Combinatorics, Probability and Computing (2008) 17, 627–639. © 2008 Cambridge University Press
doi:10.1017/S096354830800919X Printed in the United Kingdom

Sums of Dilates

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Received 10 November 2007; revised 2 April 2008; first published online 30 June 2008

The λ-dilate of a set A is $\lambda \cdot A = \{\lambda a : a \in A\}$. We give an asymptotically sharp lower bound on the size of sumsets of the form $\lambda_1 \cdot A + \cdots + \lambda_k \cdot A$ for arbitrary integers $\lambda_1, \ldots, \lambda_k$ and integer sets A. We also establish an upper bound for such sums, which is similar to, but often stronger than Plünnecke's inequality.

1. Introduction

For sets A, B in an abelian group G, their sumset is $A + B = \{a + b : a \in A,\ b \in B\}$. For $\lambda \in \mathbb{Z}$ the dilate of A by λ is $\lambda \cdot A = \{\lambda a : a \in A\}$. Expressions of the form

$$\lambda_1 \cdot A + \cdots + \lambda_k \cdot A \tag{1.1}$$

appear frequently in combinatorial number theory. For $k = 2$ they appeared in the work of Nathanson, O'Bryant, Orosz, Ruzsa and Silva on binary linear forms [8]. For small k they appeared in the proofs of sum-product estimates in $\mathbb{Z}/p\mathbb{Z}$ of Garaev, and Katz and Shen [3, 5]. They played an important part in the solution to a problem of Ruzsa on symmetric linear equations [2].

The problem of giving a lower bound on a sum of the form (1.1) first occurred in the work of Łaba and Konyagin on distances in well-distributed planar sets [6] . They treated the case of $A + \lambda \cdot A$ for $G = \mathbb{R}$ and transcendental λ. The general problem of giving a lower bound on the sum of dilates when $G = \mathbb{Z}$ was treated by Nathanson [7] who, in particular, proved that $|A + 2 \cdot A| \geqslant 3|A| - 2$ and $|A + \lambda \cdot A| \geqslant 7|A|/2 - O(1)$ for positive $\lambda \neq 1, 2$. Our first result is a sharp lower bound on the size of $A + 3 \cdot A$.

Theorem 1.1. *For every finite set $A \subset \mathbb{Z}$ we have $|A + 3 \cdot A| \geqslant 4|A| - O(1)$.*

It is interesting that there are two essentially different examples that achieve the lower bound in the theorem above. Besides the arithmetic progression $A = \{1, \ldots, n\}$, the lower

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bound is achieved by the set $A = \{1, 2, 4, 5, \dots, 3m+1, 3m+2\}$. The proof of Theorem 1.1 is an easy, but computationally involved, induction argument. However, for the reason that is explained after the proof of Theorem 1.1, any similarly structured induction argument has to become computationally even more involved for $A + \lambda \cdot A$ with $\lambda = 4$ or greater.

The main result of this paper is an almost sharp lower bound on any sum of dilates of the form (1.1). Instead of the sharp $O(1)$ error term of Theorem 1.1, it has the weaker $o(|A|)$ error term.

Theorem 1.2. *For every vector $\bar{\lambda} = (\lambda_1, \dots, \lambda_k) \in \mathbb{Z}^k$ of k coprime integers, we have*

$$|\lambda_1 \cdot A + \cdots + \lambda_k \cdot A| \geqslant (|\lambda_1| + \cdots + |\lambda_k|)|A| - o(|A|)$$

for every finite set $A \subset \mathbb{Z}$ with the error term $o(|A|)$ depending on $\bar{\lambda}$ only.

The case when $\lambda_1, \dots, \lambda_k$ are not coprime can be reduced to the case when they are coprime via the relation

$$\lambda_1 \cdot A + \cdots + \lambda_k \cdot A = \gcd(\lambda_1, \dots, \lambda_k) \cdot \left(\frac{\lambda_1}{\gcd(\lambda_1, \dots, \lambda_k)} \cdot A + \cdots + \frac{\lambda_k}{\gcd(\lambda_1, \dots, \lambda_k)} \cdot A \right).$$

Theorem 1.2 is sharp apart from the $o(|A|)$ term as witnessed by $A = \{1, \dots, n\}$.

The problem of bounding (1.1) can be seen as a special case of the problem of establishing inequalities between two or more sums of the form (1.1). For instance, if $|A + A|$ is small, how small does $A + \lambda \cdot A$ need to be? Since

$$A + \lambda \cdot A \subset \underbrace{A + \cdots + A}_{\lambda+1 \text{ times}}$$

for positive integer λ, the classical Plünnecke inequality [9, Corollary 5.2] tells us that $|A + A| \leqslant K|A|$ implies $|A + \lambda \cdot A| \leqslant K^{\lambda+1}|A|$. This estimate is far from being sharp.

Theorem 1.3. *If either $|A + A| \leqslant K|A|$ or $|A - A| \leqslant K|A|$, then $|\lambda_1 \cdot A + \cdots + \lambda_k \cdot A| \leqslant K^p|A|$, where*

$$p = 7 + 12 \sum_{i=1}^{k} \log_2(1 + |\lambda_i|).$$

The logarithmic dependence on λ_i is optimal, as seen by considering $A + \lambda \cdot A$ with $A = \{1, \dots, n\}$. The constants 7 and 12 are certainly not optimal, and the dependence on k is probably not optimal.

Theorem 1.3 allows one to strengthen the main result of [2].

Theorem 1.4. *For a symmetric linear equation $\lambda_1 x_1 + \cdots + \lambda_k x_k = \lambda_1 y_1 + \cdots + \lambda_k y_k$ let $R(N)$ be the size of the largest $A \subset \{1, \dots, N\}$ not containing a solution to the equation in distinct integers. Then, if $k \geqslant 3$ and $\lambda_i \neq 0$ for all $1 \leqslant i \leqslant k$, then*

$$R(N) = O\left(N^{\frac{1}{2} - \frac{1}{c(k) \log \|\bar{\lambda}\|_1}}\right).$$

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The proof can be obtained from the proof of Theorem 1 in [2] by replacing invocation of Plünnecke's inequality with an application of Theorem 1.3.

The rest of the paper is split into three sections. In Section 2 we gather the tools that we need from combinatorial number theory. The bulk of the paper is in Section 3, which contains the proofs of Theorems 1.1 and 1.2 about the lower bounds on sums of dilates. Section 4 contains the proof of the Plünnecke-type Theorem 1.3.

2. Notation and tools

Lemma 2.1 (Sum form of Ruzsa's triangle inequality [9], Corollary 6.2). *For any finite $A, B, C \subset \mathbb{Z}$, we have*

$$|A + C| \leqslant \frac{|A + B||B + C|}{|B|}.$$

Corollary 2.2. *For any finite sets $A, B \subset \mathbb{Z}$,*

$$|A + B| \geqslant \sqrt{|A + A||B|}.$$

In proving Theorem 1.3, besides the operation of forming a dilate $\lambda \cdot A$ we will also make use of the operation of repeated addition $\lambda * A = \{a_1 + \cdots + a_\lambda : a_1, \ldots, a_\lambda \in A\}$. Moreover, we will need to be able to bound the size of sums of the form $\lambda_1 * A + \lambda_2 * A$ from above.

Lemma 2.3 (Plünnecke's inequality). *If $|A + A| \leqslant K|A|$ or $|A - A| \leqslant K|A|$, then $|\lambda_1 * A - \lambda_2 * A| \leqslant K^{\lambda_1+\lambda_2}|A|$ for all non-negative integers λ_1, λ_2.*

Lemma 2.4 (Ruzsa's covering lemma [10]). *For any non-empty set A, B in abelian group G one can cover B by $\frac{|A+B|}{|A|}$ translates of $A - A$.*

Definition. Let G_1, G_2 be abelian groups. We say that $A_1 \subset G_1$ and $A_2 \subset G_2$ are *r-isomorphic* if there is a bijection $\phi : A_1 \to A_2$ satisfying

$$a_1 + \cdots + a_r = b_1 + \cdots + b_r \iff \phi(a_1) + \cdots + \phi(a_r) = \phi(b_1) + \cdots + \phi(b_r)$$

for all $a_1, \ldots, a_r, b_1, \ldots, b_r \in A_1$. The map ϕ is called a Freiman isomorphism of order r.

We will need the following version of Freiman's theorem which can be deduced by a similar argument to the standard Freiman's theorem.

Theorem 2.5 ([1], Theorem 1.2). *Fix $r \in \mathbb{N}$. Suppose a non-empty set $A \subset \mathbb{Z}$ satisfies $|A + A| \leqslant K|A|$. Then A is r-isomorphic to a subset of $[1, t_1] \times \cdots \times [1, t_d] \subset \mathbb{Z}^d$ of density at least $\alpha > 0$, where d and α depend only on K and r, but not on A.*

For a vector $\bar{\lambda} = (\lambda_1, \ldots, \lambda_k)$ we set $\bar{\lambda}^i = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_k)$ and let $S_{\bar{\lambda}}(A) = \lambda_1 \cdot A + \cdots + \lambda_k \cdot A$ denote the corresponding sumset involving A. The greatest common

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divisor of a set of integers $\{\lambda_1,\dots,\lambda_k\}$ is $\gcd(\lambda_1,\dots,\lambda_k)=\max\{d\in\mathbb{N}: d \mid \lambda_i \text{ for } i=1,\dots,k\}$. Integers $\{\lambda_1,\dots,\lambda_k\}$ are said to be coprime if $\gcd(\lambda_1,\dots,\lambda_k)=1$. For $\bar{\lambda}=(\lambda_1,\dots,\lambda_k)$ we abbreviate $\gcd(\lambda_1,\dots,\lambda_k)$ to $\gcd(\bar{\lambda})$. The notation $\|\bar{\lambda}\|_1$ stands for $|\lambda_1|+\cdots+|\lambda_k|$.

3. Lower bounds on sums of dilates

We start off with a lower bound on $A+3\cdot A$.

Proof of Theorem 1.1. Let $a_1<\cdots<a_n$ be the elements of A in increasing order. Let $A_k=\{a_1,\dots,a_k\}$. We will analyse how the size of $B_k=A_k+3\cdot A_k$ grows as k grows. We will prove that, for every $k\geqslant 5$, either $|B_k|-|B_{k-1}|\geqslant 4$ holds or both $|B_k|-|B_{k-1}|=3$ and $|B_{k+1}|-|B_k|\geqslant 5$ hold. The theorem will then follow.

Note that three sums $3a_{k-1}+a_k, a_{k-1}+3a_k, a_k+3a_k$ belonging to B_k are greater than any element of B_{k-1}, and thus do not belong to B_{k-1}. Moreover, the three sums are distinct since $3a_{k-1}+a_k<a_{k-1}+3a_k<a_k+3a_k$. Therefore, to complete the proof we need to analyse the case when $B_k\setminus B_{k-1}$ consists of precisely these three sums.

There are two cases: either $3a_k+a_{k-2}$ is in B_{k-1} or it is not.

Case A. If $3a_k+a_{k-2}\in B_{k-1}$, then since $3a_k+a_{k-2}>3a_{k-1}+a_{k-2}$, it follows that $3a_k+a_{k-2}=4a_{k-1}$. By dilating and translating the set A as necessary we can assume that $a_{k-2}=0$ and $a_{k-1}=3$. Then it follows that $a_k=4$. Since $a_k+3a_{k-2}=4$ is larger than $a_{k-1}+3a_{k-2}=3+3\cdot 0$, it follows that $a_k+3a_{k-2}=3a_{k-1}+a_t$ or $a_k+3a_{k-2}=3a_k+a_t$ for some $t<k$. Thus $a_t=-5$ or $a_t=-8$ for some $t\leqslant k-3$. In either case, since $a_{k-3}\geqslant -8$, it follows that $a_{k-3}+3a_k\geqslant 4>3=a_{k-1}+3a_{k-2}$. Thus $a_{k-3}+3a_k=a_{k-2}+3a_{k-1}$, and $a_{k-3}=-3$. Therefore the five largest elements of B_k are $16,15,13,12,9$.

Since $a_{k-3}\neq a_t\geqslant -8$, we also have $a_{k-4}\geqslant -8$. Hence $a_{k-4}+3a_k\geqslant 4=a_k+3a_{k-2}$, implying that either $a_{k-4}+3a_k=a_k+3a_{k-2}$ or $a_{k-4}+3a_k=a_{k-3}+3a_{k-1}$. Therefore, either $a_{k-4}=-6$ or $a_{k-4}=-8$. Since a_t does not exceed a_{k-4}, it follows that $a_t=-8$, after all.

As before, $\{3\cdot 4+a_{k+1}, 4+3a_{k+1}, a_{k+1}+3a_{k+1}\}\subset B_{k+1}\setminus B_k$. We need to exhibit two additional elements from B_{k+1} that are not in B_k. There are a few subcases to consider.

(a) $3\cdot 4+a_{k+1}=3a_{k+1}+3$. In this case $a_{k+1}=9/2$, implying that $3a_{k+1}+0=27/2$ and $3a_{k+1}-3=21/2$ are not in B_k.

(b) $3\cdot 4+a_{k+1}=3a_{k+1}+0$. In this case $a_{k+1}=6$, implying that $3a_{k+1}+a_{k-1}=21$ and $3a_{k+1}+a_t=10$ are not in B_k.

(c) $3\cdot 4+a_{k+1}=3a_{k+1}-3$. In this case $a_{k+1}=15/2$, implying that $3a_{k+1}+3=51/2$ and $3a_{k+1}+0=45/2$ are not in B_k.

If these subcases do not hold, then none of the four sums $3a_{k+1}+3$, $3a_{k+1}+0$, $3a_{k+1}-3$, $a_{k+1}+3\cdot 3$ are equal to any of the three elements $\{3\cdot 4+a_{k+1}, 4+3a_{k+1}, a_{k+1}+3a_{k+1}\}$ of $B_{k+1}\setminus B_k$ that we already counted. Moreover, all of them are greater than 9. We will show that at least two of these four sums are not in B_k. Since $3a_{k+1}+3>3a_k+3=15$, only two subcases remain.

(d) $3a_{k+1}+3=16$. In this case $a_{k+1}=13/3$, implying that $3a_{k+1}-3=10$ and $a_{k+1}+3\cdot 3=40/3$ are not in B_k.

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(e) $3a_{k+1}+3$ is not in B_k. Since $3a_{k+1} > 3a_k = 12$, then either $3a_{k+1}+0$ is not in B_k or it is equal to one of 16, 15 or 13. In the latter case $a_{k+1}+3\cdot 3$ is not in B_k being equal to 43/3, 14 and 40/3 in these three cases respectively.

Case B. If $3a_k + a_{k-2} \notin B_{k-1}$, then $3a_k + a_{k-2} = 3a_{k-1} + a_k$. By dilating and scaling we can ensure that a_{k-2}, a_{k-1}, a_k are $0, 2, 3$ respectively. Since $a_k + 3\cdot a_{k-2} = 3$ is an element of B_{k-1}, we necessarily have that $a_{k-3} \geqslant 3 - 3a_{k-1} = -3$. Indeed, suppose on the other hand that $a_{k-3} + 3a_{k-1} < 3 = a_k + 3a_{k-2}$. Then we must have that $a_k + 3a_{k-2}$ is equal to $a_{k-2} + 3a_{k-1}$ or $4a_{k-1}$, neither of which is possible. Since $3a_k + a_{k-3}$ is an element of B_{k-1} not less than 6, there are two cases.

(a) $a_{k-3} = -3$. The five largest elements of B_k are $12, 11, 9, 8, 6$. The sums $3a_{k+1}+a_{k+1}$, $3a_{k+1}+3$, $a_{k+1}+3\cdot 3$ are in $B_{k+1}\setminus B_k$. The sums $3a_{k+1}+2$, $3a_{k+1}+0$, $3a_{k+1}-3$, $a_{k+1}+6$ are each greater than 6. The several subcases are as follows.

(i) $3a_{k+1}+2 = a_{k+1}+9$. Then $3a_{k+1}+0 = 21/2$ and $3a_{k+1}-3 = 15/2$ are in $B_{k+1}\setminus B_k$.
(ii) $3a_{k+1}+2 = 12$. Then $3a_{k+1}+0 = 10$ and $3a_{k+1}-3 = 7$ are in $B_{k+1}\setminus B_k$.
(iii) $3a_{k+1}+2$ is not in B_k. Then $3a_{k+1}$ is either 11 or 12. In either case $a_{k+1}+6$ is in $B_{k+1}\setminus B_k$.

(b) $a_{k-3} = -1$. The four largest elements of B_k are $12, 11, 9, 8$. The sums $3a_{k+1}+a_{k+1}$, $3a_{k+1}+3$, $a_{k+1}+3\cdot 3$ are in $B_{k+1}\setminus B_k$. The sums $3a_{k+1}+2$, $3a_{k+1}+0$, $3a_{k+1}-1$, $a_{k+1}+6$ are each greater than 8. The subcases are as follows.

(i) $3a_{k+1}+2 = a_{k+1}+3\cdot 3$. Then $3a_{k+1}+0 = 21/2$ and $3a_{k+1}-1 = 19/2$ are in $B_{k+1}\setminus B_k$.
(ii) $3a_{k+1}+2 = 12$. Then $3a_{k+1} = 10$ and $a_{k+1}+6 = 28/3$ are in $B_{k+1}\setminus B_k$.
(iii) In the case $3a_{k+1}+2 \in B_{k+1}\setminus B_k$ the sum $3a_{k+1}+0$ is either 12 or 11. In the first case $a_{k+1}+6 = 10$ is in $B_{k+1}\setminus B_k$. In the second case $3a_{k+1}-1 = 10$ is in $B_{k+1}\setminus B_k$ and not equal to $a_{k+1}+9$. □

One might be puzzled by the two-step induction scheme in the proof above, where, with the addition of each next element, the sumset B_k either grows by the required number of elements, or it grows by more than that at the next step. However, this actually occurs for the set $A = \{0, 1, 3, 4, \dots, 3k, 3k+1\}$. Each next multiple of 3 increases the size of the sumset by 5, and every other number increases the sumset only by 3. Such examples impose a limitation on how simple such kinds of proofs can be. For instance, if one adopts this proof strategy to show that $|A + 4\cdot A| \geqslant 5|A| - O(1)$, then the example $A = \{0, 1, 2, 4, 5, 6, \dots, 4m, 4m+1, 4m+2\}$ shows that a similar proof will require a three-step induction.

For proving the lower bound on an arbitrary sum of dilates (1.1) we abandon the proof strategy above. The basis for the modified approach is the observation that the reason why $A + \lambda\cdot A$ is large in the examples above is that A can be partitioned into λ subsets $A_1, \dots, A_\lambda$ according to the residue class modulo λ, such that $A_i + \lambda\cdot A_i$ are disjoint from one another for different values of i. Thus $A + \lambda\cdot A$ is large because each of $A_i + \lambda\cdot A_i$ is large.

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For a general sum of dilates $S_{\bar\lambda}(A) = \lambda_1 \cdot A + \cdots + \lambda_k \cdot A$ it turns out that considering A modulo only λ_1 is insufficient. One needs to find a τ that is coprime with $\sum_{i=1}^k \lambda_i$. Then if $A_1 \cup \cdots \cup A_\tau$ is a partition of A into residue classes modulo τ, the $S_{\bar\lambda}(A_i)$s are disjoint.

It would have been excellent had $A_1, \ldots, A_\tau$ always turned out to be arithmetic progressions. They need not be, but under favourable circumstances at least one of the A_i is a somewhat denser set than A. The denser a set is, the closer it is to being an arithmetic progression. So we would like to keep the dense sets of the partition. As for the parts that are not dense, those will be partitioned further into more parts, at least some of which are dense. This leads to a recursive subpartition process, where at each step we partition sparse sets until only a few elements of A belong to the sparse parts. Those we will discard.

The next lemma characterizes sets A that cannot be broken into parts, at least one of which is dense, as those for which one can use induction on the number of dilates. After that Lemma 3.3 describes the basic step in the repeated subpartition process.

Lemma 3.1. *For a vector $\bar\lambda = (\lambda_1, \ldots, \lambda_k)$ of $k \geqslant 2$ coprime non-zero integers, let $\tau_i = \gcd(\bar\lambda^i)$. Then for every such $\bar\lambda$, every $\delta > 0$ and every finite set $A \subset \{1, \ldots, n\}$ at least one of the following holds.*

I) *The sumset $S = S_{\bar\lambda}(A)$ satisfies*

$$|S| \geqslant \frac{1}{k-1} \sum_{i=1}^{k} \tau_i |S_i| - 2\delta n - \tau_1,$$

where $S_i = S_{\bar\lambda^i}(A)$.

II) *There is an $i \in \{1, \ldots, k\}$ and $r^* \in \mathbb{Z}/\tau_i\mathbb{Z}$ such that the set*

$$\{a \in A : a \equiv r^* \pmod{\tau_i}\}$$

is contained either in $\big[1, \big(1 - \delta/|\lambda_i|\big)n\big]$ or in $\big[\delta n/|\lambda_i|, n\big]$.

Proof. Suppose alternative II does not hold. Then

$$\begin{aligned} l_{i,j} &= \min\{\lambda_i a : a \in A,\ a \equiv j \pmod{\tau_i}\}, \\ r_{i,j} &= \max\{\lambda_i a : a \in A,\ a \equiv j \pmod{\tau_i}\}, \\ l_i &= \min\{\lambda_i A\} = \min_j l_{i,j}, \\ r_i &= \max\{\lambda_i A\} = \max_j r_{i,j} \end{aligned}$$

satisfy $l_{i,j} - l_i \leqslant \delta n$ and $r_i - r_{i,j} \leqslant \delta n$. Set $L_i = \{l_{i,1}, \ldots, l_{i,\tau_i}\}$ and $R_i = \{r_{i,1}, \ldots, r_{i,\tau_i}\}$. Since all elements of S_i are divisible by τ_i, whereas L_i is a set of distinct integers modulo τ_i, we have that $S_i + l_{i,j}$ is disjoint from $S_i + l_{i,j'}$ for $j \neq j'$. Similarly, $S_i + r_{i,j}$ is disjoint from $S_i + r_{i,j'}$.

For a set T and $x \in \mathbb{Z}$ let $T_{\leqslant x} = \{t \in T : t \leqslant x\}$ and $T_{>x} = \{t \in T : t > x\}$. Now we will use the idea from the proof of [4, Theorem 1.1]. Namely, we make $k-1$ copies of the set S, and then mark some of the elements in each copy. We allow some elements

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to be marked more than once. We start by marking in the first copy the elements of $L_k + S_k$. They all belong to the interval $[l_1 + \cdots + l_k, r_1 + \cdots + r_{k-1} + l_k + \delta n]$. Then in the first copy mark the elements of $R_{k-1} + (S_{k-1})_{>r_1+\cdots+r_{k-2}+l_k}$. All elements in the first copy are marked at most once except possibly some of the elements in the interval $[r_1 + \cdots + r_{k-1} + l_k - \delta n, r_1 + \cdots + r_{k-1} + l_k + \delta n]$ are marked twice. This interval has length $2\delta n$.

Then, for $2 \leqslant i \leqslant k-2$, in the ith copy we mark the elements of

$$L_{k-i+1} + (S_{k-i+1})_{\leqslant r_1+\cdots+r_{k-i}+l_{k-i+2}+\cdots+l_k}$$

and of

$$R_{k-i} + (S_{k-i})_{>r_1+\cdots+r_{k-i-1}+l_{k-i+1}+\cdots+l_k}.$$

Only the elements in the interval

$$[r_1 + \cdots + r_{k-i} + l_{k-i+1} + \cdots + l_k - \delta n, r_1 + \cdots + r_{k-i} + l_{k-i+1} + \cdots + l_k + \delta n]$$

can possibly be marked twice. Finally, in the $(k-1)$th copy we mark the elements of $L_2 + (S_2)_{\leqslant r_1+l_3+\cdots+l_k}$ and of $R_1 + (S_1)_{>l_2+\cdots+l_k}$. Again elements only in $[r_1 + l_2 + \cdots + l_k - \delta n, r_1 + l_2 + \cdots + l_k + \delta n]$ can be marked twice. And again this interval is of length $2\delta n$.

Counting the number of marked elements we obtain

$$(k-1)|S| \geqslant \sum_{i=1}^{k} \tau_i |S_i| - 2(k-1)\delta n - \tau_1,$$

where the right side counts the number of elements that are marked at least once, and the left side counts the total number of elements. The term τ_1 on the right is due to the set $(S_1)_{>l_2+\cdots+l_k}$ having one fewer element than S_1. □

Corollary 3.2. *If* $\sum_{i=1}^{k} \lambda_i = 0$, *then*

$$|S_{\bar{\lambda}}(A)| \geqslant \frac{1}{k-1} \sum_{i=1}^{k} |S_{\bar{\lambda}}(A^i)| - 5 \tag{3.1}$$

for any non-empty $A \subset \mathbb{Z}$. *Moreover the vectors* $\bar{\lambda}^i$ *are coprime for every* i.

Proof. Since both sides of (3.1) are translation-invariant, we can assume that $1 \in A$, and set $n = \max A$. Since $\tau_i \mid \sum_{j \neq i} \lambda_j = -\lambda_i$ and $\bar{\lambda}$ is a coprime vector, $\tau_i = 1$ for all i. If we set $\delta = 2/n$, then alternative II does not hold, and alternative I becomes (3.1). □

Lemma 3.3. *For every* $\bar{\lambda} = (\lambda_1, \ldots, \lambda_k)$ *satisfying* $\sum_{i=1}^{k} \lambda_i \neq 0$ *there are* $\alpha > 0$ *and* $\beta > 0$ *such that, for every integer* $t \geqslant 0$ *and every finite set* $A \subset \mathbb{Z}$ *of size* $|A| \geqslant (\alpha/2)^{-\beta^t}$, *there are four families of sets* $\mathcal{D}_t$, $\mathcal{G}_t$, $\mathcal{S}_t, \mathcal{T}_t$ *satisfying the following conditions.*

(1) *The families* $\mathcal{D}_t$, $\mathcal{G}_t$, $\mathcal{S}_t, \mathcal{T}_t$ *together form a partition of* A, *i.e., the sets in* $\mathcal{D}_t$, $\mathcal{G}_t$, $\mathcal{S}_t, \mathcal{T}_t$ *are disjoint from one another and their union is* A.

(2) *If* B_1, B_2 *are any two unequal sets from* $\mathcal{D}_t \cup \mathcal{G}_t \cup \mathcal{S}_t \cup \mathcal{T}_t$ (*i.e., the sets* B_1 *and* B_2 *possibly belong to different families*), *then* $S_{\bar{\lambda}}(B_1)$ *is disjoint from* $S_{\bar{\lambda}}(B_2)$.

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(3) *The sets in* $\mathcal{D}_t$ *are dense: each set in* $\mathcal{D}_t$ *is* $\|\bar\lambda\|_1$*-isomorphic to a subset of an interval of length at least* $|A|^{\beta^t}$ *of density at least* $\alpha/2$.
(4) *The sets in* $\mathcal{G}_t$ *are growing: for each* $G \in \mathcal{G}_t$ *we have* $|S_{\bar\lambda}(G)| \geqslant \|\bar\lambda\|_1|G|$.
(5) *The sets in* $\mathcal{S}_t$ *are small, but not too small:* $|\bigcup \mathcal{S}_t| \leqslant |A|/2^t$, *but* $|S| \geqslant |A|^{\beta^t}$ *for every* $S \in \mathcal{S}_t$.
(6) *The sets in* $\mathcal{T}_t$ *are tiny:* $|\bigcup \mathcal{T}_t| \leqslant \frac{1}{\alpha}\sum_{i=1}^{t}|A|^{1-\beta^i}$.

Proof. We let α and d be as in Freiman's theorem (Theorem 2.5) when applied with $K = \|\bar\lambda\|_1^2$ and $r = \|\bar\lambda\|_1$. We set $\beta = 1/2d$.

The proof is by induction on t. For $t = 0$ we simply set $\mathcal{D}_0 = \mathcal{G}_0 = \mathcal{T}_0 = \emptyset$ and $\mathcal{S}_0 = \{A\}$. If $t \geqslant 1$, then we use induction to obtain $\mathcal{D}_{t-1}$, $\mathcal{G}_{t-1}$, $\mathcal{S}_{t-1}$ and $\mathcal{T}_{t-1}$. We will not do anything to sets in $\mathcal{D}_{t-1}$, $\mathcal{G}_{t-1}$ and $\mathcal{T}_{t-1}$: they will become members of $\mathcal{D}_t$, $\mathcal{G}_t$ and $\mathcal{T}_t$, respectively. However, the sets in $\mathcal{S}_{t-1}$ will be either moved to $\mathcal{G}_t$ or subpartitioned further into $\mathcal{D}$-, $\mathcal{S}$- and $\mathcal{T}$-sets.

Let A' be a set in $\mathcal{S}_{t-1}$. If $|\lambda_1 \cdot A' + \lambda_1 \cdot A'| \geqslant \|\bar\lambda\|_1^2|A'|$, then by Corollary 2.2

$$\begin{aligned}|\lambda_1 \cdot A' + (\lambda_2 \cdot A' + \cdots + \lambda_k \cdot A')| &\geqslant \|\bar\lambda\|_1\sqrt{|A'||\lambda_2 \cdot A' + \cdots + \lambda_k \cdot A'|}\\ &\geqslant \|\bar\lambda\|_1|A'|\end{aligned}$$

and we can move A' to $\mathcal{G}_t$.

Hence we can assume that $|A' + A'| = |\lambda_1 \cdot A' + \lambda_1 \cdot A'| < \|\bar\lambda\|_1^2|A'|$. By Freiman's theorem (Theorem 2.5) the set A' is $\|\bar\lambda\|_1$-isomorphic to A'', which is a subset of $[1, t_1] \times \cdots \times [1, t_d]$ of density at least $\alpha > 0$, where d and α as above. Since A' and A'' are $\|\bar\lambda\|_1$-isomorphic, $|S_{\bar\lambda}(A')| = |S_{\bar\lambda}(A'')|$. Without loss of generality we may assume that $t_1 \geqslant \cdots \geqslant t_d$. This assures us that $t_1 \geqslant |A'|^{1/d} \geqslant |A|^{\beta^{t-1}/d}$. For every $x \in [1, t_2] \times \cdots \times [1, t_d]$, there is a *fibre*

$$A_x = \{(a_1, \ldots, a_d) \in A'' : (a_2, \ldots, a_d) = x\}.$$

These fibres form a partition of A''. Since $\sum_{i=1}^{k}\lambda_i \neq 0$ the set $S_{\bar\lambda}(A_x)$ is disjoint from $S_{\bar\lambda}(A_y)$ for $x \neq y$.

Let $X = \{x \in [1, t_2] \times \cdots \times [1, t_d] : |A_x| \leqslant \alpha t_1/2\}$. For $x \notin X$ the fibre A_x is $\|\bar\lambda\|_1$-isomorphic to a subset of the interval $[1, t_1]$ of density at least $\alpha/2$. Since $t_1 \geqslant |A|^{2\beta^t} \geqslant \frac{2}{\alpha}|A|^{\beta^t}$, we can move any fibre A_x with $x \notin X$ to $\mathcal{D}_t$.

Let $Y = \{y \in [1, t_2] \times \cdots \times [1, t_d] : |A_y| \leqslant t_1^{1/2}\}$. Since $|A'| \geqslant \alpha t_1 \cdots t_d$ we have that $|\bigcup_{y\in Y} A_y| \leqslant (|A'|/\alpha)t_1^{-1/2}$. Therefore the total number of elements in fibres of the form A_y for $y \in Y$ for all $A' \in S$ is at most $(|A|/\alpha)|A|^{-\beta^t}$. We add $\{A_y\}_{y\in Y}$ to $\mathcal{T}_t$. Because $|\bigcup_{x\in X\setminus Y} A_x| \leqslant |\bigcup_{x\in X} A_x| \leqslant |A'|/2$, the remaining fibres A_x with $x \in X \setminus Y$ can then be moved to $\mathcal{S}_t$. □

With the previous two lemmas in our arsenal, we are ready to prove the sharp lower bound on the arbitrary sum of dilates.

Proof of Theorem 1.2. The proof is by induction on k. The case $k = 1$ is true since $\gcd(\lambda_1) = 1$ only if $\lambda_1 \in \{\pm 1\}$. Suppose we are given a vector $\bar\lambda = (\lambda_1, \ldots, \lambda_k)$ of coprime integers. Assume we have already established the theorem for all vectors of fewer than k

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integers. We can assume that $\sum \lambda_i \neq 0$ since in the case $\sum \lambda_i = 0$ Corollary 3.2 yields

$$\begin{aligned}|S_{\bar{\lambda}}(A)| &\geqslant \frac{1}{k-1}\sum_{i=1}^{k} S_{\bar{\lambda}^i}|A| - 5\\ &\geqslant \frac{1}{k-1}\sum_{i=1}^{k}\big(\|\bar{\lambda}^i\|_1|A| - o(|A|)\big) - 5\\ &= \frac{1}{k-1}\sum_{i=1}^{k}\big(\|\bar{\lambda}\|_1 - |\lambda_i|\big)|A| - o(|A|)\\ &= \|\bar{\lambda}\|_1|A| - o(|A|)\end{aligned}$$

by the induction hypothesis.

Let M be the largest number such that $|S_{\bar{\lambda}}(A)| \geqslant M|A| - o(|A|)$. Similarly, let $M(\gamma)$ be the largest number such that $|S_{\bar{\lambda}}(A)| \geqslant M(\gamma)|A| - o(|A|)$ for sets A that are subsets of density at least γ in some interval.

Claim 1. *$M \geqslant M(\alpha/2)$, where α is as in Lemma 3.3.*

Claim 2. *For every $\delta > 0$ and $0 < \gamma < 1$,*

$$M(\gamma) \geqslant \min\left(M + \big(M\big(\gamma(1+\delta/4\|\bar{\lambda}\|_\infty^2\big) - M\big)\frac{\delta}{4\|\bar{\lambda}\|_\infty^3}, \|\bar{\lambda}\|_1 - 2\delta/\gamma\right).$$

If $\gamma(1+\delta/4\|\bar{\lambda}\|_\infty^2) > 1$ the right-hand side should be interpreted as $\|\bar{\lambda}\|_1 - 2\delta/\gamma$.

Proof of Claim 1. Let α and β be as in Lemma 3.3. Fix $\epsilon > 0$ and an integer $t \geqslant 0$. Let N be so large that $|S_{\bar{\lambda}}(A)| \geqslant \big(M(\alpha/2) - \epsilon\big)|A|$ for sets A with at least N elements that are subsets of density at least $\alpha/2$ in some interval. Lemma 3.3 then implies that $|S_{\bar{\lambda}}(A)| \geqslant \big(M(\alpha/2) - \epsilon\big)(1 - 2^{-t} - \frac{1}{\alpha}\sum_{i=1}^{t}|A|^{-\beta^i})|A|$ if $|A| \geqslant N^{\beta^{-t}}$. Therefore, $M \geqslant \big(M(\alpha/2) - \epsilon\big)(1 - 2^{-t})$ for every $\epsilon > 0$ and every $t \geqslant 0$. □

Proof of Claim 2. Fix an $\epsilon > 0$. Let N be so large that $|S_{\bar{\lambda}}(A)| \geqslant \big(M(\epsilon) - \epsilon\big)|A|$ for sets A with at least N elements that are subsets of intervals of density at least ϵ. Moreover, let N be so large that $|S_{\bar{\lambda}}(A)| \geqslant \big(M\big(\gamma(1+\delta/4|\lambda_i|\tau_i)\big) - \epsilon\big)|A|$ for sets A that are subsets of density at least $\gamma(1+\delta/4|\lambda_i|\tau_i)$ in some interval.

Let A be a subset of an interval of density at least γ, and suppose $|A| \geqslant N\max_i \tau_i/\epsilon$. Without loss of generality we may assume that $A \subset \{1,\ldots,n\}$ and $1, n \in A$. Apply Lemma 3.1 with δ as in the statement of the claim. If alternative I holds, then

$$\begin{aligned}S_{\bar{\lambda}}(A) &\geqslant \frac{1}{k-1}\sum_{i=1}^{k}\tau_i|S_{\bar{\lambda}^i}(A)| - 2\delta n - \tau_1\\ &\geqslant \frac{1}{k-1}\sum_{i=1}^{k}\tau_i|A|\|\bar{\lambda}^i/\tau_i\|_1 - o(|A|) - 2\delta n - \tau_1\end{aligned}$$

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$$= \frac{1}{k-1}\sum_{i=1}^{k}\big(\|\bar\lambda\| - |\lambda_i|\big) - 2\delta n - o(n)$$
$$= \|\bar\lambda\|_1|A| - 2\delta n - o(n)$$
$$\geqslant \big(\|\bar\lambda\|_1 - 2\delta/\gamma - o(1)\big)|A|.$$

Suppose alternative II holds, and let i and r^* be given as in the alternative. For $r \in \mathbb{Z}/\tau_i\mathbb{Z}$ let $A_r = \{a \in A : a \equiv r \pmod{\tau_i}\}$. Since $\bar\lambda$ is a coprime vector, τ_i is coprime with $\sum_{j=1}^{k}\lambda_j$. Therefore $S_{\bar\lambda}(A_{r_1})$ is disjoint from $S_{\bar\lambda}(A_{r_2})$ for $r_1 \neq r_2$. Let $B_r = \{\lfloor a/\tau_i\rfloor : a \in A_r\}$. Clearly $|S_{\bar\lambda}(A_r)| = |S_{\bar\lambda}(B_r)|$. Each set B_r is contained in an interval of length $\lceil n/\tau_i\rceil$. Moreover, B_{r^*} is contained in a shorter interval of length $\lceil n(1-\delta/|\lambda_i|)/\tau_i\rceil$. Therefore the total length of the intervals containing $\{B_r\}_{r\in\mathbb{Z}/\tau_i\mathbb{Z}}$ is at most

$$n\left(1 - \frac{\delta}{|\lambda_i|\tau_i}\right) + \tau_i \leqslant n\left(1 - \frac{\delta}{2|\lambda_i|\tau_i}\right).$$

Let $R_1 = \{r \in \mathbb{Z}/\tau_i\mathbb{Z} : |B_r| \leqslant |A|\delta/4|\lambda_i|\tau_i^2\}$. Then $\sum_{r\in R_1}|B_r| \leqslant |A|\delta/4|\lambda_i|\tau_i$. Therefore there is an $r_0 \in (\mathbb{Z}/\tau_i\mathbb{Z}) \setminus R_1$ such that the density of B_{r_0} in the appropriate interval is at least

$$\frac{|A| - \sum_{r\in R_1}|B_r|}{n(1-\delta/2|\lambda_i|\tau_i)} \geqslant \gamma\frac{1-\delta/4|\lambda_i|\tau_i}{1-\delta/2|\lambda_i|\tau_i} \geqslant \gamma(1+\delta/4|\lambda_i|\tau_i).$$

Let $R_2 = \{r \in \mathbb{Z}/\tau_i\mathbb{Z} : |B_r| \leqslant \epsilon|A|/\tau_i\}$. Then

$$\begin{aligned}
S_{\bar\lambda}(A) &\geqslant \sum_{r\in\mathbb{Z}/\tau_i\mathbb{Z}}|S_{\bar\lambda}(B_r)| \\
&\geqslant \sum_{r\in(\mathbb{Z}/\tau_i\mathbb{Z})\setminus R_2}|S_{\bar\lambda}(B_r)| \\
&= |S_{\bar\lambda}(B_{r_0})| + \sum_{r\in(\mathbb{Z}/\tau_i\mathbb{Z})\setminus(R_2\cup\{r_0\})}|S_{\bar\lambda}(B_r)| \\
&\geqslant \big(M\big(\gamma(1+\delta/4|\lambda_i|\tau_i)\big) - \epsilon\big)|B_{r_0}| + \sum_{r\in(\mathbb{Z}/\tau_i\mathbb{Z})\setminus(R_2\cup\{r_0\})}(M(\epsilon)-\epsilon)|B_r| \\
&\geqslant (M(\epsilon)-\epsilon)|A|(1-\epsilon) + \big(M\big(\gamma(1+\delta/4|\lambda_i|\tau_i)\big) - M(\epsilon)\big)|B_{r_0}|.
\end{aligned}$$

Since $M(\epsilon) \geqslant M$ and ϵ can be chosen arbitrarily small, we infer

$$M(\gamma) \geqslant \min\left(M + \big(M\big(\gamma(1+\delta/4|\lambda_i|\tau_i)\big) - M\big)\frac{\delta}{4|\lambda_i|\tau_i^2}, \|\bar\lambda\|_1 - 2\delta/\gamma\right).$$

Since $\tau_i^2|\lambda_i| \leqslant \|\bar\lambda\|_\infty^3$ Claim 2 follows. □

Claims 1 and 2 imply the theorem. Indeed, fix $\delta > 0$ and assume that $M \leqslant \|\bar\lambda\|_1 - 4\delta/\alpha$. Let

$$\Gamma = \{\gamma \in [\alpha/2, 1] : M \geqslant M(\gamma)\}.$$

By Claim 1 the set Γ is non-empty. By Claim 2 $\gamma \in \Gamma$ implies that either

$$M \geqslant M(\gamma) \geqslant \|\bar\lambda\|_1 - 2\delta/\gamma,$$

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which is inconsistent with the assumption above, or that

$$M \geqslant M(\gamma) \geqslant M + \big(M\big(\gamma(1+\delta/4\|\bar{\lambda}\|_\infty^2)\big) - M\big)\frac{\delta}{4\|\bar{\lambda}\|_\infty^3},$$

implying

$$M \geqslant M\big(\gamma\big(1+\delta/4\|\bar{\lambda}\|_\infty^2\big)\big)$$

and $\gamma(1+\delta/4\|\bar{\lambda}\|_\infty^2) \in \Gamma$. However, this is a contradiction since no element in Γ exceeds 1. Thus $M \geqslant \|\bar{\lambda}\|_1 - 4\delta/\alpha$ for every $\delta > 0$, and it follows that $M \geqslant \|\bar{\lambda}\|_1$. □

4. Plünnecke-type inequalities on sums of dilates

Proof of Theorem 1.3. First we deal with the case $\lambda_1, \ldots, \lambda_k > 0$. Without loss of generality $0 \in A$. Let $r = \max_i \lfloor \log_2 \lambda_i \rfloor$. Write λ_i in base 2 as $\lambda_i = \sum_{j=0}^{r} \lambda_{i,j} 2^j$ with $\lambda_{i,j} \in \{0,1\}$. Then clearly

$$S_{\bar{\lambda}}(A) \subset \sum_{j=0}^{r}\left(\sum_{i=1}^{k}\lambda_{i,j}\right) * (2^j \cdot A). \tag{4.1}$$

Since by Plünnecke's inequality $|t * (2\cdot A) + 2\cdot A - 2\cdot A + A| \leqslant |(2t+3) * A - 2 * A| \leqslant K^{2t+5}|A|$, Lemma 2.4 implies that there are $X_1, \ldots, X_r$ satisfying

$$\left(\sum_{i=1}^{k}\lambda_{i,j}\right) * (2\cdot A) + 2\cdot A - 2\cdot A \subset A - A + X_j, \qquad |X_j| \leqslant K^{2\sum_{i=1}^{k}\lambda_{i,j}+5}. \tag{4.2}$$

Let $Y_j = 2^{j-1}\cdot X_j$ for $j = 1, \ldots, r$. Note that $|Y_j| = |X_j|$. The inclusion (4.2) with $j = r$ and (4.1) combine into

$$S_{\bar{\lambda}}(A) \subset \sum_{j=0}^{r-1}\left(\sum_{i=1}^{k}\lambda_{i,j}\right) * (2^j \cdot A) + 2^{r-1}\cdot A - 2^{r-1}\cdot A + Y_r.$$

Repeatedly using inclusion (4.2) for $j = r-1, r-2, \ldots, 1$, we obtain

$$S_{\bar{\lambda}}(A) \subset A - A + \left(\sum_{i=1}^{k}\lambda_{i,0}\right) * A + Y_1 + \cdots + Y_r,$$

implying

$$\begin{aligned} |S_{\bar{\lambda}}(A) - A| &\leqslant \left|A - A - A + \left(\sum_{i=1}^{k}\lambda_{i,0}\right) * A\right| \prod_{j=1}^{r}|X_j| \\ &\leqslant |A| K^{3+5r+2\sum_{j=0}^{r}\sum_{i=1}^{k}\lambda_{i,j}}. \end{aligned}$$

Now we turn to the case when some of the λs are negative. Say $\lambda_1, \ldots, \lambda_p$ are positive, whereas $\lambda_{p+1}, \ldots, \lambda_k$ are negative. As before we let $|\lambda_i| = \sum_{j=0}^{r}\lambda_{i,j}2^j$. Let $B = \lambda_1\cdot A + \cdots + \lambda_p \cdot A$ be the sum of positive dilates, and $C = \lambda_{p+1}\cdot A + \cdots + \lambda_k \cdot A$ be the sum of negative

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dilates. By the above,

$$|B+A| \leqslant |A|K^{4+5r+2\sum_{j=0}^{r}\sum_{i=1}^{p}\lambda_{i,j}},$$
$$|C+A| \leqslant |A|K^{3+5r+2\sum_{j=0}^{r}\sum_{i=p+1}^{k}\lambda_{i,j}}.$$

By the triangle inequality (Lemma 2.1)

$$|B+C| \leqslant \frac{|A+B||A+C|}{|A|} \leqslant |A|K^{7+10r+2\sum_{j=0}^{r}\sum_{i=1}^{k}\lambda_{i,j}}.$$

Since $|\sum_{j=0}^{r}\lambda_{i,j}| \leqslant \log_2(1+|\lambda_i|)$ and $r \leqslant \max_i \log_2(1+|\lambda_i|)$, the theorem follows. □

Observe that the actual bound obtained in the course of the proof of Theorem 1.3 involves the sum of binary digits of λ_i rather than $\log(1+|\lambda_i|)$. In particular, if $\lambda_1,\dots,\lambda_k$ are k positive integers not exceeding 2^k, each containing no more than 7 ones in binary development, then $|A+A| \leqslant K|A|$ implies $|S_{\bar{\lambda}}(A)| \leqslant K^{100k}|A|$. Since the proof of Theorem 1.3 could be easily adapted to use b-ary expansion in place of binary, similar results are true of λs that have sparse b-ary expansion at the cost of worsening the constant 100 above if b gets large. Since numbers with 7 ones in binary development are commonly believed to look quite random in almost any other base, any bound that depends on the base in which a number is written is unnatural. Perhaps a condition on the size of the λs is all one needs.

Question 4.1. *Suppose $\bar{\lambda} = (\lambda_1,\dots,\lambda_k)$ satisfies $|\lambda_i| \leqslant 2^k$. Does it follow that*

$$\frac{|\lambda_1\cdot A+\dots+\lambda_k\cdot A|}{|A|} \leqslant \left(\frac{|A+A|}{|A|}\right)^{Ck}$$

for an absolute constant C?

With Ck^2 in place of Ck the estimate follows from Theorem 1.3.

One can also use the triangle inequality for proving inequalities similar to that in Theorem 1.3.

Theorem 4.2. *For any $k \in \mathbb{N}$ and $\lambda \in \mathbb{Z}$, we have*

$$\frac{|A+\lambda^k\cdot A|}{|A|} \leqslant \left(\frac{|A+A|}{|A|}\right)^{k(|\lambda|+1)}.$$

Proof. By the triangle inequality,

$$|A+(\lambda_1\lambda_2)\cdot A| \leqslant \frac{|A+\lambda_1\cdot A||\lambda_1\cdot A+(\lambda_1\lambda_2)\cdot A|}{|\lambda_1\cdot A|} = \frac{|A+\lambda_1\cdot A||A+\lambda_2\cdot A|}{|A|},$$

and the theorem follows from Plünnecke's inequality by induction on k. □

Though a more careful argument can improve on the constants in Theorem 1.3, the simplest case of $A+2\cdot A$ seems to be out of reach.

Question 4.3. *Is $|A+2\cdot A|/|A| \leqslant (|A+A|/|A|)^p$ for some $p<3$?*

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Acknowledgements

I am grateful to Brooke Orosz, and the anonymous referee who read preliminary versions of the paper, and pointed out many inaccuracies. This work was inspired by conversations with Jacob Fox and Jacob Tsimerman.

After this paper was completed, I was informed by Manuel Silva that he with Javier Cilleruelo and Carlos Vinuesa proved that $|A + 3 \cdot A| \geqslant 4|A| - 4$ and characterized the cases when equality occurs.

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