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Quasi-Static Linear Thermo-Viscoelastic Process with Irregular Viscous Dissipation

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Abstract. We consider a mathematical model which describes the quasi-static evolution of a thermo-viscoelastic linear body with taking into account the effects of internal forces which generate a non linear viscous dissipative function. We derive a variational formulation of the system of equilibrium equation and energy equation. An existence result of weak solutions was obtained in an appropriate function space.

AMS subject classifications: 35M10, 74D05, 74F05

Key words: Thermo-viscoelasticity, viscous dissipation.

1 Introduction

The constitutive laws with internal variables have been used in various publications in order to model the effect of internal variables in the behaviour of real bodies like metals, rocks polymers and so on, for which the rate of deformation depends on the internal variables. Some of the internal state variables considered by many authors are the spatial display of dislocation, the work-hardening of materials and the absolute temperature, see for examples and details [3,7,10,11,14,16,19] and references therein for other internal state variables.

The stress-strain behaviour has a direct relation with the temperature. Indeed, the work of internal forces generates a temperature and, inversely, variations of temperature may generate deformations in rigid materials. The physical experiences have prove that mechanical properties change dramatically with temperature, going from glass-like brittle behaviour at low temperatures to a rubber-like behaviour at high temperatures. In

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general, decreasing the strain rate has the same influence on the strain-strength characteristics as increasing the temperature: the material becomes softer and more ductile.

In the study of the influence of temperature on the behaviour of rigid materials (elastic or plastic) majority of mathematicians, see for example [14, 19], assumed, to simplify the calculations, that the dissipation is a Lipschitz function of strain, strain rate and stress tensors. But physically, this assumption can not describe all the forces of collision and interaction between the particles constituting the material. In fact, the dissipation function can be written as product of the stress tensor and the dissipative part of the strain rate tensor.

The aim object of this paper is to study the quasi-static evolution of linear thermo-viscoelastic materials by considering real viscous dissipative function. For this, we consider a rate-type constitutive law of the form

$$\frac{\partial \sigma}{\partial t} = \mathcal{A}\left(\varepsilon\left(\frac{\partial \mathbf{u}}{\partial t}\right)\right) + \mu(\theta)\left(\mathcal{G}_1(\sigma) + \mathcal{G}_2(\varepsilon(\mathbf{u}))\right), \tag{1.1}$$

in which $\mathbf{u}$, σ represent, respectively, the displacement field and stress field, θ represents the absolute temperature, $\mathcal{A}$, $\mathcal{G}_1$ and $\mathcal{G}_2$ are real tensors and μ is a real function. The paper is organized as follows. In Section 2 we present the mechanical problem of the quasi-static evolution of thermo-viscoelastic materials, we introduce some notations and preliminaries and we derive the variational formulation of the problem. We prove in Section 3 the existence of solutions and we try to apply our results to the thermo-viscoelastic Maxwell model.

2 Problem statement and preliminaries

Let $\Omega \subset \mathbb{R}^n$ $(n=2,3)$ be a bounded domain with a Lipschitz boundary Γ, partitioned into two disjoint measurable parts Γ_1 and Γ_2 such that $\|\Gamma_1\| > 0$ and let $Q = \Omega \times (0,T)$. We denote by $\mathbb{S}_n$ the space of symmetric tensors on $\mathbb{R}^n$. We define the inner product and the Euclidean norm on $\mathbb{R}^n$ and $\mathbb{S}_n$, respectively, by

$$\mathbf{u}\cdot\mathbf{v} = u_i v_i, \quad \forall \mathbf{u},\mathbf{v} \in \mathbb{R}^n \quad \text{and} \quad \sigma\cdot\tau = \sigma_{ij}\tau_{ij} \quad \forall \sigma,\tau \in \mathbb{S}_n,$$
$$\|\mathbf{u}\| = (\mathbf{u}\cdot\mathbf{u})^{\frac{1}{2}}, \quad \forall \mathbf{u} \in \mathbb{R}^n \quad \text{and} \quad \|\sigma\| = (\sigma\cdot\sigma)^{\frac{1}{2}}, \quad \forall \sigma \in \mathbb{S}_n.$$

Here and below, the indices i and j run from 1 to n and the summation convention over repeated indices is used.

For the rest of this article, we will denote by c possibly different positive constants depending only on the data of the problem. Denote by p' and q' the conjugates of p and

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q, where $1\leq p<\frac{n}{n-1}$, $q\geq 1$ and let $s\geq n$. We define the function spaces

$$
\begin{aligned}
&H=L^2(\Omega)^n=\{\mathbf{u}=\{u_i\}\,|u_i\in L^2(\Omega)\},\\
&\mathcal{H}=\{\boldsymbol{\sigma}=\{\sigma_{ij}\}\,|\sigma_{ij}=\sigma_{ji}\in L^2(\Omega)\},\\
&H_1=\{\mathbf{u}\in H|\boldsymbol{\varepsilon}(\mathbf{u})\in\mathcal{H}\},\\
&\mathcal{H}_1=\{\boldsymbol{\sigma}\in\mathcal{H}|div\boldsymbol{\sigma}\in H\},\\
&\mathcal{X}=\left\{\varphi\in L^2\left(0,T;H^1(\Omega)\right)\cap\mathcal{C}^0\left([0,T],L^2(\Omega)\right),\ \frac{\partial\varphi}{\partial t}\in L^2\left(0,T;H^1(\Omega)'\right)\right\},\\
&\mathcal{Y}_{p,q}=\left\{\varphi\in L^q\left(0,T;W^{1,p}(\Omega)\right),\ \frac{\partial\varphi}{\partial t}\in L^q\left(0,T;W^{-1,p}(\Omega)\right)\right\},\\
&\mathcal{Z}_{p,q}=\left\{\varphi\in W^{1,q}\left(0,T;W^{1,p}(\Omega)\right),\ \varphi(x,T)=0\text{ in }\Omega\right\}.
\end{aligned}
$$

Here $\boldsymbol{\varepsilon}:H_1\longrightarrow\mathcal{H}$ and $div:\mathcal{H}_1\longrightarrow H$ are the deformation (small strain) and the divergence (stress) operators, respectively, defined by

$$
\boldsymbol{\varepsilon}(\mathbf{u})=\left(\varepsilon_{ij}(\mathbf{u})\right),\quad \varepsilon_{ij}(\mathbf{u})=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right),\quad div\boldsymbol{\sigma}=\left(\sigma_{ij,j}\right).
$$

H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$
\begin{aligned}
&\langle\mathbf{u},\mathbf{v}\rangle_H=\int_\Omega u_iv_idx, && \langle\boldsymbol{\sigma},\boldsymbol{\tau}\rangle_{\mathcal{H}}=\int_\Omega\sigma_{ij}\tau_{ij}dx,\\
&\langle\mathbf{u},\mathbf{v}\rangle_{H_1}=\langle\mathbf{u},\mathbf{v}\rangle_H+\langle\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v})\rangle_{\mathcal{H}}, && \langle\boldsymbol{\sigma},\boldsymbol{\tau}\rangle_{\mathcal{H}_1}=\langle\sigma,\tau\rangle_{\mathcal{H}}+\langle div\sigma,div\tau\rangle_H.
\end{aligned}
$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively.

In addition, $\mathcal{X}$, $\mathcal{Y}_{p,q}$ and $\mathcal{Z}_{p,q}$ are Banach spaces equipped, respectively, with the norms

$$
\begin{aligned}
&\|\zeta\|_{\mathcal{X}}=\|\zeta\|_{L^2(0,T;H^1(\Omega))}+\|\zeta\|_{\mathcal{C}^0([0,T],L^2(\Omega))}+\left\|\frac{\partial\zeta}{\partial t}\right\|_{L^2\left(0,T;H^1(\Omega)'\right)}, && \forall\zeta\in\mathcal{X},\\
&\|\zeta\|_{\mathcal{Y}_{p,q}}=\|\zeta\|_{L^q\left(0,T;W^{1,p}(\Omega)\right)}+\left\|\frac{\partial\zeta}{\partial t}\right\|_{L^q\left(0,T;W^{-1,p}(\Omega)\right)}, && \forall\zeta\in\mathcal{Y}_{p,q},\\
&\|\zeta\|_{\mathcal{Z}_{p,q}}=\|\zeta\|_{L^q\left(0,T;W^{1,p}(\Omega)\right)}+\left\|\frac{\partial\zeta}{\partial t}\right\|_{L^q\left(0,T;W^{1,p}(\Omega)\right)}, && \forall\zeta\in\mathcal{Z}_{p,q}.
\end{aligned}
$$

Since the boundary Γ is Lipschitz continuous, the unit outward normal vector field $\mathbf{n}$ on the boundary is defined a.e. Let

$$
H_\Gamma=\left(H^{\frac{1}{2}}(\Gamma)\right)^n
$$

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and $\gamma: H_1 \longrightarrow H_\Gamma$ be the trace map.

We also denote by $\mathcal{V}$ the closed subspace of H_1 defined by

$$\mathcal{V} = \{\mathbf{v} \in H_1 | \gamma\mathbf{v} = 0 \text{ on } \Gamma_1\}.$$

Since $\|\Gamma_1\| > 0$, Korn's inequality holds in $\mathcal{V}$ and thus there exists a positive constant C_0 depending only on Ω, Γ such that

$$\|\boldsymbol{\varepsilon}(\mathbf{v})\|_{\mathcal{H}} \geq C_0 \|\mathbf{v}\|_{H_1} \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}.$$

Moreover, the dual of H_Γ is denoted H'_Γ. If $\sigma \in \mathcal{H}_1$ there exists $\sigma\nu \in H'_\Gamma$ such that the following Green formula holds :

$$\int_\Omega \sigma \cdot \boldsymbol{\varepsilon}(\mathbf{v}) dx + \int_\Omega div\sigma \cdot \mathbf{v} dx = (\sigma\nu, \gamma\mathbf{v})_{H'_\Gamma \times H_\Gamma}, \quad \forall \mathbf{v} \in H_1.$$

In addition, if σ is sufficiently regular (say $\mathcal{C}^1$), then

$$\int_\Omega \sigma \cdot \boldsymbol{\varepsilon}(\mathbf{v}) dx + \int_\Omega div\sigma \cdot \mathbf{v} dx = \int_\Gamma \sigma\nu \cdot \gamma\mathbf{v} d\gamma, \quad \forall \mathbf{v} \in H_1,$$

where $d\gamma$ represents the surface element.

The physical setting is the following. A linear thermo-viscoelastic body occupies the domain Ω. We assume that the body is clamped on $\Gamma_1 \times (0,T)$ and therefore the displacement field vanishes there. Surface tractions of density $\mathbf{f}_0$ act on $\Gamma_2 \times (0,T)$ and a volume forces of density $\mathbf{f}$ is applied in Q. In addition, we admit a possible external heat source applied in Q, given by the function r. Moreover, we take into account the effect of internal forces that generate a non linear viscous dissipative function.

The mechanical problem may be formulated as follows.

Problem P1. Find the displacement field $\mathbf{u}: Q \longrightarrow \mathbb{R}^n$, the stress field $\sigma: Q \longrightarrow \mathbb{S}_n$ and the temperature $\theta: Q \longrightarrow \mathbb{R}$ such that

$$\frac{\partial \sigma}{\partial t} = \mathcal{A}\left(\boldsymbol{\varepsilon}\left(\frac{\partial \mathbf{u}}{\partial t}\right)\right) + \mu(\theta)(\mathcal{G}_1(\sigma) + \mathcal{G}_2(\boldsymbol{\varepsilon}(\mathbf{u}))) \qquad \text{in } Q, \tag{2.1a}$$

$$div(\sigma) + \mathbf{f} = 0 \qquad \text{in } Q, \tag{2.1b}$$

$$\frac{\partial \theta}{\partial t} - div(\mathcal{K}(\nabla\theta)) = \boldsymbol{\varepsilon}^d\left(\frac{\partial \mathbf{u}}{\partial t}\right) \cdot \sigma + r \qquad \text{in } Q, \tag{2.1c}$$

$$\mathbf{u} = 0 \qquad \text{on } \Gamma_1 \times (0,T), \tag{2.1d}$$

$$\sigma\nu = \mathbf{f}_0 \qquad \text{on } \Gamma_2 \times (0,T), \tag{2.1e}$$

$$\frac{\partial \theta}{\partial \mathbf{n}} = 0 \qquad \text{on } \Gamma \times (0,T), \tag{2.1f}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \ \sigma(0) = \sigma_0 \quad \text{and} \quad \theta(0) = \theta_0 \qquad \text{in } \Omega. \tag{2.1g}$$

This problem represents the quasistatic evolution of a linear thermo-viscoelastic material with viscous dissipation. Relation (2.1a) is the thermo-viscoelastic constitutive law

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where $\mathcal{A}$ is a real tensor describing the elastic properties of the material, μ is a real function which represents the viscosity of the material, corresponding to the thermal properties and $\mathcal{G}_1$, $\mathcal{G}_2$ are real tensors describing the viscoelastic behaviour of the material. This constitutive law is also called "Standard linear solid model", see [10]. (2.1b) is the equilibrium equation. Eq. (2.1c) represents the energy conservation where $\mathcal{K}$ is an isotropic real tensor called the thermal conductivity tensor and the term $\boldsymbol{\varepsilon}^d(\frac{\partial \mathbf{u}}{\partial t})\cdot\boldsymbol{\sigma}$ is the viscous dissipative function, generated by the work of internal forces, in which $\boldsymbol{\varepsilon}^d$ denotes the dissipative part in the strain rate tensor and r is a given volume heat source.

Equalities (2.1d) and (2.1e) are the displacement-traction boundary conditions, respectively. (2.1f) is an homogeneous Neumann boundary conditions on $\Gamma\times(0,T)$ for the temperature. Finally, the functions $\mathbf{u}_0$, σ_0 and θ_0 in (2.1g) represent the initial data.

In the study of the mechanical problem P1 we consider the following hypotheses:

$$\begin{cases} \mathcal{A}:\Omega\times\mathbb{S}_n\longrightarrow\mathbb{S}_n \text{ is a tensor verifying :}\\ \text{(a) } \mathcal{A}_{ijkh}\in L^\infty(\Omega),\quad \forall i,j,k,h=1,\cdots,n;\\ \text{(b) } \mathcal{A}(x)\boldsymbol{\sigma}\cdot\boldsymbol{\tau}=\sigma\cdot\mathcal{A}(x)\cdot\boldsymbol{\tau},\quad \forall\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}_n,\ \text{a.e. in }\ \Omega;\\ \text{(c) There exists an } \alpha_{\mathcal{A}}>0 \text{ such that}\\ \qquad \mathcal{A}(x)\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\geq\alpha_{\mathcal{A}}\|\boldsymbol{\sigma}\|^2,\quad \forall\boldsymbol{\sigma}\in\mathbb{S}_n,\ \text{a.e. in }\Omega, \end{cases} \tag{2.2a}$$

$$\begin{cases} \mathcal{G}_e:\Omega\times\mathbb{S}_n\longrightarrow\mathbb{S}_n,(e=1,2), \text{ is a tensor verifying :}\\ \text{(a) } (\mathcal{G}_e)_{ijkh}\in L^\infty(\Omega),\quad \forall i,j,k,h=1,\cdots,n;\\ \text{(b) } \mathcal{G}_e(x)\boldsymbol{\sigma}\cdot\boldsymbol{\tau}=\sigma\cdot\mathcal{G}_e(x)\boldsymbol{\tau},\quad \forall\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}_n,\ \text{a.e. in }\ \Omega, \end{cases} \tag{2.2b}$$

$$\begin{cases} \mathcal{K}:\Omega\times\mathbb{R}^n\longrightarrow\mathbb{R}^n \text{ is a tensor verifying :}\\ \text{(a) } \mathcal{K}_{ij}(x)=\mathcal{K}_{ji}(x)\in L^\infty(\Omega),\quad \forall i,j=1,\cdots,n,\ \text{a.e. in }\ \Omega;\\ \text{(b) There exists an } \alpha_{\mathcal{K}}>0 \text{ such that}\\ \qquad \mathcal{K}(x)\mathbf{v}\cdot\mathbf{v}\geq\alpha_{\mathcal{K}}\|\mathbf{v}\|^2,\quad \forall\mathbf{v}\in\mathbb{R}^n,\ \text{a.e. in }\ \Omega; \end{cases} \tag{2.2c}$$

$$\begin{cases} \mu:\mathbb{R}\longrightarrow\mathbb{R} \text{ is a real function verifying :}\\ \text{(a) } \mu\in\mathcal{C}^0(\mathbb{R});\\ \text{(b) There exists } \mu_*,\mu^*>0 \text{ such that}\\ \qquad \mu_*\leq\mu(s)\leq\mu^*,\quad \forall s\in\mathbb{R}, \end{cases} \tag{2.2d}$$

$$\begin{cases} \mathbf{f}\in W^{1,+\infty}(0,T;H),\ \mathbf{f}_0\in W^{1,+\infty}\left(0,T;L^2(\Gamma_2)^n\right),\\ r\in L^1(Q), \end{cases} \tag{2.2e}$$

$$\mathbf{u}_0\in\mathcal{V},\quad \sigma_0\in\mathcal{H}_1\quad\text{and}\quad \theta_0\in L^1(\Omega). \tag{2.2f}$$

We denote by $\mathbf{F}\in\mathcal{V}'$ the following element:

$$\langle\mathbf{F}(t),\mathbf{v}\rangle_{\mathcal{V}'\times\mathcal{V}}=\int_\Omega\mathbf{f}(t)\cdot\mathbf{v}dx+\int_{\Gamma_2}\mathbf{f}_0(t)\cdot\gamma\mathbf{v}d\gamma,\quad\forall\mathbf{v}\in\mathcal{V}. \tag{2.3}$$

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The use of (2.2e) permits us to verify that

$$\mathbf{F}\in W^{1,+\infty}\left(0,T;\mathcal{V}'\right). \tag{2.4}$$

Moreover, we remark that hypotheses (2.2a), (2.2b) and (2.2c) imply the existence of positive constants $m_{\mathcal{A}}$, m_e, $(e=1,2)$ and $m_{\mathcal{K}}$ such that

$$\|\mathcal{A}\sigma\|_{\mathcal{H}}\leq m_{\mathcal{A}}\|\sigma\|_{\mathcal{H}}, \qquad \forall\sigma\in\mathcal{H}, \tag{2.5a}$$
$$\|\mathcal{G}_e\sigma\|_{\mathcal{H}}\leq m_e\|\sigma\|_{\mathcal{H}}, \quad (e=1,2), \qquad \forall\sigma\in\mathcal{H}, \tag{2.5b}$$
$$\|\mathcal{K}\mathbf{v}\|_H\leq m_{\mathcal{K}}\|\mathbf{v}\|_H, \qquad \forall\mathbf{v}\in H. \tag{2.5c}$$

Furthermore, one can check that under the hypothesis (2.2a), the constitutive law (2.1a) can be rewritten

$$\varepsilon\left(\frac{\partial\mathbf{u}}{\partial t}\right)=\mathcal{A}^{-1}\left(\frac{\partial\sigma}{\partial t}\right)-\mu(\theta)\mathcal{A}^{-1}(\mathcal{G}_1(\sigma)+\mathcal{G}_2(\varepsilon(\mathbf{u}))) \quad \text{in } \Omega\times(0,T),$$

where $\mathcal{A}^{-1}$ denotes the inverse tensor of $\mathcal{A}$.

Consequently the non dissipative and the dissipative parts of the strain rate tensor are, respectively

$$\varepsilon^{nd}\left(\frac{\partial\mathbf{u}}{\partial t}\right)=\mathcal{A}^{-1}\left(\frac{\partial\sigma}{\partial t}\right), \tag{2.6a}$$
$$\varepsilon^{d}\left(\frac{\partial\mathbf{u}}{\partial t}\right)=-\mu(\theta)\mathcal{A}^{-1}(\mathcal{G}_1(\sigma)+\mathcal{G}_2(\varepsilon(\mathbf{u}))). \tag{2.6b}$$

Using the above notations and Green's formula, we can easily derive the following variational formulation of the mechanical problem P1.

Problem P2. Find the displacement field $\mathbf{u}:Q\longrightarrow\mathbb{R}^n$, the stress field $\sigma:Q\longrightarrow\mathrm{S}_n$ and the temperature $\theta:Q\longrightarrow\mathbb{R}$ such that

$$\frac{\partial\sigma}{\partial t}=\mathcal{A}\left(\varepsilon\left(\frac{\partial\mathbf{u}}{\partial t}\right)\right)+\mu(\theta(t))(\mathcal{G}_1(\sigma(t))+\mathcal{G}_2(\varepsilon(\mathbf{u}(t)))) \qquad \text{a.e. } t\in(0,T), \tag{2.7a}$$
$$\int_\Omega\sigma(t)\cdot\varepsilon(\mathbf{v})dx=\int_\Omega\mathbf{F}(t)\cdot\mathbf{v}dx, \quad \forall\mathbf{v}\in\mathcal{V}, \qquad \text{a.e. } t\in(0,T), \tag{2.7b}$$
$$\begin{aligned}&\int_\Omega\frac{\partial\theta}{\partial t}\varphi dx+\int_\Omega\mathcal{K}(\nabla\theta(t))\cdot\nabla\varphi dx\\&=-\int_\Omega\mu(\theta(t))\mathcal{A}^{-1}(\mathcal{G}_1(\sigma)+\mathcal{G}_2(\varepsilon(\mathbf{u})))\cdot\sigma(t)\,\varphi dx+\int_\Omega r(t)\,\varphi dx,\\&\quad\forall\varphi\in W^{1,p'}(\Omega), \qquad \text{a.e. } t\in(0,T),\end{aligned} \tag{2.7c}$$
$$\mathbf{u}(0)=\mathbf{u}_0, \quad \sigma(0)=\sigma_0 \quad \text{and} \quad \theta(0)=\theta_0 \qquad \text{in } \Omega. \tag{2.7d}$$

Remark 2.1. If $(\mathbf{u}(t),\sigma(t))$ has at least the regularity $\mathcal{V}\times\mathcal{H}$ a.e. $t\in(0,T)$, then in Eq. (2.7c), the terms on the right hand side have sense, since the injection $W^{1,p'}(\Omega)\subset\mathcal{C}^0\left(\overline{\Omega}\right)$ is continuous for $p'>n$, that is, $p<\frac{n}{n-1}$.

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Definition 2.1. We will say that a function $\theta \in \mathcal{Y}_{p,q}$ is a weak solution of the variational equation (2.7c) if

$$\begin{aligned}&-\int_Q \theta \frac{\partial \varphi}{\partial t} dxdt + \int_Q \mathcal{K}(\nabla\theta)\cdot\nabla\varphi dxdt \\ &= -\int_Q \mu(\theta)\mathcal{A}^{-1}(\mathcal{G}_1(\sigma)+\mathcal{G}_2(\varepsilon(\mathbf{u})))\cdot\sigma\varphi dxdt + \int_Q r\varphi dxdt + \int_\Omega \theta_0\varphi(0)dx, \quad \forall \varphi \in \mathcal{Z}_{p',q'}.\end{aligned}$$

We can then reformulate the variational problem P3 as follows.

Problem P3. Find the displacement field $\mathbf{u}(t) \in \mathcal{V}$, the stress field $\sigma(t) \in \mathcal{H}_1$ and the temperature $\theta(t) \in \mathcal{Y}_{p,q}$ such that

$$\frac{\partial \sigma}{\partial t} = \mathcal{A}\left(\varepsilon\left(\frac{\partial \mathbf{u}}{\partial t}\right)\right) + \mu(\theta(t))(\mathcal{G}_1(\sigma(t))+\mathcal{G}_2(\varepsilon(\mathbf{u}(t)))) \text{ a.e. } t\in(0,T), \tag{2.8a}$$

$$\int_\Omega \sigma(t)\cdot''(\mathbf{v})dx = \int_\Omega \mathbf{F}(t)\cdot\mathbf{v}dx \text{ a.e. } t\in(0,T), \quad \forall \mathbf{v}\in\mathcal{V}, \tag{2.8b}$$

$$\begin{aligned}-\int_Q \theta\frac{\partial\varphi}{\partial t}dxdt + \int_Q \mathcal{K}(\nabla\theta)\cdot\nabla\varphi dxdt = &-\int_Q \mu(\theta)\mathcal{A}^{-1}(\mathcal{G}_1(\sigma)+\mathcal{G}_2(\varepsilon(\mathbf{u})))\cdot\sigma\varphi dxdt \\ &+\int_Q r\varphi dxdt + \int_\Omega \theta_0\varphi(0)dx, \quad \forall\varphi\in\mathcal{Z}_{p',q'}.\end{aligned} \tag{2.8c}$$

3 Existence results

The main result of this section is stated by the following existence theorem.

Theorem 3.1. *Under the assumptions* (2.2a)-(2.2f), *there exists at least one solution* $(\mathbf{u},\sigma,\theta) \in W^{1,\infty}(0,T;\mathcal{V})\times W^{1,\infty}(0,T;\mathcal{H}_1)\times\mathcal{Y}_{p,q}$ *to problem P3, where q is such that*

$$\begin{cases} 1 \le q < 2, \\ \dfrac{2}{q}+\dfrac{n}{p} > n+1. \end{cases} \tag{3.1}$$

The proof will be carried up by several steps. Based on classical results of functional analysis concerning evolution problems, Banach and Kakutani-Glicksberg fixed point theorems, see [8, 16], as well as some monotonicity and compactness arguments, using two auxiliary existence results.

Let $\lambda \in \mathcal{Y}_{p,q}$ and consider the following auxiliary problem.

Problem $P_{1\lambda}$. Find the displacement field $\mathbf{u}_\lambda \in W^{1,\infty}(0,T;\mathcal{V})$ and the stress field $\sigma_\lambda \in W^{1,\infty}(0,T;\mathcal{H}_1)$ such that

$$\frac{\partial\sigma_\lambda}{\partial t} = \mathcal{A}\left(\varepsilon\left(\frac{\partial\mathbf{u}_\lambda}{\partial t}\right)\right) + \mu(\lambda(t))(\mathcal{G}_1(\sigma_\lambda(t))+\mathcal{G}_2(\varepsilon(\mathbf{u}_\lambda(t)))) \quad \text{a.e. } t\in(0,T), \tag{3.2a}$$

$$\int_\Omega \sigma_\lambda(t)\cdot\varepsilon(\mathbf{v})dx = \int_\Omega \mathbf{F}(t)\cdot\mathbf{v}dx, \quad \forall\mathbf{v}\in\mathcal{V}, \quad \text{a.e. } t\in(0,T), \tag{3.2b}$$

$$\mathbf{u}_\lambda(0)=\mathbf{u}_0, \quad \sigma_\lambda(0)=\sigma_0 \quad \text{in } \Omega. \tag{3.2c}$$

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Lemma 3.1. *For all $\lambda\in\mathcal{Y}_{p,q}$ there exists a unique solution $(\mathbf{u}_\lambda,\boldsymbol{\sigma}_\lambda)\in W^{1,\infty}(0,T;\mathcal{V})\times W^{1,\infty}(0,T;\mathcal{H}_1)$ to the auxiliary problem $(P_{1\lambda})$. Moreover, there exists two constants $d_1,d_2>0$ such that*

$$\|\mathbf{u}_\lambda\|_{W^{1,\infty}(0,T;\mathcal{V})}\leq d_1, \tag{3.3a}$$

$$\|\boldsymbol{\sigma}_\lambda\|_{W^{1,\infty}(0,T;\mathcal{H}_1)}\leq d_2. \tag{3.3b}$$

Proof. Take an arbitrary

$$\boldsymbol{\eta}\in L^\infty(0,T;\mathcal{H}), \tag{3.4}$$

and let Z_η be the function

$$Z_\eta(t)=\int_0^t\boldsymbol{\eta}(s)ds+\boldsymbol{\sigma}_0-\mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{u}_0)).$$

Considering the following auxiliary problem.

Problem $\mathrm{P}_{1\lambda}^{\eta}$. Find the displacement field $\mathbf{u}_{\lambda\eta}\in W^{1,\infty}(0,T;\mathcal{V})$ and the stress field $\boldsymbol{\sigma}_{\lambda\eta}\in W^{1,\infty}(0,T;\mathcal{H}_1)$ such that

$$\boldsymbol{\sigma}_{\lambda\eta}(t)=\mathcal{A}\left(\boldsymbol{\varepsilon}\left(\mathbf{u}_{\lambda\eta}(t)\right)\right)+Z_\eta(t) \qquad \text{a.e. } t\in(0,T), \tag{3.5a}$$

$$\int_\Omega\boldsymbol{\sigma}_{\lambda\eta}(t)\cdot\boldsymbol{\varepsilon}(\mathbf{v})dx=\int_\Omega\mathbf{F}(t)\cdot\mathbf{v}dx,\quad\forall\mathbf{v}\in\mathcal{V}, \qquad \text{a.e. } t\in(0,T), \tag{3.5b}$$

$$\mathbf{u}_{\lambda\eta}(0)=\mathbf{u}_0,\quad\boldsymbol{\sigma}_{\lambda\eta}(0)=\boldsymbol{\sigma}_0\quad\text{in }\Omega. \tag{3.5c}$$

We star by proving that the problem $(\mathrm{P}_{1\lambda}^{\eta})$ has a unique solution $\left(\mathbf{u}_{\lambda\eta},\boldsymbol{\sigma}_{\lambda\eta}\right)\in W^{1,\infty}(0,T;\mathcal{V})\times W^{1,\infty}(0,T;\mathcal{H}_1)$. To do this, we find making use (3.5a) and (3.5b)

$$\begin{aligned}&\int_\Omega\mathcal{A}\left(\boldsymbol{\varepsilon}\left(\mathbf{u}_{\lambda\eta}(t)\right)\right)\cdot\boldsymbol{\varepsilon}(\mathbf{v})dx\\&=\int_\Omega\mathbf{F}(t)\cdot\mathbf{v}dx-\int_\Omega Z_\eta(t)\cdot\boldsymbol{\varepsilon}(\mathbf{v})dx,\quad\forall\mathbf{v}\in\mathcal{V},\quad\text{a.e. } t\in(0,T).\end{aligned} \tag{3.6}$$

It is easy to see that the bilinear form

$$\begin{cases}\mathcal{V}\times\mathcal{V}\longrightarrow\mathbb{R},\\ \mathbf{u},\mathbf{v}\longmapsto\displaystyle\int_\Omega\mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{u}))\cdot\boldsymbol{\varepsilon}(\mathbf{v})dx,\end{cases}$$

is continuous and coercive. Hence, using the Lax-Milgram theorem if follows that Eq. (3.6) admits a unique solution $\mathbf{u}_{\lambda\eta}(t)\in\mathcal{V}$. In addition, the solution satisfies the estimate

$$\left\|\mathbf{u}_{\lambda\eta}(t)\right\|_{\mathcal{V}}\leq\frac{1}{C_0^2\alpha_{\mathcal{A}}}\left(\|\mathbf{F}(t)\|_{\mathcal{V}'}+\int_0^t\|\boldsymbol{\eta}(s)\|_{\mathcal{H}}ds+\|\boldsymbol{\sigma}_0\|_{\mathcal{H}}+\|\mathbf{u}_0\|_{\mathcal{V}}\right)\quad\text{a.e. } t\in(0,T). \tag{3.7}$$

Similarly, we get by differentiating equation (3.6) with respect to the time variable $t\in(0,T)$ and setting $\frac{\partial\mathbf{u}_{\lambda\eta}}{\partial t}$ as test function in the obtained equation

$$\left\|\frac{\partial\mathbf{u}_{\lambda\eta}}{\partial t}\right\|_{\mathcal{V}}\leq\frac{1}{C_0^2\alpha_{\mathcal{A}}}\left(\left\|\frac{\partial\mathbf{F}}{\partial t}\right\|_{\mathcal{V}'}+\|\boldsymbol{\eta}(t)\|_{\mathcal{H}}\right)\quad\text{a.e.}\quad t\in(0,T). \tag{3.8}$$

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Elsewhere, the existence of the stress field $\sigma_{\lambda\eta}(t)\in\mathcal{H}$ is an immediate consequence of relation (3.5a). Again (3.5a) gives

$$div\sigma_{\lambda\eta}(t)+\mathbf{f}(t)=0 \quad \text{in} \quad \mathcal{D}'(Q). \tag{3.9}$$

Which implies under the hypothesis (2.2e) that $\sigma_{\lambda\eta}(t)\in\mathcal{H}_1$ and it satisfies the estimate

$$\begin{aligned}&\left\|\sigma_{\lambda\eta}(t)\right\|_{\mathcal{H}_1}\\ &\leq m_{\mathcal{A}}\left\|\mathbf{u}_{\lambda\eta}(t)\right\|_{\mathcal{V}}+\int_0^t\|\boldsymbol{\eta}(s)\|_{\mathcal{H}}ds+\|\sigma_0\|_{\mathcal{H}}+\|\mathbf{u}_0\|_{\mathcal{V}}+\|\mathbf{f}(t)\|_H \quad \text{a.e. } t\in(0,T).\end{aligned} \tag{3.10}$$

Let us differentiate now Eq. (3.5a) with respect to the time variable $t\in(0,T)$, we find

$$\left\|\frac{\partial\sigma_{\lambda\eta}}{\partial t}\right\|_{\mathcal{H}_1}\leq m_{\mathcal{A}}\left\|\frac{\partial\mathbf{u}_{\lambda\eta}}{\partial t}\right\|_{\mathcal{V}}+\|\boldsymbol{\eta}(t)\|_{\mathcal{H}} \quad \text{a.e. } t\in(0,T). \tag{3.11}$$

Hence, the estimates (3.3a) and (3.3b) follow making use (3.7), (3.8), (3.10) and (3.11).

Introducing now the operator Λ defined as

$$\Lambda: L^\infty(0,T;\mathcal{H})\longrightarrow L^\infty(0,T;\mathcal{H}), \tag{3.12a}$$

$$\Lambda(\boldsymbol{\eta}(t))=\mu(\lambda(t))\left(\mathcal{G}_1\left(\sigma_{\lambda\eta}(t)\right)+\mathcal{G}_2\left(\boldsymbol{\varepsilon}\left(\mathbf{u}_{\lambda\eta}(t)\right)\right)\right). \tag{3.12b}$$

Let $t\in(0,T)$ and $\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\in L^\infty(0,T;\mathcal{H})$, it is easy to check that for a.e. $t\in(0,T)$

$$\begin{aligned}&\|\Lambda\boldsymbol{\eta}_1(t)-\Lambda\boldsymbol{\eta}_2(t)\|_{\mathcal{H}}\\ &\leq\mu^*\max(m_1,m_2)\left(\left\|\mathbf{u}_{\lambda\eta_1}(t)-\mathbf{u}_{\lambda\eta_2}(t)\right\|_{\mathcal{V}}+\left\|\sigma_{\lambda\eta_1}(t)-\sigma_{\lambda\eta_2}(t)\right\|_{\mathcal{H}}\right).\end{aligned} \tag{3.13}$$

Taking $\boldsymbol{\eta}=\boldsymbol{\eta}_1$, $\boldsymbol{\eta}=\boldsymbol{\eta}_2$, respectively, in Eq. (3.6), subtracting the two obtained equations we can infer by choosing $\mathbf{v}=\mathbf{u}_{\lambda\eta_1}-\mathbf{u}_{\lambda\eta_2}$ as test function that

$$\begin{aligned}&\int_\Omega\left(\mathcal{A}\left(\boldsymbol{\varepsilon}\left(\mathbf{u}_{\lambda\eta_1}(t)\right)\right)-\mathcal{A}\left(\boldsymbol{\varepsilon}\left(\mathbf{u}_{\lambda\eta_2}(t)\right)\right)\right)\cdot\left(\boldsymbol{\varepsilon}\left(\mathbf{u}_{\lambda\eta_1}(t)-\mathbf{u}_{\lambda\eta_2}(t)\right)\right)dx\\ &=-\int_\Omega\left(Z_{\eta_1}(t)-Z_{\eta_2}(t)\right)\cdot\left(\boldsymbol{\varepsilon}\left(\mathbf{u}_{\lambda\eta_1}(t)-\mathbf{u}_{\lambda\eta_2}(t)\right)\right)dx \quad \text{a.e. } t\in(0,T).\end{aligned}$$

This yields, employing Korn's inequality

$$\left\|\mathbf{u}_{\lambda\eta_1}(t)-\mathbf{u}_{\lambda\eta_2}(t)\right\|_{\mathcal{V}}\leq\frac{1}{C_0^2\alpha_{\mathcal{A}}}\int_0^t\|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_{\mathcal{H}}ds \quad \text{a.e. } t\in(0,T). \tag{3.14}$$

Moreover, taking into account (3.14) Eq. (3.5a) gives for a.e. $t\in(0,T)$,

$$\left\|\sigma_{\lambda\eta_1}(t)-\sigma_{\lambda\eta_2}(t)\right\|_{\mathcal{H}}\leq\left(\frac{m_{\mathcal{A}}}{C_0^2\alpha_{\mathcal{A}}}+1\right)\int_0^t\|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_{\mathcal{H}}ds. \tag{3.15}$$

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We conclude from (3.13)-(3.15) that

$$\|\Lambda\eta_1(t)-\Lambda\eta_2(t)\|_{\mathcal{H}}\leq C\int_0^t\|\eta_1(s)-\eta_2(s)\|_{\mathcal{H}}ds \quad \text{a.e.} \quad t\in(0,T),$$

where

$$C=\mu^*\max(m_1,m_2)\max\left(\frac{1}{C_0^2\alpha_{\mathcal{A}}},\frac{m_{\mathcal{A}}}{C_0^2\alpha_{\mathcal{A}}}+1\right).$$

This implies that

$$\|\Lambda\eta_1-\Lambda\eta_2\|_{L^\infty(0,T;\mathcal{H})}\leq CT\|\eta_1-\eta_2\|_{L^\infty(0,T;\mathcal{H})}.$$

Applying Λ another time. Similarly, we get

$$\left\|\Lambda^2\eta_1-\Lambda^2\eta_2\right\|_{L^\infty(0,T;\mathcal{H})}\leq\frac{C^2T^2}{2!}\|\eta_1-\eta_2\|_{L^\infty(0,T;\mathcal{H})}.$$

We generalize this procedure by recurrence on n, we obtain the following formula, see for more details the reference [14]

$$\|\Lambda^n\eta_1-\Lambda^n\eta_2\|_{L^\infty(0,T;\mathcal{H})}\leq\frac{C^nT^n}{n!}\|\eta_1-\eta_2\|_{L^\infty(0,T;\mathcal{H})}. \tag{3.16}$$

The sequence $(\frac{C^nT^n}{n!})_n$ converges to 0. Thus, for n sufficiently large $\frac{C^nT^n}{n!}<1$. It means that a large power n of the operator Λ is a contraction on $L^\infty(0,T;\mathcal{H})$. Then, Banach fixed point theorem asserts that Λ^n admits a unique fixed point

$$\tilde{\eta}\in L^\infty(0,T;\mathcal{H}):\Lambda^n\tilde{\eta}=\tilde{\eta}.$$

By applying the operator Λ another time we get

$$\Lambda^{n+1}\tilde{\eta}=\Lambda\tilde{\eta}.$$

Which means that

$$\Lambda^n(\Lambda\tilde{\eta})=\Lambda\tilde{\eta}.$$

Then, $\Lambda\tilde{\eta}$ is another fixed point of Λ^n. Thus, the uniqueness of fixed point of Λ^n leads to

$$\Lambda\tilde{\eta}=\tilde{\eta}.$$

Which proves that the problem $(\mathrm{P}_{1\lambda}^{\eta})$ has a unique solution $(\mathbf{u}_{\lambda\eta},\boldsymbol{\sigma}_{\lambda\eta})\in W^{1,\infty}(0,T;\mathcal{V})\times W^{1,\infty}(0,T;\mathcal{H}_1)$. We are ready now to prove the existence of solution to problem $\mathrm{P}_{1\lambda}$. To this aim, by definition of Λ we get

$$\mu(\lambda(t))\left(\mathcal{G}_1\left(\boldsymbol{\sigma}_{\lambda\tilde{\eta}}(t)\right)+\mathcal{G}_2\left(\boldsymbol{\varepsilon}\left(\mathbf{u}_{\lambda\tilde{\eta}}(t)\right)\right)\right)=\tilde{\eta}(t) \quad \text{a.e.} \quad t\in(0,T),$$

and so Eq. (3.5a) becomes

$$\boldsymbol{\sigma}_{\lambda\tilde{\eta}}(t)=\mathcal{A}\left(\boldsymbol{\varepsilon}\left(\mathbf{u}_{\lambda\tilde{\eta}}(t)\right)\right)+Z_{\tilde{\eta}}(t) \quad \text{a.e.} \quad t\in(0,T).$$

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By taking the derivative of the above equation with respect to time variable t we deduce for a.e. $t\in(0,T)$

$$\frac{\partial\sigma_{\lambda\tilde{\eta}}}{\partial t}=\mathcal{A}\left(\varepsilon\left(\frac{\partial\mathbf{u}_{\lambda\tilde{\eta}}}{\partial t}\right)\right)+\mu(\lambda(t))\left(\mathcal{G}_1\left(\sigma_{\lambda\tilde{\eta}}(t)\right)+\mathcal{G}_2\left(\varepsilon\left(\mathbf{u}_{\lambda\tilde{\eta}}(t)\right)\right)\right).$$

Then, $\left(\mathbf{u}_{\lambda\tilde{\eta}},\sigma_{\lambda\tilde{\eta}}\right)\in W^{1,\infty}(0,T;\mathcal{V})\times W^{1,\infty}(0,T;\mathcal{H}_1)$ represents the unique solution of the auxiliary problem $\mathrm{P}_{1\lambda}$. This permits us to conclude the proof of Lemma 3.1. □

Considering now the following auxiliary problem.

Problem $\mathrm{P}_{2\lambda}$. Find the temperature $\theta_\lambda\in\mathcal{Y}_{p,q}$ solution of the variational equation

$$\begin{aligned}&-\int_Q\theta_\lambda\frac{\partial\varphi}{\partial t}dxdt+\int_Q\mathcal{K}(\nabla\theta_\lambda)\cdot\nabla\varphi dxdt\\&=-\int_Q\mu(\lambda)\mathcal{A}^{-1}(\mathcal{G}_1(\sigma_\lambda)+\mathcal{G}_2(\varepsilon(\mathbf{u}_\lambda)))\cdot\sigma_\lambda\varphi dxdt+\int_Q r\varphi dxdt\\&\quad+\int_\Omega\theta_0\varphi(0)dx,\quad\forall\varphi\in\mathcal{Z}_{p',q'},\end{aligned}\tag{3.17a}$$

$$\theta_\lambda(0)=\theta_0\quad\text{ in }\Omega.\tag{3.17b}$$

Lemma 3.2. *Let $(\mathbf{u}_\lambda,\sigma_\lambda)\in W^{1,\infty}(0,T;\mathcal{V})\times W^{1,\infty}(0,T;\mathcal{H}_1)$ be the solution of problem $P_{1\lambda}$ given by Lemma 3.1. Then, there exists $\theta_\lambda\in\mathcal{Y}_{p,q}$, q given by the relation* (3.1)*, solution to the auxiliary weak problem $P_{2\lambda}$. Moreover, it satisfies the estimate*

$$\|\theta_\lambda\|_{\mathcal{Y}_{p,q}}\leq d_3,\tag{3.18}$$

where d_3 is a positive constant.

Proof. Technically, it is difficult to prove the existence of solution using variational methods. To this end, we introduce for each $m\in\mathbb{N}$ the following approximate standard weak problem

$$\begin{aligned}&\int_\Omega\frac{\partial\theta_{m\lambda}}{\partial t}\varphi dx+\int_\Omega\mathcal{K}(\nabla\theta_{m\lambda}(t))\cdot\nabla\varphi dx\\&=-\int_Q r_m(t)\varphi dx\qquad\text{a.e. } t\in(0,T),\quad\forall\varphi\in H^1(\Omega),\end{aligned}\tag{3.19a}$$

$$\theta_{m\lambda}(0)=\theta_{m0},\tag{3.19b}$$

in which

$$r_m=\frac{m\left[\mu(\lambda)\mathcal{A}^{-1}(\mathcal{G}_1(\sigma_\lambda)+\mathcal{G}_2(\varepsilon(\mathbf{u}_\lambda)))\cdot\sigma_\lambda+r\right]}{m+\mu(\lambda)\mathcal{A}^{-1}(\mathcal{G}_1(\sigma_\lambda)+\mathcal{G}_2(\varepsilon(\mathbf{u}_\lambda)))\cdot\sigma_\lambda+r}\in L^\infty(Q),\tag{3.20}$$

and $\theta_{m0}\in L^2(\Omega)$ such that it converges to θ_0 in $L^1(\Omega)$.

Let us consider the functions

$$\tilde{\theta}_{m\lambda}(t)=e^{-\alpha_{\mathcal{K}}t}\theta_{m\lambda}(t).\tag{3.21}$$

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So, Eq. (3.19a) can be rewritten

$$\int_\Omega \frac{\partial \tilde{\theta}_{m\lambda}}{\partial t}\varphi dx+\alpha_{\mathcal{K}}\int_\Omega \tilde{\theta}_{m\lambda}(t)\,\varphi dx+\int_\Omega \mathcal{K}\left(\tilde{\theta}_{m\lambda}(t)\right)\cdot\nabla\varphi dx$$
$$=-\int_\Omega r_m(t)e^{-\alpha_{\mathcal{K}}t}\varphi dx,\quad \text{a.e. } t\in(0,T),\quad \forall\varphi\in H^1(\Omega). \tag{3.22}$$

Hence, we deduce employing classical arguments of functional analysis concerning parabolic equations, see for example [4,12], that Eq. (3.22) has a unique solution $\theta_{m\lambda}\in\mathcal{X}$ such that

$$\|\theta_{m\lambda}\|_{\mathcal{X}}\leq c, \tag{3.23}$$

whither c is a positive constant.

Furthermore, the use of Hölder's inequality for $1\leq p<\frac{n}{n-1}$ and $\rho>1$, leads for a.e. $t\in(0,T)$ to

$$\int_\Omega \|\nabla\theta_{m\lambda}(x,t)\|^p dx\leq\left(\int_\Omega\frac{\|\nabla\theta_{m\lambda}(x,t)\|^2}{(1+\|\theta_{m\lambda}(x,t)\|)^\rho}dx\right)^{\frac{p}{2}}\left(\int_\Omega(1+\|\theta_{m\lambda}(x,t)\|)^{\frac{\rho p}{2-p}}dx\right)^{\frac{2-p}{2}}.$$

This eventually gives

$$\|\nabla\theta_{m\lambda}(t)\|^p_{L^p(\Omega)^n}\leq\left(\int_\Omega\frac{\|\nabla\theta_{m\lambda}\|(x,t)^2}{(1+\|\theta_{m\lambda}(x,t)\|)^\rho}dx\right)^{\frac{p}{2}}\left(\int_\Omega(1+\|\theta_{m\lambda}(x,t)\|)^{\frac{\rho p}{2-p}}dx\right)^{\frac{2-p}{2}}.$$

Raising to the power q/p where $1\leq q<2$ and using Hölder's inequality with respect to the time variable $t\in(0,T)$, the following inequality holds

$$\int_0^T\|\nabla\theta_{m\lambda}(t)\|^q_{L^p(\Omega)^n}dt\leq\left(\int_Q\frac{\|\nabla\theta_{m\lambda}(x,t)\|^2}{(1+\|\theta_{m\lambda}(x,t)\|)^\rho}dxdt\right)^{\frac{q}{2}}$$
$$\times\left\{\int_0^T\left(\left(\int_\Omega(1+\|\theta_{m\lambda}(x,t)\|)^{\frac{\rho p}{2-p}}dx\right)^{\frac{(2-p)q}{(2-q)p}}\right)dt\right\}^{\frac{2-q}{2}}. \tag{3.24}$$

On the other hand, the use of estimate (3.23) permits us to find for a.e. $t\in(0,T)$

$$\int_\Omega\frac{\|\nabla\theta_{m\lambda}(x,t)\|^2}{(1+\|\nabla\theta_{m\lambda}(x,t)\|)^\rho}dx\leq\int_\Omega\|\nabla\theta_{m\lambda}(x,t)\|^2dx\leq c. \tag{3.25}$$

Thus, by (3.23), (3.24) as well as some algebraic manipulations we can infer

$$\int_0^T\|\nabla\theta_{m\lambda}(t)\|^q_{L^p(\Omega)^n}dt\leq c\left\{1+\left(\int_0^T\|\theta_{m\lambda}(t)\|^{\frac{\rho q}{2-q}}_{L^{\frac{\rho p}{2-p}}(\Omega)}dt\right)^{\frac{2-q}{2}}\right\}. \tag{3.26}$$

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Elsewhere, making use the below Gagliardo-Nirenberg imbedding, see [4]

$$\|\theta_{m\lambda}(t)\|_{L^{\frac{\rho p}{2-p}}(\Omega)} \leq c\|\nabla\theta_{m\lambda}(t)\|^{\sigma}_{L^p(\Omega)^n}\|\theta_{m\lambda}(t)\|^{1-\sigma}_{L^1(\Omega)} \quad \text{a.e. } t\in(0,T), \tag{3.27}$$

for every σ satisfying

$$\begin{cases} \dfrac{2-p}{\rho p} = \sigma\dfrac{n-p}{np} + 1 - \sigma, \\ 0 \leq \sigma \leq 1, \end{cases} \tag{3.28}$$

and Hölder's inequality, we get, taking into account the fact that $\theta_{m\lambda} \in C^0\left([0,T], L^2(\Omega)\right)$

$$\|\theta_{m\lambda}(t)\|_{L^{\frac{\rho p}{2-p}}(\Omega)} \leq c\|\nabla\theta_{m\lambda}(t)\|^{\sigma}_{L^p(\Omega)^n}, \quad \text{a.e. } t\in(0,T).$$

Raising the inequality above to the power $\frac{q}{\sigma}$ and integrating over the interval time $(0,T)$, we obtain

$$\int_0^T \|\theta_{m\lambda}(t)\|^{\frac{q}{\sigma}}_{L^{\frac{\rho p}{2-p}}(\Omega)} dt \leq c\int_0^T \|\nabla\theta_{m\lambda}(t)\|^{q}_{L^p(\Omega)^n} dt. \tag{3.29}$$

Now, assume that

$$\frac{\rho q}{2-q} = \frac{q}{\sigma}. \tag{3.30}$$

Hence, (3.26), (3.29) and (3.30) imply

$$\int_0^T \|\nabla\theta_{m\lambda}(t)\|^{q}_{L^p(\Omega)^n} dt \leq c\left\{1 + \left(\int_0^T \|\nabla\theta_{m\lambda}(t)\|^{q}_{L^p(\Omega)^n} dt\right)^{\frac{2-q}{2}}\right\}. \tag{3.31}$$

Since $\frac{2-q}{2} < 1$, we can obtain an a priori estimate on the term $\nabla\theta_m$ in the space $L^q(0,T;L^p(\Omega))$.

Putting together (3.28) and (3.30), we obtain after some manipulations

$$\begin{cases} 1 \leq q < 2, \\ \rho = \dfrac{2p+np+nq-qp-nqp}{np}. \end{cases}$$

We deduce, keeping in mind the fact that $\rho > 1$

$$1 \leq q < 2, \quad \frac{2}{q} + \frac{n}{q} > n+1.$$

This gives sense to hypothesis (3.1). Which, together with the fact that $1 \leq p < \frac{n}{n-1}$ and standard embedding theorems between Lebesgue spaces, the following regularity holds

$$\theta_{m\lambda} \in L^q\left(0,T;W^{1,p}(\Omega)\right). \tag{3.32}$$

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We can then extract, using (3.32) a subsequence of $\theta_{m\lambda}$, still denoted $\theta_{m\lambda}$ satisfying

$$\theta_{m\lambda} \longrightarrow \theta_\lambda \text{ in } L^q\left(0,T;W^{1,p}(\Omega)\right) \text{ weakly.} \tag{3.33}$$

Furthermore, we find using hypothesis (2.2c) and (3.32)

$$div\left(\mathcal{K}(\nabla\theta_{m\lambda})\right) \in L^q\left(0,T;W^{-1,p}(\Omega)\right). \tag{3.34}$$

Moreover, by virtue of the approximate strong system

$$\begin{cases} \dfrac{\partial\theta_{m\lambda}}{\partial t} - div\left(\mathcal{K}(\nabla\theta_{m\lambda})\right) = -r_m & \text{in } Q, \\ \dfrac{\partial\theta_{m\lambda}}{\partial n} = 0 & \text{on } \Gamma\times(0,T), \\ \theta_{m\lambda}(0) = \theta_{m0} & \text{in } \Omega, \end{cases} \tag{3.35}$$

and by definition of the function r_m, we can infer employing (3.34)

$$\frac{\partial\theta_{m\lambda}}{\partial t} \in L^q\left(0,T;W^{-1,p}(\Omega)\right). \tag{3.36}$$

Making use (3.33), (3.36) and Aubin-Simon's compactness theorem, see [18], it is easy to see that

$$\theta_{m\lambda} \longrightarrow \theta_\lambda \text{ in } L^q(0,T;L^p(\Omega)) \text{ strongly and a.e. in } Q. \tag{3.37}$$

We conclude from (3.19a), (3.19b), (3.20), (3.33), (3.36) and (3.37) that problem $\mathrm{P}_{2\lambda}$ admits at least one solution $\theta_\lambda \in \mathcal{Y}_{p,q}$. Which achieves the proof of Lemma 3.2. □

Proof of Theorem 3.1. In order to apply the Kakutani-Glicksberg fixed point theorem, see [8, 16], we consider the closed convex ball

$$K = \left\{\lambda \in \mathcal{Y}_{p,q} : \|\lambda\|_{\mathcal{Y}_{p,q}} \leq d_3\right\}, \tag{3.38}$$

where d_3 is the constants given by estimate (3.18). The ball K is compact when the topological vector space is provided by the weak topology of the space $\mathcal{Y}_{p,q}$. Let us built the mapping $Laplace : K \longrightarrow 2^K$, as follows

$$\lambda \longmapsto \mathcal{L}(\lambda) \subset K, \tag{3.39}$$

whither $\mathcal{L}(\lambda)$ represents the set of solutions θ_λ of the auxiliary problem $\mathrm{P}_{2\lambda}$.

For every $\lambda \in K$, problem $\mathrm{P}_{2\lambda}$ is linear with respect to the function θ_λ. Moreover, $(\mathbf{u}_\lambda, \sigma_\lambda) \in W^{1,\infty}(0,T;\mathcal{V}) \times W^{1,\infty}(0,T;\mathcal{H}_1)$ represents the unique solution of problem $\mathrm{P}_{1\lambda}$. Consequently the set $\mathcal{L}(\lambda)$ is convex. To conclude the proof it remains to verify the closeness in $K \times K$ of the graph set

$$G(\mathcal{L}) = \{(\lambda,\theta_\lambda) \in K \times K | \theta_\lambda \in \mathcal{L}(\lambda)\}. \tag{3.40}$$

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To do so, we consider a sequence $(\lambda_m)\in K$, such that

$$\lambda_m \longrightarrow \lambda \text{ in } \mathcal{Y}_{p,q} \text{ weakly}, \tag{3.41}$$

and let $\theta_{m\lambda}\in\mathcal{L}(\lambda)$.

Remembering that $\theta_{m\lambda}$ is solution of the following equation

$$\begin{aligned}
&-\int_Q \theta_{m\lambda}\frac{\partial\varphi}{\partial t}dxdt+\int_Q \mathcal{K}(\nabla\theta_{m\lambda})\cdot\nabla\varphi dxdt\\
&=-\int_Q \mu(\lambda_m)\mathcal{A}^{-1}(\mathcal{G}_1(\sigma_{m\lambda})+\mathcal{G}_2(\boldsymbol{\varepsilon}(\mathbf{u}_{m\lambda})))\cdot\sigma_{m\lambda}\varphi dxdt\\
&\qquad+\int_Q r\varphi dxdt+\int_\Omega \theta_0\varphi(0)dx,\quad \forall\varphi\in\mathcal{Z}_{p',q'},
\end{aligned} \tag{3.42a}$$

$$\theta_\lambda(0)=\theta_0 \quad \text{in } \Omega, \tag{3.42b}$$

where $(\mathbf{u}_{m\lambda},\sigma_{m\lambda})\in W^{1,\infty}(0,T;\mathcal{V})\times W^{1,\infty}(0,T;\mathcal{H}_1)$ is the unique solution of system

$$\frac{\partial\sigma_{m\lambda}}{\partial t}=\mathcal{A}\left(\boldsymbol{\varepsilon}\left(\frac{\partial\mathbf{u}_{m\lambda}}{\partial t}\right)\right)+\mu(\lambda_m(t))(\mathcal{G}_1(\sigma_{m\lambda})+\mathcal{G}_2(\boldsymbol{\varepsilon}(\mathbf{u}_{m\lambda}))) \quad \text{a.e. } t\in(0,T), \tag{3.43a}$$

$$\int_\Omega \sigma_{m\lambda}(t)\cdot\boldsymbol{\varepsilon}(\mathbf{v})dx=\int_\Omega \mathbf{F}(t)\cdot\mathbf{v}dx,\quad \forall\mathbf{v}\in\mathcal{V},\quad t\in(0,T), \tag{3.43b}$$

$$\mathbf{u}_{m\lambda}(0)=\mathbf{u}_0,\ \ \sigma_{m\lambda}(0)=\sigma_0 \quad \text{in } \Omega. \tag{3.43c}$$

Then, by virtue of Lemmas 3.1 and 3.2 we get

$$\|\mathbf{u}_{m\lambda}\|_{W^{1,\infty}(0,T;\mathcal{V})}\le d_1,\quad \|\sigma_{m\lambda}\|_{W^{1,\infty}(0,T;\mathcal{H}_1)}\le d_2 \quad \text{and} \quad \|\theta_{m\lambda}\|_{\mathcal{Y}_{p,q}}\le d_3. \tag{3.44}$$

Thus, we can extract a subsequences, still denoted $\mathbf{u}_{m\lambda}$, $\sigma_{m\lambda}$ and $\theta_{m\lambda}$ such that

$$\mathbf{u}_{m\lambda}\longrightarrow\mathbf{u}_\lambda \text{ in } L^\infty(0,T;\mathcal{V}) \text{ weakly*}, \tag{3.45a}$$

$$\frac{\partial\mathbf{u}_{m\lambda}}{\partial t}\longrightarrow\frac{\partial\mathbf{u}_\lambda}{\partial t} \text{ in } L^\infty(0,T;\mathcal{V}) \text{ weakly*}, \tag{3.45b}$$

$$\sigma_{m\lambda}\longrightarrow\sigma_\lambda \text{ in } L^\infty(0,T;\mathcal{H}_1) \text{ weakly*}, \tag{3.45c}$$

$$\frac{\partial\sigma_{m\lambda}}{\partial t}\longrightarrow\frac{\partial\sigma_\lambda}{\partial t} \text{ in } L^\infty(0,T;\mathcal{H}_1) \text{ weakly*}, \tag{3.45d}$$

$$\theta_{m\lambda}\longrightarrow\theta_\lambda \text{ in } L^q\left(0,T;W^{1,p}(\Omega)\right) \text{ weakly}, \tag{3.45e}$$

$$\frac{\partial\theta_{m\lambda}}{\partial t}\longrightarrow\frac{\partial\theta_\lambda}{\partial t} \text{ in } L^q\left(0,T;W^{-1,p}(\Omega)\right) \text{ weakly}. \tag{3.45f}$$

These allow us, via Aubin-Simon's compactness theorems to extract subsequences, still denoted λ_m, $\mathbf{u}_{m\lambda}$ and $\theta_{m\lambda}$ such that

$$\lambda_m\longrightarrow\lambda \text{ in } L^q(0,T;L^p(\Omega)) \text{ strongly and a.e. in } Q, \tag{3.46a}$$

$$\mathbf{u}_{m\lambda}\longrightarrow\mathbf{u}_\lambda \text{ in } L^\infty(0,T;H) \text{ strongly and a.e. in } Q, \tag{3.46b}$$

$$\theta_{m\lambda}\longrightarrow\theta_\lambda \text{ in } L^q(0,T;L^p(\Omega)) \text{ strongly and a.e. in } Q. \tag{3.46c}$$

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Since the system (3.43a)-(3.43c) is linear, we can easily pass to the limit, using the convergence results (3.45a)-(3.45d), the fact that $\mu\in\mathcal{C}^0(\mathbb{R})$ and (3.46a), to obtain

$$\frac{\partial\sigma_\lambda}{\partial t}=\mathcal{A}\left(\varepsilon\left(\frac{\partial\mathbf{u}_\lambda}{\partial t}\right)\right)+\mu(\lambda(t))(\mathcal{G}_1(\sigma_\lambda(t))+\mathcal{G}_2(\varepsilon(\mathbf{u}_\lambda(t)))) \quad \text{a.e. } t\in(0,T), \tag{3.47a}$$

$$\int_\Omega\sigma_\lambda(t)\cdot\varepsilon(\mathbf{v})dx=\int_\Omega\mathbf{F}(t)\cdot\mathbf{v}dx, \quad \forall\mathbf{v}\in\mathcal{V}, \quad \text{a.e. } t\in(0,T), \tag{3.47b}$$

$$\mathbf{u}_\lambda(0)=\mathbf{u}_0, \quad \sigma_\lambda(0)=\sigma_0 \quad \text{in } \Omega. \tag{3.47c}$$

Our goal now is to prove that

$$\mathbf{u}_{m\lambda}\longrightarrow\mathbf{u}_\lambda \text{ in } L^\infty(0,T;\mathcal{V}) \text{ strongly}, \tag{3.48a}$$

$$\sigma_{m\lambda}\longrightarrow\sigma_\lambda \text{ in } L^\infty(0,T;\mathcal{H}) \text{ strongly}. \tag{3.48b}$$

To this aim, we proceed as follows. Subtracting Eqs. (3.43a) and (3.47a) and integrating over the variable $t\in(0,T)$ we find

$$\begin{aligned}
&\|\sigma_{m\lambda}(t)-\sigma_\lambda(t)\|_{\mathcal{H}}\\
\leq& m_{\mathcal{A}}\|\mathbf{u}_{m\lambda}(t)-\mathbf{u}_\lambda(t)\|_{\mathcal{V}}\\
&+\int_0^t\|(\mu(\lambda_m(s))-\mu(\lambda(s)))(\mathcal{G}_1(\sigma_\lambda(s))+\mathcal{G}_2(\varepsilon(\mathbf{u}_\lambda(s))))\|_{\mathcal{H}}ds\\
&+\mu^*\int_0^t(m_1\|\sigma_{m\lambda}(s)-\sigma_\lambda(s)\|_{\mathcal{H}}+m_2\|\mathbf{u}_{m\lambda}(s)-\mathbf{u}_\lambda(s)\|_{\mathcal{V}})ds \quad \text{a.e. } t\in(0,T).
\end{aligned} \tag{3.49}$$

Furthermore, we get by subtracting equations (3.43b) and (3.47b), making use again Eqs. (3.43a) and (3.47a) and setting $\mathbf{v}=\mathbf{u}_{m\lambda}-\mathbf{u}_\lambda$ as test function in the obtained equation

$$\begin{aligned}
&\|\mathbf{u}_{m\lambda}(t)-\mathbf{u}_\lambda(t)\|_{\mathcal{V}}\\
\leq&\frac{T}{\alpha_{\mathcal{A}}C_0^2}\int_0^t\|(\mu(\lambda_m(s))-\mu(\lambda(s)))(\mathcal{G}_1(\sigma_\lambda(s))+\mathcal{G}_2(\varepsilon(\mathbf{u}_\lambda(s))))\|_{\mathcal{H}}ds\\
&+\frac{\mu^*}{\alpha_{\mathcal{A}}C_0^2}\int_0^t(m_1\|\sigma_{m\lambda}(s)-\sigma_\lambda(s)\|_{\mathcal{H}}+m_2\|\mathbf{u}_{m\lambda}(s)-\mathbf{u}_\lambda(s)\|_{\mathcal{V}})ds \quad \text{a.e. } t\in(0,T).
\end{aligned} \tag{3.50}$$

From (3.49), (3.50) and exploiting Gronwall Lemma, we obtain for a.e. $t\in(0,T)$

$$\begin{aligned}
&\|\sigma_{m\lambda}(t)-\sigma_\lambda(t)\|_{\mathcal{H}}+\|\mathbf{u}_{m\lambda}(t)-\mathbf{u}_\lambda(t)\|_{\mathcal{V}}\\
\leq& c\int_0^t\|(\mu(\lambda_m(s))-\mu(\lambda(s)))(\mathcal{G}_1(\sigma_\lambda(s))+\mathcal{G}_2(\varepsilon(\mathbf{u}_\lambda(s))))\|_{\mathcal{H}}ds.
\end{aligned} \tag{3.51}$$

On the other hand, since $\lambda_m\in K$, we extract a subsequence still denoted λ_m such that

$$\lambda_m\longrightarrow\lambda \quad \text{a.e. in } Q. \tag{3.52}$$

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So, since $\mu\in\mathcal{C}^0(\mathbb{R})\cap L^\infty(\mathbb{R})$, we find

$$\begin{cases} \mu(\lambda_m)\longrightarrow\mu(\lambda) & \text{a.e. in } Q,\\ \mu(\lambda_m)\longrightarrow\mu(\lambda) & \text{in } L^\infty(Q) \text{ weakly* (after a new extraction).}\end{cases} \tag{3.53}$$

Hence, it follows using (3.51) and (3.53) as well as the fact that the term $\mathcal{G}_1(\sigma_\lambda)+\mathcal{G}_2(\varepsilon(\mathbf{u}_\lambda))$ belongs to a bounded of $L^\infty(Q)\subset L^1(Q)$

$$\lim\left(\|\sigma_{m\lambda}(t)-\sigma_\lambda(t)\|_{\mathcal{H}}+\|\mathbf{u}_{m\lambda}(t)-\mathbf{u}_\lambda(t)\|_{\mathcal{V}}\right)=0 \quad \text{a.e. } t\in(0,T),$$

which allows us to validate (3.48a)-(3.48b).

Elsewhere, to pass to the limit in problem (3.42a)-(3.42b) it is enough to check the right hand side. To this end, it is well known that after a possible modification on a set of measure zero, $\varphi\in\mathcal{Z}_{p',q'}$ is continuous from $[0,T]$ into $W^{1,p'}(\Omega)\subset\mathcal{C}(\overline{\Omega})$. Thus, we can write

$$\begin{aligned}&\left\|\int_Q\mu(\lambda_m)\mathcal{A}^{-1}(\mathcal{G}_1(\sigma_{m\lambda})+\mathcal{G}_2(\varepsilon(\mathbf{u}_{m\lambda})))\cdot\sigma_{m\lambda}\varphi dxdt-\right.\\&\qquad\left.\int_Q\mu(\lambda)\mathcal{A}^{-1}(\mathcal{G}_1(\sigma_\lambda)+\mathcal{G}_2(\varepsilon(\mathbf{u}_\lambda)))\cdot\sigma_\lambda\varphi dxdt\right\|\\&\leq c\operatorname*{esssup}_Q\|\varphi(x,t)\|\left[\left\|(\mu(\lambda_m)-\mu(\lambda))\mathcal{A}^{-1}\left(\mathcal{G}_1(\sigma_\lambda)+\mathcal{A}^{-1}\mathcal{G}_2(\varepsilon(\mathbf{u}_\lambda))\right)\right\|_{L^2(0,T;\mathcal{H})}\right.\\&\qquad\left.+\|\sigma_{m\lambda}-\sigma_\lambda\|_{L^\infty(0,T;\mathcal{H})}+\|\mathbf{u}_{m\lambda}-\mathbf{u}_\lambda\|_{L^\infty(0,T;\mathcal{V})}\right]\|\sigma_\lambda\|_{L^\infty(0,T;\mathcal{H})},\quad\forall\varphi\in\mathcal{Z}_{p',q'}.\end{aligned}$$

Hence, by the foregoing

$$\begin{aligned}&\int_Q\mu(\lambda_m)\mathcal{A}^{-1}(\mathcal{G}_1(\sigma_{m\lambda})+\mathcal{G}_2(\varepsilon(\mathbf{u}_{m\lambda})))\cdot\sigma_{m\lambda}\varphi dxdt\\&\longrightarrow\int_Q\mu(\lambda)\mathcal{A}^{-1}(\mathcal{G}_1(\sigma_\lambda)+\mathcal{G}_2(\varepsilon(\mathbf{u}_\lambda)))\cdot\sigma_\lambda\varphi dxdt,\quad\forall\varphi\in\mathcal{Z}_{p',q'}.\end{aligned} \tag{3.54}$$

Moreover, remarking from definition of the space function $\mathcal{Y}_{p,q}$, that $\mathcal{Y}_{p,q}\subset\mathcal{C}^0([0,T];L^p(\Omega))$. Consequently, the condition $\theta_m(0)=\theta_{0m}$ has sense in the space $L^1(\Omega)$ and we have

$$\theta_{0m}\longrightarrow\theta_0 \text{ in } L^1(\Omega) \text{ weakly.} \tag{3.55}$$

On the other hand, the condition $\varphi\in\mathcal{Z}_{q',r'}$ implies

$$\varphi(0)\in\mathcal{C}^0(\bar{\Omega}). \tag{3.56}$$

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Thus, it follows by (3.45e), (3.46c), (3.54), (3.55) and (3.56) that θ_λ solves the limit problem

$$\begin{aligned}
&-\int_Q \theta_\lambda \frac{\partial \varphi}{\partial t} dxdt + \int_Q \mathcal{K}(\nabla\theta_\lambda)\cdot\nabla\varphi dxdt \\
&= -\int_Q \mu(\lambda)\mathcal{A}^{-1}(\mathcal{G}_1(\sigma_\lambda(t)) + \mathcal{G}_2(\varepsilon(\mathbf{u}_\lambda(t))))\cdot\sigma_\lambda\varphi dxdt + \int_Q r\varphi dxdt \\
&\quad + \int_\Omega \theta_0\varphi(0)dx, \quad \forall \varphi \in \mathcal{Z}_{p',q'}, \qquad (3.57a)\\
&\theta_\lambda(0) = \theta_0 \quad \text{in } \Omega. \qquad (3.57b)
\end{aligned}$$

By virtue of Kakutani-Glicksberg's fixed point theorem, we conclude that the mapping $\mathcal{L}$ admits at least one fixed point $\theta_{\tilde{\lambda}} \in \mathcal{L}(\theta_{\tilde{\lambda}})$, which permits us to achieve the proof. □

Proposition 3.1. Let hypotheses of Theorem 3.1 hold and suppose in addition that

$$\begin{aligned}
&\left(\mathcal{A}^{-1}\mathcal{G}_1(\sigma) + \mathcal{A}^{-1}\mathcal{G}_2(\varepsilon(\mathbf{u}))\right)\cdot\sigma \le 0 && \text{a.e. in } \Omega\times(0,T), && (3.58a)\\
&r \ge 0 && \text{a.e. in } \Omega\times(0,T), && (3.58b)\\
&\theta_0 \ge 0 && \text{a.e. in } \Omega. && (3.58c)
\end{aligned}$$

Then, there exists a positive constant d depending only on the data of problem, such that

$$0 \le \int_\Omega \theta(t)dx \le d \quad \text{a.e. } t \in (0,T). \qquad (3.59)$$

Proof. We proceed by testing the Eq. (2.7c) by the constant function 1. We can infer

$$\int_\Omega \frac{\partial\theta}{\partial t}dx = -\int_\Omega \mu(\lambda(t))\mathcal{A}^{-1}(\mathcal{G}_1(\sigma(t)) + \mathcal{G}_2(\varepsilon(\mathbf{u}(t))))\cdot\sigma dx + \int_\Omega r(t)dx. \qquad (3.60)$$

Hence, from (3.58a)-(3.58c), we eventually get

$$\int_\Omega \frac{\partial\theta}{\partial t}dx \ge 0 \quad \text{a.e. } t \in (0,T). \qquad (3.61)$$

So, integrating (3.61) over the interval time $(0,t)$ one obtains for a.e. $t \in (0,T)$

$$0 \le \int_\Omega (\theta(t) - \theta_0)dx \le \frac{\mu^* \max(m_1,m_2)}{\alpha_{\mathcal{A}}} \|\sigma(t)\|^2_{L^\infty(0,T;\mathcal{H})} + \|r\|_{L^1(Q)}.$$

Consequently, (3.59) is done. □

We describe in the following example the concrete constitutive law of Maxwell's thermo-viscoelatic model which may be cast in the abstract form (1.1) and for which our main results apply.

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Example 3.1. The Maxwell model is a linear thermo-viscoelastic constitutive law of the form, see [10]

$$\frac{\partial \sigma}{\partial t} = \mathcal{A}\left(\varepsilon\left(\frac{\partial \mathbf{u}}{\partial t}\right)\right) - \mu(\theta)\mathcal{G}(\sigma) \quad \text{in } Q, \tag{3.62}$$

where $\mathcal{A}$, $\mathcal{G}$, μ verify, respectively, the hypotheses (2.2a), (2.2b) and (2.2d).

This model can be used to describe soft solids: thermoplastic polymers in the vicinity of their melting temperature, fresh concrete (neglecting its aging), numerous metals at a temperature close to their melting point.

The model (3.62) is a particular case of (1.1). In addition, we can verify that $-\mu(\theta)\mathcal{G}(\sigma)\cdot\sigma \leq 0$. Hence, the results obtained here may be applied.

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