11

Nucleosynthesis in stars

In the preceding chapter we explained how in a star like the Sun helium is steadily formed from the fusion of hydrogen. In this chapter we sketch some of the basic ideas of 'nuclear astrophysics', a subject which seeks to understand all the nuclear processes leading to energy generation in stars in the various stages of stellar evolution, and to account for the observed relative abundances of the elements in the Solar System in terms of these processes.

The accepted theory of the Universe is that it is expanding, and began with an intensely hot and dense 'big bang' between 10×10^9 and 20×10^9 years ago. A few hundred thousand years after the big bang, the expanding material had cooled sufficiently for it to condense into a gas made up of hydrogen and helium atoms in a ratio of about 100:7 by number, together with photons and neutrinos. Apart from a small amount of lithium, it is thought that the proportion of heavier elements produced in this first explosion was insignificant (essentially because there are no stable nuclei with $A = 5$ or $A = 8$). If this is so, we must conclude that all the heavier nuclei in the Solar System have been produced in previous generations of stars and then thrown out into space again, perhaps in the explosion of supernovae.

11.1 Stellar evolution

Consider a star which has condensed from the primordial hydrogen–helium mixture, and in which hydrogen burning has set in at the core. As the hydrogen in the core is consumed, the reaction rate eventually becomes insufficient to sustain the temperature, and hence the pressure,

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that prevents further gravitational contraction. Thus more material falls into the core region. If the star is massive enough, the gravitational energy released raises the temperature of the core sufficiently for helium to begin burning at a significant rate. As the helium in turn is consumed, further stages of nuclear burning set in until the most tightly bound elements, iron and nickel, are formed. At each stage a higher temperature is needed to overcome a higher Coulomb barrier; the energy for this is provided by gravitational contraction.

Before considering these later stages of nuclear burning in more detail, it is important to appreciate that there are conditions under which the central pressure can permanently balance the pressure exerted by gravity. Then contraction will cease and the temperatures for further steps in nucleosynthesis will not be reached. After completing as much burning as it can, the star will simply cool. The first contribution to the pressure that may stop contraction is the 'electron-degeneracy pressure'. Since electrons are fermions, it follows from the Pauli principle that, even in a cold star with $T \sim 0$ K, electron states are occupied up to an energy $\varepsilon_F = (\hbar^2/2m_e)k_F^2$, where (Appendix B, equation (B.5)) $k_F^3 = 3\pi^2\rho_e$, and ρ_e is the number density of electrons. Thus in matter at high density there exist electrons with high k_F and hence high kinetic energy, which necessarily exert a high pressure. To obtain a simple order-of-magnitude estimate of this effect, we set the density of matter in a star of mass M, radius R, to be constant. Then the number of electrons in the star is $N_e = (M/\mu)$, where μ is the stellar mass per electron. For material with $Z = N$, we have $\mu = 2$ amu, to a good approximation. The electron number density is $\rho_e = (M/\mu)/(4\pi R^3/3)$, giving

$$k_F^3 = 3\pi^2\rho_e = \left(\frac{9\pi}{4}\right)\left(\frac{M}{\mu}\right)\frac{1}{R^3}. \tag{11.1}$$

Assuming that the electrons can be treated non-relativistically, the total kinetic energy of N_e electrons at $T = 0$ is $(3/5)N_e\varepsilon_F$ (cf. Problem 5.2). At $T \sim 0$ K, the sum of the electron kinetic energies and the gravitational potential energy is therefore

$$E = \frac{3}{5}\left(\frac{M}{\mu}\right)\frac{\hbar^2}{2m_e}\left(\frac{9\pi M}{4\mu}\right)^{\frac{2}{3}}\frac{1}{R^2} - \frac{3}{5}\frac{GM^2}{R}. \tag{11.2}$$

The star begins its life with R large, and the electron energy is then much smaller than the gravitational energy. As the star evolves it con-

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tracts, so that our 'model' R decreases and E becomes more negative. The energy released goes into heating the interior of the star and into radiation. However, no more energy can be released by contraction when E reaches its minimum value where $\mathrm{d}E/\mathrm{d}R = 0$, at

$$\begin{aligned} R_{\min} &= \left(\frac{9\pi}{4}\right)^{\frac{2}{3}} \frac{\hbar^2}{G m_{\mathrm{e}} M^{\frac{1}{3}} \mu^{\frac{5}{3}}} \\ &= 7.2 \left(\frac{M_\odot}{M}\right)^{\frac{1}{3}} \times 10^3 \text{ km, taking } \mu = 2 \text{ amu.} \end{aligned} \tag{11.3}$$

(A calculation which does not make our assumption of constant density, but determines the density self-consistently, gives a numerical coefficient of 8.8 instead of 7.2.)

The corresponding mass density in our model at this minimum radius is

$$\begin{aligned} \rho_{\text{mass}} &= \frac{M}{(4\pi R_{\min}^3/3)} = \frac{4M^2 G^3 m_{\mathrm{e}}^3 \mu^5}{27\pi^3 \hbar^6} \\ &= 1.27 \left(\frac{M}{M_\odot}\right)^2 \times 10^9 \text{ kg m}^{-3}. \end{aligned} \tag{11.4}$$

There are many stars with masses similar to, but generally smaller than, the Sun which are close to this inert condition. They have high density and small radii, and are called white dwarfs. It may be noted that the minimum radius decreases as the mass increases.

The maximum electron momentum in our model when $R = R_{\min}$ is, using equation (11.1),

$$p_{\mathrm{F}} = \hbar k_{\mathrm{F}} = (3\pi^2)^{\frac{1}{3}} (\rho_{\text{mass}}/\mu)^{\frac{1}{3}} \hbar = 0.44 \left(\frac{M}{M_\odot}\right)^{\frac{2}{3}} \text{MeV}/c.$$

Since the rest mass of an electron is 0.511 MeV/c^2, the assumption in our calculation above that the electron can be treated non-relativistically, i.e. that $p_{\mathrm{F}} \ll m_{\mathrm{e}} c$, is clearly suspect for stars with $M \approx M_\odot$ and is certainly wrong for stars of large M. In the limit when M is large, we take

$$\varepsilon = (p^2 c^2 + m_{\mathrm{e}}^2 c^4)^{\frac{1}{2}} \approx pc = \hbar c k,$$

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for all the electrons, so that the total energy of N_e electrons at $T = 0$ becomes $(3/4)N_e(\hbar c k_F)$ (Problem 11.1). Hence, using equation (11.1) again, the expression (11.2) for the energy is replaced by

$$E \approx \left[\frac{3}{4}\hbar c\left(\frac{9\pi}{4}\right)^{\frac{1}{3}}\left(\frac{M}{\mu}\right)^{\frac{4}{3}} - \frac{3}{5}GM^2\right]\frac{1}{R}. \tag{11.5}$$

If M is sufficiently large the coefficient of $(1/R)$ is negative and there is no minimum energy: electron degeneracy alone cannot prevent the collapse of the star. Our extreme relativistic approximation becomes increasingly valid as R decreases. Equation (11.5) suggests that the critical value of M is

$$M = \frac{15}{16}\frac{(5\pi)^{\frac{1}{2}}}{\mu^2}\left(\frac{\hbar c}{G}\right)^{\frac{3}{2}} = 1.74\ M_\odot.$$

A more careful calculation takes proper account of relativistic energies and determines the density distribution self-consistently. It is then found that the electron-degeneracy pressure cannot stop the gravitational collapse of a star of mass M if $M > 1.44\ M_\odot$. This result, due to Chandrasekhar, is known as the *Chandrasekhar limit*.

At very high densities of matter it becomes energetically favourable for electrons to be captured by protons, and a Fermi gas of neutrons is formed. Thus final collapse may be prevented by neutron-degeneracy pressure. The number of neutrons in such a *neutron star* is approximately $(M/1\text{ amu})$. Putting $\mu = 1$ amu and replacing m_e by m_n in (11.3) and (11.4) suggests a neutron star has a radius of

$$R = 1.26(M_\odot/M)^{1/3} \times 10\text{ km}, \tag{11.6}$$

and a corresponding mass density $2.37\ (M/M_\odot)^2 \times 10^{17}$ kg m^{-3}. Such a mass density is comparable with the mass density of nuclear matter, so that our simple expressions, which neglect nuclear interactions, are at best only order of magnitude estimates. Nevertheless, we expect there to be a mass limit, analogous to the Chandrasekhar limit, beyond which neutron-degeneracy pressure cannot stop further gravitational collapse. Putting $\mu = 1$ amu in (11.6) suggests this limit is about four times the Chandrasekhar limit. More realistic calculations, taking into account the compressibility of nuclear matter, give the maximum possible mass

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of a neutron star to be $\sim 3M_{\odot}$. Neutron stars having a greater mass than this will collapse. Newtonian gravitation theory becomes inadequate to describe what will happen at very high mass density. According to the general theory of relativity, the star will collapse into a *black hole*, manifested only by its intense gravitational field.

11.2 From helium to silicon

We return now to the problem of nucleosynthesis beyond helium. It is clear that the fusion of hydrogen to helium already converts most of the available nuclear potential energy into heat and radiation. The binding energy per nucleon in ^{4}He is 7.1 MeV, and there is only a further 1.7 MeV per nucleon to be released in complete burning to iron. Also, as can be seen from Fig. 10.3, as the elements involved become heavier and more charged, higher and higher temperatures are required for there to be significant tunnelling through the Coulomb barriers. In fact, as we shall see, the simple fusion process is superseded by another when elements around $^{28}_{14}$Si have been produced.

A few of the important reactions associated with helium burning to oxygen, and oxygen to silicon, are listed below, along with typical temperatures and mass densities at which in a sufficiently massive star they are calculated to occur:

$$\left.\begin{aligned} &^4\mathrm{He} + {}^4\mathrm{He} \rightarrow {}^8\mathrm{Be} \\ &^4\mathrm{He} + {}^8\mathrm{Be} \rightarrow {}^{12}\mathrm{C} + \gamma + 12 \times (0.61\ \mathrm{MeV}) \\ &^4\mathrm{He} + {}^{12}\mathrm{C} \rightarrow {}^{16}\mathrm{O} + \gamma + 16 \times (0.45\ \mathrm{MeV}) \end{aligned}\right\} \quad \begin{aligned} &k_\mathrm{B}T \sim (10\text{–}20\ \mathrm{keV}) \\ &\rho \sim (10^5\text{–}10^8)\ \mathrm{kg\ m^{-3}} \end{aligned}$$

$$^{16}\mathrm{O} + {}^{16}\mathrm{O} \rightarrow {}^{28}_{14}\mathrm{Si} + {}^4\mathrm{He} + 32 \times (0.30\ \mathrm{MeV}) \quad \begin{aligned} &k_\mathrm{B}T \sim (100\text{–}200)\ \mathrm{keV} \\ &\rho \sim 10^9\ \mathrm{kg\ m^{-3}}. \end{aligned}$$

The initial stage of helium burning needs some explanation. As Table 4.2 indicated, ^{4}He has the largest binding energy per nucleon of any nucleus less massive than ^{12}C. The most stable form of nuclear material with $A < 12$ is therefore ^{4}He, and in particular $^8_4$Be does not exist as a stable nucleus. Nevertheless $^8_4$Be exists as a resonant state that is seen in the laboratory in α–α scattering at an energy of 94 keV, in the centre-of-mass frame, with a narrow width (due to the Coulomb barrier) of 2.5 eV. In a ^{4}He plasma this state is established with an equilibrium density such that the rate of production equals the rate of decay. Thus the 'mass gaps'

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at $A = 5$ and $A = 8$ can be bridged. The next step in the chain, $^{8}\mathrm{Be} + \alpha \rightarrow {}^{12}\mathrm{C} + \gamma$, is in fact enhanced because it is a resonant reaction. There is an excited state of ^{12}C at 0.29 MeV above the $^{8}\mathrm{Be} + \alpha$ threshold.

In the final stages of oxygen burning, core temperatures in the star are calculated to reach 300–400 keV, with mass densities in excess of 10^{9} kg m^{-3}.

11.3 Silicon burning

In all the preceding stages of stellar evolution, photons have always been present in thermal equilibrium with the plasma. They have played an important role in radiative heat transfer, but have been unimportant for initiating the nuclear processes we have discussed. However, a photon couples electromagnetically to a nucleus and can be readily absorbed by a nucleus to form an excited state. If the photon has an energy above the threshold for nuclear break-up of that nucleus, break-up can occur. This process is called *photodisintegration*.

As the temperature in the core of a star approaches $k_{\mathrm{B}}T = 1$ MeV, the increasing number of photons in the high-energy tail of the thermal distribution makes photodisintegration an important process. In particular, protons, neutrons and α-particles are knocked out of nuclei. Although this effectively undoes some of the nuclear binding that has gone on before, protons and α-particles, as well as neutrons, are at these temperatures readily accepted into any nucleus present, and a situation approaching thermal equilibrium is quickly established with the most tightly bound elements, iron and nickel, copiously produced.

At this stage the core of a massive star is in an unstable condition. There is no more nuclear fuel to burn to delay further gravitational contraction, so even higher densities and temperatures occur. It then becomes energetically advantageous for electrons at the top of the Fermi distribution to undergo electron capture to form neutron-rich nuclei, which on Earth would be β^{-}-unstable. This process removes heat from the core by producing neutrinos which escape, as well as removing electrons. Thus the pressure falls, hastening contraction and leading to the removal of even more electrons. Eventually there will be a catastrophic collapse of the core, an implosion which can only be stopped by nucleon pressure and the nucleon–nucleon short-range repulsion.

The cooler regions of the star outside the core will contain unburned or only partially burned material. As the core implodes, these regions will quickly fall inwards and rise in temperature so that the remaining fuel

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burns explosively, blowing the stellar outer mantle into space. This is the scenario for neutron star formation accompanied by a *supernova explosion*.

11.4 Supernovae

In a supernova, it is only the core of the star that collapses to form a neutron star or a black hole. An important feature of core collapse is the large gravitational energy release. We may understand the order of magnitude of the energies involved using the simple model described in §11.1, adapted to neutron degeneracy. If the core has collapsed to radius R, and radiated all of the heat generated, its energy in the model is

$$E(R) = \frac{A}{R^2} - \frac{B}{R},$$

where $A = \frac{3}{5}\left(\frac{9\pi}{4}\right)^{2/3} \frac{\hbar^2}{2m_\mathrm{n}} \left(\frac{M}{m_\mathrm{n}}\right)^{5/3}$, $B = \frac{3}{5}GM^2$, and M is the mass of the core.

In equilibrium, the core will adopt the radius $R_\mathrm{min} = 2A/B$, which minimises this energy. The energy taken in compressing the neutrons is A/R_min^2. Since

$$\frac{A}{R_\mathrm{min}^2} = \frac{B^2}{4A} = \frac{B}{2R_\mathrm{min}},$$

this energy is only half the gravitational energy released. (Note that the initial energies are negligible.) Almost all of the other half of the energy goes into heating the core.

As an example, we shall consider a core with a mass of $2M_\odot$. This core mass is above the Chandrasekhar limit. The core will contain $N = (2M_\odot/m_\mathrm{n}) = 2.4 \times 10^{57}$ neutrons, and $R_\mathrm{min} = 10.0$ km, using equation (11.6). The heat energy per neutron is

$$\frac{B}{2R_\mathrm{min}N} = 83 \text{ MeV}.$$

Such an energy release is almost 10% of the rest-mass energy of a nucleon, and about ten times the energy per nucleon released in the whole of the star's previous history (see §11.2)!

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After silicon burning, and at the start of collapse, about one-half of the nucleons in the core are protons. The reaction

$$e^- + p \rightarrow n + \nu_e,$$

which takes place during collapse, will therefore produce $\sim N/2 = 1.2 \times 10^{57}$ electron neutrinos. Detailed calculations show that collapse takes place in $\sim 10^{-3}$ s, and the average neutrino energy is about 10 MeV. Thus these neutrinos carry away about 6% of the gravitational energy released: the rest is left as heat in the collapsed core.

With heat energies of ~ 80 MeV per neutron, and densities approaching that of nuclear matter, a transient state of thermal equilibrium will exist, containing not only neutrons, protons, and electrons, but also electron–positron pairs, photons, neutrinos and anti-neutrinos coupled in by reactions such as

$$e^+ + n \leftrightarrow p + \bar{\nu}_e, \quad e^- + p \leftrightarrow n + \nu_e.$$

Muon and tau neutrinos and anti-neutrinos will also be present, induced by reactions such as

$$e^+ + e^- \leftrightarrow \nu_\mu + \bar{\nu}_\mu, \quad e^+ + e^- \leftrightarrow \nu_\tau + \bar{\nu}_\tau,$$

which proceed through the intermediary of the neutral Z boson.

For thermal equilibrium to be established, the mean free path of the particles participating in equilibrium must be less than the size of the system. Cross-sections for neutrino scattering will be discussed in Chapter 13. At the density of nuclear matter this condition is easily satisfied (Problem 13.3), so that neutrinos and anti-neutrinos take part in thermal equilibrium. On the other hand, their mean free path is very large compared with that of photons or electrons, so that neutrinos and anti-neutrinos take the place of photons in transferring heat from the interior to the surface of the core, and radiating it away.

The time scale for this neutrino cooling of the core is of the greatest interest. Detailed calculations suggest a neutrino will diffuse from the core to the surface in ~ 1 s (Problem 13.3), and that appreciable cooling will take place in ~ 10 s. A supernova is an impressive event: the star emits ten times as much energy in ~ 10 s as it has previously emitted in billions of years of stellar evolution. Assuming an average energy of 10 MeV for neutrinos and anti-neutrinos, the total number of all types

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emitted must be $\sim 80N$ MeV/10 MeV $\approx 2 \times 10^{58}$. This number greatly exceeds the number of electron neutrinos squeezed out in the initial collapse of the core.

Strong support for the supernova scenario we have outlined, and in particular the role played by neutrinos, was provided by the observations of a burst of neutrinos accompanying the 1987 supernova explosion in the Large Magellanic Cloud. The LMC is a nearby galaxy at a distance D of about 1.5×10^{21} m from the Solar System. This was the first, and so far is the only, local supernova explosion to have occurred while neutrino detectors have been in place. If in total N_ν neutrinos and anti-neutrinos were emitted, we should expect $N_\nu/4\pi D^2 = (N_\nu/10^{57})3.5 \times 10^{13}$ m^{-2} to arrive at Earth over a few seconds. Of the instrument types described in §10.5, the water detector is most appropriate, since it gives the vital information on the arrival times of those neutrinos and anti-neutrinos it detects.

Note also that water detectors are sensitive to energetic electron anti-neutrinos $\bar{\nu}_e$, through the reaction

$$\bar{\nu}_e + p \rightarrow n + e^+,$$

on the protons in the water. The cross-section for this reaction is of similar size to the cross-sections of neutrinos in the gallium and chlorine detectors, about two orders of magnitude greater than the cross-section for neutrino–electron scattering.

Taking $N_\nu \sim 10^{58}$, gives an expected integrated flux of $\sim 10^{14}$ m^{-2} supernova neutrinos and anti-neutrinos over a few seconds. For comparison, the flux of ^{8}B and hep neutrinos calculated using the standard solar model is $\sim 6 \times 10^{10}$ m^{-2} s^{-1}. These are the neutrinos to which the Kamiokande II detector was sensitive, and in the typical working conditions of the detector gave an event rate of about one per day.

An observed burst of twelve events within 13 s, and with energies between 6 and 36 MeV, was clearly high above any background of solar neutrino events, and correlated with the optically observed supernova explosion. The IMB detector in the USA, which is another water detector, also observed six events within 6 s, with energies between 20 MeV (the detector threshold) and 40 MeV.

Analysis of the data from the two detectors suggests that the surface temperature T of the source was of the form

$$T^4 = T_0^4 e^{-t/\tau},$$

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with $T_0 = 4.2$ MeV and $\tau = 4.6$ s, and the integrated flux of anti-neutrinos was 1.3×10^{14} m^{-2}. These measurements are entirely consistent with the picture of stellar collapse we have outlined, and we can have some confidence in identifying the source as a supernova.

11.5 Nucleosynthesis of heavy elements

The most likely process for the formation of elements heavier than those grouped around iron, produced in the silicon burning described above, is neutron capture. If a supply of free neutrons is available, they can accrete on an iron-group seed nucleus by radiative capture, unimpeded by Coulomb barriers, to build up a neutron-rich isotope. As the neutron number in the nucleus increases it will become unstable to β^--decay, thus forming a new element of atomic number $Z + 1$ from an element of atomic number Z. Successive neutron captures, interspersed with β^--decays, can eventually build up many, but not all, of the heavy stable nuclei. Since the build-up follows the neutron-rich side of the 'β stability valley' (§4.6), some of the proton-rich stable isotopes are inaccessible in this process. It is an interesting fact that such isotopes have a much smaller natural abundance than their neutron-rich neighbours.

There are two basic time scales in this scenario of heavy element synthesis by neutron accretion: the β-decay lifetimes and the time intervals between successive neutron captures (which are inversely proportional to the capture cross-sections and the neutron flux). If the rate of neutron capture is slow compared with the relevant β-decay rates (the s-process) the nuclei that are built up will follow the bottom of the β stability valley very closely. If the rate of neutron capture is rapid (the r-process) highly unstable neutron-rich isotopes will be formed which cascade down to stable nuclei, some of which are inaccessible by the s-process; thorium and uranium must have been formed in this way. The observed nuclear abundances, especially in the regions of closed-shell nuclei, suggest that both the r- and the s-processes have played a part in the synthesis of nuclei found in the Solar System and, in particular, the heavy elements found on Earth.

The site of the r-process is believed to be in supernovae explosions close to the region of neutron star formation, where over a short period of time large neutron fluxes can be expected. The s-process probably occurs during helium burning in massive stars, where a low neutron flux can be provided by a number of reactions, for example (Fig. 8.4),

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$$\alpha + {}^{13}_{6}\mathrm{C} \rightarrow {}^{16}_{8}\mathrm{O} + \mathrm{n}.$$

For nucleosynthesis of the heavy elements by the s-process there must be iron present, derived from nucleosynthesis in previous generations of stars and forming part of the gas from which the star in question condensed.

In this chapter we have attempted to provide no more than a qualitative sketch of a theory which is still being developed. Many of the basic components of the theory are probably in place but important aspects are still being investigated through laboratory measurements and theoretical estimates of reaction rates, and computer studies of reaction networks, combined with stellar models. A rich variety of facts and phenomena remains to be explained.

Problems

11.1 In a plasma with high electron number density ρ_e, using the extreme relativistic approximation in which energy and momentum are related by $E = pc$, show that the average energy of an electron is $(3/4)\,\hbar c k_F$, where k_F is given by $k_F^3 = 3\pi^2 \rho_e$.

11.2 The planet Jupiter is composed mostly of hydrogen. It has mass 1.9×10^{27} kg and mean radius $\approx 7 \times 10^7$ m. Show that if it were uniformly dense its gravitational energy per particle would be only 7 eV, too small to ignite nuclear reactions.

11.3 Estimate the mass density of (metallic) hydrogen at 0 K at which it is energetically favourable to subtract electrons from the electron gas and form neutrons by the inverse β-decay

$$\mathrm{p} + \mathrm{e}^- \rightarrow \mathrm{n} + \nu.$$

11.4 Estimate the ratio of the number of protons and electrons to the number of neutrons in a neutron star at the density of nuclear matter, in thermal equilibrium at low temperature.

11.5 The Planck radiation law states that the number of photons per unit volume in an energy range $\mathrm{d}E$ is

$$\frac{1}{\pi^2(\hbar c)^3}\frac{E^2 \mathrm{d}E}{\mathrm{e}^{E/k_B T} - 1}.$$

At a temperature $k_B T = 500$ keV, estimate the number of photons per unit volume with an energy greater than 8 MeV.

11.6 The cross-section for ^{8}Be production in α–α scattering at energy E in the centre-of-mass frame is given by the Breit–Wigner formula

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$$\sigma(E) = \frac{2\pi\hbar^2}{m_\alpha E}\frac{\Gamma^2}{(E-E_0)^2+\Gamma^2/4},$$

with $E_0 = 94$ keV, $\Gamma = 2.5$ eV. (Note the additional factor of 2 in the Breit–Wigner formula for identical particles: see Problem D.1.) In a plasma at temperatures $k_\mathrm{B}T \gg \Gamma$, the thermal energy $\overline{v\sigma}$ is dominated by energies in the neighbourhood of E_0.

(*a*) Show that

$$\overline{v\sigma} \approx 16\hbar^2\Gamma\left(\frac{\pi}{m_\alpha k_\mathrm{B}T}\right)^{\frac{3}{2}} \mathrm{e}^{-E_0/k_\mathrm{B}T}.$$

(*b*) Hence show that the density ρ_Be of ^{8}Be in equilibrium with α-particles of number density ρ_α is in the ratio

$$\frac{\rho_\mathrm{Be}}{\rho_\alpha} = \frac{\rho_\alpha}{2}\left(\frac{\hbar}{\Gamma}\right)\overline{v\sigma} = \rho_\alpha\left(\frac{4\pi\hbar^2}{m_\alpha k_\mathrm{B}T}\right)^{\frac{3}{2}} \mathrm{e}^{-E_0/k_\mathrm{B}T}.$$

(*c*) Calculate this ratio for $k_\mathrm{B}T = 15$ keV and a helium mass density of 10^6 kg m^{-3}.

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