y^2) let (x^3, y^3) be the vector obtained by setting $x(v_0, w_0) = 0$ and $x(v_0, w) = 1$. Since $ax^3 + by^3 = \alpha$, we have $a(v_0, w_0) = a(v_0, w)$ for $w \in V_2$, $w \neq w_0$. Since the same construction can be done with a node different from v_0, we have that $a(v, w_0) = a(v, w)$ for $w \in V_2$, $w \neq w_0$, for any node $v \in V_1$.

Now starting from (x^1, y^1), set $x(v, w) = 1$, $y(v) = 1$, $y(w) = 1$, for a node $v \in V_1$, $v \neq v_0$, and a node $w \in V_2$, $w \neq w_0$. Denote by (x^4, y^4) this new vector, we have that $ax^4 + by^4 = \alpha$, therefore $a(v, w) = 0$. Since the same construction can be done with nodes different from v_0 and w_0, we have that $a(v, w) = 0$ for all $v \in V_1$, and all $w \in V_2$.

Finally, starting from (x^1, y^1), set $x(u, v) = 1$, $y(v) = 1$, for a node $u \in V_0$, $v \in V_1$, $v \neq v_0$, $x(v, w_0) = 1$, $x(u, v_0) = 0$. Let (x^5, y^5) be this new vector. Since $ax^5 + by^5 = \alpha$, we have that $a(u, v) = a(u, v_0) = 0$. Since u and v can be chosen arbitrarily, we have $a(u, v) = 0$ for all $u \in V_0$ and all $v \in V_1$.

This shows that there is no equation linearly independent from (1.1) and (1.3) that is satisfied by all vectors in $TLFLP(G)$, thus its dimension is $mn+p-m+np$ and its affine hull is defined by (1.1) and (1.3). □

Recall that G_1 is the subgraph induced by $V_0 \cup V_1$, and G_2 is the subgraph induced by $V_1 \cup V_2$.

5.1. Facets associated with $UFLP(G_1)$

The following theorem shows that all facets of $UFLP(G_1)$ give facets of $TLFLP(G)$.

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Theorem 5.2. *If*

$$\sum_{(u,v)\in A_1} a(u,v)x(u,v) + \sum_{v\in V_1} b(v)y(v) \leq \alpha \tag{5.5}$$

defines a facet of $UFLP(G_1)$*, then it also defines a facet of* $TLFLP(G)$*.*

Proof. Clearly this inequality is valid for $TLFLP(G)$. Let F be the facet of $UFLP(G_1)$ defined by (5.5), and let F' be the face of $TLFLP(G)$ defined by (5.5). Let us assume that

$$F' \subset \{(x,y)|cx + \mathrm{d}y = \beta\},$$

where $cx + \mathrm{d}y \leq \beta$ is valid for $TLFLP(G)$.

Let $v \in V_1$, the inequality $y(v) \geq 0$ does not define a facet of $UFLP(G_1)$, see [9]. Thus there is a 0−1 vector (x^v, y^v) in F with $y(v) = 1$. For each node $v' \in V_1$ with $y^v(v') = 1$, we define $x^v(v', w_0) = 1$, $y^v(w_0) = 1$, where w_0 is a fixed node in V_2. Then we set $x^v(u,w) = 0$ for every other arc $(u,w) \in A_2$, and $y^v(w) = 0$ for every other node $w \in V_2$. This new vector $(\bar{x}^v, \bar{y}^v)$ belongs to F'.

Starting from $(\bar{x}^v, \bar{y}^v)$, pick a node $w \in V_2$, $w \neq w_0$, and set $y(w) = 1$. Since this new vector also belongs to F', we have $d(w) = 0$. Since the node w_0 was chosen arbitrarily, and the same construction can be done with a different node, we have that $d(w) = 0$ for all $w \in V_2$.

Now starting again from $(\bar{x}^v, \bar{y}^v)$, set $x(v, w_0) = 0$ and $x(v,w) = y(w) = 1$, where $w \in V_2$, $w \neq w_0$. Since this new vector also belongs to F', we have $c(v,w) = c(v,w_0)$. Thus we have that $c(v,w) = \lambda_v$, for all $w \in V_2$. We repeat this for all $v \in V$ and we have $c(v,w) = \lambda_v$, for all $v \in V_1$ and $w \in V_2$.

Thus $cx + \mathrm{d}y \leq \beta$ can be written as

$$\sum_{(u,v)\in A_1} c(u,v)x(u,v) + \sum_{v\in V_1} (d(v) + \lambda_v)y(v) \leq \beta. \tag{5.6}$$

Thus (5.6) defines a face of $UFLP(G_1)$. Since (5.5) defines a facet of $UFLP(G_1)$, then (5.6) can be obtained by taking a positive multiple of (5.5) and adding multiples of equations (1.1). This shows that (5.5) defines a facet of $TLFLP(G)$. □

Corollary 5.3. *The following inequalities define facets of* $TLFLP(G)$*.*

$$\begin{aligned} &x(u,v) \leq y(v), \quad \forall (u,v) \in A_1 \\ &x(u,v) \geq 0, \quad \forall (u,v) \in A_1 \\ &y(v) \leq 1, \quad \forall v \in V_1. \end{aligned}$$

Proof. These inequalities define facets of $UFLP(G_1)$, see [9]. □

Corollary 5.4. *Inequalities* (3.3) *define facets of* $TLFLP(G)$*.*

Proof. It follows from Theorems 6 and 8 in [2] that (3.3) define facets of $UFLP(G_1)$. □

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5.2. Facets associated with $UFLP(G_2)$

Let $UFLP'(G_2)$ the relaxation of $UFLP(G_2)$ obtained by replacing equations (5.1) by inequalities, that is the convex hull of the solutions of the system below

$$\begin{aligned}
&\sum_{(v,w)\in A_2} x(v,w) \leq 1, \quad \forall v \in V_1,\\
&x(v,w) \leq y(w), \quad \forall (v,w) \in A_2,\\
&x(v,w) \in \{0,1\}, \quad \forall (v,w) \in A_2,\\
&y(w) \in \{0,1\}, \quad \forall w \in V_2.
\end{aligned}$$

Notice that any facet of $UFLP(G_2)$ is contained in a facet of $UFLP'(G_2)$. In [9], it has been seen that the polytope $UFLP'(G_2)$ corresponds to a stable set polytope. Define $\bar{y}(w) = 1 - y(w)$ for each node $w \in V_2$, then the system above may be rewritten as follows:

$$\sum_{(v,w)\in A_2} x(v,w) \leq 1, \quad \forall v \in V_1, \tag{5.7}$$

$$x(v,w) + \bar{y}(w) \leq 1, \quad \forall (v,w) \in A_2, \tag{5.8}$$

$$x(v,w) \in \{0,1\}, \quad \forall (v,w) \in A_2, \tag{5.9}$$

$$y(w) \in \{0,1\}, \quad \forall w \in V_2. \tag{5.10}$$

Let A be the $0-1$ matrix associated with the constraints (5.7) and (5.8). An undirected graph H, called the *intersection graph of* A, is built containing a node for each variable $x(v,w) \in A_2$ and a node for each variable $\bar{y}$. For any two variables (v,w_1) and (v,w_2) where (v,w_1) and (v,w_2) are arcs of G_2, we have an edge between the corresponding nodes; thus there is a maximal clique $K(v)$ for each node $v \in V_1$. Also for every variable $x(v,w)$ there is an edge between the node associated with $x(v,w)$ and the node associated with $\bar{y}(w)$. We have the following well known result.

Theorem 5.5. *[9] The convex hull of the feasible solutions of (5.7)–(5.10) is exactly the stable set polytope of H.*

To establish the theorem that shows that all facets of $UFLP'(G_2)$ give facets of $TLFLP(G)$, we need the lemma below.

Lemma 5.6. *If $ax+by \leq \alpha$ defines a facet F of $UFLP'(G_2)$, then there is a $0-1$ vector $(\bar{x},\bar{y}) \in F$ such that*

$$\sum_{w} \bar{x}(v,w) = 1, \text{ for all } v \in V_1.$$

Proof. Here we follow the approach of [9] associating a stable set polytope with $UFLP'(G_2)$. Let $SSP(H)$ the stable set polytope of H. The facets of $UFLP'(G_2)$

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correspond to the facets of $SSP(H)$. For a "trivial" facet of $SSP(H)$ defined by $x(s) \geq 0$, we just pick a node different from s in every maximal clique of H.

Now we have to consider a nontrivial facet of $SSP(H)$ defined by $ax \leq \alpha$. We have that $a \geq 0$, $\alpha > 0$. Let x^W be the incidence vector of a stable set of H such that $ax^W = \alpha$. We may assume that W is a maximal stable set. Since $\sum_{w \in V_2} x(w) \leq |V_2|$ does not define a facet, we may assume that $x^W(w_0) = 0$ for some node $w_0 \in V_2$. So for every clique $K(v)$ we can try to add to W the node associated with (v, w_0), if this is not possible it is because another node in $K(v)$ is already in W. Thus we may assume that for each $v \in V_1$ there is a node in $K(v)$ that belongs to W. □

Theorem 5.7. *If*

$$\sum_{(v,w) \in A_2} a(v,w)x(v,w) + \sum_{w \in V_2} b(w)y(w) \leq \alpha \tag{5.11}$$

defines a facet of $UFLP'(G_2)$, then it also defines a facet of $TLFLP(G)$.

Proof. Clearly (5.11) is valid for $TLFLP(G)$. Let F be the facet of $UFLP'(G_2)$ defined by (5.11), and let F' be the face of $TLFLP(G)$ defined by (5.11). Let us assume that

$$F' \subset \{(x,y) | cx + \mathrm{d}y = \beta\},$$

where $cx + \mathrm{d}y \leq \beta$ is valid for $TLFLP(G)$.

Let $(\bar{x}, \bar{y})$ be the vector defined in Lemma 5.6. Starting from this vector, fix a node $v_0 \in V_1$ and set $x(u, v_0) = 1$ for all $u \in V_0$. Set $x(u, v) = 0$ for every other arc $(u, v) \in G_1$, and $y(v) = 1$ for all $v \in V_1$. Denote by $(\tilde{x}, \tilde{y})$ this new vector. We have that $(\tilde{x}, \tilde{y}) \in F'$. Starting from $(\tilde{x}, \tilde{y})$, pick a node $v_1 \in V_1$, $v_1 \neq v_0$, a node $u_0 \in V_0$ and set $x(u_0, v_0) = 0$, $x(u_0, v_1) = 1$. Denote by $(\hat{x}, \hat{y})$ this last vector. We have that $(\hat{x}, \hat{y}) \in F'$, thus $c(u_0, v_0) = c(u_0, v_1)$. Since v_1 was chosen arbitrarily, we have $c(u_0, v_0) = c(u_0, v)$ for all $v \in V_1$. Since u_0 was chosen arbitrarily, we have $c(u, v) = \lambda_u$ for all $u \in V_0$ and all $v \in V_1$.

Therefore by adding multiples of equations (1.1) to $cx + \mathrm{d}y \leq \beta$, we obtain a new inequality

$$c'x + d'y \leq \beta', \tag{5.12}$$

with $c'(u, v) = 0$ for all $u \in V_0$ and all $v \in V_1$. By adding multiples of equations (1.3), we may assume that $d'(v) = 0$ for all $v \in V_1$.

Thus (5.12) defines a face of $UFLP'(G_2)$. Since this is a full dimensional polytope, we have that (5.12) is a positive multiple of (5.11). Therefore (5.11) defines a facet of $TLFLP(G)$. □

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Corollary 5.8. *The following inequalities define facets of* $TLFLP(G)$.

$$\begin{aligned} y(u) &= \sum_{(u,v)\in A_2} x(u,v) \leq 1 \quad \forall u \in V_1 \\ x(u,v) &\leq y(v), \quad \forall (u,v) \in A_2 \\ x(u,v) &\geq 0, \quad \forall (u,v) \in A_2 \\ y(v) &\leq 1, \quad \forall v \in V_2. \end{aligned}$$

Proof. These inequalities define facets of $UFLP'(G_2)$, see [9]. □

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