COMPOSITIO MATHEMATICA

A p-adic monodromy theorem for de Rham local systems

Koji Shimizu

Compositio Math. **158** (2022), 2157–2205.

doi:10.1112/S0010437X2200776X

FOUNDATION
COMPOSITIO
MATHEMATICA



LONDON
MATHEMATICAL
SOCIETY
EST. 1865



https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

Compositio Math. 158 (2022) 2157–2205
doi:10.1112/S0010437X2200776X

A p-adic monodromy theorem for de Rham local systems

Koji Shimizu

Abstract

We study horizontal semistable and horizontal de Rham representations of the absolute Galois group of a certain smooth affinoid over a p-adic field. In particular, we prove that a horizontal de Rham representation becomes horizontal semistable after a finite extension of the base field. As an application, we show that every de Rham local system on a smooth rigid analytic variety becomes horizontal semistable étale locally around every classical point. We also discuss potentially crystalline loci of de Rham local systems and cohomologically potentially good reduction loci of smooth proper morphisms.

1. Introduction

Let p be a prime and let K be a complete discrete valuation field of characteristic zero with perfect residue field of characteristic p.

Fontaine introduced the notions of crystalline, semistable, and de Rham representations of the absolute Galois group of K to study p-adic étale cohomology of algebraic varieties over K. It is now known that Galois representations arising from geometry in this way are de Rham. He also conjectured that every de Rham representation is potentially semistable. This reflects the semistable reduction conjecture stating that every smooth proper algebraic variety over K acquires semistable reduction after a finite field extension. Fontaine's conjecture was proved by Berger [Ber02] and is often referred to as the p-adic monodromy theorem.

The present paper discusses a similar question for families of p-adic Galois representations parametrized by a variety. More precisely, let X be a smooth rigid analytic variety over K (regarded as an adic space) and let $\mathbb{L}$ be an étale $\mathbb{Z}_p$-local system on X. For every classical point $x \in X$ (i.e. a point whose residue class field $k(x)$ is finite over K) and a geometric point $\overline{x}$ above x, the stalk $\mathbb{L}_{\overline{x}}$ is a Galois representation of $k(x)$. In this way, we regard $\mathbb{L}$ as a family of Galois representations. In particular, we can ask whether $\mathbb{L}$ is de Rham, semistable, etc., at every classical point. Note that building on earlier works [Fal88, Hyo89, Fal02, Bri08, AI13], Scholze [Sch13] defines the notion of de Rham local systems, and Liu and Zhu [LZ17] prove that $\mathbb{L}$ is a de Rham local system if and only if $\mathbb{L}$ is de Rham at every classical point of X (or even at a single classical point on each connected component). In this context, we prove the following p-adic monodromy theorem.

Received 30 June 2021, accepted in final form 27 July 2022, published online 15 December 2022.
2020 Mathematics Subject Classification 14G22 (primary), 11F80, 11F85, 14D10, 14G45 (secondary).
Keywords: rigid analytic geometry, p-adic Hodge theory.
The author was supported by the National Science Foundation under Grant No. DMS-1638352 through membership of the Institute for Advanced Study.
© 2022 The Author(s). The publishing rights in this article are licensed to Foundation Compositio Mathematica under an exclusive licence.

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

Theorem 1.1 (cf. Theorem 9.2). *Let $\mathbb{L}$ be a de Rham $\mathbb{Z}_p$-local system on a smooth rigid analytic variety X over K. For every K-rational point $x \in X$, there exist an open neighborhood $U \subset X$ of x and a finite extension L of K such that $\mathbb{L}|_{U \otimes_K L}$ is semistable at every classical point. Moreover, if $\mathbb{L}_{\overline{x}}$ is potentially crystalline as a Galois representation of $k(x)$, then we can choose U and L so that $\mathbb{L}|_{U \otimes_K L}$ is crystalline at every classical point.*

We obtain a similar result for every classical point. Note that X can have a proper open subset containing all the K-rational points or even all the classical points in general. Hence, the union of open sets U in the theorem may not be the entire X.

Remark 1.2. Currently, crystalline or semistable local systems on X are defined only when we fix a nice integral or formal model of X (cf. [AI13, AI12, Fal89, Fal90, Fal02, TT19, Tsu]). This is the reason why our p-adic monodromy theorem concerns pointwise semistability.

Remark 1.3. Kisin [Kis99] studies torsion and ℓ-adic étale local systems on X for a prime $\ell \neq p$, and shows that around each K-rational point, such a local system is locally constant in the analytic topology of X. In contrast, p-adic étale local systems on X are not necessarily locally constant in the analytic topology.

Remark 1.4. Theorem 1.1 is of local nature. We may also pose a global version of the p-adic monodromy conjecture (cf. [LZ17, Remark 1.4]). Let X be a smooth rigid analytic variety over K and $\mathbb{L}$ a de Rham $\mathbb{Z}_p$-local system on X. Does there exist a rigid analytic variety X' finite étale over X such that $\mathbb{L}|_{X'}$ is semistable at every classical point? We have an affirmative answer in the following two cases:

(i) the case where $\mathbb{L}$ is the p-adic Tate module of an abelian variety over X (use Raynaud's criterion [SGA7, Exposé IX, Proposition 4.7]);
(ii) the case where $\mathbb{L}$ has a single Hodge–Tate weight [Shi18, Theorem 1.6].

On the other hand, Lawrence and Li [LL21] show that there exist an algebraic variety X over K and a de Rham $\mathbb{Z}_p$-local system $\mathbb{L}$ on X such that for every finite extension L of K, $\mathbb{L}|_{X \otimes_K L}$ is not semistable at some closed point.

Remark 1.5. It is an interesting attempt to formulate a semistable reduction conjecture in families over a field of mixed characteristic; Theorem 1.1 will be p-adic Hodge-theoretic evidence for such a conjecture. In the case of fields of characteristic zero, Abramovich and Karu [AK00] make a semistable reduction conjecture in families, and Adiprasito, Liu, and Temkin [ALT18] prove the conjecture of Abramovich and Karu in a larger generality.

Let us turn to applications of Theorem 1.1. Note that Liu and Zhu's result implies that if an étale $\mathbb{Q}_p$-local system $\mathbb{L}$ on X is de Rham at a classical point on each connected component, then $\mathbb{L}$ is de Rham at every classical point. Such a strong result does not hold for potentially crystalline representations. However, we prove that if $\mathbb{L}$ is potentially crystalline at a classical point x, then it is so at every classical point in an open neighborhood of x.

Theorem 1.6 (Theorem 10.1). *If $\mathbb{L}$ is a de Rham $\mathbb{Q}_p$-local system on a smooth rigid analytic variety X over K, then there exists an open subset $U \subset X$ such that the following assertions hold for every classical point $x \in X$:*

(i) *if $x \in U$, then $\mathbb{L}$ is potentially crystalline at x;*
(ii) *if $x \notin U$, then $\mathbb{L}$ is not potentially crystalline at x.*

The second application concerns the special case where $\mathbb{L}$ comes from geometry. For example, suppose that we have a family of elliptic curves $f\colon E \to X$. For a point $x \in X(\overline{K})$, the elliptic

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

curve E_x has good reduction if and only if the j-invariant of E_x is a p-adic integer. The set of such points may well be called the (potentially) good reduction locus of E. In general, if we have a smooth proper family of algebraic varieties, the étale cohomology of the fibers yields both ℓ-adic and p-adic étale local systems on the base. By combining known cases of ℓ-adic/p-adic monodromy-weight conjecture and the ℓ-independence with our result or Kisin's result, we obtain the following theorem:

Theorem 1.7 (Theorem 10.9). *Assume that the residue field of K is finite. Let $f\colon Y \to X$ be a smooth proper morphism between smooth algebraic varieties over K such that the relative dimension of f is at most two. For each $m \in \mathbb{N}$, there exists an open subset U of the adic space X^{ad} associated to X such that the following assertions hold for every classical point x of X^{ad}:*

(i) *if $x \in U$, then $R^m f_* \mathbb{Q}_\ell$ is potentially unramified at x for $\ell \neq p$ and $R^m f_* \mathbb{Q}_p$ is potentially crystalline at x;*
(ii) *if $x \notin U$, then $R^m f_* \mathbb{Q}_\ell$ is not potentially unramified at x for $\ell \neq p$ and $R^m f_* \mathbb{Q}_p$ is not potentially crystalline at x.*

Remark 1.8. Imai and Mieda [IM20] introduce the notion of the potentially good reduction locus of a Shimura variety using automorphic local systems. They prove the existence of the potentially good reduction loci of Shimura varieties of preabelian type, and use these loci to study the étale cohomology of Shimura varieties.

We now explain ideas behind the proof of Theorem 1.1. We start with an example that motivates our proof. Let $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$ be a smooth proper morphism of smooth p-adic formal schemes over $\operatorname{Spf} \mathcal{O}_K$, and let $f\colon Y \to X$ denote its adic generic fiber. Then $\mathbb{L} := R^m f_* \mathbb{Q}_p$ is a de Rham $\mathbb{Q}_p$-local system. In fact, we even know that it is crystalline at every classical point (hence there is nothing to prove in this case).

Let k denote the residue field of K. Let z be a k-rational point in the special fiber of $\mathfrak{X}$ and let $U \subset X$ be the residue open disk of z. For K-rational points $x_1, x_2 \in U$, there is a canonical identification of φ-modules[1]

$$D_{\mathrm{cris}}(\mathbb{L}_{\overline{x_1}}) \cong H^m_{\mathrm{cris}}(\mathfrak{Y}_z/W(k))[p^{-1}] \cong D_{\mathrm{cris}}(\mathbb{L}_{\overline{x_2}}).$$

One key property of U is that the Gauss–Manin connection on the vector bundle $H^m_{\mathrm{dR}}(Y/X)$ on X has a full set of solutions when restricted to U (cf. [Kat73] and [BO83, Remark 2.9]). Heuristically, this means that we can compare φ-modules associated to stalks of $\mathbb{L}$ at different points on U by parallel transport. Our proof articulates this idea despite the absence of integral models of X or motives for $\mathbb{L}$.

Let us return to the general setting of Theorem 1.1. Liu and Zhu [LZ17] introduce a natural functor D_{dR} from the category of étale $\mathbb{Q}_p$-local systems on X to the category of filtered vector bundles with integrable connections on X. When $X = \operatorname{Spa}(K, \mathcal{O}_K)$, this agrees with Fontaine's D_{dR}-functor for Galois representations of K. The functor commutes with arbitrary pullbacks, and an étale $\mathbb{Q}_p$-local system $\mathbb{L}$ on X is de Rham if and only if $\operatorname{rank} D_{\mathrm{dR}}(\mathbb{L}) = \operatorname{rank} \mathbb{L}$. Note that in the above geometric situation, $D_{\mathrm{dR}}(R^m f_* \mathbb{Q}_p)$ coincides with $H^m_{\mathrm{dR}}(Y/X)$ equipped with the Gauss–Manin connection and the Hodge filtration.

Suppose that $\mathbb{L}$ is a de Rham $\mathbb{Z}_p$-local system and fix a K-rational point x of X. Since X is smooth, the theory of p-adic differential equations over a polydisk (Theorem 9.7) tells us that there is an open neighborhood $U \subset X$ of x such that $D_{\mathrm{dR}}(\mathbb{L}|_U)$ has a full set of solutions.

[1] They need not be isomorphic as *filtered* φ-modules, and thus $\mathbb{L}_{\overline{x_1}}$ and $\mathbb{L}_{\overline{x_2}}$ may not be isomorphic as Galois representations of K.

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

By shrinking U if necessary, we may further assume that U is isomorphic to a rigid torus $\mathrm{Spa}(A, A^\circ)$, where $A^\circ := \mathcal{O}_K\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle$, the p-adic completion of the Laurent polynomial ring $\mathcal{O}_K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$, and $A := A^\circ[p^{-1}]$.

Let $\mathcal{G}_A$ denote the absolute Galois group of A. Following works of Colmez [Col02] and Brinon [Bri08], we define horizontal crystalline, semistable, and de Rham period rings $\mathrm{B}^{\nabla}_{\max}(A^\circ)$, $\mathrm{B}^{\nabla}_{\mathrm{st}}(A^\circ)$, $\mathrm{B}^{\nabla}_{\mathrm{dR}}(A^\circ)$. Using these period rings, we define the notions of horizontal crystalline, semistable, and de Rham representations of $\mathcal{G}_A$ (Definition 4.13). In our setting, $\mathbb{L}|_U$ corresponds to a p-adic representation V of $\mathcal{G}_A$. Since $\mathbb{L}|_U$ is de Rham and $D_{\mathrm{dR}}(\mathbb{L}|_U)$ has a full set of solutions, V is horizontal de Rham. Now the first part of Theorem 1.1 follows from a p-adic monodromy theorem for horizontal de Rham representations.

Theorem 1.9 (Theorem 7.3). *If a p-adic representation V of $\mathcal{G}_A$ is horizontal de Rham, then there exists a finite extension L of K such that $V|_{\mathcal{G}_{A_L}}$ is horizontal semistable, where $A_L := A \otimes_K L$.*

The second part of Theorem 1.1 follows from the study of the monodromy operator (cf. Theorem 7.4).

To explain the proof of Theorem 1.9, we need some more notation. Let $\mathcal{O}_{\mathcal{K}}$ denote the p-adic completion of the localization $A^\circ_{(\pi)}$, where π is a uniformizer of K. Then $\mathcal{K} := \mathcal{O}_{\mathcal{K}}[p^{-1}]$ is a complete discrete valuation field with imperfect residue field. We similarly have the notions of horizontal semistable and de Rham representations of the absolute Galois group of $\mathcal{K}$. In [Ohk13], Ohkubo shows that every horizontal de Rham representation becomes horizontal semistable after restricting to the absolute Galois group of the composite $L\mathcal{K}$ for some finite extension L of K. Hence, Theorem 1.9 follows from Ohkubo's result and the following purity theorem for horizontal semistable representations of $\mathcal{G}_A$.

Theorem 1.10 (Theorem 6.1). *If a horizontal de Rham representation of $\mathcal{G}_A$ is horizontal semistable when restricted to the absolute Galois group of $\mathcal{K}$, then it is also horizontal semistable as a representation of $\mathcal{G}_A$.*

This is the key technical result of our paper, and the proof requires detailed analysis of period rings (e.g. Proposition 2.37). Our purity theorem and its proof are inspired by similar results in different contexts: a p-adic monodromy theorem for de Rham representations of the absolute Galois group of K with values in a reduced affinoid algebra by Berger and Colmez [BC08], and a purity theorem for crystalline local systems in the case of good reduction by Tsuji [Tsu, Theorem 5.4.8].

At this point, we emphasize that the main goal of our work is to develop p-adic Hodge theory of horizontal crystalline, semistable, and de Rham representations of $\mathcal{G}_A$. These classes of representations are much more restrictive than those of crystalline and de Rham representations studied by Brinon [Bri08]. However, it is still worth studying them. In fact, as we have explained, de Rham local systems come from horizontal de Rham representations (étale) locally around each classical point. For another instance, Faltings [Fal10] studies the admissible loci of certain p-adic period domains using horizontal crystalline representations (see also [Har13, Ked10b, Far18] for relevant works).

This paper establishes two results in p-adic Hodge theory of $\mathcal{G}_A$. First, we prove the p-adic monodromy theorem for horizontal de Rham representations as we have discussed. Second, we define Fontaine's functors D_{pst} and V_{pst} in our setting (§ 5). We prove that these functors induce an equivalence of categories between the category of horizontal de Rham representations of $\mathcal{G}_A$ and that of admissible discrete filtered $(\varphi, N, \mathrm{Gal}(\overline{K}/K), A)$-modules (Theorem 5.10).

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

Finally, we explain the organization of this paper. The first part (§§ 2–7) develops p-adic Hodge theory for rings (i.e. representations of $\mathcal{G}_A$) and the second part (§§ 8–10) studies p-adic Hodge theory for rigid analytic varieties.

In the first part, we need to define horizontal period rings such as $\mathrm{B}_{\mathrm{dR}}^{\nabla}(A^{\circ})$. We define them in two steps. We define horizontal period rings such as $\mathbb{B}_{\mathrm{dR}}(\Lambda, \Lambda^{+})$ for every perfectoid Banach pair (Λ, Λ^{+}) over the completed algebraic closure of K. For a base ring R (e.g. A°), the p-adic completion $\widehat{\overline{R}}$ of the integral closure of R in the algebraic closure of $R[p^{-1}]$ gives such a perfectoid Banach pair, and we set $\mathrm{B}_{\mathrm{dR}}^{\nabla}(R) := \mathbb{B}_{\mathrm{dR}}(\widehat{\overline{R}}[p^{-1}], \widehat{\overline{R}})$.

In § 2, we define horizontal period rings for perfectoid Banach pairs, following [Col02, Bri08]. Most of the arguments in this section are standard in p-adic Hodge theory. One exception might be Proposition 2.37, which will be used crucially in the proof of the purity theorem in § 6. Section 3 studies the class of base rings R over which Brinon [Bri08] develops p-adic Hodge theory. In § 4, we define horizontal crystalline, semistable, and de Rham representations. Section 5 introduces filtered $(\varphi, N, \mathrm{Gal}(L/K), A)$-modules, and the functors D_{pst} and V_{pst}. This section is not used in later sections. After these preparations, we state and prove the purity theorem in § 6. We prove Theorem 1.9 in § 7.

The second part starts with § 8, where we review étale local systems on smooth rigid analytic varieties and period sheaves on the pro-étale site. Our p-adic monodromy theorem is proved in § 9. Finally, we discuss potentially crystalline loci and cohomologically potentially good reduction loci in § 10.

Notation 1.11. For a ring S of characteristic p, the Frobenius map $x \mapsto x^p$ is denoted by φ. We say that S is *perfect* if φ is an isomorphism. In this case, $W(S)$ denotes the ring of p-typical Witt vectors of S. By functoriality, $W(S)$ admits the Witt vector Frobenius $W(\varphi)$, which we simply denote by φ.

A rigid analytic variety over K refers to a quasi-separated adic space locally of finite type over $\mathrm{Spa}(K, \mathcal{O}_K)$ (cf. [Hub94, Proposition 4.5(iv)]). For a rigid analytic variety X over K, an open subset of X refers to an open subset of X as an adic space. In particular, open subsets of X may not correspond to admissible open subsets in the sense of [BGR84, 9.1.4 Proposition 2(i)]. We write $\mathbb{B}_K^n$ for the n-dimensional unit polydisk $\mathrm{Spa}\big(K\langle T_1, \dots, T_n\rangle, \mathcal{O}_K\langle T_1, \dots, T_n\rangle\big)$ and write $\mathbb{T}_K^n$ for the n-dimensional rigid torus $\mathrm{Spa}(K\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle, \mathcal{O}_K\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle)$. Definitions of étale $\mathbb{Z}_p$-local systems and étale $\mathbb{Q}_p$-local systems on X are explained in § 8.1.

2. Period rings

In this section, we introduce horizontal period rings $\mathbb{B}_{\mathrm{dR}}, \mathbb{B}_{\max}, \mathbb{B}_{\mathrm{st}}$, etc., and study their basic properties. We will use these period rings to define horizontal de Rham, crystalline, and semistable representations in § 4. Generalizing [Fon94a], Colmez defines these period rings for sympathetic algebras in [Col02], and we follow his approach. Proposition 2.37 is a key technical result used in the proof of the purity theorem in § 6.

2.1 Review of Banach algebras and perfectoid pairs

In this subsection, let L be a field that is complete with respect to a non-archimedean p-adic norm $|\ |$ satisfying $|p| = p^{-1}$.

Definition 2.1. For an L-Banach algebra Λ, write $|\ |_{\Lambda}$ for the norm. We denote by Λ° the set of power-bounded elements of Λ. We say that Λ is *uniform* if Λ° is bounded

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

(cf. [KL15, Definition 2.8.1]). Note that Λ is uniform if and only if the norm on Λ is equivalent to its spectral (semi)norm.[2]

An *L-Banach pair* is a pair (Λ, Λ^+) in which Λ is an L-Banach algebra and Λ^+ is a *a ring of integral elements*, that is, a subring of Λ° which is open and integrally closed in Λ.

Remark 2.2. If (Λ, Λ^+) is an L-Banach pair and if Λ is uniform, then Λ^+ is reduced, p-torsion-free, and p-adically complete and separated.

DEFINITION 2.3. Assume further that L is algebraically closed. An L-Banach pair (Λ, Λ^+) is called *perfectoid* if Λ is uniform and if the Frobenius morphism $x \mapsto x^p$ on $\Lambda^+/p\Lambda^+$ is surjective.

Remark 2.4. When L is algebraically closed, an L-Banach pair (Λ, Λ^+) is perfectoid if and only if (Λ, Λ°) is perfectoid, or equivalently, Λ is a perfectoid L-algebra in the sense of [Sch12, Definition 5.1(i)]; see [KL15, Proposition 3.6.2]. In this case, we will simply say that Λ is perfectoid (over L).

2.2 Period rings for perfectoid pairs

Let k be a perfect field of characteristic p and set $K_0 := W(k)[p^{-1}]$. Let K be a totally ramified finite extension of K_0. We fix a uniformizer π of K and an algebraic closure $\overline{K}$ of K. Let C denote the p-adic completion of $\overline{K}$. We denote by $\mathcal{O}_C$ the ring of integers of C

SET-UP 2.5. Let (Λ, Λ^+) be a perfectoid C-Banach pair such that Λ^+ is an $\mathcal{O}_C$-algebra. Note that $\Lambda = \Lambda^+[p^{-1}]$.

Remark 2.6. When we discuss a morphism $(\Lambda, \Lambda^+) \to (\Lambda', \Lambda'^+)$ of such objects, we consider a morphism of K-Banach pairs, namely, a morphism $f\colon \Lambda \to \Lambda'$ of K-Banach algebras such that $f(\Lambda^+) \subset \Lambda'^+$. For example, we will consider $\sigma\colon (C, \mathcal{O}_C) \to (C, \mathcal{O}_C)$ for $\sigma \in \mathrm{Gal}(\overline{K}/K)$.

We set $\Lambda^{+\flat} := \varprojlim_{\varphi} \Lambda^+/p\Lambda^+$, where φ is the Frobenius of $\Lambda^+/p\Lambda^+$. By [KL15, Remark 3.4.10], this is a ring of characteristic p (called the *tilt* of Λ^+) and there is a natural multiplicative monoid isomorphism $\varprojlim_{\lambda \mapsto \lambda^p} \Lambda^+ \xrightarrow{\cong} \Lambda^{+\flat}$. We denote by $\Lambda^{+\flat} \to \Lambda^+; x \mapsto x^\sharp$ the inverse of the above isomorphism followed by the first projection. Note that $\Lambda^{+\flat}$ has a norm $|\ |$ defined by $|x| := |x^\sharp|_\Lambda$ and it is complete with respect to this norm [KL15, Lemma 3.4.5].

We set

$$\mathbb{A}_{\mathrm{inf}}(\Lambda^+) = \mathbb{A}_{\mathrm{inf}}(\Lambda, \Lambda^+) := W(\Lambda^{+\flat}),$$
$$\mathbb{A}_{\mathrm{inf},K}(\Lambda^+) = \mathbb{A}_{\mathrm{inf},K}(\Lambda, \Lambda^+) := \mathcal{O}_K \otimes_{W(k)} \mathbb{A}_{\mathrm{inf}}(\Lambda^+).$$

By functoriality of p-typical Witt vectors, $\mathbb{A}_{\mathrm{inf}}(\Lambda^+)$ carries the Frobenius φ lifting $x \mapsto x^p$. There exists a unique surjective $W(k)$-algebra homomorphism

$$\theta_{\mathbb{A}_{\mathrm{inf}}(\Lambda^+)}\colon \mathbb{A}_{\mathrm{inf}}(\Lambda^+) \to \Lambda^+$$

characterized by $\theta_{\mathbb{A}_{\mathrm{inf}}(\Lambda^+)}(\sum_{i=0}^\infty [x_i]p^i) = \sum_{i=0}^\infty x_i^\sharp p^i$, where $[x_i] \in \mathbb{A}_{\mathrm{inf}}(\Lambda^+)$ is the Teichmüller lift of x_i (cf. [KL15, Lemma 3.2.2, Definition 3.4.3]). This map also extends to a surjective $\mathcal{O}_K$-algebra homomorphism $\theta_{\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)}\colon \mathbb{A}_{\mathrm{inf},K}(\Lambda^+) \to \Lambda^+$.

Choose a compatible system $(p_m)_{m\in\mathbb{N}}$ of p-power roots of p in $\overline{K}$, that is, $p_m \in \overline{K}$ with $p_0 = p$ and $p_{m+1}^p = p_m$. Set $p^\flat := (p_0 \bmod p, p_1 \bmod p, \ldots) \in \mathcal{O}_C^\flat \subset \Lambda^{+\flat}$. Similarly, choose a compatible system $(\pi_m)_{m\in\mathbb{N}}$ of p-power roots of π in $\overline{K}$ and set $\pi^\flat := (\pi_0 \bmod p, \pi_1 \bmod p, \ldots) \in \mathcal{O}_C^\flat \subset \Lambda^{+\flat}$.

[2] If Λ is uniform, then Λ with the spectral norm is uniform in the sense of [Ber90, p. 16].

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

Lemma 2.7 (cf. [KL15, Lemma 3.6.3]). *The ideal* $\operatorname{Ker}\theta_{\mathbb{A}_{\mathrm{inf}}(\Lambda^+)}$ *is principal and generated by* $[p^\flat]-p$. *Similarly,* $\operatorname{Ker}\theta_{\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)}$ *is principal with a generator* $[\pi^\flat]-\pi$.

Let us define the horizontal de Rham period rings.

Definition 2.8. We set

$$\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)=\mathbb{B}^+_{\mathrm{dR}}(\Lambda,\Lambda^+):=\varprojlim_m \mathbb{A}_{\mathrm{inf}}(\Lambda^+)[p^{-1}]/(\operatorname{Ker}\theta_{\mathbb{A}_{\mathrm{inf}}(\Lambda^+)}[p^{-1}])^m,$$

$$\mathbb{B}^+_{\mathrm{dR},K}(\Lambda^+)=\mathbb{B}^+_{\mathrm{dR},K}(\Lambda,\Lambda^+):=\varprojlim_m \mathbb{A}_{\mathrm{inf},K}(\Lambda^+)[p^{-1}]/(\operatorname{Ker}\theta_{\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)}[p^{-1}])^m.$$

Note that there is a natural ring homomorphism $\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)\to\mathbb{B}^+_{\mathrm{dR},K}(\Lambda^+)$, and we will explain that this is an isomorphism.

The map $\theta_{\mathbb{A}_{\mathrm{inf}}(\Lambda^+)}$ extends to $\theta_{\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)}\colon\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)\to\Lambda=\Lambda^+[p^{-1}]$. Since $\operatorname{Ker}\theta_{\mathbb{A}_{\mathrm{inf}}(\Lambda^+)}[p^{-1}]$ of $\mathbb{A}_{\mathrm{inf}}(\Lambda^+)[p^{-1}]$ is principal, $\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)$ is complete with respect to $\operatorname{Ker}\theta_{\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)}$. Similarly, $\theta_{\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)}$ extends to $\theta_{\mathbb{B}^+_{\mathrm{dR},K}(\Lambda^+)}\colon\mathbb{B}^+_{\mathrm{dR},K}(\Lambda^+)\to\Lambda$, and $\mathbb{B}^+_{\mathrm{dR},K}(\Lambda^+)$ is complete with respect to $\operatorname{Ker}\theta_{\mathbb{B}^+_{\mathrm{dR},K}(\Lambda^+)}$.

Choose a compatible system $(\varepsilon_m)_{m\in\mathbb{N}}$ of non-trivial p-power roots of unity in $\overline{K}$, that is, $\varepsilon_m\in\overline{K}$ with $\varepsilon_0=1$, $\varepsilon_1\neq1$, and $\varepsilon_{m+1}^p=\varepsilon_m$. Set $\varepsilon:=(\varepsilon_0 \bmod p,\varepsilon_1 \bmod p,\dots)\in\mathcal{O}_C^\flat\subset\Lambda^{+\flat}$. Note that $[\varepsilon]-1\in\mathbb{A}_{\mathrm{inf}}(\Lambda^+)$ satisfies $\theta_{\mathbb{A}_{\mathrm{inf}}(\Lambda^+)}([\varepsilon]-1)=0$. In particular, we define

$$t:=\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}([\varepsilon]-1)^m$$

as an element of $\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)$.

The ideal $\operatorname{Ker}\theta_{\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)}$ is generated by t (cf. [Bri08, Proposition 5.1.3]). Similarly, $\operatorname{Ker}\theta_{\mathbb{B}^+_{\mathrm{dR},K}(\Lambda^+)}$ is generated by t. Combining this with the completeness of $\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)$ and $\mathbb{B}^+_{\mathrm{dR},K}(\Lambda^+)$, we obtain the following lemma.

Lemma 2.9. *The natural map* $\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)\to\mathbb{B}^+_{\mathrm{dR},K}(\Lambda^+)$ *is an isomorphism.*

We identify these rings and denote them by $\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)$ in what follows. In particular, $\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)$ is a K-algebra.

Lemma 2.10. *The natural map* $\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)\to\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)$ *is injective.*

Proof. We can check that $\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)\cap(\operatorname{Ker}\theta_{\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)}[p^{-1}])^m=(\operatorname{Ker}\theta_{\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)})^m$. Moreover, since $\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)/(\pi)=\Lambda^{+\flat}$ has a norm $|\ |$ satisfying $|\pi^\flat|=|\pi|_\Lambda>0$, it follows from Lemma 2.7 that $\bigcap_m(\operatorname{Ker}\theta_{\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)})^m=0$. These claims imply the assertion. □

Lemma 2.11 (cf. [Bri08, Proposition 5.1.4]). *The ring* $\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)$ *is* t*-torsion-free.*

Definition 2.12. Set $\mathbb{B}_{\mathrm{dR}}(\Lambda^+)=\mathbb{B}_{\mathrm{dR}}(\Lambda,\Lambda^+):=\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)[t^{-1}]$. We equip $\mathbb{B}_{\mathrm{dR}}(\Lambda^+)$ with the decreasing separated and exhaustive filtration defined by $\mathrm{Fil}^m\mathbb{B}_{\mathrm{dR}}(\Lambda^+):=t^m\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)$ for $m\in\mathbb{Z}$.

Definition 2.13. Let $\mathbb{A}_{\mathrm{inf}}(\Lambda^+)[(\operatorname{Ker}\theta)/p]$ be the $\mathbb{A}_{\mathrm{inf}}(\Lambda^+)$-subalgebra of $\mathbb{A}_{\mathrm{inf}}(\Lambda^+)[p^{-1}]$ generated by $p^{-1}\operatorname{Ker}\theta_{\mathbb{A}_{\mathrm{inf}}(\Lambda^+)}$. We define $\mathbb{A}_{\mathrm{max}}(\Lambda^+)=\mathbb{A}_{\mathrm{max}}(\Lambda,\Lambda^+)$ to be the p-adic completion of $\mathbb{A}_{\mathrm{inf}}(\Lambda^+)[(\operatorname{Ker}\theta)/p]$.

Similarly, let $\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)[(\operatorname{Ker}\theta)/\pi]$ be the $\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)$-subalgebra of $\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)[p^{-1}]$ generated by $\pi^{-1}\operatorname{Ker}\theta_{\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)}$. We define $\mathbb{A}_{\mathrm{max},K}(\Lambda^+)=\mathbb{A}_{\mathrm{max},K}(\Lambda,\Lambda^+)$ to be the p-adic completion of $\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)[(\operatorname{Ker}\theta)/\pi]$.

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

LEMMA 2.14. *The Frobenius φ on $\mathbb{A}_{\mathrm{inf}}(\Lambda^+)$ extends to an endomorphism φ on $\mathbb{A}_{\mathrm{inf}}(\Lambda^+)[(\operatorname{Ker}\theta)/p]$ and thus on $\mathbb{A}_{\mathrm{max}}(\Lambda^+)$.*

Proof. Since $\operatorname{Ker}\theta_{\mathbb{A}_{\mathrm{inf}}(\Lambda^+)}$ is generated by $[p^\flat]-p$, the assertion follows from the fact that

$$\varphi\left(\frac{[p^\flat]-p}{p}\right)=\frac{[p^\flat]^p-p}{p}=p^{p-1}\left(\frac{[p^\flat]-p}{p}+1\right)^p-1\in\mathbb{A}_{\mathrm{inf}}(\Lambda^+)\left[\frac{\operatorname{Ker}\theta}{p}\right]. \qquad \square$$

LEMMA 2.15. *The rings $\mathbb{A}_{\mathrm{max}}(\Lambda^+)$ and $\mathbb{A}_{\mathrm{max},K}(\Lambda^+)$ are p-torsion-free.*

Proof. Since $\mathbb{A}_{\mathrm{inf}}(\Lambda^+)$ and $\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)$ are p-torsion-free, so are $\mathbb{A}_{\mathrm{inf}}(\Lambda^+)[(\operatorname{Ker}\theta)/p]$ and $\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)[(\operatorname{Ker}\theta)/\pi]$. Hence, the lemma follows by taking p-adic completion. $\square$

DEFINITION 2.16. Set

$$\mathbb{B}^+_{\mathrm{max}}(\Lambda^+)=\mathbb{B}^+_{\mathrm{max}}(\Lambda,\Lambda^+):=\mathbb{A}_{\mathrm{max}}(\Lambda^+)[p^{-1}],$$
$$\mathbb{B}^+_{\mathrm{max},K}(\Lambda^+)=\mathbb{B}^+_{\mathrm{max},K}(\Lambda,\Lambda^+):=\mathbb{A}_{\mathrm{max},K}(\Lambda^+)[p^{-1}].$$

LEMMA 2.17. *The series $t=\sum_{m=1}^{\infty}((-1)^{m-1}/m)([\varepsilon]-1)^m$ converges in $(1/p^r)\mathbb{A}_{\mathrm{max},K}(\Lambda^+)$ for a sufficiently large $r\in\mathbb{N}$. It also converges in $\mathbb{A}_{\mathrm{max}}(\Lambda^+)$.*

Proof. Note that $[\varepsilon]-1\in\operatorname{Ker}\theta_{\mathbb{A}_{\mathrm{max},K}(\Lambda^+)}$ and

$$\frac{(-1)^{m-1}}{m}([\varepsilon]-1)^m=\frac{(-1)^{m-1}\pi^m}{m}\left(\frac{[\varepsilon]-1}{\pi}\right)^m.$$

Then the first assertion follows from the fact that $(-1)^{m-1}\pi^m/m$ approaches zero p-adically when $m\to\infty$. The proof of the second assertion is similar. $\square$

PROPOSITION 2.18. *We have*

$$\mathcal{O}_K\otimes_{W(k)}\mathbb{A}_{\mathrm{inf}}(\Lambda^+)\left[\frac{\operatorname{Ker}\theta}{p}\right]\subset\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)\left[\frac{\operatorname{Ker}\theta}{\pi}\right]\subset\frac{1}{p}\left(\mathcal{O}_K\otimes_{W(k)}\mathbb{A}_{\mathrm{inf}}(\Lambda^+)\left[\frac{\operatorname{Ker}\theta}{p}\right]\right).$$

In particular, $K\otimes_{K_0}\mathbb{B}^+_{\mathrm{max}}(\Lambda^+)\cong\mathbb{B}^+_{\mathrm{max},K}(\Lambda^+)$.

Proof. Set $e=[K:K_0]$ and write $p=a\pi^e$ and $p^\flat=a^\flat(\pi^\flat)^e$ for some $a\in\mathcal{O}_K^\times$ and $a^\flat\in(\mathcal{O}_C^\flat)^\times$. For every $m\in\mathbb{N}$, write $m=qe+r$ with $0\le r<e$. Then we see that

$$\left(\frac{[p^\flat]}{p}\right)^m=[a^\flat]^m a^{-m}\left(\frac{[\pi^\flat]}{\pi}\right)^{em}\quad\text{and}\quad\left(\frac{[\pi^\flat]}{\pi}\right)^m=\frac{1}{p}\left(a[a^\flat]^{-1}\frac{[p^\flat]}{p}\right)^q\frac{p}{\pi^r}[\pi^\flat]^r.$$

The first assertion follows from these equalities and Lemma 2.7. Taking the p-adic completion of the first part and inverting p yields the second assertion. $\square$

Let us now turn to Colmez's description of the horizontal period rings $\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)$ and $\mathbb{B}^+_{\mathrm{max},K}(\Lambda^+)$ as K-vector spaces [Col02, § 8.4].

CONSTRUCTION 2.19. Fix a family of elements $(e_i)_{i\in I}$ of Λ^+ such that the images of the e_i in $\Lambda^+/\pi\Lambda^+$ form a k-basis of $\Lambda^+/\pi\Lambda^+$. Then every element of Λ is written uniquely as $\sum_{i\in I}a_ie_i$ with $a_i\in K$ such that for every $r>0$ there are only finitely many i with $|a_i|_\Lambda\ge r$. For each $i\in I$, choose $\widetilde{e}_i\in\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)$ such that $\theta_{\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)}(\widetilde{e}_i)=e_i$. Consider the map

$$s\colon\Lambda\to\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)[p^{-1}];\quad\sum_{i\in I}a_ie_i\mapsto\sum_{i\in I}a_i\widetilde{e}_i.$$

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

This is a K-linear map and satisfies $\theta_{\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)} \circ s = \mathrm{id}$ and $s(\Lambda^+) \subset \mathbb{A}_{\mathrm{inf},K}(\Lambda^+)$. In particular, s is continuous with respect to p-adic topology.

Set $v = \pi^{-1}([\pi^\flat] - \pi)$. Recall that $\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)$ is $\operatorname{Ker}\theta_{\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)}$-adically complete and that v generates $\operatorname{Ker}\theta_{\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)}$. For $x \in \mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)$, we define two sequences $(a_m(x))_{m\in\mathbb{N}}$ and $(b_m(x))_{m\in\mathbb{N}}$ with $a_m(x) \in \mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)$ and $b_m(x) \in \Lambda$ by

$$a_0(x) = x,$$
$$b_m(x) = \theta_{\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)}(a_m(x)), \quad a_{m+1}(x) = \frac{1}{v}(a_m(x) - s(b_m(x))).$$

Define $\widetilde{\theta}_{s,v}\colon \mathbb{B}^+_{\mathrm{dR}}(\Lambda^+) \to \Lambda[[X]]$ by $\widetilde{\theta}_{s,v}(x) := \sum_{m=0}^{\infty} b_m(x)X^m$.

Lemma 2.20.

(i) *The map $\widetilde{\theta}_{s,v}$ is a K-linear isomorphism, and the inverse map is given by $\sum_{m=0}^{\infty} c_m X^m \mapsto \sum_{m=0}^{\infty} s(c_m)v^m$.*
(ii) *For $F(X) \in K[[X]]$ and $x \in \mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)$, we have $\widetilde{\theta}_{s,v}\big(xF(v)\big) = \widetilde{\theta}_{s,v}(x)F(X)$.*
(iii) *We have an equality $\widetilde{\theta}_{s,v}\big(\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)\big) = \Lambda^+[[\pi X]]$.*
(iv) *The map $\widetilde{\theta}_{s,v}$ induces natural identifications (as K-vector spaces)*

$$\mathbb{A}_{\mathrm{max},K}(\Lambda^+) \cong \Lambda^+\langle X\rangle \quad \textit{and} \quad \mathbb{B}^+_{\mathrm{max},K}(\Lambda^+) \cong \Lambda\langle X\rangle.$$

Here $\Lambda^+\langle X\rangle$ denotes the p-adic completion of the polynomial algebra $\Lambda^+[X]$ and $\Lambda\langle X\rangle := \Lambda^+\langle X\rangle[p^{-1}]$.

Proof. Part (i) follows from $\operatorname{Ker}\theta_{\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)} = v\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)$, and part (ii) follows from the definition of $\widetilde{\theta}_{s,v}$ and the fact that $\theta_{\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)}$ and s are both K-linear.

We show part (iii). Recall from Lemma 2.7 that $\pi v = [\pi^\flat] - \pi$ generates $\operatorname{Ker}\theta_{\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)}$. For $x \in \mathbb{A}_{\mathrm{inf},K}(\Lambda^+)$, we see $b_m(x) \in \pi^m\Lambda^+$ by induction on m. Hence, $\widetilde{\theta}_{s,v}(\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)) \subset \Lambda^+[[\pi X]]$. Since $\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)$ is $(\pi, [\pi^\flat])$-adically complete and thus πv-adically complete, the other inclusion follows from part (i).

For part (iv), first note that parts (ii) and (iii) imply

$$\widetilde{\theta}_{s,v}\left(\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)\left[\frac{\operatorname{Ker}\theta}{\pi}\right]\right) = \Lambda^+[[\pi X]][X].$$

Hence, the identification $\mathbb{A}_{\mathrm{max},K}(\Lambda^+) \cong \Lambda^+\langle X\rangle$ follows from

$$\varprojlim_m \Lambda^+[[\pi X]][X]/p^m\Lambda^+[[\pi X]][X] \cong \Lambda^+\langle X\rangle.$$

By inverting p, we obtain $\mathbb{B}^+_{\mathrm{max},K}(\Lambda^+) \cong \Lambda\langle X\rangle$. □

We will show that there is an injective K-algebra homomorphism $\mathbb{B}^+_{\mathrm{max},K}(\Lambda^+) \to \mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)$. For each m, $\mathbb{B}^+_{\mathrm{dR},K}(\Lambda^+)_m := \mathbb{A}_{\mathrm{inf},K}(\Lambda^+)[p^{-1}]/(\operatorname{Ker}\theta_{\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)}[p^{-1}])^m$ is a K-Banach algebra. Hence, the map $\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)[(\operatorname{Ker}\theta)/\pi] \to \mathbb{B}^+_{\mathrm{dR},K}(\Lambda^+)_m$ extends to $\mathbb{A}_{\mathrm{max},K}(\Lambda^+) \to \mathbb{B}^+_{\mathrm{dR},K}(\Lambda^+)_m$. By taking the inverse limit with respect to m and inverting p, we obtain $\mathbb{B}^+_{\mathrm{max},K}(\Lambda^+) \to \mathbb{B}^+_{\mathrm{dR}}(\Lambda^+) = \mathbb{B}^+_{\mathrm{dR},K}(\Lambda^+)$. We deduce the injectivity of the map from the description of these period rings in terms of $\widetilde{\theta}_{s,v}$ as follows.

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

PROPOSITION 2.21. *The map $\widetilde{\theta}_{s,v}$ induces identifications (as K-vector spaces)*

$$\begin{array}{ccccc} \mathbb{A}_{\max,K}(\Lambda^+) & \longrightarrow & \mathbb{B}^+_{\max,K}(\Lambda^+) & \longrightarrow & \mathbb{B}^+_{\mathrm{dR}}(\Lambda^+) \\ \downarrow \cong & & \downarrow \cong & & \widetilde{\theta}_{v,s} \downarrow \cong \\ \Lambda^+\langle X\rangle & \hookrightarrow & \Lambda\langle X\rangle & \hookrightarrow & \Lambda[[X]] \end{array}$$

making the diagram commutative. In particular, the top horizontal maps are injective, and $\mathbb{B}^+_{\max,K}(\Lambda)$ is t-torsion-free.

Proof. We have already proved that the vertical maps are all isomorphisms. Since the bottom horizontal maps are injective, so are the top horizontal maps. Since $\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)$ is t-torsion-free, the last assertion follows. □

DEFINITION 2.22. We set

$$\mathbb{B}_{\max}(\Lambda^+) = \mathbb{B}_{\max}(\Lambda,\Lambda^+) := \mathbb{B}^+_{\max}(\Lambda^+)[t^{-1}],$$
$$\mathbb{B}_{\max,K}(\Lambda^+) = \mathbb{B}_{\max,K}(\Lambda,\Lambda^+) := \mathbb{B}^+_{\max,K}(\Lambda^+)[t^{-1}].$$

Since $\varphi(t) = pt$, the Frobenius φ on $\mathbb{A}_{\max}(\Lambda^+)$ extends to an endomorphism on $\mathbb{B}_{\max}(\Lambda^+)$, which we still call the Frobenius and denote by φ.

Let us define the horizontal semistable period rings. Consider the elements

$$\log\frac{[p^\flat]}{p} := \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\left(\frac{[p^\flat]}{p}-1\right)^m, \quad \log\frac{[\pi^\flat]}{\pi} := \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\left(\frac{[\pi^\flat]}{\pi}-1\right)^m \in \mathbb{B}^+_{\mathrm{dR}}(\Lambda^+).$$

DEFINITION 2.23. Define $\mathbb{B}^+_{\mathrm{st}}(\Lambda^+) = \mathbb{B}^+_{\mathrm{st}}(\Lambda,\Lambda^+)$ to be the $\mathbb{B}^+_{\max}(\Lambda^+)$-subalgebra of $\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)$ generated by $\log([p^\flat]/p)$. Similarly, define $\mathbb{B}^+_{\mathrm{st},K}(\Lambda^+) = \mathbb{B}^+_{\mathrm{st},K}(\Lambda,\Lambda^+)$ to be the $\mathbb{B}^+_{\max,K}(\Lambda^+)$-subalgebra of $\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)$ generated by $\log([p^\flat]/p)$.[3]

LEMMA 2.24. *The definitions of $\mathbb{B}^+_{\mathrm{st}}(\Lambda^+)$ and $\mathbb{B}^+_{\mathrm{st},K}(\Lambda^+)$ are independent of the choice of a compatible system of p-power roots (p_m) of p.*

Proof. If (p'_m) is a different choice, there exists $\alpha \in \mathbb{Z}_p$ such that $p'_m = \varepsilon_m^\alpha p_m$ for every $m \in \mathbb{N}$. It follows that $\log([p'^\flat]/p) = \log([p^\flat]/p) + \alpha t$ in $\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)$ with $\alpha t \in \mathbb{B}^+_{\max}(\Lambda^+)$. □

LEMMA 2.25. *The ring $\mathbb{B}^+_{\mathrm{st},K}(\Lambda^+)$ coincides with the $\mathbb{B}^+_{\max,K}(\Lambda^+)$-subalgebra of $\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)$ generated by $\log([\pi^\flat]/\pi)$.*

Proof. Set $e = [K : K_0]$. Write $p = a\pi^e$ and $p^\flat = a^\flat(\pi^\flat)^e$ for some $a \in \mathcal{O}_K^\times$ and $a^\flat \in (\mathcal{O}_C^\flat)^\times$. Then $[a^\flat]a^{-1} \in \mathbb{A}_{\mathrm{inf},K}(\Lambda^+)$ and $\theta_{\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)}([a^\flat]a^{-1}-1) = 0$. It follows that the series $\sum_{m=1}^{\infty}((-1)^{m-1}/m)([a^\flat]/a-1)^m$ converges in $\mathbb{A}_{\max,K}(\Lambda^+)$. Let $\log([a^\flat]/a)$ denote the limit. We see that the image of $\log([a^\flat]/a)$ in $\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)$ is given by the same series and $\log([p^\flat]/p) = \log([a^\flat]/a) + e\log([\pi^\flat]/\pi)$. Hence, the assertion follows. □

We will describe $\mathbb{B}^+_{\mathrm{st},K}(\Lambda^+)$ via $\widetilde{\theta}_{s,v}$. For this we need the following fact.

LEMMA 2.26. *Let A be a uniform $\mathbb{Q}_p$-Banach algebra. Then the power series $\log(1+X) := \sum_{m=0}^{\infty}((-1)^{m-1}/m)X^m \in A[[X]]$ is transcendental over $A\langle X\rangle$. Hence, the $A\langle X\rangle$-subalgebra*

[3] Strictly speaking, our definitions differ slightly from the ones in [Col02, § 8.6]. However, our horizontal semistable period rings are isomorphic to Colmez's period rings by Corollary 2.28.

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

$A\langle X\rangle[\log(1+X)]$ of $A[[X]]$ generated by $\log(1+X)$ is isomorphic to a polynomial algebra over $A\langle X\rangle$.

Proof. Since A is uniform, equip A with its spectral norm. Then the Gel'fand transform $A \to \prod_{x\in\mathcal{M}(A)} \mathcal{H}(x)$ is isometric with each $\mathcal{H}(x)$ a field [Ber90, Corollary 1.3.2(ii)]. We can thus reduce the general case to the case where A is a field. We may further assume that A is complete and algebraically closed. In this case, the assertion is well known. In fact, suppose that there exist $f_0(X),\dots,f_m(X) \in A\langle X\rangle$ such that $f_0(X)+f_1(X)\log(1+X)+\cdots+f_m(X)(\log(1+X))^m = 0$. If ζ is a p-power root of unity, then $X=\zeta-1$ is a root of $\log(1+X)$, and thus it is also a root of $f_0(X)$. Since any non-zero element of $A\langle X\rangle$ has at most finitely many roots in $\mathcal{O}_A$, we have $f_0(X)=0$. By repeating the same argument, we conclude that $f_0(X),\dots,f_m(X)$ are all zero. □

Proposition 2.27. *The map $\widetilde{\theta}_{s,v}$ induces identifications (as K-vector spaces)*

$$\begin{array}{ccccc}
\mathbb{B}^+_{\max,K}(\Lambda^+) & \longrightarrow & \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+) & \longrightarrow & \mathbb{B}^+_{\mathrm{dR}}(\Lambda^+) \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
\Lambda\langle X\rangle & \longrightarrow & \Lambda\langle X\rangle[\log(1+X)] & \longrightarrow & \Lambda[[X]]
\end{array}$$

sending $\log([\pi^\flat]/\pi)$ to $\log(1+X)$.

Proof. Recall that $v=[\pi^\flat]/\pi-1$. Hence, we have $\widetilde{\theta}_{s,v}((\log([\pi^\flat]/\pi))^m) = (\log(1+X))^m$ by Lemma 2.20(ii). By Lemma 2.25, $\mathbb{B}^+_{\mathrm{st},K}(\Lambda^+) = \sum_{m=0}^\infty \mathbb{B}^+_{\max,K}(\Lambda^+)(\log[\pi^\flat]/\pi)^m$ as K-vector subspaces of $\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+)$. It follows from Lemma 2.20(ii) and (iv) that $\widetilde{\theta}_{s,v}(\mathbb{B}^+_{\mathrm{st},K}(\Lambda^+)) = \sum_{m=0}^\infty \Lambda\langle X\rangle(\log(1+X))^m \subset \Lambda[[X]]$. □

Corollary 2.28. *The $\mathbb{B}^+_{\max}(\Lambda^+)$-algebra homomorphism from the polynomial ring $\mathbb{B}^+_{\max}(\Lambda^+)[u]$ to $\mathbb{B}^+_{\mathrm{st}}(\Lambda^+)$ sending u to $\log([p^\flat]/p)$ is an isomorphism. Similarly, the $\mathbb{B}^+_{\max,K}(\Lambda^+)$-algebra homomorphism $\mathbb{B}^+_{\max,K}(\Lambda^+)[u] \to \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+); u\mapsto \log([p^\flat]/p)$ is an isomorphism. In particular, $K\otimes_{K_0}\mathbb{B}^+_{\mathrm{st}}(\Lambda^+) \cong \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+)$.*

Proof. The first assertion is a special case of the second. The second assertion follows from Lemmas 2.20(ii), 2.25, 2.26, and Proposition 2.27. The last assertion follows from the first two since $K\otimes_{K_0}\mathbb{B}^+_{\max}(\Lambda^+) \cong \mathbb{B}^+_{\max,K}(\Lambda^+)$. □

Definition 2.29. We extend the Frobenius on $\mathbb{B}_{\max}(\Lambda^+)$ to the polynomial ring $\mathbb{B}^+_{\max}(\Lambda^+)[u]$ by setting $\varphi(u)=pu$. Then the $\mathbb{B}^+_{\max}(\Lambda^+)$-linear derivation $N=-d/du$ on $\mathbb{B}^+_{\max}(\Lambda^+)[u]$ satisfies $N\varphi = p\varphi N$. Via the $\mathbb{B}^+_{\max}(\Lambda^+)$-algebra isomorphism

$$\mathbb{B}^+_{\max}(\Lambda^+)[u] \xrightarrow{\cong} \mathbb{B}^+_{\mathrm{st}}(\Lambda^+); \quad u\mapsto \log\frac{[p^\flat]}{p},$$

we define the Frobenius φ and the monodromy operator N on $\mathbb{B}^+_{\mathrm{st}}(\Lambda^+)$. Note that $N\varphi=p\varphi N$, $\varphi(\log([p^\flat]/p)) = p\log([p^\flat]/p)$, and $N((\log([p^\flat]/p))^m) = -m(\log([p^\flat]/p))^{m-1}$. As in Lemma 2.24, we can check that the definitions of φ and N on $\mathbb{B}^+_{\mathrm{st}}(\Lambda^+)$ are independent of the choice of a compatible system of p-power roots of p.

Definition 2.30. Set

$$\mathbb{B}_{\mathrm{st}}(\Lambda^+) = \mathbb{B}_{\mathrm{st}}(\Lambda,\Lambda^+) := \mathbb{B}^+_{\mathrm{st}}(\Lambda^+)[t^{-1}],$$
$$\mathbb{B}_{\mathrm{st},K}(\Lambda^+) = \mathbb{B}_{\mathrm{st},K}(\Lambda,\Lambda^+) := \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+)[t^{-1}].$$

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

We have isomorphisms $\mathbb{B}_{\max}(\Lambda^+)[u] \xrightarrow{\cong} \mathbb{B}_{\mathrm{st}}(\Lambda^+)$ and $\mathbb{B}_{\max,K}(\Lambda^+)[u] \xrightarrow{\cong} \mathbb{B}_{\mathrm{st},K}(\Lambda^+)$ sending u to $\log([p^\flat]/p)$. Moreover, the Frobenius φ (respectively, the monodromy operator N) on $\mathbb{B}^+_{\mathrm{st}}(\Lambda^+)$ extends to a ring (respectively, $\mathbb{B}^+_{\max}(\Lambda^+)$-linear) endomorphism on $\mathbb{B}_{\mathrm{st}}(\Lambda^+)$, which we still denote by φ (respectively, N).

Lemma 2.31. *The sequence*

$$0 \longrightarrow \mathbb{B}_{\max}(\Lambda^+) \longrightarrow \mathbb{B}_{\mathrm{st}}(\Lambda^+) \xrightarrow{N} \mathbb{B}_{\mathrm{st}}(\Lambda^+) \longrightarrow 0$$

is exact.

Proof. This follows easily from the isomorphism $\mathbb{B}_{\max}(\Lambda^+)[u] \xrightarrow{\cong} \mathbb{B}_{\mathrm{st}}(\Lambda^+)$ commuting with N. □

We also introduce the horizontal crystalline period ring for completeness.

Definition 2.32. Let $\mathbb{A}'_{\mathrm{cris}}(\Lambda^+)$ denote the PD envelope (enveloppe à puissances divisées) of $\mathbb{A}_{\mathrm{inf}}(\Lambda^+)$ relative to the ideal $\operatorname{Ker}\theta_{\mathbb{A}_{\mathrm{inf}}(\Lambda^+)}$ compatible with the canonical PD structure on $p\mathbb{A}_{\mathrm{inf}}(\Lambda^+)$. Define $\mathbb{A}_{\mathrm{cris}}(\Lambda^+) = \mathbb{A}_{\mathrm{cris}}(\Lambda, \Lambda^+)$ to be the p-adic completion of $\mathbb{A}'_{\mathrm{cris}}(\Lambda^+)$.

Lemma 2.33. *The Frobenius φ on $\mathbb{A}_{\mathrm{inf}}(\Lambda^+)$ extends to a ring endomorphism φ on $\mathbb{A}_{\mathrm{cris}}(\Lambda^+)$.*

Proof. Since $\varphi(x) \equiv x^p \bmod p\mathbb{A}_{\mathrm{inf}}(\Lambda^+)$ for $x \in \mathbb{A}_{\mathrm{inf}}(\Lambda^+)$, the Frobenius φ on $\mathbb{A}_{\mathrm{inf}}(\Lambda^+)$ extends to a ring endomorphism on $\mathbb{A}'_{\mathrm{cris}}(\Lambda^+)$. Passing to the p-adic completion, it further extends to a ring endomorphism on $\mathbb{A}_{\mathrm{cris}}(\Lambda^+)$. □

Note that we have a natural φ-equivariant inclusion $\mathbb{A}_{\mathrm{cris}}(\Lambda^+) \hookrightarrow \mathbb{A}_{\max}(\Lambda^+)$ (cf. [Bri08, p. 62]).

Lemma 2.34. *We have $\varphi(\mathbb{A}_{\max}(\Lambda^+)) \subset \mathbb{A}_{\mathrm{cris}}(\Lambda^+) \subset \mathbb{A}_{\max}(\Lambda^+)$.*

Proof. We need to prove the first inclusion. Choose $x \in \operatorname{Ker}\theta_{\mathbb{A}_{\mathrm{inf}}(\Lambda^+)}$ and write $\varphi(x) = x^p + py$ for some $y \in \mathbb{A}_{\mathrm{inf}}(\Lambda^+)$. Then we have $\varphi(x/p) = (p-1)!x^{[p]} + y \in \mathbb{A}'_{\mathrm{cris}}(\Lambda^+)$. Hence, we have $\varphi(\mathbb{A}_{\mathrm{inf}}(\Lambda^+)[(\operatorname{Ker}\theta)/p]) \subset \mathbb{A}'_{\mathrm{cris}}(\Lambda^+)$. Taking p-adic completion yields the desired inclusion. □

Definition 2.35. Set

$$\mathbb{B}^+_{\mathrm{cris}}(\Lambda^+) = \mathbb{B}^+_{\mathrm{cris}}(\Lambda, \Lambda^+) := \mathbb{A}_{\mathrm{cris}}(\Lambda^+)[p^{-1}],$$
$$\mathbb{B}_{\mathrm{cris}}(\Lambda^+) = \mathbb{B}_{\mathrm{cris}}(\Lambda, \Lambda^+) := \mathbb{B}_{\mathrm{cris}}(\Lambda^+)[t^{-1}].$$

The Frobenius φ on $\mathbb{A}_{\mathrm{cris}}(\Lambda^+)$ naturally extends over these rings, and the previous lemma implies $\varphi(\mathbb{B}_{\max}(\Lambda^+)) \subset \mathbb{B}_{\mathrm{cris}}(\Lambda^+) \subset \mathbb{B}_{\max}(\Lambda^+)$.

Remark 2.36. The construction of all the period rings is functorial on perfectoid C-Banach pairs (Λ, Λ^+) with morphisms of K-Banach pairs.

The following proposition will be heavily used in the proof of the purity theorem for horizontal semistable representations (Theorem 6.1).

Proposition 2.37. *Let (Λ', Λ'^+) be another perfectoid C-Banach pair such that Λ'^+ is an $\mathcal{O}_C$-algebra, and let $f^+\colon \Lambda^+ \hookrightarrow \Lambda'^+$ be an injective $\mathcal{O}_K$-algebra homomorphism with*

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

p-torsion-free cokernel. Then the natural commutative diagram

$$\begin{array}{ccc} \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+) & \longrightarrow & \mathbb{B}^+_{\mathrm{dR},K}(\Lambda^+) \\ \downarrow & & \downarrow \\ \mathbb{B}^+_{\mathrm{st},K}(\Lambda'^+) & \longrightarrow & \mathbb{B}^+_{\mathrm{dR},K}(\Lambda'^+) \end{array}$$

is Cartesian with injective vertical maps.

Proof. By construction of the period rings, we have a natural commutative diagram as in the statement. We will prove that it is Cartesian.

Let $f\colon \Lambda \to \Lambda'$ denote the induced K-algebra homomorphism from f^+ by inverting p. We also change the norm on Λ (respectively, Λ') so that Λ^+ (respectively, Λ'^+) becomes the unit ball. Then f is a closed isometric embedding of K-Banach algebras; by p-torsion-freeness of $\operatorname{Coker} f^+$, the induced map $\Lambda^+/p^n\Lambda^+ \to \Lambda'^+/p^n\Lambda'^+$ is injective for all $n \in \mathbb{N}$, and thus f is closed and isometric.

The commutative diagram in question only depends on f^+ as an $\mathcal{O}_K$-algebra homomorphism and not on the entire C-Banach structures on Λ and Λ'. Hence, by replacing the C-Banach structure $C \to \Lambda'$ by precomposing an isometric K-algebra automorphism $C \to C$, we may further assume $f(\pi_n) = \pi_n$ for every $n \in \mathbb{N}$.

We will explain that Construction 2.19 is made compatibly with $f^+\colon \Lambda^+ \hookrightarrow \Lambda'^+$. Since the map $\Lambda^+/\pi\Lambda^+ \to \Lambda'^+/\pi\Lambda'^+$ is injective, choose a family of elements $(e_i)_{i\in I}$ of Λ^+ and a family of elements $(e'_j)_{j\in J}$ of Λ'^+ such that the images of the e_i in $\Lambda^+/\pi\Lambda^+$ form a k-basis of $\Lambda^+/\pi\Lambda^+$ and such that the images of the $f^+(e_i)$ and e'_j in $\Lambda'^+/\pi\Lambda'^+$ form a k-basis of $\Lambda'^+/\pi\Lambda'^+$; this is possible by lifting k-bases of $\Lambda^+/\pi\Lambda^+ \subset \Lambda'^+/\pi\Lambda'^+$. We also choose a lift $\widetilde{e}_i \in \mathbb{A}_{\mathrm{inf},K}(\Lambda^+)$ of e_i with respect to $\theta_{\mathbb{A}_{\mathrm{inf},K}(\Lambda^+)}$ and a lift $\widetilde{e}'_j \in \mathbb{A}_{\mathrm{inf},K}(\Lambda'^+)$ of e'_j with respect to $\theta_{\mathbb{A}_{\mathrm{inf},K}(\Lambda'^+)}$.

Consider the K-linear maps

$$s\colon \Lambda \to \mathbb{A}_{\mathrm{inf},K}(\Lambda^+)[p^{-1}]; \quad \sum_{i\in I} a_i e_i \mapsto \sum_{i\in I} a_i \widetilde{e}_i,$$

$$s'\colon \Lambda' \to \mathbb{A}_{\mathrm{inf},K}(\Lambda'^+)[p^{-1}]; \quad \sum_{i\in I} a_i f(e_i) + \sum_{j\in J} b_j e'_j \mapsto \sum_{i\in I} a_i \mathbb{A}_{\mathrm{inf},K}(f)(\widetilde{e}_i) + \sum_{j\in J} b_j \widetilde{e}'_j.$$

Then we have $\mathbb{A}_{\mathrm{inf},K}(f) \circ s = s' \circ f$.

Since $f(\pi_n) = \pi_n$ for every $n \in \mathbb{N}$, we have $\mathbb{A}_{\mathrm{inf},K}(f)([\pi^\flat]) = [\pi^\flat]$. Hence, we apply Construction 2.19 to the maps s and s' with $v = \pi^{-1}([\pi^\flat] - \pi)$ and obtain the following commutative diagram.

$$\begin{array}{ccc} \mathbb{B}^+_{\mathrm{dR},K}(\Lambda^+) & \xrightarrow[\cong]{\widetilde{\theta}_{s,v}} & \Lambda[[X]] \\ \downarrow & & \downarrow \\ \mathbb{B}^+_{\mathrm{dR},K}(\Lambda'^+) & \xrightarrow[\cong]{\widetilde{\theta}_{s',v}} & \Lambda'[[X]] \end{array}$$

By these maps and Proposition 2.27, it suffices to show

$$\Lambda[[X]] \cap (\Lambda'\langle X\rangle[\log(1+X)]) = \Lambda\langle X\rangle[\log(1+X)] \quad \text{inside } \Lambda'[[X]].$$

This follows from the following lemma. □

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

Lemma 2.38. *Let A be a $\mathbb{Q}_p$-Banach algebra and B a closed subalgebra of A. If $g(X) \in \bigoplus_{i=0}^{h} A\langle X\rangle(\log(1+X))^i$ satisfies $g(X) \in B[[X]]$, then $g(X) \in \bigoplus_{i=0}^{h} B\langle X\rangle(\log(1+X))^i$.*

Proof. This is stated and proved in the proof of [BC08, Lemma 6.3.1]. For the convenience of the reader, we give a variant of their proof suggested by the referee.

First we claim that if $f(X) \in A\langle X\rangle$ satisfies $f(\zeta_{p^n}-1) \in B \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_{p^n})$ for every n and every p^nth root of unity ζ_{p^n}, then $f(X) \in B\langle X\rangle$. To see this, take any $a \in A \smallsetminus B$. By a consequence of the Hahn–Banach theorem (e.g. [Sch02, Corollary 9.3]), there exists a continuous $\mathbb{Q}_p$-linear map of the form $l\colon A \to A/B \to \mathbb{Q}_p$ such that $l(a) \neq 0$. Extend l coefficientwise to a continuous $\mathbb{Q}_p$-linear map $l\colon A\langle X\rangle \to \mathbb{Q}_p\langle X\rangle$. By assumption and construction, $l(f)(X) \in \mathbb{Q}_p\langle X\rangle$ satisfies $l(f)(\zeta_{p^n}-1)=0$ for every n and ζ_{p^n}. Hence, $l(f)(X)=0$ and we conclude that $f(X) \in B\langle X\rangle$.

Take any

$$g(X) = f^{(0)}(X) + f^{(1)}(X)\log(1+X) + \cdots + f^{(h)}(X)(\log(1+X))^h \in \bigoplus_{i=0}^{h} A\langle X\rangle(\log(1+X))^i$$

with $g(X) \in B[[X]]$. Then we see that $f^{(0)}(\zeta_{p^n}-1) = g(\zeta_{p^n}-1) \in B \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_{p^n})$ for every n and ζ_{p^n}. By the above claim, we obtain $f^{(0)}(X) \in B\langle X\rangle$. Applying the same argument to $(g(X)-f^{(0)}(X))/\log(1+X)$ yields $f^{(1)}(X) \in B\langle X\rangle$. By induction, we conclude $f^{(i)}(X) \in B\langle X\rangle$ for every i. □

3. Preliminaries on base rings

In this section, we review the class of base rings which is considered in [Bri08]. As in the previous section, let k be a perfect field of characteristic p and set $K_0 := W(k)[p^{-1}]$. Let K be a totally ramified finite extension of K_0 and fix a uniformizer π of K. We denote by C the p-adic completion of an algebraic closure $\overline{K}$ of K.

Let $\mathcal{O}_K\langle T_1^{\pm 1},\dots,T_n^{\pm 1}\rangle$ denote the p-adic completion of $\mathcal{O}_K[T_1^{\pm 1},\dots,T_n^{\pm 1}]$.

Set-up 3.1 [Bri08, p. 7]. Let $\widetilde{R}$ be a ring obtained from $\mathcal{O}_K\langle T_1^{\pm 1},\dots,T_n^{\pm 1}\rangle$ by a finite number of iterations of the following operations:

- p-adic completion of an étale extension;
- p-adic completion of a localization;
- completion with respect to an ideal containing p.

We further assume that either $\mathcal{O}_K\langle T_1^{\pm 1},\dots,T_n^{\pm 1}\rangle \to \widetilde{R}$ has geometrically regular fibers or $\widetilde{R}$ has Krull dimension less than two, and that $k \to \widetilde{R}/\pi\widetilde{R}$ is geometrically integral. Let R be an $\widetilde{R}$-algebra such that:

(i) R is an integrally closed domain that is finite and flat over $\widetilde{R}$;
(ii) $R[p^{-1}]$ is étale over $\widetilde{R}[p^{-1}]$;
(iii) K is algebraically closed in $R[p^{-1}]$.

Remark 3.2. For the proof of the main theorem (Theorem 9.2), one only needs the case where $R = \mathcal{O}_K\langle T_1^{\pm 1},\dots,T_n^{\pm 1}\rangle$. See § 9.4.

Definition 3.3. Following [Bri08, p. 9], we say that R satisfies (BR) if $R = \widetilde{R}$. This is a condition on good reduction (*bonne réduction*).

Set $R_K := R[p^{-1}]$. Fix an algebraic closure $\overline{\operatorname{Frac} R}$ of $\operatorname{Frac} R$ containing $\overline{K}$. Let $\overline{R}$ be the union of all finite R-subalgebras R' of $\overline{\operatorname{Frac} R}$ such that $R'[p^{-1}]$ is étale over R_K. Note that $\overline{R}$ is

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

a normal domain. Since $\overline{R}[p^{-1}]$ is the union of all finite Galois extensions of R_K, we set[4]

$$\mathcal{G}_{R_K} := \mathrm{Gal}(\overline{R}[p^{-1}]/R_K).$$

Let $\widehat{\overline{R}}$ denote the p-adic completion of $\overline{R}$. Then $\widehat{\overline{R}}$ is a p-torsion-free $\mathcal{O}_C$-algebra, the map $\overline{R} \to \widehat{\overline{R}}$ is injective, and $\overline{R} \cap p\widehat{\overline{R}} = p\overline{R}$ [Bri08, Proposition 2.0.3]. The action of $\mathcal{G}_{R_K}$ on $\overline{R}$ extends to $\widehat{\overline{R}}$ continuously.

We will explain that $(\widehat{\overline{R}}[p^{-1}], \widehat{\overline{R}})$ is naturally a perfectoid C-Banach pair and thus the results in § 2 apply. For $x \in \widehat{\overline{R}}[p^{-1}]$, set $|x| = \inf\{|a|_C \mid a \in C,\ x \in a\widehat{\overline{R}}\}$. Here we normalize the norm $|\ |_C$ on C by $|p|_C = p^{-1}$. The proof of the following lemma is clear and left to the reader.

Lemma 3.4. *The function $|\ |$ on $\widehat{\overline{R}}[p^{-1}]$ is a C-algebra norm making $\widehat{\overline{R}}[p^{-1}]$ a C-Banach algebra. Moreover, $\widehat{\overline{R}}$ is an open subring of $\widehat{\overline{R}}[p^{-1}]$, and the induced topology on $\widehat{\overline{R}}$ agrees with the p-adic topology.*

Lemma 3.5. *The C-Banach algebra $\widehat{\overline{R}}[p^{-1}]$ is uniform.*

Proof. Since $p^{-1}\widehat{\overline{R}}$ is a bounded subset, it suffices to show $(\widehat{\overline{R}}[p^{-1}])^\circ \subset p^{-1}\widehat{\overline{R}}$.

First we prove $(\widehat{\overline{R}}[p^{-1}])^\circ \cap \overline{R}[p^{-1}] \subset p^{-1}\overline{R}$. Take $x \in (\widehat{\overline{R}}[p^{-1}])^\circ \cap \overline{R}[p^{-1}]$. Then the sequence $(p^n x^n)_n$ converges to zero as $n \to \infty$. In particular, $(px)^n \in \overline{R}$ for some n. Since $\overline{R}$ is a normal domain, we have $px \in \overline{R}$ and thus $x \in p^{-1}\overline{R}$.

For $x \in (\widehat{\overline{R}}[p^{-1}])^\circ$, write $x = u + v$ ($u \in \overline{R}[p^{-1}]$, $v \in \widehat{\overline{R}}$). Then $u = x - v \in (\widehat{\overline{R}}[p^{-1}])^\circ$. Hence, $u \in p^{-1}\overline{R}$ by what we have proved, and thus $x \in p^{-1}\widehat{\overline{R}}$. □

Lemma 3.6. *The subring $\widehat{\overline{R}}$ of $\widehat{\overline{R}}[p^{-1}]$ is a ring of integral elements.*

Proof. We need to show that $\widehat{\overline{R}}$ is integrally closed in $\widehat{\overline{R}}[p^{-1}]$.

Let $x \in \widehat{\overline{R}}[p^{-1}]$ be an element that is integral over $\widehat{\overline{R}}$ and suppose that x satisfies $x^m + a_{m-1}x^{m-1} + \cdots + a_0 = 0$ for some $a_0, \ldots, a_{m-1} \in \widehat{\overline{R}}$. For each $l \in \mathbb{N}$, there exist $y_l \in \overline{R}[p^{-1}]$ and $z_l \in p^l\widehat{\overline{R}}$ such that $x = y_l + z_l$. Similarly, for each $i = 0, \ldots, m-1$, there exists $a_{i,l} \in \overline{R}$ such that $a_i - a_{i,l} \in p^l\widehat{\overline{R}}$. We compute

$$\begin{aligned}
&|y_l^m + a_{m-1,l}y_l^{m-1} + \cdots + a_{0,l}| \\
&\quad \le \max\{|(y_l^m + a_{m-1,l}y_l^{m-1} + \cdots + a_{0,l}) - (x^m + a_{m-1,l}x^{m-1} + \cdots + a_{0,l})|, \\
&\qquad\qquad |(x^m + a_{m-1,l}x^{m-1} + \cdots + a_{0,l}) - (x^m + a_{m-1}x^{m-1} + \cdots + a_0)|\} \\
&\quad \le |p^l| \max\{1, |x|^{m-1}\}.
\end{aligned}$$

Since $\{y \in \widehat{\overline{R}}[p^{-1}] \mid |y| < 1\} \subset \widehat{\overline{R}}$ and $\overline{R}[p^{-1}] \cap \widehat{\overline{R}} = \overline{R}$, we see that $y_l^m + a_{m-1,l}y_l^{m-1} + \cdots + a_{0,l} \in \overline{R}$ for $l \gg 0$. For such l, we have $y_l \in \overline{R}$ by normality of $\overline{R}$ and thus $x = y_l + z_l \in \widehat{\overline{R}}$. □

Lemma 3.7. *The C-Banach pair $(\widehat{\overline{R}}[p^{-1}], \widehat{\overline{R}})$ is perfectoid.*

Proof. It remains to prove that the Frobenius is surjective on $\widehat{\overline{R}}/p\widehat{\overline{R}} = \overline{R}/p\overline{R}$. Let $\overline{x} \in \overline{R}/p\overline{R}$ and choose a lift $x \in \overline{R}$ of $\overline{x}$. Set $R' := R[x][T]/(T^{p^2} + pT - x)$. Then R' is finite over R and $R'[p^{-1}]$ is étale over R_K. Choose an embedding of R' into $\overline{\mathrm{Frac}\, R}$ and let y denote the image of T. Then $y \in \overline{R}$ and it satisfies $(y^p)^p \equiv x \bmod p\overline{R}$. Hence, the Frobenius on $\overline{R}/p\overline{R}$ is surjective. □

[4] Our $\mathcal{G}_{R_K}$ is denoted by $\mathcal{G}_R$ in [Bri08].

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

4. Horizontal crystalline, semistable, and de Rham representations

In this section, we define horizontal crystalline, semistable, and de Rham representations of $\mathcal{G}_{R_K}$, and study their basic properties. We mainly follow [Bri08, § 8], where horizontal crystalline and de Rham representations are studied.

4.1 Admissible representations

We review the notion of admissible representations. Let E be a finite extension of $\mathbb{Q}_p$ and G be a topological group.

DEFINITION 4.1. An *E-representation* of G is a finite-dimensional E-vector space equipped with a continuous action of G. A morphism of E-representations of G is an E-linear map commuting with actions of G. We denote the category of E-representations of G by $\mathrm{Rep}_E(G)$. It is naturally a Tannakian category.

Remark 4.2. Note that [Fon94b, § 1] deals with an abstract field E and an abstract group G but that properties discussed there continue to hold in our setting.

DEFINITION 4.3 (cf. [Fon94b, 1.3]).

(i) An *(E, G)-ring* is an E-algebra B equipped with an E-linear action of G.
(ii) Let B be an (E, G)-ring and let $V \in \mathrm{Rep}_E(G)$. We set

$$D_B(V) := (B \otimes_E V)^G,$$

and we let

$$\alpha_B(V)\colon B \otimes_{B^G} D_B(V) \to B \otimes_E V$$

denote the B-linear map induced from the inclusion $D_B(V) \subset B \otimes_E V$. We say that V is *B-admissible* if $\alpha_B(V)$ is an isomorphism of B-modules.

LEMMA 4.4. *Let $\alpha\colon G' \to G$ be a continuous group homomorphism between topological groups. Let B be an (E, G)-ring and B' an (E, G')-ring equipped with an E-algebra homomorphism $\beta\colon B \to B'$ satisfying*

$$\beta(\alpha(g') \cdot b) = g' \cdot \beta(b) \quad \text{for all } g' \in G' \text{ and } b \in B.$$

Let $V \in \mathrm{Rep}_E(G)$ and let $V|_{G'}$ denote the corresponding E-representation of G'. Assume that V is B-admissible and $\alpha_{B'}(V|_{G'})$ is injective. Then $V|_{G'}$ is B'-admissible. Moreover, if $(B')^{G'}$ is a field, then the natural map $(B')^{G'} \otimes_{B^G} D_B(V) \to D_{B'}(V|_{G'})$ is an isomorphism.

Proof. Consider the following commutative diagram.

$$\begin{array}{ccc} B \otimes_{B^G} D_B(V) & \xrightarrow[\cong]{\alpha_B(V)} & B \otimes_E V \\ \downarrow & & \downarrow \\ B' \otimes_{(B')^{G'}} D_{B'}(V|_{G'}) & \xhookrightarrow{\alpha_{B'}(V|_{G'})} & B' \otimes_E V \end{array}$$

It follows that $V \subset B' \otimes_E V$ is contained in the image of $\alpha_{B'}(V|_{G'})$. Since $\alpha_{B'}(V|_{G'})$ is injective and B'-linear, it is an isomorphism. Hence, $V|_{G'}$ is B'-admissible. It also follows that the left

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

vertical map induces a G'-equivariant isomorphism

$$B' \otimes_{(B')^{G'}} ((B')^{G'} \otimes_{B^G} D_B(V)) = B' \otimes_B (B \otimes_{B^G} D_B(V)) \xrightarrow{\cong} B' \otimes_{(B')^{G'}} D_{B'}(V|_{G'}).$$

Now assume that $(B')^{G'}$ is a field. Then taking G'-invariants of the above isomorphism yields an isomorphism $(B')^{G'} \otimes_{B^G} D_B(V) \xrightarrow{\cong} D_{B'}(V|_{G'})$. □

4.2 Horizontal crystalline, semistable, and de Rham representations

Let R be an $\mathcal{O}_K$-algebra satisfying the conditions in Set-up 3.1. Recall that we set $\mathcal{G}_{R_K} := \mathrm{Gal}(\overline{R_K}/R_K)$ and that $(\widehat{\overline{R}}[p^{-1}], \widehat{\overline{R}})$ is a perfectoid C-Banach pair.

Definition 4.5. Set

$$\begin{aligned}
&\mathrm{B}^{\nabla}_{\max}(R) := \mathbb{B}_{\max}(\widehat{\overline{R}}), \quad \mathrm{B}^{\nabla}_{\max,K}(R) := \mathbb{B}_{\max,K}(\widehat{\overline{R}}), \quad \mathrm{B}^{\nabla}_{\mathrm{cris}}(R) := \mathbb{B}_{\mathrm{cris}}(\widehat{\overline{R}}),\\
&\mathrm{B}^{\nabla}_{\mathrm{st}}(R) := \mathbb{B}_{\mathrm{st}}(\widehat{\overline{R}}), \quad \mathrm{B}^{\nabla}_{\mathrm{st},K}(R) := \mathbb{B}_{\mathrm{st},K}(\widehat{\overline{R}}),\\
&\mathrm{B}^{\nabla}_{\mathrm{dR}}(R) := \mathbb{B}_{\mathrm{dR}}(\widehat{\overline{R}}) = \mathbb{B}_{\mathrm{dR},K}(\widehat{\overline{R}}), \quad \text{and} \quad \mathrm{B}^{\nabla+}_{\mathrm{dR}}(R) := \mathbb{B}^{+}_{\mathrm{dR}}(\widehat{\overline{R}}).
\end{aligned}$$

By functoriality of the construction, all these period rings have an action of $\mathcal{G}_{R_K}$. In particular, they are $(\mathbb{Q}_p, \mathcal{G}_{R_K})$-rings. To simplify the notation, for $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$, we write $D^{\nabla}_{\max}(V)$ for $D_{\mathrm{B}^{\nabla}_{\max}(R)}(V)$ and $\alpha^{\nabla}_{\max}(V)$ for $\alpha_{\mathrm{B}^{\nabla}_{\max}(R)}(V)$. We use similar conventions for other period rings.

Remark 4.6.

(i) Our period rings $\mathrm{B}^{\nabla}_{\max}(R)$, $\mathrm{B}^{\nabla}_{\mathrm{cris}}(R)$, and $\mathrm{B}^{\nabla}_{\mathrm{dR}}(R)$ coincide with the ones defined in [Bri08, Définitions 5.1.2, 6.1.2] and denoted by the same symbols.

(ii) The period rings $\mathrm{B}^{\nabla}_{\max}(R)$ and $\mathrm{B}^{\nabla}_{\mathrm{cris}}(R)$ admit the Frobenius φ. Similarly, $\mathrm{B}^{\nabla}_{\mathrm{st}}(R)$ admits the Frobenius φ and the monodromy operator N satisfying $N\varphi = p\varphi N$. The action of $\mathcal{G}_{R_K}$ on $\mathrm{B}^{\nabla}_{\max}(R)$, $\mathrm{B}^{\nabla}_{\mathrm{cris}}(R)$, and $\mathrm{B}^{\nabla}_{\mathrm{st}}(R)$ commutes with these endomorphisms. In particular, for $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$, the modules $D^{\nabla}_{\max}(V)$ and $D^{\nabla}_{\mathrm{cris}}(V)$ (respectively, $D^{\nabla}_{\mathrm{st}}(V)$) are equipped with the Frobenius φ (respectively, the Frobenius φ and the monodromy operator N). We have $\varphi(D^{\nabla}_{\max}(V)) \subset D^{\nabla}_{\mathrm{cris}}(V) \subset D^{\nabla}_{\max}(V)$ and $(D^{\nabla}_{\mathrm{st}}(V))^{N=0} = D^{\nabla}_{\max}(V)$.

Proposition 4.7 [Bri08, Corollary 5.3.7]. *We have* $(\mathrm{B}^{\nabla}_{\mathrm{dR}}(R))^{\mathcal{G}_{R_K}} = K$.

Corollary 4.8. *We have*

$$\begin{aligned}
(\mathrm{B}^{\nabla}_{\max,K}(R))^{\mathcal{G}_{R_K}} &= (\mathrm{B}^{\nabla}_{\mathrm{st},K}(R))^{\mathcal{G}_{R_K}} = K,\\
(\mathrm{B}^{\nabla}_{\mathrm{cris}}(R))^{\mathcal{G}_{R_K}} &= (\mathrm{B}^{\nabla}_{\max}(R))^{\mathcal{G}_{R_K}} = (\mathrm{B}^{\nabla}_{\mathrm{st}}(R))^{\mathcal{G}_{R_K}} = K_0.
\end{aligned}$$

Proof. Since we have $\mathcal{G}_{R_K}$-equivariant K-algebra homomorphisms

$$K \otimes_{K_0} \mathrm{B}^{\nabla}_{\max}(R) \xrightarrow{\cong} \mathrm{B}^{\nabla}_{\max,K}(R) \hookrightarrow \mathrm{B}^{\nabla}_{\mathrm{dR}}(R),$$

we obtain $(\mathrm{B}^{\nabla}_{\max,K}(R))^{\mathcal{G}_{R_K}} = K$, and $(\mathrm{B}^{\nabla}_{\max}(R))^{\mathcal{G}_{R_K}} = K_0$. By the inclusions $\varphi(\mathrm{B}^{\nabla}_{\max}(R)) \subset \mathrm{B}^{\nabla}_{\mathrm{cris}}(R) \subset \mathrm{B}^{\nabla}_{\max}(R)$, we have $(\mathrm{B}^{\nabla}_{\mathrm{cris}}(R))^{\mathcal{G}_{R_K}} = K_0$. We can prove the remaining assertions in the same way. □

The following result is clear from what we have discussed.

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

LEMMA 4.9. *Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$.*

(i) *$D^{\nabla}_{\mathrm{max}}(V)$ and $D^{\nabla}_{\mathrm{st}}(V)$ are K_0-vector spaces, and $D^{\nabla}_{\mathrm{max},K}(V)$, $D^{\nabla}_{\mathrm{st},K}(V)$, and $D^{\nabla}_{\mathrm{dR}}(V)$ are K-vector spaces.*

(ii) *We have $K \otimes_{K_0} D^{\nabla}_{\mathrm{max}}(V) \cong D^{\nabla}_{\mathrm{max},K}(V)$ and $K \otimes_{K_0} D^{\nabla}_{\mathrm{st}}(V) \cong D^{\nabla}_{\mathrm{st},K}(V)$. Moreover, $\alpha^{\nabla}_{\mathrm{max}}(V)$ (respectively, $\alpha^{\nabla}_{\mathrm{st}}(V)$) is an isomorphism if and only if $\alpha^{\nabla}_{\mathrm{max},K}(V)$ (respectively, $\alpha^{\nabla}_{\mathrm{st},K}(V)$) is an isomorphism.*

(iii) *We have injective K-linear maps $D^{\nabla}_{\mathrm{max},K}(V) \hookrightarrow D^{\nabla}_{\mathrm{st},K}(V) \hookrightarrow D^{\nabla}_{\mathrm{dR}}(V)$.*

LEMMA 4.10. *For every $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$, the maps $\alpha^{\nabla}_{\mathrm{cris}}(V)$, $\alpha^{\nabla}_{\mathrm{max}}(V)$, $\alpha^{\nabla}_{\mathrm{max},K}(V)$, $\alpha^{\nabla}_{\mathrm{st}}(V)$, $\alpha^{\nabla}_{\mathrm{st},K}(V)$, and $\alpha^{\nabla}_{\mathrm{dR}}(V)$ are all injective. Moreover, we have*

$$\begin{aligned}\dim_{K_0} D^{\nabla}_{\mathrm{max}}(V) = \dim_K D^{\nabla}_{\mathrm{max},K}(V) &\le \dim_{K_0} D^{\nabla}_{\mathrm{st}}(V) = \dim_K D^{\nabla}_{\mathrm{st},K}(V)\\ &\le \dim_K D^{\nabla}_{\mathrm{dR}}(V) \le \dim_{\mathbb{Q}_p} V.\end{aligned}$$

Proof. The map $\alpha^{\nabla}_{\mathrm{dR}}(V)$ is injective by [Bri08, Proposition 8.2.9]. Consider the following commutative diagram.

$$\begin{array}{ccc}
K \otimes_{K_0} (\mathrm{B}^{\nabla}_{\mathrm{max}}(R) \otimes_{K_0} D^{\nabla}_{\mathrm{max}}(V)) & \xrightarrow{K \otimes_{K_0} \alpha^{\nabla}_{\mathrm{max}}(V)} & K \otimes_{K_0} (\mathrm{B}^{\nabla}_{\mathrm{max}}(R) \otimes_{\mathbb{Q}_p} V) \\
\| & & \| \\
\mathrm{B}^{\nabla}_{\mathrm{max},K}(R) \otimes_K D^{\nabla}_{\mathrm{max},K}(V) & \xrightarrow{\alpha^{\nabla}_{\mathrm{max},K}(V)} & \mathrm{B}^{\nabla}_{\mathrm{max},K}(R) \otimes_{\mathbb{Q}_p} V \\
\downarrow & & \downarrow \\
\mathrm{B}^{\nabla}_{\mathrm{dR}}(R) \otimes_K D^{\nabla}_{\mathrm{dR}}(V) & \xrightarrow{\alpha^{\nabla}_{\mathrm{dR}}(V)} & \mathrm{B}^{\nabla}_{\mathrm{dR}}(R) \otimes_{\mathbb{Q}_p} V
\end{array}$$

The vertical equalities follow from Lemma 4.9(ii). Hence, the injectivity of $\alpha^{\nabla}_{\mathrm{dR}}(V)$ implies that of $\alpha^{\nabla}_{\mathrm{max}}(V)$ and $\alpha^{\nabla}_{\mathrm{max},K}(V)$. The same argument works for the injectivity of $\alpha^{\nabla}_{\mathrm{st}}(V)$ and $\alpha^{\nabla}_{\mathrm{st},K}(V)$. The injectivity of $\alpha^{\nabla}_{\mathrm{cris}}(V)$ follows from that of $\alpha^{\nabla}_{\mathrm{max}}(V)$ and the inclusion $\mathrm{B}^{\nabla}_{\mathrm{cris}}(R) \subset \mathrm{B}^{\nabla}_{\mathrm{max}}(R)$ by a similar argument. The second assertion follows from Lemma 4.9(iii). □

LEMMA 4.11. *For $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$, $\alpha^{\nabla}_{\mathrm{max}}(V)$ is an isomorphism if and only if $\alpha^{\nabla}_{\mathrm{cris}}(V)$ is an isomorphism. Similarly, $\alpha^{\nabla}_{\mathrm{st}}(V)$ is an isomorphism if and only if $\alpha_{\mathrm{B}^{\nabla}_{\mathrm{cris}}(R)[\log([p^b]/p)]}(V)$ is an isomorphism.*

Proof. The sufficiency follows from Lemma 4.4. Since the proof of the necessity of the two assertions is similar, we deal with the necessity for the first. Assume that $\alpha^{\nabla}_{\mathrm{max}}(V)$ is an isomorphism. Then the Frobenius $\varphi\colon \mathrm{B}^{\nabla}_{\mathrm{max}}(R) \to \mathrm{B}^{\nabla}_{\mathrm{max}}(R)$ induces the following commutative diagram.

$$\begin{array}{ccc}
\mathrm{B}^{\nabla}_{\mathrm{max}}(R) \otimes_{K_0} D^{\nabla}_{\mathrm{max}}(V) & \xrightarrow[\cong]{\alpha^{\nabla}_{\mathrm{max}}(V)} & \mathrm{B}^{\nabla}_{\mathrm{max}}(R) \otimes_{\mathbb{Q}_p} V \\
\downarrow & & \downarrow \\
\mathrm{B}^{\nabla}_{\mathrm{cris}}(R) \otimes_{K_0} D^{\nabla}_{\mathrm{cris}}(V) & \xhookrightarrow{\alpha^{\nabla}_{\mathrm{cris}}(V)} & \mathrm{B}^{\nabla}_{\mathrm{cris}}(R) \otimes_{\mathbb{Q}_p} V \\
\downarrow & & \downarrow \\
\mathrm{B}^{\nabla}_{\mathrm{max}}(R) \otimes_{K_0} D^{\nabla}_{\mathrm{max}}(V) & \xrightarrow[\cong]{\alpha^{\nabla}_{\mathrm{max}}(V)} & \mathrm{B}^{\nabla}_{\mathrm{max}}(R) \otimes_{\mathbb{Q}_p} V
\end{array}$$

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

Here we also use the fact that $\varphi\colon \mathrm{B}^{\nabla}_{\max}(R) \to \mathrm{B}^{\nabla}_{\max}(R)$ factors through $\mathrm{B}^{\nabla}_{\mathrm{cris}}(R) \subset \mathrm{B}^{\nabla}_{\max}(R)$. In particular, the image of $\alpha^{\nabla}_{\mathrm{cris}}(V)$ contains V. Since $\alpha^{\nabla}_{\mathrm{cris}}(V)$ is injective and $\mathrm{B}^{\nabla}_{\mathrm{cris}}(R)$-linear, it is an isomorphism. □

LEMMA 4.12. *For $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$, $\alpha^{\nabla}_{\max}(V)$ is an isomorphism if and only if $\alpha^{\nabla}_{\mathrm{st}}(V)$ is an isomorphism and the monodromy operator N on $D^{\nabla}_{\mathrm{st}}(V)$ is zero.*

Proof. The necessity follows from Lemma 4.4 and $(D^{\nabla}_{\mathrm{st}}(V))^{N=0} = D^{\nabla}_{\max}(V)$. We prove the sufficiency. Note that the multiplication map $\mathrm{B}^{\nabla}_{\mathrm{st}}(R) \otimes_{K_0} \mathrm{B}^{\nabla}_{\mathrm{st}}(R) \to \mathrm{B}^{\nabla}_{\mathrm{st}}(R)$ is compatible with $N \otimes \mathrm{id} + \mathrm{id} \otimes N$ on the source and N on the target. Since $\alpha^{\nabla}_{\mathrm{st}}(V)$ is an isomorphism, this yields the following diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{B}^{\nabla}_{\max} \otimes_{\mathbb{Q}_p} V & \longrightarrow & \mathrm{B}^{\nabla}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V & \xrightarrow{N\otimes \mathrm{id}} & \mathrm{B}^{\nabla}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V & \longrightarrow & 0 \\
 & & \uparrow \cong & & \alpha^{\nabla}_{\mathrm{st}}(V) \uparrow \cong & & \alpha^{\nabla}_{\mathrm{st}}(V) \uparrow \cong & & \\
0 & \longrightarrow & \mathrm{Ker}(f) & \longrightarrow & \mathrm{B}^{\nabla}_{\mathrm{st}} \otimes_{K_0} D^{\nabla}_{\mathrm{st}}(V) & \xrightarrow{f} & \mathrm{B}^{\nabla}_{\mathrm{st}} \otimes_{K_0} D^{\nabla}_{\mathrm{st}}(V) & \longrightarrow & 0
\end{array}$$

where $\mathrm{B}^{\nabla}_{\max} = \mathrm{B}^{\nabla}_{\max}(R)$, $\mathrm{B}^{\nabla}_{\mathrm{st}} = \mathrm{B}^{\nabla}_{\mathrm{st}}(R)$, and f denotes $N \otimes \mathrm{id} + \mathrm{id} \otimes N$. Since the monodromy operator N on $D^{\nabla}_{\mathrm{st}}(V)$ is assumed to be zero, we conclude that $\mathrm{Ker}\, f = \mathrm{B}^{\nabla}_{\max}(R) \otimes_{K_0} D^{\nabla}_{\max}(V)$ and the left vertical isomorphism is $\alpha^{\nabla}_{\max}(V)$. □

DEFINITION 4.13. Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$. We say that V is *horizontal crystalline* if $\alpha^{\nabla}_{\max}(V)$ is an isomorphism. Similarly, we say that V is *horizontal semistable* (respectively, *horizontal de Rham*) if $\alpha^{\nabla}_{\mathrm{st}}(V)$ (respectively, $\alpha^{\nabla}_{\mathrm{dR}}(V)$) is an isomorphism.

See Lemmas 4.9(ii), 4.11, and 4.12 for equivalent definitions. By Lemma 4.4, a horizontal crystalline representation is horizontal semistable, and a horizontal semistable representation is horizontal de Rham.

Remark 4.14. Our definition of horizontal de Rham representations coincides with that of [Bri08, p. 117]. By Lemma 4.11, our definition of horizontal crystalline representations coincides with the definition of horizontal R_0-crystalline representations in [Bri08, p. 117].

Remark 4.15. The full subcategory of horizontal crystalline (respectively, semistable, de Rham) representations of $\mathcal{G}_{R_K}$ is stable under taking tensor products, subquotients, and exterior products. This follows from [Bri06, Proposition 3.5] since the period rings $\mathrm{B}^{\nabla}_{\max}(R), \mathrm{B}^{\nabla}_{\mathrm{st}}(R)$, and $\mathrm{B}^{\nabla}_{\mathrm{dR}}(R)$ all satisfy the assumptions of the proposition.

LEMMA 4.16. *The full subcategory of horizontal de Rham representations of $\mathcal{G}_{R_K}$ is stable under taking duals. If R satisfies condition (BR), then the full subcategory of horizontal crystalline (respectively, semistable) representations of $\mathcal{G}_{R_K}$ is stable under taking duals.*

Proof. By [Bri06, Proposition 3.7], it suffices to show the assertion for one-dimensional representations. For horizontal de Rham representations (respectively, horizontal crystalline representations under condition (BR)), it is [Bri08, Theorem 8.4.2]. In general, if $D^{\nabla}_{\mathrm{st}}(V)$ is one-dimensional over K_0 for $V \in \mathrm{Rep}_{\mathbb{Q}_P}(\mathcal{G}_{R_K})$, then $N = 0$ by the relation $N\varphi = p\varphi N$. In particular, every one-dimensional horizontal semistable representation is horizontal crystalline, and thus so is its dual. □

PROPOSITION 4.17. *The sequence*

$$0 \to \mathbb{Q}_p \to \mathrm{B}^{\nabla}_{\max}(R)^{\varphi=1} \to \mathrm{B}^{\nabla}_{\mathrm{dR}}(R)/\mathrm{B}^{\nabla+}_{\mathrm{dR}}(R) \to 0$$

is exact.

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

Proof. Since $\varphi(\mathrm{B}^{\nabla}_{\max}(R)) \subset \mathrm{B}^{\nabla}_{\mathrm{cris}}(R) \subset \mathrm{B}^{\nabla}_{\max}(R)$, we get $\mathrm{B}^{\nabla}_{\mathrm{cris}}(R)^{\varphi=1} = \mathrm{B}^{\nabla}_{\max}(R)^{\varphi=1}$. Hence, the assertion is [Bri08, Proposition 6.2.24]. □

The exact sequence in the proposition is called *the fundamental exact sequence*.

4.3 Functoriality

Let R be an $\mathcal{O}_K$-algebra satisfying the conditions in Set-up 3.1 and fix an algebraic closure $\overline{\mathrm{Frac}\, R}$ of $\mathrm{Frac}\, R$ as before. Let us discuss functorial aspects of our formalism in two special cases.

Let R' be another $\mathcal{O}_R$-algebra satisfying the conditions in Set-up 3.1 and let $f\colon R \to R'$ be an $\mathcal{O}_K$-algebra homomorphism. Recall that

$$\mathcal{G}_{R_K} = \mathrm{Gal}(\overline{R}[p^{-1}]/R_K) = \pi_1^{\text{ét}}(\mathrm{Spec}\, R_K, \overline{\mathrm{Frac}\, R}).$$

Note that $R_K \to R'_K \to \overline{\mathrm{Frac}\, R'}$ defines another geometric point of $\mathrm{Spec}\, R_K$. Choose a path on $\mathrm{Spec}\, R_K$ from $\overline{\mathrm{Frac}\, R'}$ to $\overline{\mathrm{Frac}\, R}$ (cf. [SGA1, Exposé V, 7]). Concretely, this is equivalent to choosing an $\mathcal{O}_K$-algebra homomorphism $\overline{f}\colon \overline{R} \to \overline{R'}$ extending f. The path defines a group homomorphism

$$\begin{aligned}\mathcal{G}_{R'_K} = \pi_1^{\text{ét}}(\mathrm{Spec}\, R'_K, \overline{\mathrm{Frac}\, R'}) &\xrightarrow{\mathrm{Spec}\, f_*} \pi_1^{\text{ét}}(\mathrm{Spec}\, R_K, \overline{\mathrm{Frac}\, R'}) \\ &\longrightarrow \pi_1^{\text{ét}}(\mathrm{Spec}\, R_K, \overline{\mathrm{Frac}\, R}) = \mathcal{G}_{R_K}.\end{aligned}$$

In what follows (e.g. Proposition 4.20), we always choose a path so that $\overline{f}$ is an $\mathcal{O}_{\overline{K}}$-algebra homomorphism. Then $\overline{f}$ gives rise to a homomorphism $(\widehat{\overline{R}}[p^{-1}], \widehat{\overline{R}}) \to (\widehat{\overline{R'}}[p^{-1}], \widehat{\overline{R'}})$ of C-Banach pairs and induces homomorphisms

$$\mathrm{B}^{\nabla}_{\max}(R) \to \mathrm{B}^{\nabla}_{\max}(R'), \quad \mathrm{B}^{\nabla}_{\mathrm{st}}(R) \to \mathrm{B}^{\nabla}_{\mathrm{st}}(R'), \quad \text{and} \quad \mathrm{B}^{\nabla}_{\mathrm{dR}}(R) \to \mathrm{B}^{\nabla}_{\mathrm{dR}}(R').$$

Moreover, they are compatible with $\mathcal{G}_{R'_K} \to \mathcal{G}_{R_K}$.

LEMMA 4.18. *If a $\mathbb{Q}_p$-representation V of $\mathcal{G}_{R_K}$ is horizontal crystalline (respectively, semistable, de Rham), then so is the representation $V|_{\mathcal{G}_{R'_K}}$ of $\mathcal{G}_{R'_K}$.*

Proof. This follows from Lemma 4.4. □

We also consider the case of changing the base field. Let L be a finite extension of K, and set $R_{\mathcal{O}_L} := R \otimes_{\mathcal{O}_K} \mathcal{O}_L$ and $R_L := R_{\mathcal{O}_L}[p^{-1}] = R_K \otimes_K L$. Observe that $R_{\mathcal{O}_L}$ satisfies the conditions in Set-up 3.1 with respect to L. Since $\overline{\mathrm{Frac}\, R}$ is an algebraic closure of $\mathrm{Frac}\, R_{\mathcal{O}_L}$, we have $\overline{R_{\mathcal{O}_L}} = \overline{R}$. It follows that $\mathcal{G}_{R_L}$ is an open subgroup of $\mathcal{G}_{R_K}$ of finite index.

LEMMA 4.19. *Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$.*

(i) *If V is horizontal crystalline, then the representation $V|_{\mathcal{G}_{R_L}}$ of $\mathcal{G}_{R_L}$ is also horizontal crystalline. Similarly, if V is horizontal semistable, then so is $V|_{\mathcal{G}_{R_L}}$.*

(ii) *V is horizontal de Rham if and only if $V|_{\mathcal{G}_{R_L}}$ is horizontal de Rham.*

Proof. Part (i) and the necessity of part (ii) follow from Lemma 4.4. Assume that $V|_{\mathcal{G}_{R_L}}$ is horizontal de Rham. Using the necessity, we may reduce to the case where L is Galois over K. Then $\mathcal{G}_{R_L}$ is a normal subgroup of $\mathcal{G}_{R_K}$ with quotient $\mathrm{Gal}(L/K)$. Since $\mathrm{B}^{\nabla}_{\mathrm{dR}}(R) = \mathrm{B}^{\nabla}_{\mathrm{dR}}(R_{\mathcal{O}_L})$, the finite-dimensional L-vector space $D^{\nabla}_{\mathrm{dR}}(V|_{\mathcal{G}_{R_{\mathcal{O}_L}}})$ has a semilinear $\mathrm{Gal}(L/K)$-action and $D^{\nabla}_{\mathrm{dR}}(V) = D^{\nabla}_{\mathrm{dR}}(V|_{\mathcal{G}_{R_L}})^{\mathrm{Gal}(L/K)}$.

By Galois descent, we have $L \otimes_K D^{\nabla}_{\mathrm{dR}}(V) \xrightarrow{\cong} D^{\nabla}_{\mathrm{dR}}(V|_{\mathcal{G}_{R_L}})$. It follows that $\alpha^{\nabla}_{\mathrm{dR}}(V) = \alpha^{\nabla}_{\mathrm{dR}}(V|_{\mathcal{G}_{R_L}})$ and it is an isomorphism by assumption. □

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

By combining these two cases, we can consider the following setting.

Proposition 4.20. *Let L be a finite extension of K and R' an $\mathcal{O}_L$-algebra satisfying the conditions in Set-up 3.1 with respect to L. Let $f\colon R \to R'$ be an $\mathcal{O}_K$-algebra homomorphism. Fix a path on* $\operatorname{Spec} R_K$ *from* $\overline{\operatorname{Frac} R'}$ *to* $\overline{\operatorname{Frac} R}$ *and consider the associated homomorphism $\mathcal{G}_{R'_L} \to \mathcal{G}_{R_K}$. Let $V \in \operatorname{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$ and set $f^*V := V|_{\mathcal{G}_{R'_L}}$. If V is horizontal semistable and f^*V is horizontal crystalline, then V is horizontal crystalline.*

Proof. Denote the residue field of L by k_L and set $L_0 = W(k_L)[p^{-1}]$. By Lemma 4.4, we have $D^{\nabla}_{\mathrm{st}}(f^*V) \cong L_0 \otimes_{K_0} D^{\nabla}_{\mathrm{st}}(V)$. Moreover, it follows from the construction that this isomorphism is compatible with the monodromy operator N on the left-hand side and $\mathrm{id} \otimes N$ on the right-hand side. Since f^*V is horizontal crystalline, we have $N = 0$ on $D^{\nabla}_{\mathrm{st}}(f^*V)$. By faithful flatness of $K_0 \to L_0$, we also have $N = 0$ on $D^{\nabla}_{\mathrm{st}}(V)$. Hence, V is horizontal crystalline by Lemma 4.12. □

Remark 4.21. We will use the proposition when $R' = \mathcal{O}_L$. In this case, it is informally phrased as follows: if a horizontal semistable representation is crystalline at one classical point, then it is horizontal crystalline.

4.4 Horizontal de Rham representations and de Rham representations

In this subsection, we review the de Rham period ring studied in [Bri08]. This ring will be used in the next section to define the V_{st}-functor. Let R be an $\mathcal{O}_K$-algebra satisfying the conditions in Set-up 3.1. Set $\widehat{\Omega}_R := \varprojlim_m \Omega^1_{R/\mathbb{Z}}/p^m\Omega^1_{R/\mathbb{Z}}$.

Definition 4.22. Let $\mathrm{B}_{\mathrm{dR}}(R)$ denote the de Rham period ring of R defined in [Bri08, Définition 5.1.5]. We briefly recall the definition below.

The map $\theta_{\mathbb{A}_{\mathrm{inf}}(\widehat{R})}\colon \mathbb{A}_{\mathrm{inf}}(\widehat{R}) \to \widehat{R}$ extends R-linearly to $\theta_R\colon R \otimes_{\mathbb{Z}} \mathbb{A}_{\mathrm{inf}}(\widehat{R}) \to \widehat{R}$. Let $\mathrm{A}_{\mathrm{inf}}(\widehat{R}/R)$ denote the $\theta_R^{-1}(p\widehat{R})$-adic completion of $R \otimes_{\mathbb{Z}} \mathbb{A}_{\mathrm{inf}}(\widehat{R})$ [Bri08, Définition 5.1.3]. It is equipped with a map $\theta_R\colon \mathrm{A}_{\mathrm{inf}}(\widehat{R}/R)[p^{-1}] \to \widehat{R}[p^{-1}]$. We set

$$\mathrm{B}^+_{\mathrm{dR}}(R) := \varprojlim_m \mathrm{A}_{\mathrm{inf}}(\widehat{R}/R)[p^{-1}]/(\operatorname{Ker}\theta_R)^m,$$

and denote the natural map $\mathrm{B}^+_{\mathrm{dR}}(R) \to \widehat{R}[p^{-1}]$ by θ_R. For $m \in \mathbb{N}$, we set $\operatorname{Fil}^m \mathrm{B}^+_{\mathrm{dR}}(R) := (\operatorname{Ker}\theta_R)^m$. Define $\mathrm{B}_{\mathrm{dR}}(R) := \mathrm{B}^+_{\mathrm{dR}}(R)[t^{-1}]$. The ring has an action of $\mathcal{G}_{R_K}$, a decreasing separated and exhaustive filtration $\operatorname{Fil}^\bullet$ given by

$$\operatorname{Fil}^0 \mathrm{B}_{\mathrm{dR}}(R) = \sum_{i=0}^{\infty} t^{-i} \operatorname{Fil}^i \mathrm{B}^+_{\mathrm{dR}}(R) \quad \text{and} \quad \operatorname{Fil}^m \mathrm{B}_{\mathrm{dR}}(R) = t^m \operatorname{Fil}^0 \mathrm{B}_{\mathrm{dR}}(R) \quad (m \in \mathbb{Z}),$$

and a $\mathrm{B}^{\nabla}_{\mathrm{dR}}(R)$-linear integrable connection $\nabla\colon \mathrm{B}_{\mathrm{dR}}(R) \to \mathrm{B}_{\mathrm{dR}}(R) \otimes_R \widehat{\Omega}_R$ satisfying the Griffiths transversality $\nabla(\operatorname{Fil}^m \mathrm{B}_{\mathrm{dR}}(R)) \subset \operatorname{Fil}^{m-1} \mathrm{B}_{\mathrm{dR}}(R) \otimes_R \widehat{\Omega}_R$. We also have $(\mathrm{B}_{\mathrm{dR}}(R))^{\mathcal{G}_{R_K}} = R_K$ and $\mathrm{B}^{\nabla}_{\mathrm{dR}}(R) = \mathrm{B}_{\mathrm{dR}}(R)^{\nabla=0}$ [Bri08, Propositions 5.2.12, 5.3.3]. The ring $\mathrm{B}_{\mathrm{dR}}(R)$ is a $(\mathbb{Q}_p, \mathcal{G}_{R_K})$-ring.

Definition 4.23. For $V \in \operatorname{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$, set $D_{\mathrm{dR}}(V) := D_{\mathrm{B}_{\mathrm{dR}}(R)}(V)$ and $\alpha_{\mathrm{dR}}(V) := \alpha_{\mathrm{B}_{\mathrm{dR}}(R)}(V)$. We say that V is *de Rham* if it is $\mathrm{B}_{\mathrm{dR}}(R)$-admissible, that is, $\alpha_{\mathrm{dR}}(V)$ is an isomorphism. Note that $\alpha_{\mathrm{dR}}(V)$ is always injective by [Bri08, Proposition 8.2.4].

The R_K-module $D_{\mathrm{dR}}(V)$ is projective of rank at most $\dim_{\mathbb{Q}_p} V$ by [Bri08, Proposition 8.3.1], and it is equipped with induced connection $\nabla\colon D_{\mathrm{dR}}(V) \to D_{\mathrm{dR}}(V) \otimes_R \widehat{\Omega}_R$

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

and filtration $\mathrm{Fil}^\bullet D_{\mathrm{dR}}(V)$. We also have a natural R_K-linear map

$$\beta_{\mathrm{dR}}(V)\colon R_K \otimes_K D^\nabla_{\mathrm{dR}}(V) \to D_{\mathrm{dR}}(V).$$

LEMMA 4.24 [Bri08, Proposition 8.2.10]. *For $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$, it is horizontal de Rham if and only if it is de Rham and $\beta_{\mathrm{dR}}(V)$ is an R_K-linear isomorphism.*

LEMMA 4.25 [Bri08, Proposition 8.4.3]. *Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$. If V is de Rham, then the isomorphism $\alpha_{\mathrm{dR}}(V)\colon \mathrm{B}_{\mathrm{dR}}(R) \otimes_{R_K} D_{\mathrm{dR}}(V) \xrightarrow{\cong} \mathrm{B}_{\mathrm{dR}}(R) \otimes_{\mathbb{Q}_p} V$ is compatible with $\mathcal{G}_{R_K}$-actions and connections, and strictly compatible with filtrations.*

LEMMA 4.26 [Bri08, Corollary 5.2.11]. *For $m \in \mathbb{Z}$, we have*

$$\mathrm{B}^\nabla_{\mathrm{dR}}(R) \cap \mathrm{Fil}^m \mathrm{B}_{\mathrm{dR}}(R) = t^m \mathrm{B}^{\nabla+}_{\mathrm{dR}}(R).$$

Remark 4.27. Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$ be a horizontal de Rham representation. The filtration $\mathrm{Fil}^m \mathrm{B}^\nabla_{\mathrm{dR}}(R) := t^m \mathrm{B}^{\nabla+}_{\mathrm{dR}}(R)$ on $\mathrm{B}^\nabla_{\mathrm{dR}}(R)$ induces a filtration on $D^\nabla_{\mathrm{dR}}(V)$. The maps $\alpha^\nabla_{\mathrm{dR}}(V)$ and $\beta_{\mathrm{dR}}(V)$ preserve filtrations, that is,

$$\alpha^\nabla_{\mathrm{dR}}(V)(\mathrm{Fil}^m(\mathrm{B}^\nabla_{\mathrm{dR}}(R) \otimes_K D^\nabla_{\mathrm{dR}}(V))) \subset \mathrm{Fil}^m \mathrm{B}^\nabla_{\mathrm{dR}}(R) \otimes_{\mathbb{Q}_p} V,$$
$$\beta_{\mathrm{dR}}(V)(\mathrm{Fil}^m(R_K \otimes_K D^\nabla_{\mathrm{dR}}(V))) \subset \mathrm{Fil}^m D_{\mathrm{dR}}(V).$$

However, neither $\alpha^\nabla_{\mathrm{dR}}(V)$ nor $\beta_{\mathrm{dR}}(V)$ is strictly compatible with filtrations in general, and the claim on $\alpha^\nabla_{\mathrm{dR}}(V)$ in [Bri08, Proposition 8.4.3] is incorrect. See the following example.

Example 4.28. Assume $K = K_0$ for simplicity. Let $R = \mathcal{O}_K\langle T^{\pm}\rangle$ and let $h(T) = 1 + pT \in R$. Fix a compatible system $(h, h^{1/p}, \ldots)$ of p-power roots of h and set $h^\flat := (h \bmod p, h^{1/p} \bmod p, \ldots) \in \widehat{\overline{R}}^\flat$.

Let $V = \mathbb{Q}_p e_1 + \mathbb{Q}_p e_2$ be a representation of $\mathcal{G}_{R_K}$ corresponding to $(\varepsilon, h^\flat)$; $g \in \mathcal{G}_{R_K}$ acts on V by $ge_1 = \chi(g)e_1$ and $ge_2 = \eta(g)e_1 + e_2$, where $\chi\colon \mathcal{G}_{R_K} \to \mathrm{Gal}(\overline{K}/K) \to \mathbb{Z}_p^\times$ is the p-adic cyclotomic character defined by $g(\varepsilon_m) = \varepsilon_m^{\chi(g)}$, and $\eta\colon \mathcal{G}_{R_K} \to \mathbb{Z}_p$ is a map defined by $g(h^{1/p^m}) = \varepsilon_m^{\eta(g)} h^{1/p^m}$.

We show that V is horizontal crystalline. Note that the series $\log[h^\flat] := \sum_{m=1}^\infty (-1)^{m-1}(([h^\flat] - 1)^m/m)$ converges in $\mathrm{A}^\nabla_{\max}(R) := \mathbb{A}_{\max}(\widehat{\overline{R}})$. Moreover, $\log[h^\flat] \in \mathrm{Fil}^0 \mathrm{B}^\nabla_{\mathrm{dR}}(R) \smallsetminus \mathrm{Fil}^1 \mathrm{B}^\nabla_{\mathrm{dR}}(R)$. Set

$$f_1 = \frac{1}{t} e_1, \quad f_2 = e_2 - \frac{\log[h^\flat]}{t} e_1 \in \mathrm{B}^\nabla_{\max}(R) \otimes_{\mathbb{Q}_p} V.$$

Then we see that V is horizontal crystalline with $D^\nabla_{\max}(V) = Kf_1 + Kf_2$. Note that $D^\nabla_{\mathrm{dR}}(V) = D^\nabla_{\max}(V)$, and if we equip $D^\nabla_{\mathrm{dR}}(V)$ with a decreasing filtration induced from $\mathrm{Fil}^m(\mathrm{B}^\nabla_{\mathrm{dR}}(R) \otimes_{\mathbb{Q}_p} V) := t^m \mathrm{B}^{\nabla+}_{\mathrm{dR}}(R) \otimes_{\mathbb{Q}_p} V$ we have

$$\mathrm{Fil}^{-1} D^\nabla_{\mathrm{dR}}(V) = D^\nabla_{\mathrm{dR}}(V), \quad \mathrm{Fil}^0 D^\nabla_{\mathrm{dR}}(V) = 0.$$

In particular, the $\mathrm{B}^\nabla_{\mathrm{dR}}(R)$-linear isomorphism

$$\alpha^\nabla_{\mathrm{dR}}(V)\colon \mathrm{B}^\nabla_{\mathrm{dR}}(R) \otimes_K D^\nabla_{\mathrm{dR}}(V) \xrightarrow{\cong} \mathrm{B}^\nabla_{\mathrm{dR}}(R) \otimes_{\mathbb{Q}_p} V$$

is not strictly compatible with filtrations. Indeed, we have $e_2 = 1 \otimes f_2 + \log[h^\flat] \otimes f_1 \notin \mathrm{Fil}^0(\mathrm{B}^\nabla_{\mathrm{dR}}(R) \otimes_K D^\nabla_{\mathrm{dR}}(V))$ while $e_2 \in \mathrm{Fil}^0(\mathrm{B}^\nabla_{\mathrm{dR}}(R) \otimes_{\mathbb{Q}_p} V)$.

Let us turn to $D_{\mathrm{dR}}(V) = R_K f_1 + R_K f_2$. We will give another R_K-basis. Since h is invertible in R and $[h^\flat]/h - 1 \in \mathrm{Ker}\,\theta_R$, the series $\log([h^\flat]/h) := \sum_{m=1}^\infty (-1)^{m-1}(([h^\flat]/h - 1)^m/m)$ converges in $\mathrm{B}^+_{\mathrm{dR}}(R)$. Let $\log h := \sum_{m=1}^\infty (-1)^{m-1}((h-1)^m/m) = \sum_{m=1}^\infty (-1)^{m-1}(p^m/m)T^m \in R$.

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

We can check $\log[h^\flat] = \log([h^\flat]/h) + \log h$ in $\mathrm{B}^+_{\mathrm{dR}}(R)$. Set

$$f_3 = e_2 - \frac{\log[h^\flat]/h}{t} e_1 = f_2 + \frac{\log h}{t} e_1 \in \mathrm{B}_{\mathrm{dR}}(R) \otimes_{\mathbb{Q}_p} V.$$

Then f_3 is fixed by $\mathcal{G}_{R_K}$, and $D_{\mathrm{dR}}(V) = R_K f_1 + R_K f_3$. Consider the filtration on $D_{\mathrm{dR}}(V)$ induced from $\mathrm{Fil}^m(\mathrm{B}_{\mathrm{dR}}(R) \otimes_{\mathbb{Q}_p} V) := (\mathrm{Fil}^m \mathrm{B}_{\mathrm{dR}}(R)) \otimes_{\mathbb{Q}_p} V$. Since $\log([h^\flat]/h) \in \mathrm{Fil}^1 \mathrm{B}_{\mathrm{dR}}(R)$, we see that

$$\mathrm{Fil}^{-1} D_{\mathrm{dR}}(V) = D_{\mathrm{dR}}(V), \quad \mathrm{Fil}^0 D_{\mathrm{dR}}(V) = R_K f_3, \quad \mathrm{Fil}^1 D_{\mathrm{dR}}(V) = 0.$$

It follows that the natural R_K-linear isomorphism

$$\beta_{\mathrm{dR}}(V) \colon R_K \otimes_K D^\nabla_{\mathrm{dR}}(V) \xrightarrow{\cong} D_{\mathrm{dR}}(V)$$

is not strictly compatible with filtrations. Note that the $\mathrm{B}_{\mathrm{dR}}(R)$-linear isomorphism

$$\alpha_{\mathrm{dR}}(V) \colon \mathrm{B}_{\mathrm{dR}}(R) \otimes_{R_K} D_{\mathrm{dR}}(V) \xrightarrow{\cong} \mathrm{B}_{\mathrm{dR}}(R) \otimes_{\mathbb{Q}_p} V$$

is strictly compatible with filtrations.

5. Filtered $(\varphi, N, \mathrm{Gal}(L/K), R_K)$-modules

In this section, we follow [Fon94b] and define filtered $(\varphi, N, \mathrm{Gal}(L/K), R_K)$-modules and the functors D_{pst} and V_{pst}.

As before, let k be a perfect field of characteristic p and set $K_0 := W(k)[p^{-1}]$. Let K be a totally ramified finite extension of K_0. We denote the maximal unramified extension of K_0 inside $\overline{K}$ by K_0^{ur}. Let L be a Galois extension of K inside $\overline{K}$. Let L_0 be the maximal unramified extension of K_0 in L and σ the absolute Frobenius on L_0. Note that $(\overline{K})_0 = K_0^{\mathrm{ur}}$.

Let us first recall the definition of $(\varphi, N, \mathrm{Gal}(L/K))$-modules.

Definition 5.1 [Fon94b, 4.2.1]. A $(\varphi, N, \mathrm{Gal}(L/K))$-*module* is a finite-dimensional L_0-vector space D equipped with

(i) an injective σ-semilinear map $\varphi \colon D \to D$ (*Frobenius*),
(ii) an L_0-linear endomorphism $N \colon D \to D$ (*monodromy operator*), and
(iii) a semilinear action of $\mathrm{Gal}(L/K)$,

such that they satisfy the compatibilities

– $N\varphi = p\varphi N$,
– for every $g \in \mathrm{Gal}(L/K)$, $g\varphi = \varphi g$ and $gN = Ng$.

When $L = K$, we simply call it a (φ, N)-module.

A morphism of $(\varphi, N, \mathrm{Gal}(L/K))$-modules is an L_0-linear map that commutes with φ, N, and $\mathrm{Gal}(L/K)$-action.

We say that a $(\varphi, N, \mathrm{Gal}(L/K))$-module D is *discrete* if the action of $\mathrm{Gal}(L/K)$ on D is discrete, that is, for every $d \in D$, the stabilizer subgroup of d is open in $\mathrm{Gal}(L/K)$.

Let R be an $\mathcal{O}_K$-algebra satisfying the conditions in Set-up 3.1. We set $R_{\mathcal{O}_L} := \mathcal{O}_L \otimes_{\mathcal{O}_K} R$ and $R_L := L \otimes_K R_K$. Note that $R_{\mathcal{O}_L}$ is p-adically complete if L is finite over K but that this is not the case in general. For an L_0-vector space D, we write D_{R_L} for $R_L \otimes_{L_0} D$.

Definition 5.2 (cf. [Fon94b, 4.3.2]). A *filtered* $(\varphi, N, \mathrm{Gal}(L/K), R_K)$-*module* is a pair consisting of

(i) a $(\varphi, N, \mathrm{Gal}(L/K))$-module D and
(ii) a $\mathrm{Gal}(L/K)$-stable separated and exhaustive decreasing filtration $(\mathrm{Fil}^i D_{R_L})_{i \in \mathbb{Z}}$ of D_{R_L},

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

such that each $\mathrm{gr}^i D_{R_L}$ is a projective R_L-submodule of D_{R_L}. When $L = K$, we simply call it a (φ, N, R_K)-module.

A morphism of filtered $(\varphi, N, \mathrm{Gal}(L/K), R_K)$-modules is a morphism of $(\varphi, N, \mathrm{Gal}(L/K))$-modules whose scalar extension from L_0 to R_L preserves filtrations. A filtered $(\varphi, N, \mathrm{Gal}(L/K), R_K)$-module is called *discrete* if it is discrete as a $(\varphi, N, \mathrm{Gal}(L/K))$-module.

We write $\mathrm{MF}_{L/K}(\varphi, N, R_K)$ for the full subcategory of discrete filtered $(\varphi, N, \mathrm{Gal}(L/K), R_K)$-modules.

PROPOSITION 5.3. *Let L be a finite Galois extension of K and let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$. If $V|_{\mathcal{G}_{R_L}}$ is horizontal semistable, then $D^\nabla_{\mathrm{st}}(V|_{\mathcal{G}_{R_L}})$ is a $(\varphi, N, \mathrm{Gal}(L/K))$-module. Moreover, if we equip $D^\nabla_{\mathrm{st}}(V|_{\mathcal{G}_{RL}})$ with the filtration induced from $D_{\mathrm{dR}}(V|_{\mathcal{G}_{R_L}})$ via the isomorphism*

$$R_L \otimes_{L_0} D^\nabla_{\mathrm{st}}(V|_{\mathcal{G}_{R_L}}) \xrightarrow{\cong} D_{\mathrm{dR}}(V|_{\mathcal{G}_{R_L}}),$$

then $D^\nabla_{\mathrm{st}}(V|_{\mathcal{G}_{R_L}})$ is a filtered $(\varphi, N, \mathrm{Gal}(L/K), R_K)$-module.

Proof. Recall that $\mathrm{B}^\nabla_{\mathrm{st}}(R_{\mathcal{O}_L}) \otimes_{\mathbb{Q}_p} V$ admits an injective σ-semilinear map $\varphi \otimes \mathrm{id}$ and an L_0-linear endomorphism $N \otimes \mathrm{id}$ satisfying $(N \otimes \mathrm{id})(\varphi \otimes \mathrm{id}) = p(\varphi \otimes \mathrm{id})(N \otimes \mathrm{id})$. Moreover, by the identification $\mathrm{B}^\nabla_{\mathrm{st}}(R_{\mathcal{O}_L}) \otimes_{\mathbb{Q}_p} V = \mathrm{B}^\nabla_{\mathrm{st}}(R) \otimes_{\mathbb{Q}_p} V$, it is equipped with diagonal semilinear $\mathcal{G}_{R_K}$-action commuting with $\varphi \otimes \mathrm{id}$ and $N \otimes \mathrm{id}$. The first assertion follows from these remarks and $\mathcal{G}_{R_K}/\mathcal{G}_{R_L} \cong \mathrm{Gal}(L/K)$. The second assertion follows from the first and [Bri08, Proposition 8.3.4(a)]. □

DEFINITION 5.4. Assume that L is finite over K. Recall the natural identifications $\mathrm{B}^\nabla_{\mathrm{st}}(R_{\mathcal{O}_L}) = \mathrm{B}^\nabla_{\mathrm{st}}(R)$ and $\mathrm{B}^\nabla_{\mathrm{dR}}(R_{\mathcal{O}_L}) = \mathrm{B}^\nabla_{\mathrm{dR}}(R)$. For a filtered $(\varphi, N, \mathrm{Gal}(L/K), R_K)$-module D, set

$$V_{\mathrm{st},L}(D) := (\mathrm{B}^\nabla_{\mathrm{st}}(R) \otimes_{L_0} D)^{\varphi=1, N=0} \cap \mathrm{Fil}^0(\mathrm{B}_{\mathrm{dR}}(R) \otimes_{R_L} D_{R_L}),$$

where φ (respectively, N) on $\mathrm{B}^\nabla_{\mathrm{st}}(R) \otimes_{L_0} D$ denotes $\varphi \otimes \varphi$ (respectively, $N \otimes \mathrm{id} + \mathrm{id} \otimes N$) and $\mathrm{Fil}^\bullet$ on $\mathrm{B}_{\mathrm{dR}}(R) \otimes_{R_L} D_{R_L}$ denotes $\sum_{m \in \mathbb{Z}} \mathrm{Fil}^m \mathrm{B}_{\mathrm{dR}}(R) \otimes_{R_L} \mathrm{Fil}^{\bullet - m} D_{R_L}$. It is a $\mathbb{Q}_p$-vector space equipped with a $\mathbb{Q}_p$-linear action of $\mathcal{G}_{R_K}$.

PROPOSITION 5.5. *Let L be a finite Galois extension of K and let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$. If $V|_{\mathcal{G}_{R_L}}$ is horizontal semistable, then the natural map $V \to V_{\mathrm{st},L}(D^\nabla_{\mathrm{st}}(V|_{\mathcal{G}_{R_L}}))$ is a $\mathcal{G}_{R_K}$-equivariant isomorphism.*

Proof. We use the identification $\mathrm{B}^\nabla_{\mathrm{st}}(R_{\mathcal{O}_L}) = \mathrm{B}^\nabla_{\mathrm{st}}(R)$. For simplicity, we write V_L for $V|_{\mathcal{G}_{R_L}}$. Since V_L is horizontal semistable, we have the following commutative diagram.

$$\begin{array}{ccc} \mathrm{B}^\nabla_{\mathrm{st}}(R) \otimes_{L_0} D^\nabla_{\mathrm{st}}(V_L) & \xrightarrow[\cong]{\alpha^\nabla_{\mathrm{st}}(V_L)} & \mathrm{B}^\nabla_{\mathrm{st}}(R) \otimes_{\mathbb{Q}_p} V \\ \downarrow & & \downarrow \\ \mathrm{B}_{\mathrm{dR}}(R) \otimes_{R_L} D_{\mathrm{dR}}(V_L) & \xrightarrow[\cong]{\alpha_{\mathrm{dR}}(V_L)} & \mathrm{B}_{\mathrm{dR}}(R) \otimes_{\mathbb{Q}_p} V \end{array}$$

Furthermore, the isomorphism $\alpha^\nabla_{\mathrm{st}}(V_L)$ is compatible with $\mathcal{G}_{R_K}$-actions, Frobenii, and monodromy operators, and the isomorphism $\alpha_{\mathrm{dR}}(V_L)$ is compatible with $\mathcal{G}_{R_K}$-actions and strictly compatible with filtrations. Note also that $D^\nabla_{\mathrm{st}}(V_L)_{R_L} \cong D_{\mathrm{dR}}(V_L)$. Hence, we have

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

$\mathcal{G}_{R_K}$-equivariant isomorphisms

$$\begin{aligned} V_{\mathrm{st}}(D^{\nabla}_{\mathrm{st}}(V_L)) &= (\mathrm{B}^{\nabla}_{\mathrm{st}}(R) \otimes_{L_0} D^{\nabla}_{\mathrm{st}}(V_L))^{\varphi=1,N=0} \cap \mathrm{Fil}^0(\mathrm{B}_{\mathrm{dR}}(R) \otimes_{R_L} D^{\nabla}_{\mathrm{st}}(V_L)_{R_L}) \\ &\cong (\mathrm{B}^{\nabla}_{\mathrm{st}}(R) \otimes_{L_0} D^{\nabla}_{\mathrm{st}}(V_L))^{\varphi=1,N=0} \cap \mathrm{Fil}^0(\mathrm{B}_{\mathrm{dR}}(R) \otimes_{R_L} D_{\mathrm{dR}}(V_L)) \\ &\cong (\mathrm{B}^{\nabla}_{\mathrm{st}}(R) \otimes_{\mathbb{Q}_p} V)^{\varphi=1,N=0} \cap \mathrm{Fil}^0(\mathrm{B}_{\mathrm{dR}}(R) \otimes_{\mathbb{Q}_p} V) \\ &\cong \left((\mathrm{B}^{\nabla}_{\mathrm{st}}(R))^{\varphi=1,N=0} \cap \mathrm{Fil}^0(\mathrm{B}_{\mathrm{dR}}(R))\right) \otimes_{\mathbb{Q}_p} V. \end{aligned}$$

By Lemma 4.26 and Proposition 4.17, we have

$$\begin{aligned} (\mathrm{B}^{\nabla}_{\mathrm{st}}(R))^{\varphi=1,N=0} \cap \mathrm{Fil}^0(\mathrm{B}_{\mathrm{dR}}(R)) &= (\mathrm{B}^{\nabla}_{\mathrm{st}}(R))^{\varphi=1,N=0} \cap (\mathrm{B}^{\nabla}_{\mathrm{dR}}(R) \cap \mathrm{Fil}^0(\mathrm{B}_{\mathrm{dR}}(R))) \\ &= (\mathrm{B}^{\nabla}_{\mathrm{max}}(R))^{\varphi=1} \cap \mathrm{B}^{\nabla+}_{\mathrm{dR}}(R) = \mathbb{Q}_p. \end{aligned}$$

Hence, $((\mathrm{B}^{\nabla}_{\mathrm{st}}(R))^{\varphi=1,N=0} \cap \mathrm{Fil}^0(\mathrm{B}_{\mathrm{dR}}(R))) \otimes_{\mathbb{Q}_p} V$ is $\mathcal{G}_{R_K}$-equivariantly isomorphic to V. □

Definition 5.6. For $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$, set

$$D^{\nabla}_{\mathrm{pst}}(V) := \varinjlim_{L} D^{\nabla}_{\mathrm{st}}(V|_{\mathcal{G}_{R_L}}),$$

where L ranges over all the finite Galois extensions of K inside $\overline{K}$. It is a vector space over K_0^{ur} equipped with a Frobenius φ, a monodromy operator N, and a semilinear action of $\mathrm{Gal}(\overline{K}/K)$. By unwinding the definitions, we see that $D^{\nabla}_{\mathrm{pst}}(V)^{\mathrm{Gal}(\overline{K}/L)} = D^{\nabla}_{\mathrm{st}}(V|_{\mathcal{G}_{R_L}})$ for every finite Galois extension L of K inside $\overline{K}$.

Let $\mathrm{Rep}^{\nabla}_{\mathrm{pst}}(\mathcal{G}_{R_K})$ be the full subcategory of $\mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$ consisting of objects V with the following property: there exists a finite Galois extension L of K such that $V|_{\mathcal{G}_{R_L}}$ is horizontal semistable.

Remark 5.7. Every object of $\mathrm{Rep}^{\nabla}_{\mathrm{pst}}(\mathcal{G}_{R_K})$ is horizontal de Rham by Lemma 4.19(ii). We will prove that $\mathrm{Rep}^{\nabla}_{\mathrm{pst}}(\mathcal{G}_{R_K})$ is exactly the full subcategory of horizontal de Rham representations if R satisfies condition (BR) (Theorem 7.3). In this case, $\mathrm{Rep}^{\nabla}_{\mathrm{pst}}(\mathcal{G}_{R_K})$ is a Tannakian subcategory of $\mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$ by Remark 4.15 and Lemma 4.16.

Lemma 5.8. *Let $V \in \mathrm{Rep}^{\nabla}_{\mathrm{pst}}(\mathcal{G}_{R_K})$. Let L be a finite Galois extension of K inside $\overline{K}$ such that $V|_{\mathcal{G}_{R_L}}$ is horizontal semistable. Then there is a natural identification $D^{\nabla}_{\mathrm{pst}}(V)_{R_{\overline{K}}} \cong R_{\overline{K}} \otimes_L D_{\mathrm{dR}}(V|_{\mathcal{G}_{R_L}})$. Moreover, if we set*

$$\mathrm{Fil}^{\bullet} D^{\nabla}_{\mathrm{pst}}(V)_{R_{\overline{K}}} := R_{\overline{K}} \otimes_{R_L} \mathrm{Fil}^{\bullet} D_{\mathrm{dR}}(V|_{\mathcal{G}_{R_L}}),$$

then $D^{\nabla}_{\mathrm{pst}}(V)$ is a discrete filtered $(\varphi, N, \mathrm{Gal}(\overline{K}/K), R_K)$-module and the filtration is independent of the choice of L.

Proof. For every finite Galois extension L' of K containing L, $V|_{\mathcal{G}_{R_{L'}}}$ is horizontal semistable and we have

$$\begin{aligned} D^{\nabla}_{\mathrm{st}}(V|_{\mathcal{G}_{R_{L'}}}) &\cong L'_0 \otimes_{L_0} D^{\nabla}_{\mathrm{st}}(V|_{\mathcal{G}_{R_L}}), \\ D_{\mathrm{dR}}(V|_{\mathcal{G}_{R_{L'}}}) &\cong R_{L'} \otimes_{R_L} D_{\mathrm{dR}}(V|_{\mathcal{G}_{R_L}}) \cong R_{L'} \otimes_{L_0} D^{\nabla}_{\mathrm{st}}(V|_{\mathcal{G}_{R_L}}). \end{aligned}$$

Moreover, the first isomorphism is compatible with Frobenii, monodromy operators and $\mathrm{Gal}(L'/K)$-actions, and the second isomorphism is compatible with $\mathrm{Gal}(L'/K)$-actions and strictly compatible with filtrations. The assertions follow from these remarks and Proposition 5.3. □

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

DEFINITION 5.9. Let D be a filtered $(\varphi, N, \mathrm{Gal}(\overline{K}/K), R_K)$-module. Set

$$V_{\mathrm{pst}}(D) := (\mathrm{B}_{\mathrm{st}}^{\nabla}(R) \otimes_{K_0^{\mathrm{ur}}} D)^{\varphi=1, N=0} \cap \mathrm{Fil}^0(\mathrm{B}_{\mathrm{dR}}(R) \otimes_{R_{\overline{K}}} D_{R_{\overline{K}}}),$$

where φ (respectively, N) on $\mathrm{B}_{\mathrm{st}}^{\nabla}(R) \otimes_{K_0^{\mathrm{ur}}} D$ denotes $\varphi \otimes \varphi$ (respectively, $N \otimes \mathrm{id} + \mathrm{id} \otimes N$) and $\mathrm{Fil}^{\bullet}$ on $\mathrm{B}_{\mathrm{dR}}(R) \otimes_{R_{\overline{K}}} D_{R_{\overline{K}}}$ denotes $\sum_{m \in \mathbb{Z}} \mathrm{Fil}^m \mathrm{B}_{\mathrm{dR}}(R) \otimes_{R_{\overline{K}}} \mathrm{Fil}^{\bullet - m} D_{R_{\overline{K}}}$. It is a $\mathbb{Q}_p$-vector space equipped with a $\mathbb{Q}_p$-linear action of $\mathcal{G}_{R_K}$.

We say that D is *admissible* if it is isomorphic to $D_{\mathrm{pst}}^{\nabla}(V)$ for some $V \in \mathrm{Rep}_{\mathrm{pst}}^{\nabla}(\mathcal{G}_{R_K})$. Let $\mathrm{MF}_{\overline{K}/K}^{\mathrm{ad}}(\varphi, N, R_K)$ be the full subcategory of $\mathrm{MF}_{\overline{K}/K}(\varphi, N, R_K)$ consisting of admissible objects.

THEOREM 5.10. *Assume that R satisfies condition (BR). Then the functor*

$$D_{\mathrm{pst}}^{\nabla} \colon \mathrm{Rep}_{\mathrm{pst}}^{\nabla}(\mathcal{G}_{R_K}) \to \mathrm{MF}_{\overline{K}/K}^{\mathrm{ad}}(\varphi, N, R_K)$$

is an equivalence of categories with quasi-inverse given by V_{pst}.

Proof. By Proposition 5.5, we see that the $\mathbb{Q}_p$-linear map $V \to V_{\mathrm{pst}}(D_{\mathrm{pst}}(V))$ is a $\mathcal{G}_{R_K}$-equivariant isomorphism. For $V_1, V_2 \in \mathrm{Rep}_{\mathrm{pst}}^{\nabla}(\mathcal{G}_{R_K})$, this result for $V = \mathrm{Hom}(V_1, V_2) \in \mathrm{Rep}_{\mathrm{pst}}^{\nabla}(\mathcal{G}_{R_K})$ implies that D_{pst} is fully faithful. Hence, the assertion follows. □

Remark 5.11. It is desirable to characterize the admissible filtered $(\varphi, N, \mathrm{Gal}(\overline{K}/K), R_K)$-module intrinsically. When $R = K$, Colmez and Fontaine [CF00] prove that admissibility is equivalent to weak admissibility. Results on p-adic period domains suggest that admissibility will not be characterized by weak admissibility after every specialization of R to a finite extension of K (see [Har13, Example 6.7]).

6. Purity for horizontal semistable representations

6.1 Statement

We first recall our set-up. Let k be a perfect field of characteristic p and set $K_0 := W(k)[p^{-1}]$. Let K be a totally ramified finite extension of K_0 and fix a uniformizer π of K. We denote by C the p-adic completion of an algebraic closure $\overline{K}$ of K. Let R be an $\mathcal{O}_K$-algebra satisfying the conditions in Set-up 3.1. We further assume that R satisfies condition (BR) of Definition 3.3. In particular, $(\pi) \subset R$ is a prime ideal.

Consider the localization $R_{(\pi)}$ of R at the prime ideal (π). Let $\mathcal{O}_{\mathcal{K}}$ denote the p-adic completion of $R_{(\pi)}$. It is a complete discrete valuation ring with residue field admitting a finite p-basis. Set $\mathcal{K} := \mathcal{O}_{\mathcal{K}}[p^{-1}]$ and fix an algebraic closure $\overline{\mathcal{K}}$ of $\mathcal{K}$. Let $\mathcal{O}_{\overline{\mathcal{K}}}$ denote the ring of integers of $\overline{\mathcal{K}}$. We denote the p-adic completion of $\mathcal{O}_{\overline{\mathcal{K}}}$ by $\widehat{\mathcal{O}_{\overline{\mathcal{K}}}}$.

Representations of $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$ are studied in [Bri06, Mor14, Ohk13]. In particular, we can discuss whether such a representation is horizontal crystalline, semistable, or de Rham (cf. [Ohk13]). Note that these classes of representations are defined by admissibility of the period rings $\mathbb{B}_{\mathrm{cris}}(\Lambda^+)$ (or equivalently $\mathbb{B}_{\mathrm{max}}(\Lambda^+)$), $\mathbb{B}_{\mathrm{st}}(\Lambda^+)$, and $\mathbb{B}_{\mathrm{dR}}(\Lambda^+)$ for $\Lambda^+ = \widehat{\mathcal{O}_{\overline{\mathcal{K}}}}$, respectively (see Lemma 4.11 and [Ohk13, § 4] for horizontal semistable representations).

Let T denote the set of prime ideals of $\overline{R}$ above $(\pi) \subset R$. Note that $\mathcal{G}_{R_K}$ acts transitively on T. For $\mathfrak{p} \in T$, consider the decomposition subgroup

$$\mathcal{G}_{R_K}(\mathfrak{p}) := \{g \in \mathcal{G}_{R_K} \mid g(\mathfrak{p}) = \mathfrak{p}\}.$$

We now fix an R-algebra embedding $\overline{R} \hookrightarrow \mathcal{O}_{\overline{\mathcal{K}}}$. This determines a prime ideal of $\overline{R}$ above $(\pi) \subset R$, which we denote by $\mathfrak{p}_0$, and it gives rise to a homomorphism

$$\widehat{\mathcal{G}}_{R_K}(\mathfrak{p}_0) := \mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K}) \to \mathcal{G}_{R_K}.$$

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

Note that the map factors as $\widehat{\mathcal{G}}_{R_K}(\mathfrak{p}_0) \twoheadrightarrow \mathcal{G}_{R_K}(\mathfrak{p}_0) \hookrightarrow \mathcal{G}_{R_K}$ by [Bri08, Lemma 3.3.1]. We also remark that a different choice of the embedding $\overline{R} \hookrightarrow \mathcal{O}_{\overline{\mathcal{K}}}$ defines another ideal $\mathfrak{p}_0' \in T$ and it changes the homomorphism $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K}) \to \mathcal{G}_{R_K}$ with its conjugate by an element g of $\mathcal{G}_{R_K}$ such that $g(\mathfrak{p}_0) = \mathfrak{p}_0'$.

Theorem 6.1. *Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$. If V is horizontal de Rham and $V|_{\widehat{\mathcal{G}}_{R_K}(\mathfrak{p}_0)}$ is horizontal semistable, then V is horizontal semistable.*

Remark 6.2. The converse of Theorem 6.1 also holds by Lemma 4.4.

The rest of this section is devoted to the proof of Theorem 6.1. Our proof is modeled after Tsuji's work on a purity theorem on crystalline local systems [Tsu, Theorem 5.4.8].

6.2 Preliminaries on relevant period rings

Lemma 6.3. *Consider $d\colon \mathcal{K} \to \mathcal{K} \otimes_{\mathcal{O}_{\mathcal{K}}} \widehat{\Omega}_{\mathcal{O}_{\mathcal{K}}}$, where $\widehat{\Omega}_{\mathcal{O}_{\mathcal{K}}} := \varprojlim_m \Omega^1_{\mathcal{O}_{\mathcal{K}}/\mathbb{Z}}/p^m\Omega^1_{\mathcal{O}_{\mathcal{K}}/\mathbb{Z}}$. We have $\mathrm{Ker}(d\colon \mathcal{K} \to \mathcal{K} \otimes_{\mathcal{O}_{\mathcal{K}}} \widehat{\Omega}_{\mathcal{O}_{\mathcal{K}}}) = K$.*

Proof. Since K is algebraically closed in R, it is also algebraically closed in $\mathcal{K}$. Hence, the lemma follows from [Bri06, Proposition 2.28]. □

Remark 6.4. In [Ohk13], this kernel is denoted by $\mathcal{K}_{\mathrm{can}}$ (cf. [Ohk13, Definition 1.3(i), Remark 1.4(i)]).

Note that $\widehat{\mathcal{O}_{\overline{\mathcal{K}}}}[p^{-1}]$ is a complete algebraically closed field containing C. Hence, the pair $(\widehat{\mathcal{O}_{\overline{\mathcal{K}}}}[p^{-1}], \widehat{\mathcal{O}_{\overline{\mathcal{K}}}})$ is a perfectoid C-Banach pair.

For $\mathfrak{p} \in T$, let $\widehat{\overline{R}_{\mathfrak{p}}}$ denote the p-adic completion of the localization $\overline{R}_{\mathfrak{p}}$ of $\overline{R}$ at $\mathfrak{p}$. It is naturally an $\mathcal{O}_C$-algebra. Since $\overline{R}_{\mathfrak{p}}$ can be written as the union of one-dimensional Noetherian normal local domains with p non-unit, $\widehat{\overline{R}_{\mathfrak{p}}}[p^{-1}]$ is a complete valuation field with respect to p-adic valuation In particular, $\widehat{\overline{R}_{\mathfrak{p}}}[p^{-1}]$ is a uniform C-Banach algebra and $\widehat{\overline{R}_{\mathfrak{p}}}$ is a ring of integral elements.

Lemma 6.5. *The C-Banach pair $(\widehat{\overline{R}_{\mathfrak{p}}}[p^{-1}], \widehat{\overline{R}_{\mathfrak{p}}})$ is perfectoid.*

Proof. Since $\widehat{\overline{R}_{\mathfrak{p}}}/p\widehat{\overline{R}_{\mathfrak{p}}} = \overline{R}_{\mathfrak{p}}/p\overline{R}_{\mathfrak{p}}$ is the localization of $\overline{R}/p\overline{R}$ at the ideal $\mathfrak{p}$ mod $p\overline{R}$, the Frobenius is surjective on $\widehat{\overline{R}_{\mathfrak{p}}}/p\widehat{\overline{R}_{\mathfrak{p}}}$ as it is so on $\overline{R}/p\overline{R}$. □

To simplify our notation, set

$$\Lambda_R^+ := \widehat{\overline{R}}, \quad \Lambda_{\mathfrak{p}}^+ := \widehat{\overline{R}_{\mathfrak{p}}} \quad (\mathfrak{p} \in T), \quad \text{and} \quad \Lambda_{\mathcal{K}}^+ := \widehat{\mathcal{O}_{\overline{\mathcal{K}}}}.$$

We also set $\Lambda_* := \Lambda_*^+[p^{-1}]$ for $* \in \{R, \mathfrak{p}, \mathcal{K}\}$.

Consider the $\widehat{\mathcal{G}}_{R_K}(\mathfrak{p}_0)$-equivariant inclusion $\overline{R}_{\mathfrak{p}_0} \hookrightarrow \mathcal{O}_{\overline{\mathcal{K}}}$, where $\widehat{\mathcal{G}}_{R_K}(\mathfrak{p}_0)$ acts on $\overline{R}_{\mathfrak{p}_0}$ via $\widehat{\mathcal{G}}_{R_K}(\mathfrak{p}_0) \twoheadrightarrow \mathcal{G}_{R_K}(\mathfrak{p}_0)$. This extends to $\widehat{\mathcal{G}}_{R_K}(\mathfrak{p}_0)$-equivariant injective maps $\Lambda_{\mathfrak{p}_0}^+ \hookrightarrow \Lambda_{\mathcal{K}}^+$ and $\Lambda_{\mathfrak{p}_0} \hookrightarrow \Lambda_{\mathcal{K}}$. Note that the former is a map of complete valuation rings of rank one and thus it has p-torsion-free cokernel.

Lemma 6.6. *The natural $\widehat{\mathcal{G}}_{R_K}(\mathfrak{p}_0)$-equivariant commutative diagram*

$$\begin{array}{ccc} \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0}) & \longrightarrow & \mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_{\mathfrak{p}_0}) \\ \downarrow & & \downarrow \\ \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_{\mathcal{K}}) & \longrightarrow & \mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_{\mathcal{K}}) \end{array}$$

is Cartesian.

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

Proof. This follows from Proposition 2.37. □

Let us recall the notion of G-regularity. We use the notation in § 4.1.

Definition 6.7 (cf. [Fon94b, 1.4.1]). Let E be a finite extension of $\mathbb{Q}_p$ and G a topological group. An (E, G)-ring B is called *G-regular* if the following three conditions hold:

($G{\cdot}R_1$) The ring B is reduced.
($G{\cdot}R_2$) For all $V \in \mathrm{Rep}_E(G)$, the map $\alpha_B(V)$ is injective.
($G{\cdot}R_3$) Every one-dimensional E-vector subspace of B that is stable under G-action is generated by an invertible element of B over E.

Proposition 6.8. *Let E be a finite extension of $\mathbb{Q}_p$ and G a topological group. Let B be a G-regular (E, G)-ring. Then $F := B^G$ is a field, and the following assertions hold.*

(i) *For every $V \in \mathrm{Rep}_E(G)$, $\dim_F D_B(V) \leq \dim_E V$.*
(ii) *If $\dim_F D_B(V) = \dim_E V$, then the map $\alpha_B(V)$ is an isomorphism.*

Proof. The proof of [Fon94b, 1.4.2] works in this setting. □

Lemma 6.9. *We have $(\mathbb{B}_{\mathrm{st}}(\Lambda_{\mathcal{K}}^+))^{\widehat{\mathcal{G}}_{R_K}(\mathfrak{p}_0)} = K_0$ and $(\mathbb{B}_{\mathrm{dR}}(\Lambda_{\mathcal{K}}^+))^{\widehat{\mathcal{G}}_{R_K}(\mathfrak{p}_0)} = K$. Moreover, the $(\mathbb{Q}_p, \widehat{\mathcal{G}}_R(\mathfrak{p}_0))$-rings $\mathbb{B}_{\mathrm{st},K}(\Lambda_{\mathcal{K}}^+)$ and $\mathbb{B}_{\mathrm{dR}}(\Lambda_{\mathcal{K}}^+)$ are $\widehat{\mathcal{G}}_{R_K}(\mathfrak{p}_0)$-regular.*

Proof. The first assertion follows from [Ohk13, Corollary 4.3, third], Lemma 6.3, and the inclusion $K \otimes_{K_0} \mathbb{B}_{\mathrm{st}}^+(\Lambda_{\mathcal{K}}^+) \subset \mathbb{B}_{\mathrm{dR}}^+(\Lambda_{\mathcal{K}}^+)$. Since $\Lambda_{\mathcal{K}}$ is a field, so is $\mathbb{B}_{\mathrm{dR}}(\Lambda_{\mathcal{K}}^+)$. Hence, $\mathbb{B}_{\mathrm{dR}}(\Lambda_{\mathcal{K}}^+)$ is $\widehat{\mathcal{G}}_{R_K}(\mathfrak{p}_0)$-regular by [Fon94b, Proposition 1.6.3].

By [Fon94b, Proposition 1.6.5], it remains to verify that $\mathbb{B}_{\mathrm{st},K}(\Lambda_{\mathcal{K}}^+)$ satisfies condition $(G \cdot R_3)$. We follow [Ohk13, Lemma 4.5] and [Fon94b, Lemma 5.1.3(ii)].

There exists a complete valuation field $\mathcal{K}^{\mathrm{pf}} \subset \overline{\mathcal{K}}$ containing $\mathcal{K}$ such that the relative ramification index of $\mathcal{K}^{\mathrm{pf}}$ over $\mathcal{K}$ is one and such that the residue field of $\mathcal{K}^{\mathrm{pf}}$ is the perfect closure of $\mathcal{K}$ inside the residue field of $\overline{\mathcal{K}}$ (cf. [Ohk13, 1G]). Let $\mathcal{L}$ denote the p-adic completion of the maximal unramified extension of $\mathcal{K}^{\mathrm{pf}}$ inside $\Lambda_{\mathcal{K}} = \widehat{\mathcal{O}_{\overline{\mathcal{K}}}}[p^{-1}]$, and let P_0 denote the fraction field of the ring of Witt vectors with coefficients in the residue field of $\mathcal{L}$. Then $\mathcal{L}$ is naturally a subfield of $\Lambda_{\mathcal{K}}$, and $\mathbb{B}_{\mathrm{st}}(\Lambda_{\mathcal{K}}^+)$ is a P_0-algebra. Note that $\mathcal{L} \cong K \otimes_{K_0} P_0$ and $\mathbb{B}_{\mathrm{st},K}(\Lambda_{\mathcal{K}}^+) = \mathcal{L} \otimes_{P_0} \mathbb{B}_{\mathrm{st}}(\Lambda_{\mathcal{K}}^+)$. Let $\overline{\mathcal{L}}$ denote the algebraic closure of $\mathcal{L}$ in $\Lambda_{\mathcal{K}}$. So $\Lambda_{\mathcal{K}}$ is the p-adic completion of $\overline{\mathcal{L}}$.

We will show that every one-dimensional $\mathcal{L}$-vector subspace Δ of $\mathbb{B}_{\mathrm{st},K}(\Lambda_{\mathcal{K}}^+)$ that is stable by $\mathrm{Gal}(\overline{\mathcal{L}}/\mathcal{L})$ is written as $\mathcal{L}t^i$ for some $i \in \mathbb{Z}$. Since t is invertible in $\mathbb{B}_{\mathrm{st},K}(\Lambda_{\mathcal{K}}^+)$ and $\mathrm{Gal}(\overline{\mathcal{L}}/\mathcal{L})$ can be identified with a subgroup of $\widehat{\mathcal{G}}_{R_K}(\mathfrak{p}_0)$, this will complete the proof.

Let $b \in \Delta$ be a generator. By replacing b by bt^{-i} for some $i \in \mathbb{Z}$, we may assume that $b \in \mathbb{B}_{\mathrm{dR}}^+(\Lambda_{\mathcal{K}}^+) \setminus t\mathbb{B}_{\mathrm{dR}}^+(\Lambda_{\mathcal{K}}^+)$. It is enough to show that $b \in \mathcal{L}$. Consider $\theta = \theta_{\mathbb{B}_{\mathrm{dR}}^+(\Lambda_{\mathcal{K}}^+)} \colon \mathbb{B}_{\mathrm{dR}}^+(\Lambda_{\mathcal{K}}^+) \to \Lambda_{\mathcal{K}}$. The image $\theta(\Delta)$ is a one-dimensional $\mathcal{L}$-vector subspace generated by $\theta(b)$ and stable by $\mathrm{Gal}(\overline{\mathcal{L}}/\mathcal{L})$. By a result of Sen (cf. [Fon94b, Remark 3.3(ii)]), we have $\theta(b) \in \overline{\mathcal{L}} \subset \Lambda_{\mathcal{K}}$. In particular, the action of $\mathrm{Gal}(\overline{\mathcal{L}}/\mathcal{L})$ on $\theta(\Delta)$ factors through $\mathrm{Gal}(\mathcal{L}'/\mathcal{L})$ for some finite Galois extension $\mathcal{L}'$ of $\mathcal{L}$ inside $\overline{\mathcal{L}}$, and so does its action on Δ.

Let $\mathcal{L}(b)$ denote the $\mathcal{L}$-subalgebra of $\mathbb{B}_{\mathrm{st},K}(\Lambda_{\mathcal{K}}^+)$ generated by b. Since $b \in \mathbb{B}_{\mathrm{dR}}^+(\Lambda_{\mathcal{K}}^+) \setminus t\mathbb{B}_{\mathrm{dR}}^+(\Lambda_{\mathcal{K}}^+)$, we see that $\mathcal{L}(b)$ is $\mathrm{Gal}(\overline{\mathcal{L}}/\mathcal{L})$-equivariantly isomorphic to its image $\theta(\mathcal{L}(b))$. It follows that $\mathcal{L}(b)$ is isomorphic to a subextension of $\mathcal{L}$ inside $\mathcal{L}'$. This implies that $\mathcal{L}' \otimes_{\mathcal{L}} \mathcal{L}(b)$ is not an integral domain unless $\mathcal{L}(b) = \mathcal{L}$. On the other hand, we know that $\mathcal{L}' \otimes_{\mathcal{L}} \mathbb{B}_{\mathrm{st},K}(\Lambda_{\mathcal{K}}^+) = \mathcal{L}' \otimes_{P_0} \mathbb{B}_{\mathrm{st}}(\Lambda_{\mathcal{K}}^+)$ embeds into $\mathbb{B}_{\mathrm{dR}}^+(\Lambda_{\mathcal{K}}^+)$ by Corollary 2.28 (with respect to $\mathcal{L}'$ instead of K) and thus it is an integral domain. Hence, we conclude that $b \in \mathcal{L}$. □

Lemma 6.10. *We have $(\mathbb{B}_{\mathrm{st}}(\Lambda_{\mathfrak{p}}^+))^{\mathcal{G}_{R_K}(\mathfrak{p})} = K_0$ and $(\mathbb{B}_{\mathrm{dR}}(\Lambda_{\mathfrak{p}}^+))^{\mathcal{G}_{R_K}(\mathfrak{p})} = K$. Moreover, the $(\mathbb{Q}_p, \mathcal{G}_{R_K}(\mathfrak{p}))$-rings $\mathbb{B}_{\mathrm{st},K}(\Lambda_{\mathfrak{p}}^+)$ and $\mathbb{B}_{\mathrm{dR}}(\Lambda_{\mathfrak{p}}^+)$ are $\mathcal{G}_{R_K}(\mathfrak{p})$-regular.*

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

Proof. Without loss of generality, we may assume that $\mathfrak{p} = \mathfrak{p}_0$. The first assertion follows from Lemmas 6.6 and 6.9. Since $\Lambda_{\mathfrak{p}_0}$ is a field, so is $\mathbb{B}_{\mathrm{dR}}(\Lambda^+_{\mathfrak{p}_0})$. Hence, $\mathbb{B}_{\mathrm{dR}}(\Lambda_{\mathfrak{p}_0})$ is $\mathcal{G}_{R_K}(\mathfrak{p}_0)$-regular. It remains to verify that $\mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0})$ satisfies condition $(G \cdot R_3)$ by [Fon94b, Proposition 1.6.5]. This follows from the $\widehat{\mathcal{G}}_{R_K}(\mathfrak{p}_0)$-equivariant Cartesian diagram in Lemma 6.6 and the fact that $\mathbb{B}_{\mathrm{dR}}(\Lambda_{\mathfrak{p}_0})$, $\mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathcal{K}})$, and $\mathbb{B}_{\mathrm{dR}}(\Lambda^+_{\mathcal{K}})$ satisfy condition $(G \cdot R_3)$. □

6.3 Proof of Theorem 6.1

Let us return to the setting of Theorem 6.1. Since the proof is long and technical, we start by explaining its main idea. We need to show that $\alpha^{\nabla}_{\mathrm{st},K}(V)\colon \mathbb{B}^{\nabla}_{\mathrm{st},K}(\Lambda^+_R) \otimes_{K_0} D_{\mathrm{st},K}(V) \hookrightarrow \mathbb{B}^{\nabla}_{\mathrm{st},K}(\Lambda^+_R) \otimes_{\mathbb{Q}_p} V$ is surjective. Heuristically, this means that $\mathbb{B}^{\nabla}_{\mathrm{st},K}(\Lambda^+_R) \otimes_{\mathbb{Q}_p} V$ has enough Galois invariant elements. By assumption, we know that $\mathbb{B}^{\nabla}_{\mathrm{dR}}(\Lambda^+_R) \otimes_{\mathbb{Q}_p} V$ has enough $\mathcal{G}_{R_K}$-invariants and that $\mathbb{B}^{\nabla}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0}) \otimes_{\mathbb{Q}_p} V$ has enough $\widehat{\mathcal{G}}_{R_K}(\mathfrak{p}_0)$-invariants. Hence, one might hope for an equality $\mathbb{B}^{\nabla}_{\mathrm{dR}}(\Lambda^+_R) \cap \mathbb{B}^{\nabla}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0}) = \mathbb{B}^{\nabla}_{\mathrm{st},K}(\Lambda^+_R)$ and deduce that $\mathbb{B}^{\nabla}_{\mathrm{st},K}(\Lambda^+_R) \otimes_{\mathbb{Q}_p} V$ has enough Galois invariant elements. Although such an argument does not work literally, our proof pursues this naive idea by using Proposition 2.37 and its consequences (Lemmas 6.6 and 6.14).

Concretely, we first show that $\mathbb{B}^{\nabla}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}}) \otimes_{\mathbb{Q}_p} V$ has enough $\mathcal{G}_{R_K}(\mathfrak{p})$-invariants for every $\mathfrak{p} \in T$ (Lemma 6.12). Then we study the period ring $\mathbb{B}^{\nabla}_{\mathrm{st},K}(\Lambda^+_T)$ where $\Lambda^+_T := \prod_{\mathfrak{p}\in T} \Lambda^+_{\mathfrak{p}}$: we analyze the map $\mathbb{B}^{+}_{\mathrm{st},K}(\Lambda^+_T) \to \prod_{\mathfrak{p}\in T} \mathbb{B}^{+}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}})$ and deduce that $\mathbb{B}^{+}_{\mathrm{st},K}(\Lambda^+_T) \otimes_{\mathbb{Q}_p} V$ has enough Galois invariants (Lemma 6.15). Combining this with the Cartesian diagram in Lemma 6.14, we prove that $\mathbb{B}^{+}_{\mathrm{st},K}(\Lambda_R) \otimes_{\mathbb{Q}_p} V$ has enough Galois invariants and that $\alpha^{\nabla}_{\mathrm{st},K}(V)$ is surjective.

Let us now start the proof of Theorem 6.1. By assumption, $V|_{\widehat{\mathcal{G}}_R(\mathfrak{p}_0)}$ is $\mathbb{B}_{\mathrm{st}}(\Lambda^+_{\mathcal{K}})$-admissible and thus $\mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathcal{K}})$-admissible. By replacing V by its Tate twist $V(-n)$ for $n \gg 0$, we may assume

$$(\mathbb{B}^{+}_{\mathrm{dR}}(\Lambda^+_R) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K}} \cong (\mathbb{B}_{\mathrm{dR}}(\Lambda^+_R) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K}}$$

and

$$(\mathbb{B}^{+}_{\mathrm{st},K}(\Lambda^+_{\mathcal{K}}) \otimes_{\mathbb{Q}_p} V)^{\widehat{\mathcal{G}}_{R_K}(\mathfrak{p}_0)} \xrightarrow{\cong} (\mathbb{B}^{+}_{\mathrm{dR}}(\Lambda^+_{\mathcal{K}}) \otimes_{\mathbb{Q}_p} V)^{\widehat{\mathcal{G}}_{R_K}(\mathfrak{p}_0)} \xrightarrow{\cong} (\mathbb{B}_{\mathrm{dR}}(\Lambda^+_{\mathcal{K}}) \otimes_{\mathbb{Q}_p} V)^{\widehat{\mathcal{G}}_{R_K}(\mathfrak{p}_0)}.$$

In particular, the natural map $(\mathbb{B}^{+}_{\mathrm{dR}}(\Lambda^+_R) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K}} \to (\mathbb{B}^{+}_{\mathrm{dR}}(\Lambda^+_{\mathcal{K}}) \otimes_{\mathbb{Q}_p} V)^{\widehat{\mathcal{G}}_{R_K}(\mathfrak{p}_0)}$ is an isomorphism.

Lemma 6.11. *The representation $V|_{\mathcal{G}_{R_K}(\mathfrak{p}_0)}$ is $\mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0})$-admissible. Moreover,* $(\mathbb{B}^{+}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0}) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K}(\mathfrak{p}_0)} = (\mathbb{B}^{+}_{\mathrm{dR}}(\Lambda^+_{\mathfrak{p}_0}) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K}(\mathfrak{p}_0)}$.

Proof. Consider the following commutative diagram.

$$\begin{array}{ccc}
(\mathbb{B}^{+}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0}) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K}(\mathfrak{p}_0)} & \hookrightarrow & (\mathbb{B}^{+}_{\mathrm{dR}}(\Lambda^+_{\mathfrak{p}_0}) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K}(\mathfrak{p}_0)} \\
\downarrow & & \downarrow \\
(\mathbb{B}^{+}_{\mathrm{st},K}(\Lambda^+_{\mathcal{K}}) \otimes_{\mathbb{Q}_p} V)^{\widehat{\mathcal{G}}_{R_K}(\mathfrak{p}_0)} & \xrightarrow{\cong} & (\mathbb{B}^{+}_{\mathrm{dR}}(\Lambda^+_{\mathcal{K}}) \otimes_{\mathbb{Q}_p} V)^{\widehat{\mathcal{G}}_{R_K}(\mathfrak{p}_0)}
\end{array}$$

It follows from the remark before the lemma that the right vertical map is an isomorphism of K-vector spaces of dimension $\dim_{\mathbb{Q}_p} V$. By Lemma 6.6, we conclude that all four K-vector spaces in the diagram are equal, and we obtain the second assertion. We also have

$$\dim_{\mathbb{Q}_p} V = \dim_K (\mathbb{B}^{+}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0}) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K}(\mathfrak{p}_0)} \leq \dim_K (\mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0}) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K}(\mathfrak{p}_0)}.$$

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

By Proposition 6.8 and Lemma 6.10, we have $\dim_K(\mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0})\otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K}(\mathfrak{p}_0)} = \dim_{\mathbb{Q}_p} V$ and thus $V|_{\mathcal{G}_{R_K}(\mathfrak{p}_0)}$ is $\mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0})$-admissible. □

Observe that each $\sigma \in \mathcal{G}_{R_K}$ induces an R-algebra isomorphism $\Lambda^+_{\mathfrak{p}} \to \Lambda^+_{\sigma(\mathfrak{p})}$ for $\mathfrak{p} \in T$, and $\mathcal{G}_{R_K}(\sigma(\mathfrak{p})) = \sigma\mathcal{G}_{R_K}(\mathfrak{p})\sigma^{-1}$ in $\mathcal{G}_{R_K}$.

LEMMA 6.12. *For each $\mathfrak{p} \in T$, the representation $V|_{\mathcal{G}_{R_K}(\mathfrak{p})}$ is $\mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}})$-admissible. Moreover, $(\mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}})\otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K}(\mathfrak{p})} = (\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_{\mathfrak{p}})\otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K}(\mathfrak{p})}$.*

Proof. Choose $\sigma \in \mathcal{G}_{R_K}$ with $\sigma(\mathfrak{p}_0) = \mathfrak{p}$. Hence, $\mathcal{G}_{R_K}(\mathfrak{p}) = \sigma\,\mathcal{G}_{R_K}(\mathfrak{p}_0)\,\sigma^{-1}$ in $\mathcal{G}_{R_K}$, and σ induces isomorphisms $\sigma\colon \Lambda^+_{\mathfrak{p}_0} \xrightarrow{\cong} \Lambda^+_{\mathfrak{p}}$ and $\sigma\colon \mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0}) \xrightarrow{\cong} \mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}})$. These maps are compatible with the group homomorphism $\mathcal{G}_{R_K}(\mathfrak{p}_0) \to \mathcal{G}_{R_K}(\mathfrak{p})$ given by σ-conjugation.

Hence, for every $\tau \in \mathcal{G}_R(\mathfrak{p}_0)$, we have a commutative diagram as follows.

$$\begin{array}{ccc}
\mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0})\otimes_{\mathbb{Q}_p} V & \xrightarrow[\cong]{\sigma\otimes\sigma} & \mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}})\otimes_{\mathbb{Q}_p} V \\
{\scriptstyle \tau\otimes\tau}\downarrow & & \downarrow{\scriptstyle \sigma\tau\sigma^{-1}\otimes\sigma\tau\sigma^{-1}} \\
\mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0})\otimes_{\mathbb{Q}_p} V & \xrightarrow[\cong]{\sigma\otimes\sigma} & \mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}})\otimes_{\mathbb{Q}_p} V
\end{array}$$

In particular, if $x \in \mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0})\otimes_{\mathbb{Q}_p} V$ is $\mathcal{G}_{R_K}(\mathfrak{p}_0)$-invariant, then $(\sigma\otimes\sigma)(x) \in \mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}})\otimes_{\mathbb{Q}_p} V$ is $\mathcal{G}_{R_K}(\mathfrak{p})$-invariant. It follows that $\sigma\otimes\sigma$ restricts to a K-linear isomorphism $D_{\mathbb{B}_{\mathrm{st},K}(\Lambda_{\mathfrak{p}_0})}(V|_{\mathcal{G}_{R_K}(\mathfrak{p}_0)}) \xrightarrow{\cong} D_{\mathbb{B}_{\mathrm{st},K}(\Lambda_{\mathfrak{p}})}(V|_{\mathcal{G}_{R_K}(\mathfrak{p})})$. Hence, we have

$$\dim_K D_{\mathbb{B}_{\mathrm{st},K}(\Lambda_{\mathfrak{p}})}(V|_{\mathcal{G}_{R_K}(\mathfrak{p})}) = \dim_{\mathbb{Q}_p} V.$$

By Proposition 6.8 and Lemma 6.10, we conclude that $V|_{\mathcal{G}_{R_K}(\mathfrak{p})}$ is $\mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}})$-admissible.

Similarly, the second assertion follows from the one for $\mathfrak{p}_0$ via $\sigma\otimes\sigma$. □

Set

$$\Lambda^+_T := \prod_{\mathfrak{p}\in T}\Lambda^+_{\mathfrak{p}} = \prod_{\mathfrak{p}\in T}\widehat{\overline{R}_{\mathfrak{p}}} \quad\text{and}\quad \Lambda_T := \Lambda^+_T[p^{-1}].$$

Hence, Λ_T is the product of $\Lambda_{\mathfrak{p}}$ over $\mathfrak{p} \in T$ as a C-Banach algebra.

LEMMA 6.13. *The map $\Lambda^+_R \to \Lambda^+_T$ is injective with p-torsion-free cokernel.*

Proof. Recall that $\overline{R}$ can be written as the union of Noetherian normal domains finite over R. Since $R/\pi R$ is an integral domain, we deduce that the natural map $\overline{R}/\pi\overline{R} \to \prod_{\mathfrak{p}\in T}\overline{R}_{\mathfrak{p}}/\pi\overline{R}_{\mathfrak{p}}$ is injective. It follows that $\overline{R}/\pi^m\overline{R} \to \prod_{\mathfrak{p}\in T}\overline{R}_{\mathfrak{p}}/\pi^m\overline{R}_{\mathfrak{p}}$ is injective for every $m \in \mathbb{N}$. By taking the inverse limit over m, we see that $\Lambda^+_R \to \prod_{\mathfrak{p}\in T}\Lambda^+_{\mathfrak{p}} = \Lambda^+_T$ is injective with π-torsion-free cokernel. Hence, the cokernel is also p-torsion-free. □

Each $\sigma \in \mathcal{G}_{R_K}$ induces an R-algebra isomorphism $\Lambda^+_{\mathfrak{p}} \to \Lambda^+_{\sigma(\mathfrak{p})}$ for $\mathfrak{p} \in T$. This gives rise to a natural $\mathcal{G}_{R_K}$-action on Λ^+_T making the injection $\Lambda^+_R \to \Lambda^+_T$ $\mathcal{G}_{R_K}$-equivariant (cf. [Bri08, Remark 3.3.2]).

Since products commute with inverse limits and the Witt vector functor, we have $\mathbb{A}_{\mathrm{inf}}(\Lambda^+_T) = \prod_{\mathfrak{p}\in T}\mathbb{A}_{\mathrm{inf}}(\Lambda^+_{\mathfrak{p}})$. Since $\mathcal{O}_K$ is finite free over $W(k)$, we conclude that

$$\mathbb{A}_{\mathrm{inf},K}(\Lambda^+_T) = \prod_{\mathfrak{p}\in T}\mathbb{A}_{\mathrm{inf},K}(\Lambda^+_{\mathfrak{p}}),$$

and $\theta_{\mathbb{A}_{\mathrm{inf},K}(\Lambda^+_T)}$ is the product of $\theta_{\mathbb{A}_{\mathrm{inf},K}(\Lambda^+_{\mathfrak{p}})}$ over $\mathfrak{p} \in T$.

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

Lemma 6.14. *The following commutative diagram is Cartesian.*

$$\begin{array}{ccc} \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_R) & \longrightarrow & \mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_R) \\ \downarrow & & \downarrow \\ \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_T) & \longrightarrow & \mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_T) \end{array}$$

Proof. This follows from Proposition 2.37 and Lemma 6.13. □

For each $\mathfrak{p} \in T$, choose $\sigma_{\mathfrak{p}} \in \mathcal{G}_{R_K}$ such that $\sigma_{\mathfrak{p}}(\mathfrak{p}_0) = \mathfrak{p}$ and $\sigma_{\mathfrak{p}}|_{\overline{K}} = \mathrm{id}$. This is possible since $R/\pi R$ is geometrically integral over k. We also assume that $\sigma_{\mathfrak{p}_0} = \mathrm{id}$. We have an isomorphism

$$\sigma_{\mathfrak{p}} \otimes \sigma_{\mathfrak{p}} \colon \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0}) \otimes_{\mathbb{Q}_p} V \xrightarrow{\cong} \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}}) \otimes_{\mathbb{Q}_p} V$$

compatible with the homomorphism $\mathcal{G}_{R_K}(\mathfrak{p}_0) \to \mathcal{G}_{R_K}(\mathfrak{p})$ given by $\sigma_{\mathfrak{p}}$-conjugation.

Lemma 6.15.

(i) *The map*

$$\mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_T) \to \prod_{\mathfrak{p} \in T} \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}})$$

is injective.

(ii) *If $x = (x_{\mathfrak{p}})_{\mathfrak{p}} \in \prod_{\mathfrak{p} \in T}(\mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}}) \otimes_{\mathbb{Q}_p} V)$ satisfies $(\sigma_{\mathfrak{p}} \otimes \sigma_{\mathfrak{p}})(x_{\mathfrak{p}_0}) = x_{\mathfrak{p}}$ for every $\mathfrak{p} \in T$, then $x \in \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_T) \otimes_{\mathbb{Q}_p} V$.*

Proof. Consider $\theta_{\mathbb{A}_{\mathrm{inf},K}(\Lambda^+_{\mathfrak{p}_0})} \colon \mathbb{A}_{\mathrm{inf},K}(\Lambda^+_{\mathfrak{p}_0}) \to \Lambda^+_{\mathfrak{p}_0}$ and choose an $\mathcal{O}_K$-linear section $s_{\mathfrak{p}_0} \colon \Lambda^+_{\mathfrak{p}_0} \to \mathbb{A}_{\mathrm{inf},K}(\Lambda^+_{\mathfrak{p}_0})$ of $\theta_{\mathbb{A}_{\mathrm{inf},K}(\Lambda^+_{\mathfrak{p}_0})}$ as in Construction 2.19. By the argument there, it defines a K-linear isomorphism

$$\widetilde{\theta}_{v,\mathfrak{p}_0} \colon \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0}) \xrightarrow{\cong} \Lambda_{\mathfrak{p}_0}\langle X\rangle[\log(1+X)].$$

For each $\mathfrak{p} \in T$, we set

$$s_{\mathfrak{p}} := \sigma_{\mathfrak{p}} \circ s_{\mathfrak{p}_0} \circ \sigma_{\mathfrak{p}}^{-1} \colon \Lambda^+_{\mathfrak{p}} \to \Lambda^+_{\mathfrak{p}_0} \to \mathbb{A}_{\mathrm{inf},K}(\Lambda^+_{\mathfrak{p}_0}) \to \mathbb{A}_{\mathrm{inf},K}(\Lambda^+_{\mathfrak{p}}).$$

Then $s_{\mathfrak{p}}$ is an $\mathcal{O}_K$-linear section of $\theta_{\mathbb{A}_{\mathrm{inf},K}(\Lambda^+_{\mathfrak{p}})}$ satisfying $s_{\mathfrak{p}} \circ \sigma_{\mathfrak{p}} = \sigma_{\mathfrak{p}} \circ s_{\mathfrak{p}_0}$. It also yields a K-linear isomorphism

$$\widetilde{\theta}_{v,\mathfrak{p}} \colon \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}}) \xrightarrow{\cong} \Lambda_{\mathfrak{p}}\langle X\rangle[\log(1+X)].$$

Let $\sigma_{\mathfrak{p}} \colon \Lambda_{\mathfrak{p}_0}\langle X\rangle[\log(1+X)] \to \Lambda_{\mathfrak{p}}\langle X\rangle[\log(1+X)]$ denote the continuous K-linear map extending $\sigma_{\mathfrak{p}} \colon \Lambda_{\mathfrak{p}_0} \to \Lambda_{\mathfrak{p}}$ by $\sigma_{\mathfrak{p}}(X^l(\log(1+X))^m) = X^l(\log(1+X))^m$. We see that $\widetilde{\theta}_{v,\mathfrak{p}} \circ \sigma_{\mathfrak{p}} = \sigma_{\mathfrak{p}} \circ \widetilde{\theta}_{v,\mathfrak{p}_0}$ by Lemma 2.20(ii), Proposition 2.27, and the assumption $\sigma_{\mathfrak{p}}|_{\overline{K}} = \mathrm{id}$.

Similarly,

$$\prod_{\mathfrak{p} \in T} s_{\mathfrak{p}} \colon \Lambda^+_T = \prod_{\mathfrak{p} \in T} \Lambda^+_{\mathfrak{p}} \to \prod_{\mathfrak{p} \in T} \mathbb{A}_{\mathrm{inf},K}(\Lambda^+_{\mathfrak{p}}) = \mathbb{A}_{\mathrm{inf},K}(\Lambda^+_T)$$

is an $\mathcal{O}_K$-linear section of $\theta_{\mathbb{A}_{\mathrm{inf},K}(\Lambda^+_T)}$, and it yields a K-linear isomorphism

$$\widetilde{\theta}_{v,T} \colon \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_T) \xrightarrow{\cong} \Lambda_T\langle X\rangle[\log(1+X)].$$

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

Then we a commutative diagram as follows.

$$\begin{array}{ccc} \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_T) & \longrightarrow & \prod_{\mathfrak{p}\in T}\mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}}) \\ \downarrow{\scriptstyle \widetilde{\theta}_{v,T}}\ \cong & & \downarrow{\scriptstyle \prod\widetilde{\theta}_{v,\mathfrak{p}}}\ \cong \\ \Lambda_T\langle X\rangle[\log(1+X)] & \longrightarrow & \prod_{\mathfrak{p}\in T}\Lambda_{\mathfrak{p}}\langle X\rangle[\log(1+X)] \end{array}$$

For $h\in\mathbb{N}$ and $*\in\{T,\mathfrak{p}\}$, we denote the $\mathbb{A}_{\max,K}(\Lambda^+_*)$-submodule $\bigoplus_{i=0}^h \mathbb{A}_{\max,K}(\Lambda^+_*)(\log[\pi^\flat]/\pi)^i$ of $\mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_*)$ by $\mathbb{A}^{\le h}_{\mathrm{st},K}(\Lambda^+_*)$. Note that $\mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_*)=\bigcup_h \mathbb{A}^{\le h}_{\mathrm{st},K}(\Lambda^+_*)[p^{-1}]$. By Lemma 2.20(ii) and (iv), we have the functorial identification

$$\widetilde{\theta}_{v,*}\colon \mathbb{A}^{\le h}_{\mathrm{st},K}(\Lambda^+_*)\xrightarrow{\cong}\bigoplus_{i=0}^h \Lambda^+_*\langle X\rangle(\log(1+X))^i =: \Lambda^+_*\langle X\rangle[\log(1+X)]^{\le h}$$

as $\mathcal{O}_K$-modules.

Recall that

$$\Lambda^+_*\langle X\rangle=\left\{\sum_{i=0}^\infty \lambda_i X^i\in\Lambda^+_*[[X]]\mid \forall\, l\in\mathbb{N},\ \lambda_i\in p^l\Lambda^+_*\ \text{for all but finitely many } i\right\}.$$

In particular, $\Lambda^+_T\langle X\rangle\to\prod_{\mathfrak{p}\in T}(\Lambda^+_{\mathfrak{p}}\langle X\rangle)$ is injective.

We now prove part (i). By the above remark and the transcendence of $\log(1+X)$ (Lemma 2.26), we see that the map

$$\Lambda^+_T\langle X\rangle[\log(1+X)]^{\le h}\to\prod_{\mathfrak{p}\in T}(\Lambda^+_{\mathfrak{p}}\langle X\rangle[\log(1+X)]^{\le h})$$

is injective. Hence, the map $\mathbb{A}^{\le h}_{\mathrm{st},K}(\Lambda^+_T)\to\prod_{\mathfrak{p}\in T}\mathbb{A}^{\le h}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}})$ is injective and part (i) follows.

Turning to part (ii), consider the following commutative diagram.

$$\begin{array}{ccc} \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0})\otimes_{\mathbb{Q}_p}V & \xrightarrow[\cong]{\widetilde{\theta}_{v,\mathfrak{p}_0}\otimes\mathrm{id}} & \Lambda_{\mathfrak{p}_0}\langle X\rangle[\log(1+X)]\otimes_{\mathbb{Q}_p}V \\ \downarrow{\scriptstyle \sigma_{\mathfrak{p}}\otimes\sigma_{\mathfrak{p}}} & & \downarrow{\scriptstyle \sigma_{\mathfrak{p}}\otimes\sigma_{\mathfrak{p}}} \\ \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}})\otimes_{\mathbb{Q}_p}V & \xrightarrow[\cong]{\widetilde{\theta}_{v,\mathfrak{p}}\otimes\mathrm{id}} & \Lambda_{\mathfrak{p}}\langle X\rangle[\log(1+X)]\otimes_{\mathbb{Q}_p}V \end{array}$$

Then the assertion is equivalent to saying that if $x=(x_{\mathfrak{p}})_{\mathfrak{p}}\in\prod_{\mathfrak{p}\in T}(\Lambda_{\mathfrak{p}}\langle X\rangle[\log(1+X)]\otimes_{\mathbb{Q}_p}V)$ satisfies $(\sigma_{\mathfrak{p}}\otimes\sigma_{\mathfrak{p}})(x_{\mathfrak{p}_0})=x_{\mathfrak{p}}$ for every $\mathfrak{p}\in T$, then $x\in\Lambda_T\langle X\rangle[\log(1+X)]\otimes_{\mathbb{Q}_p}V$.

Fix a $\mathcal{G}_{R_K}$-stable $\mathbb{Z}_p$-lattice $V_{\mathbb{Z}_p}$ of V and choose such $x=(x_{\mathfrak{p}})_{\mathfrak{p}}$. There exist $m,h\in\mathbb{N}$ such that $p^m x_{\mathfrak{p}_0}\in\Lambda^+_{\mathfrak{p}_0}\langle X\rangle[\log(1+X)]^{\le h}\otimes_{\mathbb{Z}_p}V_{\mathbb{Z}_p}$. Then for every $\mathfrak{p}\in T$, we have

$$p^m x_{\mathfrak{p}}=(\sigma_{\mathfrak{p}}\otimes\sigma_{\mathfrak{p}})(p^m x_{\mathfrak{p}_0})\in\Lambda^+_{\mathfrak{p}}\langle X\rangle[\log(1+X)]^{\le h}\otimes_{\mathbb{Z}_p}V_{\mathbb{Z}_p}.$$

Moreover, we see that if there exist $l\in\mathbb{N}$ and $f(X)\in\Lambda^+_{\mathfrak{p}_0}[X][\log(1+X)]^{\le h}\otimes_{\mathbb{Z}_p}V_{\mathbb{Z}_p}$ (i.e. a polynomial in X and $\log(1+X)$) such that

$$p^m x_{\mathfrak{p}_0}-f(X)\in p^l(\Lambda^+_{\mathfrak{p}_0}\langle X\rangle[\log(1+X)]^{\le h}\otimes_{\mathbb{Z}_p}V_{\mathbb{Z}_p}),$$

then

$$p^m x_{\mathfrak{p}}-(\sigma_{\mathfrak{p}}\otimes\sigma_{\mathfrak{p}})(f(X))\in p^l(\Lambda^+_{\mathfrak{p}}\langle X\rangle[\log(1+X)]^{\le h}\otimes_{\mathbb{Z}_p}V_{\mathbb{Z}_p}).$$

This implies that $p^m x\in\Lambda^+_T\langle X\rangle[\log(1+X)]^{\le h}\otimes_{\mathbb{Z}_p}V_{\mathbb{Z}_p}$, and thus $x\in\Lambda_T\langle X\rangle[\log(1+X)]^{\le h}\otimes_{\mathbb{Q}_p}V$. □

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

Remark 6.16. With the notation as in the proof, the same argument as in the last part of the proof shows the following assertion. Let $D_{\mathcal{O}_K}$ be a finite free $\mathcal{O}_K$-module and let $W \subset \mathbb{A}^{\leq h}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0}) \otimes_{\mathcal{O}_K} D_{\mathcal{O}_K}$ be a finite $\mathbb{Z}_p$-submodule. Then the submodule $\prod_{\mathfrak{p}\in T}((\sigma_{\mathfrak{p}} \otimes \mathrm{id})(W))$ of $\prod_{\mathfrak{p}\in T}(\mathbb{A}^{\leq h}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}}) \otimes_{\mathcal{O}_K} D_{\mathcal{O}_K})$ lands in $\mathbb{A}^{\leq h}_{\mathrm{st},K}(\Lambda^+_T) \otimes_{\mathcal{O}_K} D_{\mathcal{O}_K}$. In particular, if $x = (x_{\mathfrak{p}}) \in \prod_{\mathfrak{p}\in T}(\mathbb{A}^{\leq h}_{\mathrm{st},K}(\Lambda) \otimes_{\mathcal{O}_K} D_{\mathcal{O}_K})$ satisfies $x_{\mathfrak{p}} \in (\sigma_{\mathfrak{p}} \otimes \mathrm{id})(W) \subset \mathbb{A}^{\leq h}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}}) \otimes_{\mathcal{O}_K} D_{\mathcal{O}_K}$ for every $\mathfrak{p} \in T$, then $x \in \mathbb{A}^{\leq h}_{\mathrm{st},T}(\Lambda^+_T) \otimes_{\mathcal{O}_K} D_{\mathcal{O}_K}$.

Lemma 6.17. *Let $x \in (\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_R) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K}}$ and let $x_{\mathfrak{p}}$ denote the image of x in $(\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_{\mathfrak{p}}) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K(\mathfrak{p})}}$. Then for each $\mathfrak{p} \in T$, we have $(\sigma_{\mathfrak{p}} \otimes \sigma_{\mathfrak{p}})(x_{\mathfrak{p}_0}) = x_{\mathfrak{p}}$.*

Proof. This results from the following commutative diagram.

$$\begin{array}{ccc} \mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_R) \otimes_{\mathbb{Q}_p} V & \xrightarrow{\sigma_{\mathfrak{p}}\otimes\sigma_{\mathfrak{p}}} & \mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_R) \otimes_{\mathbb{Q}_p} V \\ \downarrow & & \downarrow \\ \mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_{\mathfrak{p}_0}) \otimes_{\mathbb{Q}_p} V & \xrightarrow{\sigma_{\mathfrak{p}}\otimes\sigma_{\mathfrak{p}}} & \mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_{\mathfrak{p}}) \otimes_{\mathbb{Q}_p} V \end{array}$$

Namely, we have $(\sigma_{\mathfrak{p}} \otimes \sigma_{\mathfrak{p}})(x_{\mathfrak{p}_0}) = ((\sigma_{\mathfrak{p}} \otimes \sigma_{\mathfrak{p}})(x))_{\mathfrak{p}} = x_{\mathfrak{p}}$. □

Lemma 6.18. *The inclusion*

$$(\mathbb{B}^+_{\mathrm{st},K}(\Lambda_R) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K}} \subset (\mathbb{B}^+_{\mathrm{dR}}(\Lambda_R) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K}}$$

is an equality.

Proof. Let $x \in (\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_R) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K}}$. For each $\mathfrak{p} \in T$, let $x_{\mathfrak{p}}$ denote the image of x in $(\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_{\mathfrak{p}}) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K(\mathfrak{p})}}$. Then we have $(\sigma_{\mathfrak{p}} \otimes \sigma_{\mathfrak{p}})(x_{\mathfrak{p}_0}) = x_{\mathfrak{p}}$ by the previous lemma. Since $(\mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}}) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K(\mathfrak{p})}} = (\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_{\mathfrak{p}}) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K(\mathfrak{p})}}$ by Lemma 6.12, we have $(x_{\mathfrak{p}})_{\mathfrak{p}} \in \prod_{\mathfrak{p}\in T}(\mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}}) \otimes_{\mathbb{Q}_p} V)$. Hence, we get $x \in \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_T) \otimes_{\mathbb{Q}_p} V$ by Lemma 6.15(ii). Since the commutative diagram

$$\begin{array}{ccc} \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_R) \otimes_{\mathbb{Q}_p} V & \longrightarrow & \mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_R) \otimes_{\mathbb{Q}_p} V \\ \downarrow & & \downarrow \\ \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_T) \otimes_{\mathbb{Q}_p} V & \longrightarrow & \mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_T) \otimes_{\mathbb{Q}_p} V \end{array}$$

is Cartesian by Lemma 6.14, we conclude that $x \in \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_R) \otimes_{\mathbb{Q}_p} V$. □

Let us complete the proof of Theorem 6.1. Set

$$D := (\mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_R) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K}} = (\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_R) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K}} = (\mathbb{B}_{\mathrm{dR}}(\Lambda^+_R) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K}}.$$

This is a K-vector space and $\dim_K D = \dim_{\mathbb{Q}_p} V$. Consider the following commutative diagram.

$$\begin{array}{ccc} \mathbb{B}_{\mathrm{st},K}(\Lambda^+_R) \otimes_K D & \xrightarrow{\alpha^{\nabla}_{\mathrm{st},K}(V)} & \mathbb{B}_{\mathrm{st},K}(\Lambda^+_R) \otimes_{\mathbb{Q}_p} V \\ \downarrow & & \downarrow \\ \mathbb{B}_{\mathrm{dR}}(\Lambda^+_R) \otimes_K D & \xrightarrow[\cong]{\alpha^{\nabla}_{\mathrm{dR}}(V)} & \mathbb{B}_{\mathrm{dR}}(\Lambda^+_R) \otimes_{\mathbb{Q}_p} V \end{array}$$

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

We need to show that $\alpha^{\nabla}_{\mathrm{st},K}(V)$ is surjective. Choose $m \in \mathbb{N}$ such that the $\mathbb{Q}_p$-vector subspace $\mathbb{Q}_p t^m \otimes_{\mathbb{Q}_p} V$ of $\mathbb{B}_{\mathrm{dR}}(\Lambda^+_R) \otimes_{\mathbb{Q}_p} V$ is contained in the image of $\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_R) \otimes_K D$ under $\alpha^{\nabla}_{\mathrm{dR}}(V)$. Fix a $\mathcal{G}_{R_K}$-stable $\mathbb{Z}_p$-lattice $V_{\mathbb{Z}_p}$ of V and set $V'_{\mathbb{Z}_p} := \mathbb{Z}_p t^m \otimes_{\mathbb{Z}_p} V_{\mathbb{Z}_p}$. It suffices to prove that $V'_{\mathbb{Z}_p}$, as a submodule of $\mathbb{B}_{\mathrm{st},K}(\Lambda^+_R) \otimes_{\mathbb{Q}_p} V$, is contained in the image of $\alpha^{\nabla}_{\mathrm{st},K}(V)$.

For $h \in \mathbb{N}$ and $* \in \{R, T, \mathfrak{p}\}$, consider the $\mathbb{A}_{\max,K}(\Lambda^+_*)$-submodule

$$\mathbb{A}^{\le h}_{\mathrm{st},K}(\Lambda^+_*) := \bigoplus_{i=0}^{h} \mathbb{A}_{\max,K}(\Lambda^+_*) \left(\log \frac{[\pi^\flat]}{\pi} \right)^i \subset \mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_*).$$

Note that $\mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_*) = \bigcup_h \mathbb{A}^{\le h}_{\mathrm{st},K}(\Lambda^+_*)[p^{-1}]$. Observe that each $\mathbb{A}^{\le h}_{\mathrm{st},K}(\Lambda^+_R)[p^{-1}]$ is stable under the action of $\mathcal{G}_{R_K}$ and

$$(\mathbb{A}^{\le h}_{\mathrm{st},K}(\Lambda^+_R)[p^{-1}])^{\mathcal{G}_{R_K}} = (\mathbb{A}^{\le h}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}})[p^{-1}])^{\mathcal{G}_{R_K}(\mathfrak{p})} = K.$$

Since D is finite-dimensional over K, there exists $h \in \mathbb{N}$ such that

$$D = (\mathbb{A}^{\le h}_{\mathrm{st},K}(\Lambda^+_R)[p^{-1}] \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K}}.$$

It follows that for each $\mathfrak{p} \in T$, we have the isomorphism

$$D \xrightarrow{\cong} (\mathbb{A}^{\le h}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}})[p^{-1}] \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K}(\mathfrak{p})} = (\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_{\mathfrak{p}}) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_{R_K}(\mathfrak{p})}.$$

By increasing h if necessary, we may further assume that $V'_{\mathbb{Z}_p}$, as a submodule of $\mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0}) \otimes_{\mathbb{Q}_p} V$, is contained in the image of $\mathbb{A}^{\le h}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0})[p^{-1}] \otimes_K D$ under the isomorphism

$$\alpha_{\mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0})}(V|_{\mathcal{G}_{R_K}(\mathfrak{p}_0)}) \colon \mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0}) \otimes_K D \xrightarrow{\cong} \mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0}) \otimes_{\mathbb{Q}_p} V.$$

Choose an $\mathcal{O}_K$-lattice $D_{\mathcal{O}_K}$ of D such that $V'_{\mathbb{Z}_p}$, as a submodule of $\mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0}) \otimes_{\mathbb{Q}_p} V$, is contained in the image of $\mathbb{A}^{\le h}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0}) \otimes_{\mathcal{O}_K} D_{\mathcal{O}_k}$ under $\alpha_{\mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0})}(V|_{\mathcal{G}_{R_K}(\mathfrak{p}_0)})$.

For each $\mathfrak{p} \in T$, the functoriality of the construction gives a commutative diagram as follows.

$$\begin{array}{ccc}
\mathbb{A}^{\le h}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0}) \otimes_{\mathcal{O}_K} D_{\mathcal{O}_K} & \xrightarrow{\alpha_{\mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0})}(V|_{\mathcal{G}_{R_K}(\mathfrak{p}_0)})} & \mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}_0}) \otimes_{\mathbb{Q}_p} V \\
\downarrow \sigma_{\mathfrak{p}} \otimes \mathrm{id} & & \downarrow \sigma_{\mathfrak{p}} \otimes \sigma_{\mathfrak{p}} \\
\mathbb{A}^{\le h}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}}) \otimes_{\mathcal{O}_K} D_{\mathcal{O}_K} & \xrightarrow{\alpha_{\mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}})}(V|_{\mathcal{G}_{R_K}(\mathfrak{p})})} & \mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}}) \otimes_{\mathbb{Q}_p} V
\end{array}$$

Since $V'_{\mathbb{Z}_p}$ is $\mathcal{G}_{R_K}$-stable, we conclude that $V'_{\mathbb{Z}_p}$, as a submodule of $\mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}}) \otimes_{\mathbb{Q}_p} V$, is contained in the image of $\mathbb{A}^{\le h}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}}) \otimes_{\mathcal{O}_K} D_{\mathcal{O}_k}$ under $\alpha_{\mathbb{B}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}})}(V|_{\mathcal{G}_{R_K}(\mathfrak{p})})$.

Consider the isomorphism

$$\alpha^{\nabla}_{\mathrm{dR}}(V) \colon \mathbb{B}_{\mathrm{dR}}(\Lambda^+_R) \otimes_K D \xrightarrow{\cong} \mathbb{B}_{\mathrm{dR}}(\Lambda^+_R) \otimes_{\mathbb{Q}_p} V$$

and let $W \subset \mathbb{B}_{\mathrm{dR}}(\Lambda^+_R) \otimes_K D$ denote the inverse image of $V'_{\mathbb{Z}_p}$, as a submodule of $\mathbb{B}_{\mathrm{dR}}(\Lambda^+_R) \otimes_{\mathbb{Q}_p} V$, under $\alpha^{\nabla}_{\mathrm{dR}}(V)$. Observe that $W \subset \mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_R) \otimes_K D$. By the above discussion, the image of W under the map

$$\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_R) \otimes_K D \to \prod_{\mathfrak{p} \in T} (\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_{\mathfrak{p}}) \otimes_K D)$$

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

lands in the subspace $\prod_{\mathfrak{p}\in T}(\mathbb{A}^{\leq h}_{\mathrm{st},K}(\Lambda^+_{\mathfrak{p}})\otimes_{\mathcal{O}_K} D_{\mathcal{O}_K})$. It follows from Remark 6.16 that the image of W under the map

$$\mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_R)\otimes_K D\to \mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_T)\otimes_K D$$

lands in the subspace $\mathbb{A}^{\leq h}_{\mathrm{st},K}(\Lambda^+_T)\otimes_{\mathcal{O}_K} D_{\mathcal{O}_K}$ and thus in $\mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_T)\otimes_K D$. By Lemma 6.14, we conclude that W lands in the subspace

$$\mathbb{B}^+_{\mathrm{st},K}(\Lambda^+_R)\otimes_K D\subset \mathbb{B}^+_{\mathrm{dR}}(\Lambda^+_R)\otimes_K D.$$

This implies that $V'_{\mathbb{Z}_p}$, as a submodule of $\mathbb{B}_{\mathrm{st},K}(\Lambda^+_R)\otimes_{\mathbb{Q}_p} V$, is contained in the image of $\alpha^{\nabla}_{\mathrm{st},K}(V)$. Hence, V is a horizontal semistable representation of $\mathcal{G}_{R_K}$. This completes the proof of Theorem 6.1.

7. p-adic monodromy theorem for horizontal de Rham representations

7.1 Horizontal version of p-adic monodromy theorem for Galois representations

Let us recall a result of Ohkubo. We use a slightly different notation from [Ohk13]. Let $\mathcal{K}$ be a complete discrete valuation field of mixed characteristic $(0,p)$ and let $k_{\mathcal{K}}$ denote the residue field of $\mathcal{K}$.[5] Fix an algebraic closure $\overline{\mathcal{K}}$ of $\mathcal{K}$. Define the subfield $\mathcal{K}_{\mathrm{can}}$ of $\mathcal{K}$ as the algebraic closure of $W(k_{\mathcal{K}}^{p^\infty})[p^{-1}]$ in $\mathcal{K}$, where $k_{\mathcal{K}}^{p^\infty}:=\bigcap_{m\in\mathbb{N}} k_{\mathcal{K}}^{p^m}$ We use a similar notation for finite extensions of $\mathcal{K}$.

Ohkubo proves the p-adic monodromy theorem for horizontal de Rham representations of $\mathcal{K}$.

Theorem 7.1 [Ohk13, Theorem 7.4]. *Let $V_{\mathcal{K}}$ be a horizontal de Rham representation of $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$. Then there exists a finite extension L of $\mathcal{K}_{\mathrm{can}}$ such that $V_{\mathcal{K}}|_{\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K}L)}$ is horizontal semistable, where $\mathcal{K}L$ is the composite field of $\mathcal{K}$ and L in $\overline{\mathcal{K}}$.*

Remark 7.2. [Ohk13, Theorem 7.6] claims that every horizontal de Rham representation of $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$ comes from a de Rham representation of $\mathrm{Gal}(\overline{\mathcal{K}_{\mathrm{can}}}/\mathcal{K}_{\mathrm{can}})$ via the pullback along $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})\to\mathrm{Gal}(\overline{\mathcal{K}_{\mathrm{can}}}/\mathcal{K}_{\mathrm{can}})$, which is not correct. The error comes from an incorrect argument in the proof of [Ohk13, Proposition 7.3] (cf. Remark 4.27 and Example 4.28). However, these errors are irrelevant to the proof of the horizontal p-adic monodromy theorem [Ohk13, Theorem 7.4], and the latter theorem is correct.

7.2 p-adic monodromy theorem for horizontal de Rham representations

Let K be a complete discrete valuation field of characteristic zero with perfect residue field k of characteristic p. Fix a uniformizer π of K. Let R be an $\mathcal{O}_K$-algebra satisfying the conditions in Set-up 3.1. We further assume that R satisfies condition (BR) of Definition 3.3. For each finite extension L of K, we denote $\mathcal{O}_L\otimes_{\mathcal{O}_K} R$ by $R_{\mathcal{O}_L}$. Then $R_{\mathcal{O}_L}$ also satisfies the same properties with respect to L. As before, we write R_K (respectively, R_L) for $R[p^{-1}]$ (respectively, $R_{\mathcal{O}_L}[p^{-1}]$).

Theorem 1.9 is a special case of the following p-adic monodromy theorem for horizontal de Rham representations.

Theorem 7.3. *Let $V\in\mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{R_K})$. If V is horizontal de Rham, then there exists a finite extension L of K such that $V|_{\mathcal{G}_{R_L}}$ is horizontal semistable.*

Proof. We use the notation in § 6.1. Consider the complete discrete valuation field $\mathcal{K}:=\widehat{R_{(\pi)}}[p^{-1}]$. Fix an algebraic closure $\overline{\mathcal{K}}$ of $\mathcal{K}$ and an R-algebra embedding $\overline{R}\hookrightarrow\overline{\mathcal{K}}$. Note $\mathcal{K}_{\mathrm{can}}=K$ by Lemma 6.3 and Remark 6.4.

[5] [Ohk13] uses K for our $\mathcal{K}$. In our paper, K is a finite extension of $\mathbb{Q}_p$.

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

Since V is a horizontal de Rham representation of $\mathcal{G}_{R_K}$, $V_{\mathcal{K}} := V|_{\mathrm{Gal}(\overline{K}/\mathcal{K})}$ is a horizontal de Rham representation of $\mathrm{Gal}(\overline{K}/\mathcal{K})$. Hence, by Theorem 7.1, there exists a finite extension L of K such that $V_{\mathcal{K}}|_{\mathrm{Gal}(\overline{K}/L\mathcal{K})}$ is horizontal semistable.

We prove that $V|_{\mathcal{G}_{R_L}}$ is horizontal semistable. Let π_L denote a uniformizer of L. Since $R/\pi R$ is geometrically integral over k, $(\pi_L) \subset R_{\mathcal{O}_L}$ is a prime ideal, and thus the ring of integers of $\mathcal{K}L$ coincides with the p-adic completion $\widehat{(R_{\mathcal{O}_L})_{(\pi_L)}}$ of the localization of $R_{\mathcal{O}_L}$ at (π_L). Hence, $\mathcal{K}L = \widehat{(R_{\mathcal{O}_L})_{(\pi_L)}}[p^{-1}]$.

We know that $V|_{\mathcal{G}_{R_L}}|_{\mathrm{Gal}(\overline{K}/L\mathcal{K})} = V_{\mathcal{K}}|_{\mathrm{Gal}(\overline{K}/L\mathcal{K})}$ is a horizontal semistable representation of $\mathrm{Gal}(\overline{K}/L\mathcal{K})$. The embedding $\overline{R_{\mathcal{O}_L}} = \overline{R} \hookrightarrow \overline{\mathcal{K}}$ determines a prime ideal $\mathfrak{p}_0$ of $\overline{R_{\mathcal{O}_L}}$ above $(\pi_L) \subset R_{\mathcal{O}_L}$, and $\mathrm{Gal}(\overline{\mathcal{K}}/L\mathcal{K})$ coincides with $\widehat{\mathcal{G}}_{R_L}(\mathfrak{p}_0)$. Therefore $V|_{\mathcal{G}_{R_L}}$ is horizontal semistable by Theorem 6.1. □

THEOREM 7.4. *Let V be a horizontal de Rham representation of $\mathcal{G}_{R_K}$. Assume that there exists a finite extension K' of K and a map $f\colon R \to K'$ such that f^*V is a potentially crystalline representation of $\mathrm{Gal}(\overline{K}/K')$. Then there exists a finite extension L of K such that $V|_{\mathcal{G}_{R_L}}$ is horizontal crystalline.*

Proof. By extending K' if necessary, we may assume that the representation f^*V of $\mathrm{Gal}(\overline{K}/K')$ is crystalline (equivalently, horizontal crystalline). By Theorem 7.3, there exists a finite extension L of K' such that $V|_{\mathcal{G}_{R_L}}$ is horizontal semistable. Then the assertion follows from Proposition 4.20. □

8. *p*-adic Hodge theory for rigid analytic varieties

In this section, we review p-adic local systems and period sheaves. Let K be a complete discrete valuation field of characteristic zero with perfect residue field of characteristic p.

We work on the étale site $X_{\text{ét}}$ (cf. [Hub96, Definition 2.3.1]) and the pro-étale site $X_{\text{proét}}$. For the pro-étale site, we use the one defined in [Sch13, Sch16], and thus we have the projection

$$\nu\colon X_{\text{proét}} \to X_{\text{ét}}.$$

Recall that $U \in X_{\text{proét}}$ is called *affinoid perfectoid* if U has a pro-étale presentation $U = \varprojlim U_i \to X$ by affinoid $U_i = \mathrm{Spa}(A_i, A_i^+)$ such that, denoting by A^+ the p-adic completion of $\varinjlim A_i^+$, and $A = A^+[p^{-1}]$, the pair (A, A^+) is a perfectoid affinoid (L, L^+)-algebra for a perfectoid field L over K and an open and bounded valuation subring $L^+ \subset L$ (cf. [Sch13, Definition 4.3(i)]). In this case, we write $\widehat{U}$ for $\mathrm{Spa}(A, A^+)$, which is independent of the pro-étale presentation $U = \varprojlim U_i$.

8.1 *p*-adic local systems on $X_{\text{ét}}$

Let us clarify our terminology of local systems.

DEFINITION 8.1 (cf. [Sch13, Definition 8.1], [KL15, §§ 1.4 and 8.4]). A *$\mathbb{Z}_p$-local system* on $X_{\text{ét}}$ (or an *étale $\mathbb{Z}_p$-local system* on X) is an inverse system $\mathbb{L} = (\mathbb{L}_n)$ of sheaves of $\mathbb{Z}/p^n$-modules $\mathbb{L}_n$ on $X_{\text{ét}}$ such that each $\mathbb{L}_n$ is locally on $X_{\text{ét}}$ a constant sheaf associated with a finitely generated flat $\mathbb{Z}/p^n$-module and such that this inverse system is isomorphic in the procategory to an inverse system for which $\mathbb{L}_{n+1}/p^n \cong \mathbb{L}_n$.

DEFINITION 8.2 (cf. [KL15, §§ 1.4 and 8.4]). An *isogeny $\mathbb{Z}_p$-local system* on $X_{\text{ét}}$ is an object of the isogeny category of $\mathbb{Z}_p$-local systems on $X_{\text{ét}}$. A *$\mathbb{Q}_p$-local system* on $X_{\text{ét}}$ (or an *étale $\mathbb{Q}_p$-local system* on X) is an object of the stack associated to the fibered category of isogeny $\mathbb{Z}_p$-local systems on $X_{\text{ét}}$.

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

Remark 8.3. Assume that $X = \mathrm{Spa}(A, A^+)$ is an affinoid. In this case, the natural functor from the category of lisse $\mathbb{Z}_p$-sheaves on the étale site $(\mathrm{Spec}\, A)_{\text{ét}}$ to the category of $\mathbb{Z}_p$-local systems on $\mathrm{Spa}(A, A^+)_{\text{ét}}$ is an equivalence of categories by [Hub96, Example 1.6.6 (ii)] (cf. [KL15, Remark 8.4.5]).

Moreover, if A^+ satisfies conditions in Set-up 3.1 relative to K, then the category $\mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_A)$ is equivalent to the category of isogeny $\mathbb{Z}_p$-local systems on $\mathrm{Spa}(A, A^+)_{\text{ét}}$ (cf. [KL15, Remark 1.4.4]).

Set $\widehat{\mathbb{Z}}_p = \varprojlim \mathbb{Z}/p^n$ and $\widehat{\mathbb{Q}}_p = \widehat{\mathbb{Z}}_p[p^{-1}]$ as sheaves on $X_{\text{proét}}$. For a $\mathbb{Z}_p$-local system $\mathbb{L} = (\mathbb{L}_n)$ on $X_{\text{ét}}$, we set $\widehat{\mathbb{L}} = \varprojlim \nu^* \mathbb{L}_n$. Then $\widehat{\mathbb{L}}$ is a lisse $\widehat{\mathbb{Z}}_p$-sheaf on $X_{\text{proét}}$ in the sense of [Sch13, Definition 8.1]. This functor is an equivalence of categories. Similarly, we can naturally associate to every $\mathbb{Q}_p$-local system $\mathbb{L}$ on $X_{\text{ét}}$ a lisse $\widehat{\mathbb{Q}}_p$-sheaf on $X_{\text{proét}}$, which we denote by $\widehat{\mathbb{L}}$.

8.2 Period sheaves and the Riemann–Hilbert functor D_{dR}

We will use the following sheaves on $X_{\text{proét}}$: the de Rham sheaf $\mathbb{B}_{\mathrm{dR}}$ (cf. [Sch13, Definition 6.1(iii)]) and the structural de Rham sheaf $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ (cf. [Sch13, Sch16]).

By [Sch13, Theorem 6.5(i)], for an affinoid perfectoid $U \in X_{\text{proét}}$ with $\widehat{U} = \mathrm{Spa}(\Lambda, \Lambda^+)$, we have

$$\mathbb{B}_{\mathrm{dR}}(U) = \mathbb{B}_{\mathrm{dR}}(\Lambda, \Lambda^+),$$

where the latter is the period ring defined in § 2, which agrees with the one defined in [Sch13, p. 49].

The sheaf $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ is a $\mathbb{B}_{\mathrm{dR}}$-algebra equipped with a filtration and a $\mathbb{B}_{\mathrm{dR}}$-linear connection $\nabla\colon \mathcal{O}\mathbb{B}_{\mathrm{dR}} \to \mathcal{O}\mathbb{B}_{\mathrm{dR}} \otimes_{\mathcal{O}_X} \Omega^1_X$ satisfying the Griffiths transversality and $(\mathcal{O}\mathbb{B}_{\mathrm{dR}})^{\nabla=0} = \mathbb{B}_{\mathrm{dR}}$.

We have the following description of $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$. Let $\mathrm{Spa}(A, A^+)$ be an affinoid admitting standard étale maps to both X and $\mathbb{T}^n_L$ for some finite extension L over K. If we replace L with its algebraic closure in A, then $R = A^+$ satisfies conditions in Set-up 3.1 relative to L. Write $\overline{R} = \varinjlim_{i \in I} A_i^+$ where A_i^+ is a finite normal R-subalgebra of $\overline{\mathrm{Frac}\, R}$ such that $A_i = A_i^+[p^{-1}]$ is étale over $A = R_K$. Set $U := \varprojlim_i \mathrm{Spa}(A_i, A_i^+) \in X_{\text{proét}}$; this is affinoid perfectoid with $\widehat{U} = \mathrm{Spa}(\widehat{\overline{R}}[p^{-1}], \widehat{\overline{R}})$. In this setting, $\mathbb{B}_{\mathrm{dR}}(U) = \mathbb{B}_{\mathrm{dR}}(\widehat{\overline{R}}) = \mathrm{B}^{\nabla}_{\mathrm{dR}}(R)$, and there is a natural $\mathrm{B}^{\nabla}_{\mathrm{dR}}(R)$-linear isomorphism

$$\mathcal{O}\mathbb{B}_{\mathrm{dR}}(U) \cong \mathrm{B}_{\mathrm{dR}}(R),$$

compatible with filtrations and connections, where the right-hand side is the de Rham period ring recalled in § 4.4. To see this, note that $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ is defined to be the sheafification of the presheaf defined in [Sch16, (3)]. From the definitions, we have a $\mathrm{B}^{\nabla}_{\mathrm{dR}}(R)$-algebra homomorphism $\mathrm{B}_{\mathrm{dR}}(R) \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}(U)$. By comparing [Sch13, Proposition 6.10], [Sch16, (3)], and [Bri08, Proposition 5.2.2], we conclude that this map is an isomorphism compatible with filtrations and connections.

Definition 8.4 (cf. [Sch13, Definition 8.3]). Let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on $X_{\text{ét}}$. We say that $\mathbb{L}$ is *de Rham* if there exists a filtered $\mathcal{O}_X$-module $\mathcal{E}$ with an integrable connection such that there is an isomorphism of sheaves on $X_{\text{proét}}$,

$$\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR}} \cong \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR}},$$

compatible with filtrations and connections. Similarly, we say that a(n isogeny) $\mathbb{Z}_p$-local system on $X_{\text{ét}}$ is de Rham if the associated $\mathbb{Q}_p$-local system is de Rham.

Let us recall an important theorem of Liu and Zhu.

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

Theorem 8.5 [LZ17, Theorem 3.9]. *For a $\mathbb{Q}_p$-local system $\mathbb{L}$ on $X_{\text{ét}}$, set*

$$D_{\mathrm{dR}}(\mathbb{L}) := \nu_*(\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR}}).$$

(i) *$D_{\mathrm{dR}}(\mathbb{L})$ is a vector bundle on $X_{\text{ét}}$ equipped with an integrable connection*

$$\nabla_{\mathbb{L}} := \nu_*(\mathrm{id} \otimes \nabla) \colon D_{\mathrm{dR}}(\mathbb{L}) \to D_{\mathrm{dR}}(\mathbb{L}) \otimes_{\mathcal{O}_{X_{\text{ét}}}} \Omega_{X_{\text{ét}}}$$

and a decreasing filtration $\mathrm{Fil}^\bullet$ satisfying the Griffiths transversality.

(ii) *The functor $\mathbb{L} \mapsto (D_{\mathrm{dR}}(\mathbb{L}), \nabla_{\mathbb{L}}, \mathrm{Fil}^\bullet)$ commutes with pullbacks.*

(iii) *The following conditions are equivalent for $\mathbb{L}$:*
   (a) *$\mathbb{L}$ is de Rham;*
   (b) *$\operatorname{rank} D_{\mathrm{dR}}(\mathbb{L}) = \operatorname{rank} \mathbb{L}$;*
   (c) *for each connected component, there exist a classical point x of X and a geometric point $\overline{x}$ above x such that the stalk $\mathbb{L}_{\overline{x}}$ is a de Rham representation of the absolute Galois group of $k(x)$.*

   In this case, the natural map $\nu^* D_{\mathrm{dR}}(\mathbb{L}) \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR}} \to \widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR}}$ is an isomorphism of sheaves on $X_{\text{proét}}$ compatible with filtrations and connections.

Remark 8.6. The category of vector bundles on X (with the analytic topology) is equivalent to the category of vector bundles on $X_{\text{ét}}$ by [Sch13, Lemma 7.3]. Hence, we also regard $D_{\mathrm{dR}}(\mathbb{L})$ as a vector bundle on X.

Lemma 8.7. *Assume that $X = \mathrm{Spa}(A, A^+)$ is affinoid such that $R = A^+$ satisfies conditions in Set-up 3.1 relative to K. Let $\mathbb{L}$ be an isogeny $\mathbb{Z}_p$-local system on $X_{\text{ét}}$ and let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_A)$ be the corresponding representation. Then the vector bundle $D_{\mathrm{dR}}(\mathbb{L})$ is associated to the A-module $D_{\mathrm{dR}}(V)$ and this identification is compatible with connections and filtrations. Moreover, $\mathbb{L}$ is de Rham if and only if V is a de Rham representation.*

Proof. Since $D_{\mathrm{dR}}(\mathbb{L})$ is a coherent sheaf on X, it suffices to show that $\Gamma(X, D_{\mathrm{dR}}(\mathbb{L}))$ is naturally identified with $D_{\mathrm{dR}}(V)$. By definition, we have

$$\Gamma(X, D_{\mathrm{dR}}(\mathbb{L})) = \Gamma(X_{\text{proét}}, \widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR}}).$$

Write $\overline{R} = \varinjlim_{i \in I} A_i^+$, where A_i^+ is a finite normal R-subalgebra of $\overline{\mathrm{Frac}\, R}$ such that $A_i = A_i^+[p^{-1}]$ is étale over $A = R_K$. Set $U := \varprojlim_i \mathrm{Spa}(A_i, A_i^+) \in X_{\text{proét}}$; this is affinoid perfectoid with $\widehat{U} = \mathrm{Spa}(\widehat{\overline{R}}[p^{-1}], \widehat{\overline{R}})$. Since every finite étale A-algebra splits over $\overline{R}[p^{-1}]$, the lisse $\widehat{\mathbb{Q}}_p$-sheaf $\widehat{\mathbb{L}}$ is trivial over U and identified with $\widehat{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} V$. Hence, $\Gamma(U, \widehat{\mathbb{L}}) = V$ and $\Gamma(U, \widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR}}) = V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}(R)$.

Note that the map $U \to X$ is a profinite étale Galois cover with Galois group $\mathcal{G}_A$. Regard the profinite group $\mathcal{G}_A$ as an object of $\mathrm{Spa}(K, \mathcal{O}_K)_{\text{profét}}$ (and thus an object of $X_{\text{proét}}$ by the pullback) and consider $U \times \mathcal{G}_A \in X_{\text{proét}}$. Then we can check that there is a natural isomorphism $U \times_X U \cong U \times \mathcal{G}_A$ compatible with the first projection to U. Under this identification, we have $\Gamma(U \times_X U, \widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR}}) = \prod_{\mathcal{G}_A} (V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}(R))$. Moreover, the two maps $p_1^*, p_2^* \colon V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}(R) \to \prod_{\mathcal{G}_A} (V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}(R))$ induced by $p_1, p_2 \colon U \times_X U \to U$ send $v \in V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}(R)$ to

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

$(v)_{\sigma\in\mathcal{G}_A}$ and $(\sigma(v))_{\sigma\in\mathcal{G}_A}$. Hence, we compute

$$\begin{aligned}\Gamma(X, D_{\mathrm{dR}}(\mathbb{L})) &= \Gamma(X_{\mathrm{pro\acute{e}t}}, \widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR}}) \\ &= \mathrm{Ker}(\Gamma(U, \widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR}}) \xrightarrow{p_1^*-p_2^*} \Gamma(U \times_X U, \widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR}})) \\ &= \mathrm{Ker}(V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}(R) \xrightarrow{p_1^*-p_2^*} \prod_{\mathcal{G}_A} (V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}(R))) \\ &= H^0_{\mathrm{cont}}(\mathcal{G}_A, V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}(R)) = D_{\mathrm{dR}}(V).\end{aligned}$$

We can easily see that this identification is compatible with connections and filtrations.

If V is a de Rham representation, then $D_{\mathrm{dR}}(V)$ is a finite projective A-module of rank equal to $\dim_{\mathbb{Q}_p} V = \mathrm{rank}\,\mathbb{L}$. Hence, $\mathbb{L}$ is de Rham. Conversely, if $\mathbb{L}$ is de Rham, we have an isomorphism $\nu^* D_{\mathrm{dR}}(\mathbb{L}) \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR}} \to \widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR}}$ of sheaves on $X_{\mathrm{pro\acute{e}t}}$. Evaluating both sides at $U \in X_{\mathrm{pro\acute{e}t}}$ yields $D_{\mathrm{dR}}(V) \otimes_A \mathrm{B}_{\mathrm{dR}}(R) \xrightarrow{\cong} V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}(R)$ and this is identified with $\alpha_{\mathrm{dR}}(V)$. Hence, V is a de Rham representation. □

9. A p-adic monodromy theorem for de Rham local systems

Let K be a complete discrete valuation field of characteristic zero with perfect residue field of characteristic p. For a rigid analytic variety X over K (i.e. a quasi-separated adic space locally of finite type over $\mathrm{Spa}(K, \mathcal{O}_K)$), we call a point x of X *classical* if the residue class field $k(x)$ of x is finite over K. For a complete extension L of K, we write X_L for $X \times_{\mathrm{Spa}(K,\mathcal{O}_K)} \mathrm{Spa}(L, \mathcal{O}_L)$.

9.1 Statement of the theorem

Definition 9.1. Let X be a smooth rigid analytic variety over K, and let $\mathbb{L}$ be an isogeny $\mathbb{Z}_p$-local system on $X_{\text{ét}}$.

(i) We say that $\mathbb{L}$ is *horizontal de Rham* if $\mathbb{L}$ is de Rham and $(D_{\mathrm{dR}}(\mathbb{L}), \nabla)$ has a full set of solutions, that is, $(D_{\mathrm{dR}}(\mathbb{L}), \nabla) \cong (\mathcal{O}_X, d)^{\mathrm{rank}\,\mathbb{L}}$.

(ii) Let x be a classical point of X. We say that $\mathbb{L}$ is *semistable* (respectively, *(potentially) crystalline*) at x if for some (equivalently, any) geometric point $\overline{x}$ above x, the stalk $\mathbb{L}_{\overline{x}}$ is a semistable (respectively, (potentially) crystalline) representation of the absolute Galois group of $k(x)$.

Theorem 1.1 is a special case of the following p-adic monodromy theorem for de Rham local systems.

Theorem 9.2. *Let X be a smooth rigid analytic variety over K, and let $\mathbb{L}$ be a de Rham isogeny $\mathbb{Z}_p$-local system on $X_{\text{ét}}$. Then for every classical point $x \in X$, there exist an open neighborhood $U \subset X_{k(x)}$ of x and a finite extension L of $k(x)$ such that $\mathbb{L}|_{U_L}$ is horizontal de Rham and semistable at every classical point. Moreover, if $\mathbb{L}_{\overline{x}}$ is potentially crystalline, then U and L can be chosen in such a way that $\mathbb{L}|_{U_L}$ is horizontal de Rham and crystalline at every classical point.*

Let us first make a remark and discuss its consequence.

Remark 9.3. If x runs over all the classical points of X, the images in X of the U in the theorem form a collection of open subsets of X. In general, the union of such open subsets may not be the entire X.

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

Corollary 9.4. *Assume that the residue field of K is finite. Let X be a quasi-compact smooth rigid analytic variety over K, and let $\mathbb{L}$ be a de Rham $\mathbb{Q}_p$-local system on $X_{\text{ét}}$. Then there exists a finite extension L of K such that for every K-rational point $x \in X$ and every geometric point $\overline{x}$ above x, the representation $\mathbb{L}_{\overline{x}}$ of the absolute Galois group of $k(x)$ becomes semistable over L.*

Proof of Corollary 9.4. Note that the set of K-valued points of X is quasi-compact with respect to the analytic topology; the general case is reduced to the case where X is a closed polydisk, and the assertion is obvious in this case.

If $\mathbb{L}$ is an isogeny $\mathbb{Z}_p$-local system, then the assertion follows from Theorem 9.2 and the quasi-compactness of $X(K)$. In general, take an étale covering $\{U_i \to X\}_{i \in I}$ such that $\mathbb{L}|_{U_i}$ is an isogeny $\mathbb{Z}_p$-local system on $U_{i,\text{ét}}$ for each $i \in I$. Since X is quasi-compact, we may assume that each U_i is quasi-compact and that the index set I is finite. Set $U := \bigsqcup_{i \in I} U_i$ and let $f \colon U \to X$ denote the structure morphism. Note that f is étale and quasi-compact. Hence, there exists $n \in \mathbb{N}$ such that every geometric fiber of f has cardinality at most n. Now let K' be the composite of all the finite extensions of K of degree at most n. Since K is assumed to have finite residue field, K' is finite over K. By applying the above discussion to $(U, \mathbb{L}|_U, K')$, we obtain a finite extension L of K' such that for every $x \in U(K')$ and every geometric point $\overline{x}$ above x, $\mathbb{L}_{\overline{x}}$ becomes semistable over L. By construction, $X(K)$ is contained in the image of $U(K')$ under f. Hence, this L satisfies the requirement. □

We prove Theorem 9.2 in the rest of this section. To this end, we show that $(D_{\text{dR}}(\mathbb{L}), \nabla)$ is analytic locally horizontal de Rham. Then the assertion follows from Theorems 7.3 and 7.4.

9.2 Preliminaries on smooth rigid analytic varieties

The following lemma is easy to prove and we omit the proof. Recall that $\mathbb{B}^n_K$ denotes the n-dimensional closed unit polydisk $\mathrm{Spa}(K\langle T_1, \dots, T_n\rangle, \mathcal{O}_K\langle T_1, \dots, T_n\rangle)$.

Lemma 9.5. *Let $x \in \mathbb{B}^n_K$ be a point with residue class field K. For every open neighborhood V of x in $\mathbb{B}^n_K$, there exists an open immersion of the form $\mathbb{B}^n_K \hookrightarrow V$ sending 0 to x.*

Proposition 9.6. *Let X be an n-dimensional smooth rigid analytic variety over K. Let L be a finite extension of K and $x \in X$ a classical point with residue class field L. Then there exists an open immersion $\mathbb{B}^n_L \to X_L$ sending $0 \in \mathbb{B}^n_L$ to $x \in X_L$.*

Proof. By replacing X by X_L, we may assume $L = K$. By shrinking X if necessary, we may assume that the structure morphism $X \to \mathrm{Spa}(K, \mathcal{O}_K)$ factors as $X \xrightarrow{f} \mathbb{B}^n_K \to \mathrm{Spa}(K, \mathcal{O}_K)$, where f is étale and the second is the structure morphism [Hub96, Corollary 1.6.10]. We may further assume that $f(x) = 0$ by translation.

By [Hub96, Corollary 1.7.4, Lemma 1.6.4], the map of local rings $f^* \colon \mathcal{O}_{\mathbb{B}^n_K, 0} \to \mathcal{O}_{X,x}$ is finite with the same residue class field K. Recall that $\mathcal{O}_{\mathbb{B}^n_K, 0}$ is Henselian (cf. [Hou60, p. 9, Corollary 2], [Ber94, Theorem 2.1.5]). Hence, we conclude that $\mathcal{O}_{\mathbb{B}^n_K, 0} \cong \mathcal{O}_{X,x}$.

It follows that there exist an affinoid open neighborhood U of $x \in X$, an affinoid open neighborhood V of $0 \in \mathbb{B}^n_K$, and a morphism $g \colon V \to U$ such that $f \circ g$ is the inclusion $V \subset \mathbb{B}^n_K$. Note that g is injective and that $g^* \colon \mathcal{O}_{U,x} \to \mathcal{O}_{V,0}$ is an isomorphism. After shrinking V and U if necessary, we may further assume that g is an isomorphism by [BGR84, 7.3.3 Proposition 5]. By Lemma 9.5, we get an open immersion $\mathbb{B}^n_K \to V \xrightarrow{g} U \to X$ sending 0 to x. □

9.3 Vector bundles with integrable connections on the polydisk $\mathbb{B}^n_K$

Let $A := K\langle T_1, \dots, T_n\rangle$ and let $d \colon A \to \Omega_A$ denote the universal continuous K-derivation of A [Hub96, Definition 1.6.1]. Concretely, Ω_A is a finite free A-module of rank n with generators

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

$dT_1, \dots, dT_n$, and $d\colon A \to \Omega_A$ sends f to $df = \sum_{i=1}^n (\partial f/\partial T_i)\, dT_i$ for $f \in A$ [Hub96, 1.6.2]. In other words, $\Omega_A = A \otimes_{A^\circ} \widehat{\Omega}_{A^\circ}$ (cf. § 4.4).

The following result will be well known to experts. For example, see [DGS94, Appendix III], [Ked10a, Proposition 9.3.3] for the case $n = 1$.

Theorem 9.7. *Let (M, ∇) be a finite free A-module equipped with an integrable connection. Then there exists $l \in \mathbb{N}$ such that the induced connection on*

$$M \otimes_A K\langle T_1/p^l, \dots, T_n/p^l\rangle$$

has a full set of solutions, that is, it is isomorphic to $(K\langle T_1/p^l, \dots, T_n/p^l\rangle, d)^{\operatorname{rank}_A M}$.

Proof. Consider the natural inclusion $M \subset M \otimes_A K[[T_1, \dots, T_n]]$ and extend the integrable connection to the latter. By a classical result (cf. [Kat70, Proposition 8.9]), we know that $M \otimes_A K[[T_1, \dots, T_n]]$ has a full set of solutions. In fact, we can construct an explicit K-linear surjection

$$P\colon M \to (M \otimes_A K[[T_1, \dots, T_n]])^{\nabla=0}$$

using the Taylor expansion, which we now explain.

For each $i = 1, \dots, n$, we set $D_i := \nabla_{\partial/\partial T_i}\colon M \to M$. Then the integrability of ∇ is equivalent to $D_i \circ D_{i'} = D_{i'} \circ D_i$ for every $i, i' = 1, \dots, n$. For each $\underline{j} = (j_1, \dots, j_n) \in \mathbb{N}^n$, we set $|\underline{j}| = j_1 + \cdots + j_n$, $\underline{j}! := (j_1!) \cdots (j_n!)$, $T^{\underline{j}} := T_1^{j_1} \cdots T_n^{j_n}$, and $D_{\underline{j}} := \prod_{i=1}^n (D_i)^{j_i}\colon M \to M$.

We define $P\colon M \to M \otimes_A K[[T_1, \dots, T_n]]$ by

$$P(f) := \sum_{\underline{j} \in \mathbb{N}^n} \frac{(-1)^{|\underline{j}|}}{\underline{j}!} T^{\underline{j}} D_{\underline{j}}(f).$$

Then we can check that $P(f)(0, \dots, 0) = f(0, \dots, 0)$, $\operatorname{Im} P \subset (M \otimes_A K[[T_1, \dots, T_n]])^{\nabla=0}$, $\operatorname{Ker} P = (T_1, \dots, T_n)M$, and that P induces an isomorphism

$$P\colon M/(T_1, \dots, T_n) \xrightarrow{\cong} (M \otimes_A K[[T_1, \dots, T_n]])^{\nabla=0}.$$

Let $r = \operatorname{rank}_A M$ and choose an A-basis of M so that we identify $M = A^r$. For each $f \in A^r$, we can uniquely write $P(f)$ as

$$P(f) = \sum_{\underline{j} \in \mathbb{N}^n} A_{\underline{j}}(f) T^{\underline{j}}, \quad A_{\underline{j}}(f) \in K^r.$$

More concretely, we see that

$$A_{\underline{j}}(f) = \left(\left(\frac{\partial}{\partial T}\right)^{\underline{j}} (P(f))\right)\Big|_{(0,\dots,0)} = \left(\left(\frac{\partial}{\partial T}\right)^{\underline{j}} \left(\sum_{\underline{j}' \le \underline{j}} \frac{(-1)^{|\underline{j}'|}}{\underline{j}'!} T^{\underline{j}'} D_{\underline{j}'}(f)\right)\right)\Big|_{(0,\dots,0)},$$

where $(\partial/\partial T)^{\underline{j}} := (\partial/\partial T_1)^{j_1} \cdots (\partial/\partial T_n)^{j_n}$ for $\underline{j} = (j_1, \dots, j_n)$, and $\underline{j}' \le \underline{j}$ refers to the inequality entrywise. In this way, we get a K-linear map $A_{\underline{j}}\colon A^r \to K^r$.

Consider the Gauss norm on A, and equip $M = A^r$ and $M_r(A)$ with the norm defined by the maximum of the Gauss norm of each entry. Similarly, equip K^r with the p-adic norm. We normalize these norms so that the norm of p is p^{-1}. We will estimate the operator norm $\|A_{\underline{j}}\|$. For each $i = 1, \dots, n$, write

$$D_i = \frac{\partial}{\partial T_i} + B_i, \quad B_i \in M_r(A).$$

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

Choose $C' > 0$ such that $\|B_i\| \leq C'$ for $i = 1, \ldots, n$. Set $C = \max\{1, C'\}$. Note that both $\partial/\partial T_i \colon A^r \to A^r$ and the multiplication by T_i on A^r have operator norm one and that the p-adic norm of $\underline{j}!$ is greater than or equal to $(1/p^{1/(p-1)})^{|\underline{j}|}$. It follows that

$$\left\| \frac{(-1)^{|\underline{j}|}}{\underline{j}!} T^{\underline{j}} D_{\underline{j}} \right\| \leq C^{|\underline{j}|} p^{|\underline{j}|/(p-1)},$$

and thus

$$\|A_{\underline{j}}\| = \left\| \left(\left(\frac{\partial}{\partial T} \right)^{\underline{j}} \left(\sum_{\underline{j}' \leq \underline{j}} \frac{(-1)^{|\underline{j}'|}}{\underline{j}'!} T^{\underline{j}'} D_{\underline{j}'} \right) \right) \Big|_{(0,\ldots,0)} \right\| \leq C^{|\underline{j}|} p^{|\underline{j}|/(p-1)}.$$

Choose $l \in \mathbb{N}$ such that $Cp^{1/(p-1)} p^{-l} < 1$. By the estimate on $\|A_{\underline{j}}\|$, we have $\operatorname{Im} P \subset M \otimes_A K\langle T_1/p^l, \ldots, T_n/p^l \rangle$. In other words, we have

$$(M \otimes_A K[[T_1, \ldots, T_n]])^{\nabla=0} \subset M \otimes_A K\langle T_1/p^l, \ldots, T_n/p^l \rangle.$$

We again identify $M = A^r$ and $M \otimes_A K\langle T_1/p^l, \ldots, T_n/p^l \rangle = K\langle T_1/p^l, \ldots, T_n/p^l \rangle^r$. Choose $s_1 = (s_{11}, \ldots, s_{1r}), \ldots, s_r = (s_{r1}, \ldots, s_{rr}) \in M \otimes_A K\langle T_1/p^l, \ldots, T_n/p^l \rangle$ such that $s_1, \ldots, s_r$ form a K-basis of $(M \otimes_A K[[T_1, \ldots, T_n]])^{\nabla=0}$ and such that the values at $(T_1, \ldots, T_n) = (0, \ldots, 0)$ satisfy

$$s_1(0, \ldots, 0) = (1, 0, \ldots, 0), \ldots, s_r(0, \ldots, 0) = (0, \ldots, 0, 1).$$

Set $s := \det(s_{ij})_{i,j=1,\ldots,r} \in K\langle T_1/p^l, \ldots, T_n/p^l \rangle$. Then s satisfies $s(0, \ldots, 0) = 1 \neq 0$. In particular, there exists $l' \geq l$ such that s is invertible in $K\langle T_1/p^{l'}, \ldots, T_n/p^{l'} \rangle$.

Let us consider the induced connection ∇ on $M \otimes_A K\langle T_1/p^{l'}, \ldots, T_n/p^{l'} \rangle$. Since $s_1, \ldots, s_r \in M \otimes_A K\langle T_1/p^{l'}, \ldots, T_n/p^{l'} \rangle$, we have

$$(M \otimes_A K\langle T_1/p^{l'}, \ldots, T_n/p^{l'} \rangle)^{\nabla=0} \subset M \otimes_A K\langle T_1/p^{l'}, \ldots, T_n/p^{l'} \rangle.$$

Moreover, since s is invertible in $K\langle T_1/p^{l'}, \ldots, T_n/p^{l'} \rangle$, we conclude that the map

$$(M \otimes_A K\langle T_1/p^{l'}, \ldots, T_n/p^{l'} \rangle)^{\nabla=0} \otimes_K K\langle T_1/p^{l'}, \ldots, T_n/p^{l'} \rangle \to M \otimes_A K\langle T_1/p^{l'}, \ldots, T_n/p^{l'} \rangle$$

is an isomorphism. This means that $(M \otimes_A K\langle T_1/p^{l'}, \ldots, T_n/p^{l'} \rangle, \nabla)$ has a full set of solutions. □

9.4 Proof of Theorem 9.2

We prove Theorem 9.2. By Proposition 9.6 and Theorem 9.7, there exists an open neighborhood $U \subset X_{k(x)}$ of x such that $U \cong \mathbb{B}^n_{k(x)}$ and $(D_{\mathrm{dR}}(\mathbb{L})|_U, \nabla)$ has a full set of solutions. Hence, $\mathbb{L}|_U$ is horizontal de Rham because $D_{\mathrm{dR}}(\mathbb{L})|_U = D_{\mathrm{dR}}(\mathbb{L}|_U)$. By shrinking U again, we may assume that U is isomorphic to $\mathbb{T}^n_{k(x)}$. Write $\mathbb{T}^n_{k(x)} = \mathrm{Spa}(A, A^\circ)$. Then $R = A^\circ$ satisfies the conditions in Set-up 3.1 relative to $k(x)$. Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_A)$ be the representation corresponding to $\mathbb{L}|_U$. Then V is horizontal de Rham by Lemma 8.7. By Theorem 7.3, there exists a finite extension L of $k(x)$ such that $V|_{\mathcal{G}_{A_L}}$ is horizontal semistable. In particular, $\mathbb{L}|_{U_L}$ is horizontal de Rham and semistable at every classical point. The second assertion follows from Theorem 7.4.

10. Potentially crystalline loci and potentially good reduction loci

As an application of our p-adic monodromy theorem, we discuss potentially crystalline loci of de Rham local systems and cohomologically potentially good reduction loci of smooth proper families of relative dimension at most two.

Let K be a complete discrete valuation field of characteristic zero with perfect residue field of characteristic p. In §§ 10.2 and 10.3, we will further assume that the residue field of K is finite.

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

10.1 Potentially crystalline loci

Theorem 10.1. *For every smooth rigid analytic variety X over K and every de Rham $\mathbb{Q}_p$-local system $\mathbb{L}$ on $X_{\text{ét}}$, there exists an open subset $U \subset X$ such that the following assertions hold for every classical point $x \in X$:*

(i) *if $x \in U$, then $\mathbb{L}$ is potentially crystalline at x;*
(ii) *if $x \notin U$, then $\mathbb{L}$ is not potentially crystalline at x.*

We call the largest open subset satisfying these properties the potentially crystalline locus of $\mathbb{L}$.

Proof. Observe that if there exists an étale covering $\{X_i\}_i$ of X such that each $\mathbb{L}|_{X_i}$ admits an open $U_i \subset X_i$ satisfying (i) and (ii) for X_i, then the union of the images of the U_i in X satisfies (i) and (ii) for X. By passing to an étale cover of X, we may assume that $\mathbb{L}$ is a de Rham isogeny $\mathbb{Z}_p$-local system on $X_{\text{ét}}$.

Let x be a classical point of X such that $\mathbb{L}$ is potentially crystalline at x. By Theorem 9.2, there exists an open neighborhood $U'_x \subset X_{k(x)}$ of x such that $\mathbb{L}|_{U'_x}$ is potentially crystalline at every classical point. Let U_x denote the image of U'_x in X; it is an open subset of X. Then the union of such U_x satisfies the desired properties. □

10.2 The monodromy-weight conjecture and ℓ-independence

We will discuss cohomologically potentially good reduction loci of smooth proper maps in the next subsection. For this purpose, let us briefly review basic results and conjectures on cohomologies of smooth proper varieties over p-adic fields.

Let k denote the residue field of K and assume that k is *finite* of cardinality $q = p^h$. Set $P_0 := W(k)[p^{-1}]$ and write $\sigma\colon P_0 \to P_0$ for the Witt vector Frobenius lifting $x \mapsto x^p$.

Set $\mathrm{Gal}_K := \mathrm{Gal}(\overline{K}/K)$. Let I_K denote the inertia subgroup of Gal_K and let W_K denote the Weil group of K; they fit in the exact sequence $0 \longrightarrow I_K \longrightarrow W_K \xrightarrow{v} h\mathbb{Z} \longrightarrow 0$, where v is normalized so that $h \in h\mathbb{Z} \subset h\widehat{\mathbb{Z}} = \mathrm{Gal}_k$ is the geometric Frobenius $a \mapsto a^{-q}$ of k. Set $W_K^+ := \{g \in W_K \mid v(g) \geq 0\}$.

Let X be a smooth proper scheme purely of dimension n over K. For each prime ℓ, the ℓ-adic étale cohomology $H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is an ℓ-adic representation of Gal_K.

First, assume $\ell \neq p$. Let $t_\ell\colon I_K \to \mathbb{Z}_\ell(1)$ be the map sending $g \in I_K$ to $(g(\pi^{\ell^{-m}})/\pi^{\ell^{-m}})_m \in \mathbb{Z}_\ell(1)$, where π is a uniformizer of K. By Grothendieck's monodromy theorem, there exist a nilpotent endomorphism $N \in \mathrm{End}(H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell))(-1)$ and an open subgroup $I \subset I_K$ such that for $g \in I$, the action of g on $H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is given by $\exp(t_\ell(g)N)$. The endomorphism N defines the monodromy filtration $\mathrm{Fil}^N_\bullet$ on $H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$; it is the unique separated and exhaustive increasing filtration such that $N(\mathrm{Fil}^N_s) \subset \mathrm{Fil}^N_{s-2}(-1)$ for $s \in \mathbb{Z}$ and such that the induced map $N^s\colon \mathrm{gr}^N_s H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell) \to \mathrm{gr}^N_{-s} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)(-s)$ is an isomorphism for $s \geq 0$.

Conjecture 10.2 (ℓ-adic monodromy-weight conjecture for $\ell \neq p$). For $s \in \mathbb{Z}$ and $g \in W_K$, the eigenvalues of g on $\mathrm{gr}^N_s H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ are algebraic numbers and the complex absolute values of their conjugates are $p^{(m+s)v(g)/2}$.

Remark 10.3. When X has good reduction, the conjecture follows from Deligne's purity theorem on weights in [Del80]. The conjecture is also known for $m \leq 2$; When X has a strictly semistable model and $\dim X = 2$, it is [RZ82, Satz 2.13]. Using [dJ96], we can reduce the general case to this case (see [Sai03, Lemma 3.9]).

Next, assume $\ell = p$. Let us recall the general recipe of Fontaine [Fon94c].

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

Let V be a de Rham $\mathbb{Q}_p$-representation of Gal_K. By the p-adic monodromy theorem [Ber02, Theorem 0.7], V is potentially semistable. We set

$$\widehat{D}_{\mathrm{pst}}(V) := \varinjlim_{H \subset I_K} (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^H,$$

where H runs over open subgroups of I_K. It is a P_0-vector space of dimension $\dim_{\mathbb{Q}_p} V$ equipped with a semilinear action of Gal_K, a Frobenius φ, and a monodromy operator N satisfying $N\varphi = p\varphi N$. Such an object is called a *p-module of Deligne* in [Fon94c, 1.1]. It follows that φ is bijective and that N is nilpotent.

We associate to $\widehat{D}_{\mathrm{pst}}(V)$ a Weil–Deligne representation $W\widehat{D}_{\mathrm{pst}}(V)$ over P_0 as follows. Set $W\widehat{D}_{\mathrm{pst}}(V) := \widehat{D}_{\mathrm{pst}}(V)$ as a P_0-vector space equipped with the monodromy operator N. Define a P_0-linear action of W_K on $W\widehat{D}_{\mathrm{pst}}(V)$ by letting $g \in W_K$ act as $g \cdot \varphi^{v(g)}$. Since N acts nilpotently, there exists a unique separated and exhaustive increasing filtration $\mathrm{Fil}^N_\bullet$ on $W\widehat{D}_{\mathrm{pst}}(V)$ such that $N(\mathrm{Fil}^N_s) \subset \mathrm{Fil}^N_{s-2}$ for $s \in \mathbb{Z}$ and such that $N^s\colon \mathrm{gr}^N_s \to \mathrm{gr}^N_{-s}$ is an isomorphism for every $s \geq 0$.

Let us return to the p-adic representation $V = H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$. It is de Rham by [Tsu99] and [dJ96] (cf. [Tsu02, Theorem A1]). We denote by $H^m_{\mathrm{pst},p}(X)$ the associated Weil–Deligne representation $W\widehat{D}_{\mathrm{pst}}(H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ of K.

The following statement is part of the conjecture of [Jan88, p. 347] and [Jan10, Conjecture 5.1].

Conjecture 10.4 (p-adic monodromy-weight conjecture). For $s \in \mathbb{Z}$ and $g \in W_K$, the eigenvalues of g on $\mathrm{gr}^N_s H^m_{\mathrm{pst},p}(X)$ are algebraic numbers and the complex absolute values of their conjugates are $p^{(m+s)v(g)/2}$.

The next result is well known to experts, but we give a proof for completeness.

Theorem 10.5. *Conjecture 10.4 holds for $m \leq 2$.*

Proof. We follow the proof of [Sai03, Lemma 3.9]. Note that for a finite extension L of K, $H^m_{\mathrm{pst},p}(X)$ and $H^m_{\mathrm{pst},p}(X_L)$ are isomorphic as Weil–Deligne representations of L (after rescaling the monodromy operator; see [Fon94c, 2.2.6]). It follows that the assertion for $H^m_{\mathrm{pst},p}(X)$ is equivalent to the one for $H^m_{\mathrm{pst},p}(X_L)$.

We may assume that X is geometrically connected over K. By [dJ96, Theorem 6.5], there exist a finite extension L of K inside $\overline{K}$, a smooth projective scheme X' over L, and a proper surjective and generically finite morphism $f\colon X' \to X$. The morphism f induces a Gal_L-equivariant injection $f^*\colon H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \to H^m_{\text{ét}}(X'_{\overline{K}}, \mathbb{Q}_p)$ and thus an injective map $f^*\colon H^m_{\mathrm{pst},p}(X_L) \to H^m_{\mathrm{pst},p}(X')$ of Weil–Deligne representations of L. In this case, it suffices to prove the assertion for the latter, and thus we may further assume that X is projective by replacing X by X' and K by L. By replacing X by a general hyperplane section, we may also assume $\dim X \leq 2$.

By applying [dJ96, Theorem 6.5] again, we may assume that X admits a projective and strictly semistable model $\mathcal{X}$ over $\mathcal{O}_K$. Denote the special fiber $\mathcal{X} \otimes_{\mathcal{O}_K} k$ by Y. Let $M_{\mathcal{X}}$ be the fine log structure on $\mathcal{X}$ associated to the simple normal crossing divisor $Y \subset \mathcal{X}$ and let M_Y be the inverse image of $M_{\mathcal{X}}$ on Y. Write $H^m_{\log\mathrm{cris}}(Y/W)$ for the log-crystalline cohomology of $(Y, M_Y)/(k, \mathbb{N} \oplus k^\times)$ over $W := W(k)$ and set $H^m_{\log\mathrm{cris}}(\mathcal{X}) := H^m_{\log\mathrm{cris}}(Y/W) \otimes_W K_0$, where $K_0 := W[p^{-1}]$. Then $H^m_{\log\mathrm{cris}}(\mathcal{X})$ has a natural (φ, N)-module structure relative to K.[6] By [Tsu99, Theorem 0.2], we have an isomorphism $D_{\mathrm{st}}(H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H^m_{\log\mathrm{cris}}(\mathcal{X})$ of

[6] A choice of a uniformizer of K equips $H^m_{\log\mathrm{cris}}(\mathcal{X})$ with a filtration over K by the comparison isomorphism to the log-de Rham cohomology, but we do not need this structure.

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

(φ, N)-modules relative to K.[7] It follows that $\widehat{D}_{\mathrm{pst}}(H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H^m_{\log\,\mathrm{cris}}(\mathcal{X}) \otimes_{K_0} P_0$ as p-modules of Deligne.

Note that φ^h acts K_0-linearly on $H^m_{\log\,\mathrm{cris}}(\mathcal{X})$ and the monodromy operator defines the monodromy filtration $\mathrm{Fil}^N_\bullet$ on $H^m_{\log\,\mathrm{cris}}(\mathcal{X})$. By the definition of the Weil–Deligne representation $H^m_{\mathrm{pst},p}(X)$ and the above isomorphism of p-modules of Deligne, it suffices to prove that the eigenvalues of φ^h on $\mathrm{gr}^N_s H^m_{\log\,\mathrm{cris}}(\mathcal{X})$ are algebraic numbers and the complex absolute values of their conjugates are $p^{(m+s)h/2} = q^{(m+s)/2}$.

For $r \geq 1$, let $Y^{(r)}$ denote the disjoint union of all r-fold intersections of the different irreducible components of Y. Then we have the p-adic weight spectral sequence

$$E_1^{-l,m+l} = \bigoplus_{j \geq \max\{-l,0\}} H^{m-2j-l}_{\mathrm{cris}}(Y^{(2j+l+1)}/W)(-j-l) \otimes_W K_0 \implies H^m_{\log\,\mathrm{cris}}(\mathcal{X}).$$

This spectral sequence is φ-equivariant (cf. [Nak05, (9.11.1)] and [Nak06, § 2 (B)]). By the purity of weights of the E_1-term [KM74], the spectral sequence degenerates at E_2, inducing the filtration $\mathrm{Fil}^W_\bullet$ on $H^m_{\log\,\mathrm{cris}}(\mathcal{X})$ such that the eigenvalues of φ^h on $\mathrm{gr}^W_s H^m_{\log\,\mathrm{cris}}(\mathcal{X})$ are algebraic numbers whose complex conjugates have absolute value $q^{(m+s)/2}$. By [Mok93, Theorem 5.3, Corollary 6.2.3] and [Nak06, Theorem 8.3, Corollary 8.4], two filtrations $\mathrm{Fil}^N_\bullet$ and $\mathrm{Fil}^W_\bullet$ on $H^m_{\log\,\mathrm{cris}}(\mathcal{X})$ agree. This completes the proof. □

To compare the results for ℓ and p, let us recall a theorem on ℓ-independence.

Theorem 10.6 [Och99, Theorems B and 3.1]. *Let X be a smooth proper scheme over K of dimension n. For every $g \in W_K^+$, $\sum_{m=0}^{2n} \mathrm{tr}(g^*; H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell))$ is a rational integer that is independent of ℓ with $\ell \neq p$. Moreover, it is equal to $\sum_{m=0}^{2n} \mathrm{tr}(g^*; H^m_{\mathrm{pst},p}(X))$.*

Corollary 10.7. *Let X be a smooth proper scheme over K of dimension at most two. For every $m \in \mathbb{N}$ and $g \in W_K^+$, the trace $\mathrm{tr}(g^*; H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell))$ is a rational integer that is independent of ℓ with $\ell \neq p$. Moreover, it is equal to $\mathrm{tr}(g^*; H^m_{\mathrm{pst},p}(X))$.*

Proof. The cases $m = 0, \dim X$ are obvious. The same assertion for an abelian variety with $m = 1$ is also known (cf. [Fon94c, Remark 2.4.6(iii)] and [Noo17, Corollary 2.2]). From these results, it remains to deal with the case where $\dim X = 2$ and $m = 2$; this follows from Theorem 10.6. □

Corollary 10.8. *Let X be a smooth proper scheme of dimension at most two over K and let $m \in \mathbb{N}$. For a rational prime $\ell \neq p$, the ℓ-adic representation $H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is potentially unramified if and only if the p-adic representation $H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is potentially crystalline.*

Proof. The former condition in the assertion is equivalent to the condition that $N = 0$ on $H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$. Since the ℓ-adic monodromy-weight conjecture is known for X (Remark 10.3), it is also equivalent to the condition that, for every $g \in W_K^+$, the eigenvalues of g on $H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$, which are algebraic numbers, satisfy the property that the complex absolute values of their conjugates are $p^{mv(g)/2}$. Similarly, by Theorem 10.5, the latter condition in the assertion is equivalent to the condition that, for every $g \in W_K^+$, the eigenvalues of g on $H^m_{\mathrm{pst},p}(X)$, which are algebraic numbers, satisfy the property that the complex absolute values of their conjugates are $p^{mv(g)/2}$. Hence, the assertion easily follows from Corollary 10.7. □

[7] Depending on the convention of the monodromy operators, we may need to change the sign of one of the monodromy operators, but this does not affect our argument.

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

10.3 Cohomologically potentially good reduction locus

Let K be a complete discrete valuation field of characteristic zero with *finite* residue field of characteristic p.

Theorem 10.9. *Let $f\colon Y \to X$ be a smooth proper morphism between smooth algebraic varieties over K and let X^{ad} denote the adic space associated to X. Assume that the relative dimension of f is at most two. Then for each $m \in \mathbb{N}$, there exists an open subset U of X^{ad} such that the following assertions hold for every classical point x of X^{ad} and every geometric point $\overline{x}$ above x:*

(i) *if $x \in U$, then $(R^m f_* \mathbb{Q}_\ell)_{\overline{x}}$ is potentially unramified for $\ell \neq p$ and $(R^m f_* \mathbb{Q}_p)_{\overline{x}}$ is potentially crystalline as Galois representations of $k(x)$;*

(ii) *if $x \notin U$, then $(R^m f_* \mathbb{Q}_\ell)_{\overline{x}}$ is not potentially unramified for $\ell \neq p$ and $(R^m f_* \mathbb{Q}_p)_{\overline{x}}$ is not potentially crystalline as Galois representations of $k(x)$.*

We call the largest open subset satisfying these properties the cohomologically potentially good reduction locus *of f of degree m.*

Proof. By Corollary 10.8, the theorem follows from Theorem 10.1 or [Kis99, Proposition 5.2 (2)]. □

Acknowledgements

I would like to thank Takeshi Tsuji for sending the preprint [Tsu] and Shun Ohkubo for email correspondence on the paper [Ohk13]. I am grateful to Alexander Betts, Hélène Esnault, Mark Kisin, Teruhisa Koshikawa, and Alex Youcis for valuable discussions and comments about this work. I would also like to thank the anonymous referee for a careful reading of the paper and helpful corrections and suggestions.

References

AK00 D. Abramovich and K. Karu, *Weak semistable reduction in characteristic* 0, Invent. Math. **139** (2000), 241–273.

ALT18 K. Adiprasito, G. Liu and M. Temkin, *Semistable reduction in characteristic* 0, Preprint (2018), arXiv:1810.03131v2.

AI12 F. Andreatta and A. Iovita, *Semistable sheaves and comparison isomorphisms in the semistable case*, Rend. Semin. Mat. Univ. Padova **128** (2012), 131–285.

AI13 F. Andreatta and A. Iovita, *Comparison isomorphisms for smooth formal schemes*, J. Inst. Math. Jussieu **12** (2013), 77–151.

Ber02 L. Berger, *Représentations p-adiques et équations différentielles*, Invent. Math. **148** (2002), 219–284.

BC08 L. Berger and P. Colmez, *Familles de représentations de de Rham et monodromie p-adique*, Astérisque **319** (2008), 303–337, Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (ϕ, Γ)-modules.

Ber90 V. G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, vol. 33 (American Mathematical Society, Providence, RI, 1990).

Ber94 V. G. Berkovich, *Étale cohomology for non-Archimedean analytic spaces*, Publ. Math. Inst. Hautes Études Sci. **78** (1993), 5–161 (1994).

BO83 P. Berthelot and A. Ogus, *F-isocrystals and de Rham cohomology. I*, Invent. Math. **72** (1983), 159–199.

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

BGR84 S. Bosch, U. Güntzer and R. Remmert, *A systematic approach to rigid analytic geometry*, in *Non-Archimedean analysis*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 261 (Springer, Berlin, 1984).

Bri06 O. Brinon, *Représentations cristallines dans le cas d'un corps résiduel imparfait*, Ann. Inst. Fourier (Grenoble) **56** (2006), 919–999.

Bri08 O. Brinon, *Représentations p-adiques cristallines et de de Rham dans le cas relatif*, Mém. Soc. Math. Fr. (N.S.) **112** (2008.

Col02 P. Colmez, *Espaces de Banach de dimension finie*, J. Inst. Math. Jussieu **1** (2002), 331–439.

CF00 P. Colmez and J.-M. Fontaine, *Construction des représentations p-adiques semi-stables*, Invent. Math. **140** (2000), 1–43.

dJ96 A. J. de Jong, *Smoothness, semi-stability and alterations*, Publ. Math. Inst. Hautes Études Sci. **83** (1996), 51–93.

Del80 P. Deligne, *La conjecture de Weil. II*, Publ. Math. Inst. Hautes Études Sci. **52** (1980), 137–252.

DGS94 B. Dwork, G. Gerotto and F. J. Sullivan, *An introduction to G-functions*, Annals of Mathematics Studies, vol. 133 (Princeton University Press, Princeton, NJ, 1994).

Fal88 G. Faltings, *p-adic Hodge theory*, J. Amer. Math. Soc. **1** (1988), 255–299.

Fal89 G. Faltings, *Crystalline cohomology and p-adic Galois-representations*, in *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)* (Johns Hopkins University Press, Baltimore, MD, 1989), 25–80.

Fal90 G. Faltings, *F-isocrystals on open varieties: results and conjectures*, in *The Grothendieck Festschrift, Vol. II*, Progress in Mathematics, vol. 87 (Birkhäuser, Boston, MA, 1990), 219–248.

Fal02 G. Faltings, *Almost étale extensions*, in *Cohomologies p-adiques et applications arithmétiques (II)*, Astérisque, vol. 279 (Société mathématique de France, 2002), 185–270.

Fal10 G. Faltings, *Coverings of p-adic period domains*, J. Reine Angew. Math. **643** (2010), 111–139.

Far18 L. Fargues, *La courbe*, in *Proceedings of the International Congress of Mathematicians (Rio de Janeiro 2018): Vol. II, Invited lectures* (World Scientific Publishing, Hackensack, NJ, 2018), 291–319.

Fon94a J.-M. Fontaine, *Le corps des périodes p-adiques*, in *Périodes p-adiques – Séminaire de Bures sur Yvette, 1988*, Astérisque, vol. 223 (Société mathématique de France, 1994), 59–111.

Fon94b J.-M. Fontaine, *Représentations p-adiques semi-stables*, in *Périodes p-adiques – Séminaire de Bures sur Yvette, 1988*, Astérisque, vol. 223 (Société mathématique de France, 1994), 113–184.

Fon94c J.-M. Fontaine, *Représentations ℓ-adiques potentiellement semi-stables*, in *Périodes p-adiques – Séminaire de Bures sur Yvette, 1988*, Astérisque, vol. 223 (Société mathématique de France, 1994), 321–347.

Har13 U. Hartl, *On a conjecture of Rapoport and Zink*, Invent. Math. **193** (2013), 627–696.

Hou60 C. Houzel, *Géométrie analytique locale, II. Théorie des morphismes finis*, Séminaire Henri Cartan **13** (1960–1961), talk no. 19.

Hub94 R. Huber, *A generalization of formal schemes and rigid analytic varieties*, Math. Z. **217** (1994), 513–551.

Hub96 R. Huber, *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, vol. E30 (Friedr. Vieweg & Sohn, Braunschweig, 1996).

Hyo89 O. Hyodo, *On variation of Hodge-Tate structures*, Math. Ann. **284** (1989), 7–22.

IM20 N. Imai and Y. Mieda, *Potentially good reduction loci of Shimura varieties*, Tunis. J. Math. **2** (2020), 399–454.

Jan88 U. Jannsen, *On the Galois cohomology of ℓ-adic representations attached to varieties over local or global fields*, in *Séminaire de Théorie des Nombres, Paris 1986–87*, Progress in Mathematics, vol. 75 (Birkhäuser, Boston, MA, 1988), 165–182.

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

Jan10 U. Jannsen, *Weights in arithmetic geometry*, Jpn. J. Math. **5** (2010), 73–102.

Kat70 N. M. Katz, *Nilpotent connections and the monodromy theorem: applications of a result of Turrittin*, Publ. Math. Inst. Hautes Études Sci. **39** (1970), 175–232.

Kat73 N. M. Katz, *Travaux de Dwork*, in *Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 409*, Lecture Notes in Mathematics, vol. 317 (Springer, Berlin, 1973), 167–200.

KM74 N. M. Katz and W. Messing, *Some consequences of the Riemann hypothesis for varieties over finite fields*, Invent. Math. **23** (1974), 73–77.

Ked10a K. S. Kedlaya, *p-adic differential equations*, Cambridge Studies in Advanced Mathematics, vol. 125 (Cambridge University Press, Cambridge, 2010).

Ked10b K. S. Kedlaya, *Relative p-adic Hodge theory and Rapoport-Zink period domains*, in *Proceedings of the International Congress of Mathematicians. Volume II* (Hindustan Book Agency, New Delhi, 2010), 258–279.

KL15 K. S. Kedlaya and R. Liu, *Relative p-adic Hodge theory: foundations*, Astérisque **371** (2015), 239.

Kis99 M. Kisin, *Local constancy in p-adic families of Galois representations*, Math. Z. **230** (1999), 569–593.

LL21 B. Lawrence and S. Li, *A counterexample to an optimistic guess about étale local systems*, C. R. Math. **359** (2021), 923–924 (en).

LZ17 R. Liu and X. Zhu, *Rigidity and a Riemann-Hilbert correspondence for p-adic local systems*, Invent. Math. **207** (2017), 291–343.

Mok93 A. Mokrane, *La suite spectrale des poids en cohomologie de Hyodo-Kato*, Duke Math. J. **72** (1993), 301–337.

Mor14 K. Morita, *Crystalline and semi-stable representations in the imperfect residue field case*, Asian J. Math. **18** (2014), 143–157.

Nak05 Y. Nakkajima, *p-adic weight spectral sequences of log varieties*, J. Math. Sci. Univ. Tokyo **12** (2005), 513–661.

Nak06 Y. Nakkajima, *Signs in weight spectral sequences, monodromy-weight conjectures, log Hodge symmetry and degenerations of surfaces*, Rend. Semin. Mat. Univ. Padova **116** (2006), 71–185.

Noo17 R. Noot, *The p-adic representation of the Weil-Deligne group associated to an abelian variety*, J. Number Theory **172** (2017), 301–320.

Och99 T. Ochiai, *l-independence of the trace of monodromy*, Math. Ann. **315** (1999), 321–340.

Ohk13 S. Ohkubo, *The p-adic monodromy theorem in the imperfect residue field case*, Algebra Number Theory **7** (2013), 1977–2037.

RZ82 M. Rapoport and Th. Zink, *Über die lokale Zetafunktion von Shimuravarietäten. Monodromiefiltration und verschwindende Zyklen in ungleicher Charakteristik*, Invent. Math. **68** (1982), 21–101.

Sai03 T. Saito, *Weight spectral sequences and independence of ℓ*, J. Inst. Math. Jussieu **2** (2003), 583–634.

Sch02 P. Schneider, *Nonarchimedean functional analysis*, Springer Monographs in Mathematics (Springer, Berlin, 2002).

Sch12 P. Scholze, *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 245–313.

Sch13 P. Scholze, *p-adic Hodge theory for rigid-analytic varieties*, Forum Math. Pi **1** (2013), e1.

Sch16 P. Scholze, *p-adic Hodge theory for rigid-analytic varieties—corrigendum*, Forum Math. Pi **4** (2016), e6.

SGA1 A. Grothendieck, *Revêtements étales et groupe fondamental, Séminaire de Géométrie Algébrique du Bois-Marie 1960–1961 (SGA 1)*, Lecture Notes in Mathematics, vol. 224 (Springer, Berlin, 1971).

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press

SGA7 A. Grothendieck, *Groupes de monodromie en géométrie algébrique I, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I)*, Lecture Notes in Mathematics, vol. 288 (Springer, Berlin, 1972).

Shi18 K. Shimizu, *Constancy of generalized Hodge-Tate weights of a local system*, Compos. Math. **154** (2018), 2606–2642.

TT19 F. Tan and J. Tong, *Crystalline comparison isomorphisms in p-adic Hodge theory: the absolutely unramified case*, Algebra Number Theory **13** (2019), 1509–1581.

Tsu T. Tsuji, *Crsytalline sheaves and filtered convergent F-isocrystals on log schemes*, Preprint.

Tsu99 T. Tsuji, *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case*, Invent. Math. **137** (1999), 233–411.

Tsu02 T. Tsuji, *Semi-stable conjecture of Fontaine-Jannsen: a survey*, in *Cohomologies p-adiques et applications arithmétiques (II)*, Astérisque, vol. 279 (Société mathématique de France, 2002), 323–370.

Koji Shimizu shimizu@tsinghua.edu.cn
Department of Mathematics, UC Berkeley, 943 Evans Hall, Berkeley, CA 94720-3840, USA

Current address: Yau Mathematical Sciences Center, Tsinghua University, W10 Ningzhai, Haidian, Beijing 100084, China

https://doi.org/10.1112/S0010437X2200776X Published online by Cambridge University Press