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Explicit characterization of the polarization tensor for rough thin layers

C. POIGNARD

INRIA Bordeaux-Sud-Ouest, Institut de Mathématiques de Bordeaux, CNRS UMR 5251 & Université de Bordeaux1, 351 cours de la Libération, F - 33405 Talence Cedex, France
email: clair.poignard@inria.fr

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In the paper 'Approximate transmission conditions through a rough thin layer. The case of periodic roughness' (Eur. J. Appl. Math. 2010; 21: 51–75), Ciuperca *et al.* derive transmission conditions that are equivalent to a rough thin layer for the conductivity problem. These conditions involved a pair of vector fields (A^0, a^0) and two constants D^1 and D^2 that are given implicitly by solving a partial differential equation in the infinite strip $\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$. We give here an explicit expression of a^0 in terms of A^0, which shows that D^1 equals zero. We infer an explicit characterization of the polarization tensor as given by Capdeboscq and Vogelius (ESAIM:M2AN. 2003; 37: 159–173).

1 Introduction and notation

The aim of this article is to provide a characterization of the constant a^0, which is the boundary layer corrector defined in Ciuperca *et al.* [2]. We first present the conventions used throughout this note.

Notation 1.1

- *All the closed curves are trigonometrically (counterclockwise) oriented.*
- *We generically denote by n the normal to a closed smooth curve of* $\mathbb{R}^2$ *outwardly directed to the domain enclosed by the curve. Moreover* $n^\perp$ *is the tangent vector to* Γ.
- *Let* $\mathscr{C}$ *be a curve of* $\mathbb{R}^2$, *and let u be a function defined in a tubular neighbourhood of* $\mathscr{C}$. *We define* $u|_{\mathscr{C}^\pm}$ *by*

$$\forall x \in \mathscr{C}, \quad u|_{\mathscr{C}^\pm}(x) = \lim_{t \to 0^+} u(x \pm tn(x)),$$

moreover if u is differentiable, we define $\partial_n u|_{\mathscr{C}^\pm}$ *and* $\partial_t u|_{\mathscr{C}^\pm}$ *by*

$$\forall x \in \mathscr{C}, \quad \partial_n u|_{\mathscr{C}^\pm}(x) = \lim_{t \to 0^+} \nabla u(x \pm tn(x)).n(x),$$
$$\partial_t u|_{\mathscr{C}^\pm}(x) = \lim_{t \to 0^+} \nabla u(x \pm tn(x)).n^\perp(x),$$

where . denotes the Euclidean scalar product of $\mathbb{R}^2$.

We recall the problem considered in [2]. Let Ω be a smooth bounded domain of $\mathbb{R}^2$ with connected boundary $\partial\Omega$. For $\varepsilon > 0$, we split Ω into three subdomains: $\mathscr{D}^1$, $\mathscr{D}^m_\varepsilon$ and $\mathscr{D}^0_\varepsilon$. $\mathscr{D}^1$

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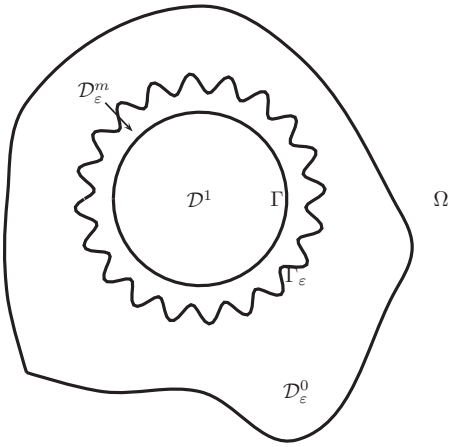


FIGURE 1. Geometry of the problem.

is a smooth domain strictly embedded in Ω (see Figure 1). We denote by Γ its connected smooth boundary. Parameterize Γ by its curvilinear abscissa denoted by Ψ. Let κ be the curvature of Γ and n its normal vector. Let f be a smooth 1–periodic and positive function. For sake of simplicity, we suppose that $1/2 \leqslant f \leqslant 3/2$. The domain $\mathscr{D}_\varepsilon^m$ is a thin oscillating layer surrounding $\mathscr{D}^1$. We denote by Γ_ε the oscillating boundary of $\mathscr{D}_\varepsilon^m$:

$$\Gamma_\varepsilon = \partial\mathscr{D}_\varepsilon^m \setminus \Gamma = \{\Psi(\theta) + \varepsilon f(\theta/\varepsilon) n(\theta), \quad \theta \in \mathbb{T}\}.$$

The domain $\mathscr{D}_\varepsilon^0$ is defined by

$$\mathscr{D}_\varepsilon^0 = \Omega \setminus \overline{\left(\mathscr{D}^1 \cup \mathscr{D}_\varepsilon^m\right)}.$$

We also write

$$\mathscr{D}^0 = \Omega \setminus \overline{\mathscr{D}^1}.$$

We define the piecewise-constant function $\sigma : \Omega \to \mathbb{R}$ by

$$\sigma(z) = \begin{cases} \sigma_1, & \text{if } z \in \mathscr{D}^1, \\ \sigma_m, & \text{if } z \in \mathscr{D}_\varepsilon^m, \\ \sigma_0, & \text{if } z \in \mathscr{D}_\varepsilon^0, \end{cases}$$

where σ_1, σ_m and σ_0 are given positive constants. Let g belong to $H^s(\partial\Omega)$ for $s \geqslant 1/2$ and denote by u^ε the unique function satisfying

$$\begin{aligned} &\nabla.(\sigma\nabla u^\varepsilon) = 0, \quad \text{in } \Omega, \\ &u^\varepsilon|_{\partial\Omega} = g. \end{aligned}$$

To derive the approximate transmission conditions that avoid the mesh of the rough thin layer, we have to solve a partial differential equation in the infinite strip $\mathbb{R} \times \mathbb{T}$ (here, the torus $\mathbb{T}$ is just $\mathbb{R}/2\pi\mathbb{Z}$). Denote by $\mathscr{C}_1$ and $\mathscr{C}_0$ the closed curves defined by

$$\mathscr{C}_0 = \{0\} \times \mathbb{T}, \quad \mathscr{C}_1 = \{(f(y), y), \forall y \in \mathbb{T}\}.$$

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The exterior normal to $\mathscr{C}_1$ is denoted by $n_{\mathscr{C}_1}$ and $n_{\mathscr{C}_0}$ is the exterior normal to $\mathscr{C}_0$:

$$n_{\mathscr{C}_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad n_{\mathscr{C}_1} = \frac{1}{\sqrt{1 + (f'(y))^2}} \begin{pmatrix} 1 \\ -f'(y) \end{pmatrix}.$$

Let $\tilde{\sigma}$ be the function defined by

$$\tilde{\sigma}(x,y) = \begin{cases} \sigma_0, & \text{in } \{(x,y), \quad y \in \mathbb{T},\ x > f(y)\}, \\ \sigma_m, & \text{in } \{(x,y), \quad y \in \mathbb{T},\ 0 < x < f(y)\}, \\ \sigma_1, & \text{in } \{(x,y), \quad y \in \mathbb{T},\ x < 0\}. \end{cases}$$

Define now the couple (A^0, a^0), where A^0 is a continuous vector field and a^0 is a constant vector, which are given by the unique solution of the following problem

$$\Delta A^0 = 0, \text{ in } \mathbb{R} \times \mathbb{T} \setminus (\mathscr{C}_0 \cup \mathscr{C}_1), \tag{1.1a}$$

$$\sigma_0 \partial_n A^0|_{\mathscr{C}_1^+} - \sigma_m \partial_n A^0|_{\mathscr{C}_1^-} = (\sigma_m - \sigma_0) n_{\mathscr{C}_1}, \tag{1.1b}$$

$$\sigma_m \partial_n A^0|_{\mathscr{C}_0^+} - \sigma_1 \partial_n A^0|_{\mathscr{C}_0^-} = -(\sigma_m - \sigma_0) n_{\mathscr{C}_0}, \tag{1.1c}$$

$$A^0 \to_{x \to -\infty} 0,\ A^0 \to_{x \to +\infty} a^0. \tag{1.1d}$$

Remark 1.2 The existence and the uniqueness of the couple (A^0, a^0) is proved in [2]. We recall that the convergence rates are exponential.

In Ciuperca *et al.* [2] we have shown that $u^\varepsilon = u^0 + \varepsilon u^1 + o(\varepsilon)$, where u^0 and u^1 satisfy

$$\begin{aligned} &\Delta u^0 = 0, \ \text{ in } \mathscr{D}^0 \cup \mathscr{D}^1, \quad u^0|_{\partial\Omega} = g, \text{ on } \partial\Omega \\ &\sigma_0 \partial_n u^0|_{\Gamma^+} = \sigma_1 \partial_n u^0|_{\Gamma^-}, \\ &u^0|_{\Gamma^+} = u^0|_{\Gamma^-}, \end{aligned}$$

and

$$\begin{aligned} &\Delta u^1 = 0, \ \text{ in } \mathscr{D}^0 \cup \mathscr{D}^1, \quad u^1|_{\partial\Omega} = 0, \ \text{ on } \partial\Omega \\ &\sigma_0 \partial_n u^1|_{\Gamma^+} = \sigma_1 \partial_n u^1|_{\Gamma^-} - \kappa D^1 \cdot \begin{pmatrix} \partial_n u^0|_{\Gamma+} \\ \partial_t u^0|_{\Gamma^+} \end{pmatrix} + D^2 \cdot \partial_t \begin{pmatrix} \partial_n u^0|_{\Gamma+} \\ \partial_t u^0|_{\Gamma^+} \end{pmatrix}, \\ &u^1|_{\Gamma^+} = u^1|_{\Gamma^-} + a^0 \cdot \begin{pmatrix} \partial_n u^0|_{\Gamma+} \\ \partial_t u^0|_{\Gamma^+} \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} D^1 &= (\sigma_0 - \sigma_m) \left[\int_0^1 f(y)\, dy\, n_{\mathscr{C}_0} + \int_0^1 A^0(f(y), y)\, dy \right] \\ &\quad + (\sigma_m - \sigma_1) \int_0^1 A^0(0,y)\, dy - \sigma_0 a^0, \\ D^2 &= (\sigma_m - \sigma_0) \left[\int_0^1 A^0(f(y), y) f'(y)\, dy - \int_0^1 f(y)\, dy \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]. \end{aligned} \tag{1.2}$$

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In this article, we show that $D^1 = 0$, and we provide an explicit characterization of the polarization tensor as defined by Capdeboscq and Vogelius [1].

2 Expression of a_0

Let $M > 2$, by integrating by parts (1.1) we have

$$\forall v \in \left(H^1([-M,M]\times\mathbb{T})\right)^2, \quad \mathscr{A}(\alpha, v) = (\sigma_m - \sigma_0)\,\mathscr{B}(v),$$

where $\mathscr{A}$ and $\mathscr{B}$ are given by

$$\begin{aligned}\mathscr{A}(A^0, v) = &-\int_{[-M,M]\times\mathbb{T}} \tilde{\sigma}\nabla A^0.\nabla v \,\mathrm{d}x\,\mathrm{d}y + \sigma_0 \int_0^1 \partial_x A^0|_{x=M}\; v|_{x=M}\,\mathrm{d}\sigma \\ &- \int_0^1 \sigma_1 \partial_x A^0|_{x=-M}\; v|_{x=-M}\,\mathrm{d}\sigma, \\ \mathscr{B}(v) = &\int_{\mathscr{C}_1} n_{\mathscr{C}_1}(\sigma) v(\sigma)\,\mathrm{d}\sigma - \int_{\mathscr{C}_0} n_{\mathscr{C}_0}(\sigma) v(\sigma)\,\mathrm{d}\sigma.\end{aligned}$$

Moreover, if $v \in H^2(\Omega)$ is piecewise harmonic, by integrating by parts once again we infer

$$\begin{aligned}\mathscr{A}(A^0, v) = &-\sigma_0 \int_0^1 A^0(M,y)\partial_x v|_M\,\mathrm{d}y + \sigma_1 \int_0^1 A^0(-M,y)\partial_x v|_{-M}\,\mathrm{d}y \\ &+ \sigma_0 \int_0^1 \partial_x A^0(M,y) v|_M\,\mathrm{d}y - \sigma_1 \int_0^1 \partial_x A^0(-M,y) v|_{-M}\,\mathrm{d}y \\ &+ \int_{\mathscr{C}_1} A^0[\sigma\partial_n v]_{\mathscr{C}_1}\,\mathrm{d}\sigma + \int_{\mathscr{C}_0} A^0[\sigma\partial_n v]_{\mathscr{C}_0}\,\mathrm{d}\sigma \\ &+ \int_{[-M,M]\times\mathbb{T}} \tilde{\sigma} A^0 \cdot \Delta v\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{2.1}$$

Let ψ be the function defined by

$$\forall (x,y) \in (0,M)\times\mathbb{T}, \quad \psi(x,y) = x,$$

We choose $v = (\psi, \psi)^T$ in (2.1). Denoting by R_M the terms

$$\begin{aligned}R_M = &\sigma_1 \int_0^1 A^0(-M,y)\,\mathrm{d}y + \sigma_0 M \int_0^1 \partial_x A^0(M,y)\,\mathrm{d}y \\ &+ \sigma_1 M \int_0^1 \partial_x A^0(-M,y)\,\mathrm{d}y,\end{aligned}$$

we infer that for all $M > 2$,

$$\begin{aligned}\int_0^1 A^0(M,y)\,\mathrm{d}y = &\frac{\sigma_0 - \sigma_m}{\sigma_0}\left[\int_0^1 A^0(f(y),y)\,\mathrm{d}y + \int_0^1 f(y)\,\mathrm{d}y n_{\mathscr{C}_0}\right] \\ &+ \frac{\sigma_m - \sigma_1}{\sigma_0}\int_0^1 A^0(0,y)\,\mathrm{d}y + \frac{1}{\sigma_0} R_M.\end{aligned}$$

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We now let M tend to infinity. Since the convergence rates in (1.1d) are exponential, the coefficient a_0 equals

$$a^0 = \frac{\sigma_0 - \sigma_m}{\sigma_0}\left[\int_0^1 A^0(f(y), y)\,\mathrm{d}y + \int_0^1 f(y)\,\mathrm{d}y n_{\mathscr{C}_0}\right] + \frac{\sigma_m - \sigma_1}{\sigma_0}\int_0^1 A^0(0, y)\,\mathrm{d}y. \tag{2.2}$$

According to (1.2) we deduce that D^1 is identically equal to zero.

In addition, let w be

$$\forall x \in \mathbb{R},\ w(x) = \begin{cases} x, & \text{if } x > 0, \\ \dfrac{\sigma_0}{\sigma_1}x, & \text{if } x < 0, \end{cases} \tag{2.3}$$

and defined by

$$\forall (x, y) \in \mathbb{R} \times \mathbb{T}, \quad V(x, y) = \begin{pmatrix} -A_2^0(x, y) \\ w(x) + A_1^0(x, y) \end{pmatrix},$$

where A_i^0 denote the i^{th} component of the vector function A^0 defined by (1.1), for $i = 1, 2$. Observe that according to (1.1), V satisfies the following properties:

$$\Delta V = 0, \text{ in } \mathbb{R} \times \mathbb{T} \backslash (\mathscr{C}_0 \cup \mathscr{C}_1)$$

$$\lim_{M \to +\infty} \partial_x V|_M = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$[\sigma \partial_n V]_{\mathscr{C}_1} = (\sigma_m - \sigma_0)\begin{pmatrix} f'/\sqrt{1 + (f')^2} \\ 0 \end{pmatrix}, \quad [\sigma \partial_n V]_{\mathscr{C}_0} = 0.$$

Take $v = V$ in (2.1), use the definition of D^2 given in (1.3) and let M tend to infinity to infer

$$-\sigma_0 a_2^0 + D_1^2 = (\sigma_m - \sigma_0)\left(\int_0^1 (V_1(f(y), y) - V_1(0, y))\,\mathrm{d}y - \int_0^1 f'(y) V_2(f(y), y)\,\mathrm{d}y\right).$$

But similar calculations lead to

$$\begin{aligned} -\int_{\mathscr{C}_1} A_2^0[\sigma \partial_n A_1^0]_{\mathscr{C}_1} - \int_{\mathscr{C}_0} A_2^0[\sigma \partial_n A_1^0]_{\mathscr{C}_0} &= (\sigma_m - \sigma_0)\int_0^1 f'(y) A_1^0(f(y), y)\,\mathrm{d}y, \\ &= (\sigma_m - \sigma_0)\int_0^1 f'(y) V_2(f(y), y)\,\mathrm{d}y, \end{aligned}$$

and on the other hand using the jumps (1.1*b*)–(1.1*c*) we infer

$$\begin{aligned} -\int_{\mathscr{C}_1} A_2^0[\sigma \partial_n A_1^0]_{\mathscr{C}_1} - \int_{\mathscr{C}_0} A_2^0[\sigma \partial_n A_1^0]_{\mathscr{C}_0} &= -(\sigma_m - \sigma_0)\int_0^1 (A_2^0(f(y), y) - A_2^0(0, y))\,\mathrm{d}y, \\ &= (\sigma_m - \sigma_0)\int_0^1 (V_1(f(y), y) - V_1(0, y))\,\mathrm{d}y. \end{aligned}$$

Hence we deduce

$$D_1^2 = \sigma_0 a_2^0. \tag{2.4}$$

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3 Explicit characterization of the polarization tensor

For $y \in \partial\Omega$, we denote by G_y the Neumann function, that is the continuous function defined by

$$\begin{aligned}
&\Delta G_y = \delta_y, \quad \text{in } \mathscr{D}^0 \cup \mathscr{D}^1,\\
&\sigma_0 \partial_n G_y|_{\Gamma^+} = \sigma_1 \partial_n G_y|_{\Gamma^-} \quad \text{on } \Gamma\\
&\sigma_0 \partial_n G_y|_{\partial\Omega} = \frac{1}{\text{mes}(\partial\Omega)}, \quad \text{on } \partial\Omega, \text{ with the Gauge condition } \int_{\partial\Omega} G_y(x)\,\mathrm{d}x = 0.
\end{aligned}$$

By integration by parts, we infer

$$\forall y \in \partial\Omega, \quad u^1(y) = (\sigma_m - \sigma_0) \int_\Gamma \begin{pmatrix} \partial_n G_y|_{\Gamma^+} \\ \nabla_\Gamma G_y|_{\Gamma^+} \end{pmatrix} \cdot M(\sigma) \begin{pmatrix} \partial_n u^0|_{\Gamma^+} \\ \nabla_\Gamma u^0|_{\Gamma^+} \end{pmatrix},$$

where M is the 2×2 matrix defined by

$$(\sigma_m - \sigma_0) M = \begin{pmatrix} \sigma_0 a_1^0 & \sigma_0 a_2^0 \\ D_1^2 & D_2^2 \end{pmatrix}. \tag{3.1}$$

Therefore we have provided an explicit characterization of M that is the polarization tensor as defined by Capdeboscq and Vogelius [1]. Observe that according to (2.4), M is symmetric as proved in [1].

4 Conclusion

In this article we have determined the polarization tensor for a periodically rough thin layer. According to (3.1) the polarization tensor is a full 2×2 matrix, whereas for smooth and for very rough thin layers the respective polarization tensors are diagonal 2×2 matrices [1, 3, 4]. This feature can be seen as a coupling between the thickness of the layer and the period of the oscillations that are in the same order of magnitude.

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