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A non-local formulation of rotational water waves

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The classical equations of irrotational water waves have recently been reformulated as a system of two equations, one of which is an explicit non-local equation for the wave height and for the velocity potential evaluated on the free surface. Here, in the two-dimensional case: (*a*) we generalize the relevant formulation to the case of constant vorticity, as well as to the case where the free surface is described by a multivalued function; (*b*) in the case of travelling waves we derive an upper bound for the free surface; (*c*) in the case of constant vorticity we construct a sequence of nearly Hamiltonian systems which provide an approximation in the asymptotic limit of certain physical small parameters. In particular, the explicit dependence of the vorticity on the coefficients of the Korteweg–de Vries equation is clarified.

Key words: waves/free-surface flows

1. Introduction

We consider the classical problem in hydrodynamics concerning the propagation of surface waves generated by an incompressible fluid with free surface. In the case of irrotational flow we consider the problem in $n = 2, 3$ (or higher) spatial dimensions, whereas in the rotational case we confine attention to $n = 2$. We consider the problem in its full generality so that the depth of the fluid may not be constant, and consider *all* solutions, not only those that arise under the travelling wave assumption. In addition, we include the effect of surface tension so that capillary-type waves are included in our study.

We denote by $\mathscr{B}_h$ the bottom surface:

$$\mathscr{B}_h = \{(x, y) : x \in \boldsymbol{R}^{n-1}, y = -h_0 + h(x)\}, \tag{1.1}$$

where h_0 is constant and $h(x)$ is a real-valued function. We denote by $\mathscr{S}_\eta$ the free surface:

$$\mathscr{S}_\eta = \{(x, y) : x \in \boldsymbol{R}^{n-1}, y = \eta(x, t)\} \quad \text{for } t \geqslant 0. \tag{1.2}$$

We refer to η as the height of the wave and we assume $\eta + h_0 > h$ for each $x \in \boldsymbol{R}^{n-1}$. The domain of the problem is the region between $\mathscr{B}_h$ and $\mathscr{S}_\eta$, which is denoted by Ω.

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1.1. *The irrotational case in $n \geqslant 2$ dimensions*

In the irrotational case we introduce the velocity potential ϕ, where $u = \nabla\phi$, and then the governing equations become

$$\Delta\phi = 0 \quad \text{in } \Omega, \tag{1.3a}$$

$$\nabla\phi \cdot N_{\mathscr{B}} = 0 \quad \text{on } \mathscr{B}_h, \tag{1.3b}$$

$$\nabla\phi \cdot N_{\mathscr{S}} = \eta_t \quad \text{on } \mathscr{S}_\eta, \tag{1.3c}$$

$$\phi_t + \tfrac{1}{2}|\nabla\phi|^2 + g\eta = f(\eta) \quad \text{on } \mathscr{S}_\eta, \tag{1.3d}$$

where g is the acceleration due to gravity, ∇ denotes the usual vector gradient, $N_{\mathscr{B}}$ is the exterior normal to $\mathscr{B}_h$ and $N_{\mathscr{S}}$ is the exterior normal to $\mathscr{S}_\eta$, i.e.

$$\nabla = (\nabla_x, \partial_y), \quad N_{\mathscr{B}} = (-\nabla_x h, -1), \quad N_{\mathscr{S}} = (-\nabla_x \eta, 1), \tag{1.4a}$$

where $\nabla_x = (\partial_{x_1}, \ldots, \partial_{x_{n-1}})$ is the horizontal gradient. The right-hand side of (1.3*d*) is functionally dependent on η through

$$f(\eta) = \frac{\sigma}{\rho}\nabla_x \cdot \left(\frac{\nabla_x \eta}{\sqrt{1 + |\nabla_x \eta|^2}} \right), \tag{1.4b}$$

which is a measure of the effect of surface tension (σ and ρ denote the constant surface tension and density, respectively). Equation (1.3*a*) is a consequence of incompressibility, (1.3*b*) is the Neumann condition satisfied on $\mathscr{B}_h$ and (1.3*c*) and (1.3*d*) are the kinematic and Bernoulli conditions, respectively, on $\mathscr{S}_\eta$.

We reformulate the problem in terms of the functions η and q where $q(x, t) = \phi(x, \eta, t)$, i.e. q represents the potential on the free surface $\mathscr{S}_\eta$. It was shown by Zakharov in his classical paper (Zakharov 1968) that in the case $n = 2$ the pair (η, q) constitutes a canonically conjugate pair in the Hamiltonian formulation of this problem. The generalization of Zakarov's result to higher dimensions is straightforward. Indeed, in the case of constant $\mathscr{B}_h$ and zero surface tension equations (1.3) admit a Hamiltonian formulation with respect to the Hamiltonian

$$H = \iint_\Omega \frac{1}{2}|\nabla\phi|^2 \, \mathrm{d}x\,\mathrm{d}y + \int \frac{1}{2} g\eta^2 \, \mathrm{d}x \tag{1.5}$$

and the standard symplectic structure. Here and throughout $\mathrm{d}x = \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1}$ denotes the Lebesgue measure on $\boldsymbol{R}^{n-1}$.

Applying the chain rule to the expression $q(x, t) = \phi(x, \eta, t)$ we find the following relations that are valid on the free surface $y = \eta(x, t)$:

$$\nabla_x q = \nabla_x \phi + (\partial_y \phi)\nabla_x \eta, \tag{1.6a}$$

$$q_t = \phi_t + \phi_y \eta_t. \tag{1.6b}$$

It is possible to solve for $\nabla\phi$ in terms of the q and η following a similar calculation with that of Ablowitz, Fokas & Musslimani (2006). Indeed, using (1.3*c*) in (1.6*a*) we find the following non-singular set of equations for $\nabla_x\phi$:

$$\nabla_x q - \eta_t \nabla_x \eta = (\boldsymbol{I} + \nabla_x \eta \otimes \nabla_x \eta) \cdot \nabla_x \phi \tag{1.7}$$

and then (1.3*c*) gives ϕ_y in terms of q and η. In the new variables the dynamic boundary condition on $\mathscr{S}_\eta$, i.e. (1.3*d*), becomes

$$q_t + \frac{1}{2}|\nabla_x q|^2 + g\eta - \frac{(\eta_t + \nabla_x \eta \cdot \nabla_x q)^2}{2(1 + |\nabla_x \eta|^2)} = f(\eta). \tag{1.8}$$

In what follows, the notation $\int \mathrm{d}x$ will be used to denote an integral over $\boldsymbol{R}^{n-1}$.

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1.2. *The rotational case in two dimensions*

In this section we derive the equations for the free surface problem for a fluid with constant vorticity. Our approach utilizes a stream function so we confine attention to two-dimensional flows. On this point we note that Constantin & Kartashova (2009) have proved that capillary wave trains can only exist on flows with non-zero constant vorticity in two dimensions. So in the case of capillary waves, we lose nothing by restricting attention to two dimensions.

Denoting the velocity of the flow by (u, v), the Euler equations for inviscid flow are written as follows:

$$u_t + uu_x + vu_y = -P_x, \tag{1.9a}$$

$$v_t + uv_x + vv_y = -P_y - g, \tag{1.9b}$$

$$u_x + v_y = 0, \tag{1.9c}$$

where $P = P(x, y)$ is the fluid pressure. Let ω denote the vorticity of the fluid, i.e.

$$\omega = v_x - u_y. \tag{1.10}$$

Eliminating P from (1.9*a*) and (1.9*b*) we find

$$\frac{\partial \omega}{\partial t} + (u\partial_x + v\partial_y)\omega = 0. \tag{1.11}$$

We restrict attention to the case in which $\omega = \gamma$ is constant throughout Ω, so that (1.11) is satisfied identically. The domain Ω is simply connected, therefore we can introduce a globally defined stream function $\psi(t, x, y)$ so that

$$u = \psi_y, \quad v = -\psi_x, \quad (x, y) \in \Omega. \tag{1.12}$$

Replacing in (1.10) u by ψ_y and v by $-\psi_x$ we find $\Delta\psi = -\gamma$. Thus, the function defined by $\psi^h = \psi + 1/2y^2\gamma$, is harmonic in Ω. In fact, ψ^h is only defined up to the addition of an arbitrary function of time, so by abuse of notation we write ψ^h to mean $[\psi^h]$, i.e. the equivalence class of such functions that differ only by a function of time. Let φ be the harmonic conjugate of ψ^h, i.e.

$$\varphi_x = \psi_y + \omega y, \tag{1.13a}$$

$$\varphi_y = -\psi_x. \tag{1.13b}$$

The function φ is harmonic throughout Ω.

Adding and subtracting in (1.9*a*) and (1.9*b*) the terms vv_x and uu_y, respectively, the equations become

$$u_t + \tfrac{1}{2}\partial_x(u^2 + v^2) - \gamma v = -P_x, \tag{1.14a}$$

$$v_t + \tfrac{1}{2}\partial_y(u^2 + v^2) + \gamma u = -P_y - g. \tag{1.14b}$$

Using in (1.14) the identities

$$u_t = (\psi_y)_t = (\varphi_x)_t, \quad v = -\psi_x, \quad v_t = -(\psi_x)_t = (\varphi_y)_t, \quad u = \psi_y, \tag{1.15}$$

equation (1.14) can be integrated to give the following equation:

$$\varphi_t + \tfrac{1}{2}|\nabla\psi|^2 + \gamma\psi + P + gy = \alpha(t), \quad (x, y) \in \Omega, \tag{1.16}$$

where $\alpha(t)$ is some function of time. Since we are still dealing with an equivalence class of functions ψ and φ we can absorb a function of time into them, so we arrive at

$$\varphi_t + \tfrac{1}{2}|\nabla\psi|^2 + \gamma\psi + gy = P_{atm} - P, \quad (x, y) \in \Omega, \tag{1.17}$$

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where P_{atm} is the atmospheric pressure above the free surface $\mathscr{S}_\eta$. On the free surface $\mathscr{S}_\eta$ we have the dynamic boundary condition

$$P_{atm} - P = f(\eta), \quad f(\eta) \stackrel{def}{=} \frac{\sigma}{\rho}\left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right)_x. \tag{1.18}$$

The stream function ψ is not constant on the free surface. In particular, if we introduce the function $\Psi(x,t) = \psi(x,\eta(x,t),t)$ to be the stream function defined on the free surface, we see

$$\begin{aligned}\Psi_x &= \psi_x + \eta_x\psi_y \\ &= -v + u\eta_x \\ &= -\eta_t,\end{aligned} \tag{1.19}$$

which follows from the kinematic boundary condition (1.3*c*) on the free surface. So up to an additional function of time, which again we simply absorb into the definition of ψ, we have

$$\psi|_{\mathscr{S}} = \Psi(x,t) = -\int_{-\infty}^{x} \eta_t(y,t)\,\mathrm{d}y \equiv -\partial_x^{-1}\eta_t. \tag{1.20}$$

Hence, (1.17) and (1.18) imply the following nonlinear boundary condition:

$$\partial_t\varphi - \gamma\partial_x^{-1}\eta_t + \tfrac{1}{2}|\nabla\psi|^2 + g\eta = f(\eta) \quad \text{on } \mathscr{S}_\eta. \tag{1.21}$$

The kinematic condition on $\mathscr{S}_\eta$ and the condition on $\mathscr{B}_h$ are the same as in the irrotational case.

In summary, if the vorticity equals the constant γ, then the functions $\varphi(x,y,t)$ and $\eta(x,t)$ satisfy the following boundary value problem:

$$\Delta\varphi = 0 \quad \text{in } \Omega, \tag{1.22a}$$

$$(\varphi_x - \gamma y, \varphi_y)\cdot N_{\mathscr{B}} = 0 \quad \text{on } \mathscr{B}_h, \tag{1.22b}$$

$$(\varphi_x - \gamma y, \varphi_y)\cdot N_{\mathscr{S}} = \eta_t \quad \text{on } \mathscr{S}_\eta, \tag{1.22c}$$

$$\varphi_t - \gamma\partial_x^{-1}\eta_t + \tfrac{1}{2}\varphi_y^2 + \tfrac{1}{2}(\varphi_x - \gamma y)^2 + gy = f(\eta) \quad \text{on } \mathscr{S}_\eta, \tag{1.22d}$$

where $N_{\mathscr{B}}$, $N_{\mathscr{S}}$ and $f(\eta)$ are defined in (1.4).

As in the irrotational case, we formulate the problem in terms of η and ξ where $\xi(x,t) = \varphi(x,\eta,t)$, i.e. ξ represents the pseudopotential φ on the free surface. Computing ξ_x using the chain rule (compare with (1.6*a*)) and invoking (1.22*c*) we find the following pair of equations for (φ_x, φ_y) on $\mathscr{S}_\eta$:

$$\xi_x = \varphi_x + \eta_x\varphi_y, \tag{1.23}$$

$$\eta_t - \gamma\eta\eta_x = \varphi_y - \eta_x\varphi_x. \tag{1.24}$$

Solving these equations for (φ_x, φ_y) we find the following relations on $\mathscr{S}_\eta$:

$$(1+\eta_x^2)\varphi_x = \xi_x - \eta_t\eta_x + \gamma\eta\eta_x^2, \tag{1.25a}$$

$$(1+\eta_x^2)\varphi_y = \eta_t + \eta_x\xi_x - \gamma\eta\eta_x. \tag{1.25b}$$

Furthermore, using (1.25) in the boundary condition (1.22*d*) on $\mathscr{S}_\eta$ we find

$$\xi_t + \frac{1}{2}\xi_x^2 + g\eta - \gamma\partial_x^{-1}\eta_t + \frac{\gamma\eta(2\eta_t\eta_x - 2\xi_x + \gamma\eta)}{2(1+\eta_x^2)} - \frac{(\eta_t + \eta_x\xi_x)^2}{2(1+\eta_x^2)} = f(\eta). \tag{1.26}$$

It is reassuring to note that (1.26) reduces to (1.8) when $\gamma = 0$.

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1.2.1. *Bounds on wave height*

Let us temporarily confine attention to travelling wave solutions to (1.26), so that $\eta = \eta(x - ct)$ and $\xi = \xi(x - ct)$. Setting $z = x - ct$ and denoting differentiation with respect to z with a prime, the equation (1.26) becomes

$$-c\xi' + \frac{1}{2}(\xi')^2 + (g + \gamma c)\eta + \frac{\gamma\eta(-2c(\eta')^2 - 2\xi' + \gamma\eta)}{2(1 + (\eta')^2)} - \frac{(-c\eta' + \eta'\xi')^2}{2(1 + (\eta')^2)} = f(\eta). \tag{1.27}$$

Following Deconinck & Oliveras (2011), we observe (1.27) is actually a quadratic in the unknown ξ', which can be solved to give

$$\xi'(z) = \gamma\eta + c - \sqrt{\left(1 + (\eta')^2\right)\left(c^2 + f(\eta) - 2g\eta\right)}, \tag{1.28}$$

where we have discarded the second solution which fails to decay for large z.

Equation (1.28) immediately gives the estimate

$$\eta \leqslant \frac{c^2}{2g} + \frac{\sigma}{2\rho g}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\eta'}{\sqrt{1 + (\eta')^2}}\right). \tag{1.29}$$

In the absence of vorticity and surface tension, this reproduces the upper bound derived in Craig & Sternberg (1988). With the assumption that $|\eta'|$ is bounded, we can integrate this inequality to establish the following estimate for the average wave height:

$$\langle\eta\rangle \leqslant \frac{c^2}{2g}, \tag{1.30}$$

which holds in the presence of arbitrary surface tension.

1.3. *Multivalued free surface*

In the above analysis we have assumed that $\eta(x, t)$ is a single-valued function. However, in general, this assumption can be violated. In particular, it is known that in the case of water waves with constant vorticity, $\eta(x, t)$ quickly becomes a multivalued function, as observed by Da Silva & Peregrine (1988). In what follows, we show that it is conceptually straightforward to modify our analysis so that it can address the case of a multivalued free surface.

The free surface is assumed to be a one-dimensional C^2-differentiable manifold, so features such as cusps and self-intersections are ignored. We set

$$\mathscr{S} = \{(x, y) \in \boldsymbol{R}^2 : x = X(\lambda, t), y = Y(\lambda, t)\}, \quad t \geqslant 0, \tag{1.31}$$

where $\lambda \in \boldsymbol{R}$ is the arc-length along the curve and $X(\lambda, \cdot)$, $Y(\lambda, \cdot)$ are assumed to be C^2. The outward normal is now given by $N_{\mathscr{S}} = (-\dot{Y}, \dot{X})$, where the dot denotes differentiation with respect to λ. The kinematic boundary condition in (1.22*c*) now becomes

$$(\varphi_x - \gamma y - X_t, \varphi_y - Y_t) \cdot N_{\mathscr{S}} = 0, \quad \text{on } \mathscr{S}. \tag{1.32}$$

Again introducing $\Psi(\lambda, t) = \psi(X(\lambda, t), Y(\lambda, t), t)$ to be the stream function evaluated

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on the free surface, the chain rule and the new kinematic boundary condition (1.32) gives

$$\dot{\Psi} = \dot{X}Y_t - \dot{Y}X_t \equiv \frac{\partial(X, Y)}{\partial(\lambda, t)}. \tag{1.33}$$

Using analogous reasoning as in the case of the single-valued free surface, this gives

$$\psi|_{\mathscr{S}} = -\partial_\lambda^{-1}(\dot{X}Y_t - \dot{Y}X_t). \tag{1.34}$$

Again, we introduce a function ξ to represent the pseudopotential on the free surface, i.e. $\xi(\lambda, t) = \varphi(X(\lambda, t), Y(\lambda, t), t)$. In analogy with (1.6), we note the following relations on $\mathscr{S}$:

$$\dot{\xi} = \dot{X}\varphi_x + \dot{Y}\varphi_y, \tag{1.35a}$$

$$\xi_t = X_t\varphi_x + Y_t\varphi_y + \varphi_t. \tag{1.35b}$$

Equations (1.32) and (1.35) constitute a non-singular system for the functions $\{\varphi_x, \varphi_y, \varphi_t\}$ on the free surface $\mathscr{S}$, so we can invert these relationships to express these functions in terms of $\{X, Y, \eta\}$. After some algebra, one finds the following relations on $\mathscr{S}$:

$$\varphi_x = \frac{\dot{\xi}\dot{X} + \dot{Y}(\dot{Y}(\gamma Y + X_t) - \dot{X}Y_t)}{(\dot{X}^2 + \dot{Y}^2)} \tag{1.36a}$$

$$\varphi_y = \frac{\dot{\xi}\dot{Y} - \dot{X}(\dot{Y}(\gamma Y + X_t) - \dot{X}Y_t)}{(\dot{X}^2 + \dot{Y}^2)} \tag{1.36b}$$

$$\varphi_t = \xi_t - \frac{\dot{\xi}(\dot{X}X_t + \dot{Y}Y_t) + (X_t\dot{Y} - Y_t\dot{X})(\dot{Y}(\gamma Y + X_t) - \dot{X}Y_t)}{(\dot{X}^2 + \dot{Y}^2)}. \tag{1.36c}$$

Using these expressions on $\mathscr{S}$ and the relation in (1.34), the left-hand side of the Bernoulli condition in (1.22*d*) becomes

$$\begin{aligned}\xi_t + gY - \frac{1}{2}X_t^2 - \frac{1}{2}Y_t^2 - \gamma\partial_\lambda^{-1}\left(\dot{X}Y_t - \dot{Y}X_t\right) \\ + \frac{\gamma Y(2\dot{X}\dot{Y}Y_t - 2\dot{X}^2X_t - 2\dot{\xi}\dot{X} + \gamma\dot{X}^2Y)}{2(\dot{X}^2 + \dot{Y}^2)} + \frac{(\dot{\xi} - \dot{X}X_t - \dot{Y}Y_t)^2}{2(\dot{X}^2 + \dot{Y}^2)}\end{aligned} \tag{1.37}$$

and the term due to surface tension is

$$\frac{\sigma}{\rho}\frac{\dot{X}\ddot{Y} - \dot{Y}\ddot{X}}{(\dot{X}^2 + \dot{Y}^2)^{3/2}}. \tag{1.38}$$

Here the coefficient of σ/ρ is the intrinsic curvature of the surface $\mathscr{S}$. Combining these expressions gives the Bernoulli condition on $\mathscr{S}$ in terms of the surface parameters (X, Y) and the potential on the free surface, $\xi(\lambda, t)$:

$$\begin{aligned}\xi_t + gY - \frac{1}{2}X_t^2 - \frac{1}{2}Y_t^2 - \gamma\partial_\lambda^{-1}\left(\dot{X}Y_t - \dot{Y}X_t\right) \\ + \frac{\gamma Y(2\dot{X}\dot{Y}Y_t - 2\dot{X}^2X_t - 2\dot{\xi}\dot{X} + \gamma\dot{X}^2Y)}{2(\dot{X}^2 + \dot{Y}^2)} \\ + \frac{(\dot{\xi} - \dot{X}X_t - \dot{Y}Y_t)^2}{2(\dot{X}^2 + \dot{Y}^2)} - \frac{\sigma}{\rho}\frac{\dot{X}\ddot{Y} - \dot{Y}\ddot{X}}{(\dot{X}^2 + \dot{Y}^2)^{3/2}} = 0.\end{aligned} \tag{1.39}$$

We note that in the case $X = \lambda$, $Y = Y(X, t)$ this equation reduces to (1.26).

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1.3.1. *Bounds on wave height*

One can also consider travelling wave solutions in this setting and achieve an analogous upper bound to (1.29). In this case

$$X(\lambda, t) \mapsto X(\lambda) - ct \tag{1.40}$$

$$Y(\lambda, t) \mapsto Y(\lambda). \tag{1.41}$$

Setting $\xi = \varphi(X - ct, Y)$ with $X = X(\lambda)$ and $Y = Y(\lambda)$ being independent of time, (1.39) reduces to a quadratic in $\dot{\xi}$. Examining the discriminant of this quadratic reveals the estimate

$$Y \leqslant \frac{c^2}{2g} + \frac{\sigma}{2\rho g}\left[\frac{\dot{X}\ddot{Y} - \dot{Y}\ddot{X}}{(\dot{X}^2 + \dot{Y}^2)^{3/2}}\right]. \tag{1.42}$$

We deduce that the estimate in (1.29) for the wave height is valid for all travelling waves, including those which are truly multivalued.

The above analysis shows that although it is straightforward to incorporate the case of a multivalued free surface, the relevant formulae become more complicated, so for convenience we present most of our results assuming that free surface is single valued.

2. The non-local formulation

A novel non-local formulation in two and three dimensions for irrotational water waves was presented by Ablowitz *et al.* (2006). Here we extend this formulation to rotational water waves with constant vorticity in two dimensions and for irrotational water waves in an arbitrary number of dimensions.

The formulation by Ablowitz *et al.* (2006) is based on the general approach for studying boundary value problems for a large class of partial differential equations (PDEs) introduced by one of the authors, see Fokas (1997, 2000). A crucial role in this approach is played by a certain non-local equation called the global relation. The global relation is a consequence of the following fact: suppose that the functions u and v are harmonic in $\Omega \subset \boldsymbol{R}^{n-1} \times \boldsymbol{R}$. Then,

$$\nabla_x \cdot ((\partial_y u)\nabla_x v + (\partial_y v)\nabla_x u) + \partial_y((\partial_y u)(\partial_y v) - \nabla_x u \cdot \nabla_x v) = 0 \tag{2.1}$$

for each $(x, y) \in \Omega$. This can be verified by expanding out the left-hand side of (2.1) to find

$$(\partial_y v)\Delta u + (\partial_y u)\Delta v, \tag{2.2}$$

which vanishes in Ω since both u and v are harmonic in Ω.

2.1. *Irrotational case*

In this section we proceed formally: we assume $\phi(x, y, t)$ and $\eta(x, t)$ have sufficient decay as $|x| \to \infty$ for each (y, t) in order for the integrals that follow to exist. Then, later, we classify the relevant function spaces so that the integral equations arising in the non-local formulation are well-defined.

Suppose that $k \in \boldsymbol{R}^{n-1}$ and define $v \in C^\infty(\Omega)$ by

$$v(x, y) = \exp(-\mathrm{i}k \cdot x + \kappa y), \quad \kappa = \pm\sqrt{k_1^2 + \cdots + k_{n-1}^2}. \tag{2.3}$$

Then v is harmonic in Ω and the following holds for $(x, y) \in \Omega$:

$$\nabla_x \cdot \{v(-\mathrm{i}k\partial_y\phi + \kappa\nabla_x\phi)\} + \partial_y\{v(\kappa\partial_y\phi + \mathrm{i}k \cdot \nabla_x\phi)\} = 0 \tag{2.4}$$

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Integrating this expression over Ω and applying the divergence theorem gives

$$\int_{\mathscr{S}_\eta} \mathrm{e}^{-\mathrm{i}k\cdot x+\kappa y}(-\mathrm{i}k\partial_y\phi+\kappa\nabla_x\phi,\kappa\partial_y\phi+\mathrm{i}k\cdot\nabla_x\phi)\cdot N_{\mathscr{S}}\,\mathrm{d}x + \int_{\mathscr{B}_h} \mathrm{e}^{-\mathrm{i}k\cdot x+\kappa y}(-\mathrm{i}k\partial_y\phi+\kappa\nabla_x\phi,\kappa\partial_y\phi+\mathrm{i}k\cdot\nabla_x\phi)\cdot N_{\mathscr{B}}\,\mathrm{d}x=0. \tag{2.5}$$

We have discarded the contributions from $\partial\boldsymbol{R}^{n-1}$ using our assumption about the decay of the fields. The contribution from $\mathscr{S}_\eta$ in (2.5), is given by the following integral:

$$\int_{\mathscr{S}_\eta} \mathrm{e}^{-\mathrm{i}k\cdot x+\kappa y}[\kappa(\partial_y\phi-\nabla_x\phi\cdot\nabla_x\eta)+\mathrm{i}k\cdot(\nabla_x\phi+(\partial_y\phi)\nabla_x\eta)]\,\mathrm{d}x. \tag{2.6}$$

From (1.3*c*) the first term in the above integrand becomes η_t, and by (1.6), the second term becomes $\nabla_x q$. This gives the expression

$$\int \mathrm{e}^{-\mathrm{i}k\cdot x+\kappa\eta}(\kappa\eta_t+\mathrm{i}k\cdot\nabla_x q)\,\mathrm{d}x. \tag{2.7}$$

Now we consider the contribution from $\mathscr{B}_h$ in (2.5). We introduce $Q(x,t)=\phi(x,-h_0-h,t)$, which represents the potential on the bottom $\mathscr{B}_h$. In analogy with (1.6), an application of the chain rule yields

$$\nabla_x Q=\nabla_x\phi-(\partial_y\phi)\nabla_x h. \tag{2.8}$$

Recalling that $N_{\mathscr{B}}=(-\nabla_x h,-1)$, the contribution from $\mathscr{B}_h$ in (2.5) gives

$$-\int_{\mathscr{B}_h} \mathrm{e}^{-\mathrm{i}k\cdot x+\kappa y}[\kappa(\partial_y\phi+\nabla_x\phi\cdot\nabla_x h)+\mathrm{i}k\cdot(\nabla_x\phi-(\partial_y\phi)\nabla_x h)]\,\mathrm{d}x. \tag{2.9}$$

Equation (1.3*b*) implies that $\nabla\phi\cdot N_{\mathscr{B}}=0$ on $\mathscr{B}_h$, so the first term in the above integrand vanishes. Using the result in (2.8) we find that the expression in (2.9) becomes

$$\int \mathrm{e}^{-\mathrm{i}k\cdot x-\kappa(h_0+h)}(-\mathrm{i}k\cdot\nabla_x Q)\,\mathrm{d}x. \tag{2.10}$$

Now combining the results in (2.7), (2.10) and (2.5) we find that equation (2.5) reduces to the following equation:

$$\int \mathrm{e}^{-\mathrm{i}k\cdot x+\kappa\eta}(\kappa\eta_t+\mathrm{i}k\cdot\nabla_x q)\,\mathrm{d}x-\int \mathrm{e}^{-\mathrm{i}k\cdot x-\kappa(h_0+h)}(\mathrm{i}k\cdot\nabla_x Q)\,\mathrm{d}x=0. \tag{2.11}$$

By evaluating (2.11) at $\pm|\kappa|$ and adding/subtracting the resulting equations, we arrive at the following result.

PROPOSITION 1 (*Irrotational waves*). *The boundary value problem given in (1.3a)–(1.3c) is equivalent to the pair of integrodifferential equations:*

$$\int \mathrm{e}^{-\mathrm{i}k\cdot x}[\kappa\eta_t\sinh(\kappa\eta)+\mathrm{i}k\cdot\nabla_x q\cosh(\kappa\eta)-\mathrm{i}k\cdot\nabla_x Q\cosh(\kappa(h+h_0))]\,\mathrm{d}x=0 \tag{2.12}$$

and

$$\int \mathrm{e}^{-\mathrm{i}k\cdot x}[\kappa\eta_t\cosh(\kappa\eta)+\mathrm{i}k\cdot\nabla_x q\sinh(\kappa\eta)+\mathrm{i}k\cdot\nabla_x Q\sinh(\kappa(h+h_0))]\,\mathrm{d}x=0 \tag{2.13}$$

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valid for each $k \in \boldsymbol{R}^{n-1}$ *. These equations, together with the Bernoulli condition (1.8) constitute three equations for the three unknowns* (η, q, Q).

At this point we discuss some rigorous aspects of our approach. We consider appropriate function spaces so that (2.12) and (2.13) are well-defined. Of course, one could impose some artificial restrictions on the functions (η, q, Q) so that both integrals make sense, but it is important to derive the appropriate function spaces from a more global point of view. For instance, one could impose that both $\nabla_x q$, $\nabla_x Q$ and η_t should be in $L^1(\boldsymbol{R}^{n-1})$, and that both η and h be in $L^\infty(\boldsymbol{R}^{n-1})$, but there would be no motivation for these constraints in relation to the boundary value problem (1.3). Instead, we provide some physically motivated constraints to the problem so that the integral equations of the non-local approach make sense. We use some standard results from the theory of Sobolev spaces, and refer the unfamiliar reader to the comprehensive account given in Adams (1975).

Our starting point is one based on physical grounds: we want to consider solutions to the water wave problem for which the configuration of the system has finite energy. That is to say, we are only interested in water waves for which the total energy

$$\iint_{\Omega} \frac{1}{2}|\nabla\phi|^2 \,\mathrm{d}x\,\mathrm{d}y + \int \left(\frac{1}{2}g\eta^2 + \frac{\sigma}{\rho}(\sqrt{1+|\nabla_x\eta|^2}-1)\right)\mathrm{d}x, \tag{2.14}$$

is finite. Clearly this requires $\nabla\phi \in L^2(\Omega)$, and noting

$$\sqrt{1+t^2}-1 \leqslant t^2, \quad t \in \boldsymbol{R}, \tag{2.15}$$

we see that it is sufficient that $\eta \in H^1(\boldsymbol{R}^{n-1})$. With this in mind, we seek to find functions spaces such that the finite energy conditions are satisfied and our integral equations are well-defined, given the *a priori* regularity conditions on η and $\nabla\phi$.

For concreteness, we first set the regularity of the free surface amplitude, η. The finite energy condition means we need $\eta \in H^s(\boldsymbol{R}^{n-1})$ for $s \geqslant 1$. It is inevitable that the regularity of η will change the required regularity of the other functions, so we proceed by adjusting s accordingly so that the other estimates can be assessed.

With regards to the potential ϕ: it would be beneficial, from our standpoint, to choose $\nabla\phi$ in a specific function space so that $\nabla_x q$ and $\nabla_x Q$ are both in $L^1(\boldsymbol{R}^{n-1})$, owing to the way in which the functions appear in (2.12) and (2.13). That is to say, we want to classify X so that $\nabla\phi \in X$ guarantees that the trace of $\nabla\phi$, i.e. $\nabla\phi|_{\partial\Omega}$, belongs to $L^1(\partial\Omega)$. To ensure the finite energy condition, we would need $X \subset L^2(\Omega)$. Now it is known (e.g. Adams 1975) that *if* $\partial\Omega$ is sufficiently nice, the trace operator $\mathrm{tr} f = f|_{\partial\Omega}$ provides the following continuous map:

$$\mathrm{tr} : H^r(\Omega) \to H^{r-1/2}(\partial\Omega) \quad \text{for } r > \tfrac{1}{2}. \tag{2.16}$$

In this case, 'sufficiently nice' would be for $\partial\Omega$ to be C^k with $r \leqslant k$. To enforce this smoothness on the boundary, and hence η, we must choose $s > \frac{1}{2}(n-1)+k$. It would be enough for us to choose $\nabla\phi \in H^r(\Omega)$ with r sufficiently large, so that $H^{r-1/2}(\partial\Omega) \subset L^1(\partial\Omega)$. Now the standard embedding result

$$H^{m/2}(\boldsymbol{R}^m) \hookrightarrow L^1(\boldsymbol{R}^m) \tag{2.17}$$

suggests that we should choose $r \geqslant n/2$. This is enough to make sure our non-local formulation holds.

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LEMMA 1. *The global relations (2.12) and (2.13) are well-defined if* $\nabla\phi \in H^{s_1}(\Omega)$ *and* $\eta \in H^{s_1}(\boldsymbol{R}^{n-1})$ *, where* $s_1 \geqslant n/2$ *and* $s_2 > n - (1/2)$ *. In particular, we have the estimates*

$$\|\nabla_x q\|_{L^1} \leqslant c_1 \|\nabla\phi\|_{H^{s_1}} \tag{2.18}$$

$$\|\nabla_x Q\|_{L^1} \leqslant c_2 \|\nabla\phi\|_{H^{s_1}} \tag{2.19}$$

$$\|\eta_t\|_{L^1} \leqslant c_3 \|\nabla\phi\|_{H^{s_1}}, \tag{2.20}$$

where c_i, $i = 1, 2, 3$ *are constants depending on the dimension* n *and* $\|\eta\|_{H^{s_2}}$.

PROOF. Note first that since $s_2 > n - (1/2) = 1/2(n-1) + (n/2)$, it follows that $\eta \in C^{[n/2]}(\boldsymbol{R}^{n-1})$, where the square bracket denote the integer part. In particular, this tells us that both $\nabla_x\eta$ and η are bounded, from which we deduce that the global relations (2.12) and (2.13) are well-defined if all of η_t, $\nabla_x q$, $\nabla_x Q$ are to be in $L^1(\boldsymbol{R}^{n-1})$. We now establish the estimates in the lemma.

Given $\nabla\phi \in H^{s_1}(\Omega)$, we know that $\operatorname{tr}\nabla\phi \in H^{s_1-1/2}(\partial\Omega)$ since our assumption on the regularity of η, hence $\partial\Omega$, are enough to ensure the trace operator maps $H^{s_1}(\Omega)$ to $H^{s_1-1/2}(\partial\Omega)$. This gives us the following estimates:

$$\begin{aligned}\|\nabla_x q\|_{H^{s_1-1/2}} &\leqslant \|\operatorname{tr}\nabla_x\phi\|_{H^{s_1-1/2}} + \|\nabla_x\eta \operatorname{tr}\phi_y\|_{H^{s_1-1/2}} \\ &\leqslant \|\operatorname{tr}\nabla_x\phi\|_{H^{s_1-1/2}} + \|\nabla_x\eta\|_{L^\infty} \|\operatorname{tr}\phi_y\|_{H^{s_1-1/2}} \\ &\leqslant \max\{\|\nabla_x\eta\|_{L^\infty}, 1\} \|\operatorname{tr}\nabla\phi\|_{H^{s_1-1/2}} \\ &\leqslant c_1' \|\nabla\phi\|_{H^{s_1}}. \end{aligned} \tag{2.21}$$

The result stated in the lemma follows from the embedding (2.17) and $s_1 \geqslant n/2$.

Finally, we see that the regularity of η_t is determined through (1.3), in particular we have $\eta_t = \operatorname{tr}\nabla\phi \cdot N_{\mathscr{S}}$, i.e.

$$\eta_t = \operatorname{tr}\phi_y - \nabla_x\eta \cdot \operatorname{tr}\nabla_x\phi. \tag{2.22}$$

Using similar estimates as those for $\nabla_x q$, we find the required result.

Similar arguments can be used for the case of rotational water waves in two dimensions (see § 2.2).

2.2. *The rotational two-dimensional case*

Let $n = 2$ and also confine attention to the case in which $\mathscr{B}_h = \mathscr{B}_0$ is constant. Using a similar approach to the irrotational case, we find the non-local integrodifferential equation for the pseudopotential $\varphi(x, t)$ and the wave height $\eta(x, t)$:

$$\int_{\mathscr{S}_\eta} \mathrm{e}^{-\mathrm{i}kx\mp ky}[\mathrm{i}(\varphi_x + \varphi_y\eta_x) \mp (\eta_x\varphi_x - \varphi_y)]\,\mathrm{d}x + \int_{\mathscr{B}_0} \mathrm{e}^{-\mathrm{i}kx\mp ky}[\mathrm{i}\varphi_x \pm \varphi_y]\,\mathrm{d}x = 0, \tag{2.23}$$

which is valid for $k \in \boldsymbol{R}$. Invoking the boundary conditions (1.22*b*), (1.22*c*) and (1.25) this expression becomes

$$\int \mathrm{e}^{-\mathrm{i}kx\mp k\eta}[\mathrm{i}\xi_x \mp (\gamma\eta\eta_x - \eta_t)]\,\mathrm{d}x + \int \mathrm{e}^{-\mathrm{i}kx\pm kh_0}\mathrm{i}\varphi_x(x, -h_0, t)\,\mathrm{d}x = 0. \tag{2.24}$$

Subtracting the above two expressions eliminates the $\mathscr{B}_0$ integral completely and we are left with the global relation for two-dimensional water waves with constant vorticity γ:

$$\int \mathrm{e}^{-\mathrm{i}kx}[\xi_x \sinh(k(\eta + h)) - \mathrm{i}(\eta_t - \gamma\eta\eta_x)\cosh(k(\eta + h))]\,\mathrm{d}x = 0, \tag{2.25}$$

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where we have dropped the subscript on h. This leads us to the following proposition.

PROPOSITION 2 (*Two-dimensional water waves with constant vorticity*). *The boundary value problem in (1.22a)–(1.22d) for* (φ, η) *is equivalent to the following pair of integrodifferential equations for* (ξ, η)*:*

$$\int \mathrm{e}^{-\mathrm{i}kx}[\xi_x \sinh(k(\eta+h)) - \mathrm{i}(\eta_t - \gamma\eta\eta_x)\cosh(k(\eta+h))]\,\mathrm{d}x = 0, \tag{2.26a}$$

$$\xi_t + \frac{1}{2}\xi_x^2 - \gamma\partial_x^{-1}\eta_t + g\eta + \frac{\gamma\eta(2\eta_t\eta_x - 2\xi_x + \gamma\eta)}{2(1+\eta_x^2)} - \frac{(\eta_t + \eta_x\xi_x)^2}{2(1+\eta_x^2)} = f(\eta), \tag{2.26b}$$

where $k \in \boldsymbol{R}$, $\xi(x,t) = \varphi(x, \eta(x,t), t)$ *is the pseudopotential evaluated on* $\mathscr{S}_\eta$ *and* $f(\eta)$ *is defined in (1.4).*

A similar formulation can be developed if $\mathscr{S}$ is allowed to become multivalued. Indeed, if we assume $\mathscr{S}$ is a differentiable manifold embedded in $\boldsymbol{R}^2$ via $x = X(\lambda, t)$ and $y = Y(\lambda, t)$, with $|Y| \to 0$ at ∞, then a calculation similar to the previous result gives us the following result.

PROPOSITION 3 (*Two-dimensional water waves with constant vorticity and multivalued free surface*). *Let the free surface* $\mathscr{S}$ *be a one-dimensional* C^2*-manifold, and let its embedding in* $\boldsymbol{R}^2$ *be parameterized by* $X(\lambda, t)$ *and* $Y(\lambda, t)$*. Then the boundary value problem in (1.22a)–(1.22d) for* $(\varphi, \mathscr{S})$ *is equivalent to the following pair of integrodifferential equations for* (ξ, X, Y)*:*

$$\int \mathrm{e}^{-\mathrm{i}kX}[\dot{\xi}\sinh(k(Y+h)) - \mathrm{i}(\dot{X}Y_t - \dot{Y}X_t - \gamma Y\dot{Y})\cosh(k(Y+h))]\,\mathrm{d}\lambda = 0, \tag{2.27a}$$

$$\begin{aligned}\xi_t + gY - \frac{1}{2}X_t^2 - \frac{1}{2}Y_t^2 - \gamma\partial_\lambda^{-1}\left(\dot{X}Y_t - \dot{Y}X_t\right) \\ + \frac{\gamma Y(2\dot{X}\dot{Y}Y_t - 2\dot{X}^2X_t - 2\dot{\xi}\dot{X} + \gamma\dot{X}^2Y)}{2(\dot{X}^2+\dot{Y}^2)} + \frac{(\dot{\xi} - \dot{X}X_t - \dot{Y}Y_t)^2}{2(\dot{X}^2+\dot{Y}^2)} \\ - \frac{\sigma}{\rho}\frac{\dot{X}\ddot{Y} - \dot{Y}\ddot{X}}{(\dot{X}^2+\dot{Y}^2)^{3/2}} = 0,\end{aligned} \tag{2.27b}$$

where $k \in \boldsymbol{R}$, $\xi(\lambda, t) = \varphi(X(\lambda,t), Y(\lambda,t), t)$ *is the pseudopotential evaluated on* $\mathscr{S}$.

2.3. *Travelling water waves*

Let us consider again travelling wave solutions to (2.26), so that $\eta = \eta(x - ct)$ and $\xi = \xi(x - ct)$. Setting $z = x - ct$ and denoting differentiation with respect to z with a prime, the (2.26) become

$$\int \mathrm{e}^{-\mathrm{i}kx}[\xi'\sinh(k(\eta+h)) + \mathrm{i}(c\eta' + \gamma\eta\eta')\cosh(k(\eta+h))]\,\mathrm{d}x = 0, \tag{2.28a}$$

$$\begin{aligned}-c\xi' + \frac{1}{2}(\xi')^2 + (g+\gamma c)\eta + \frac{\gamma\eta(-2c(\eta')^2 - 2\xi' + \gamma\eta)}{2(1+(\eta')^2)} \\ - \frac{(-c\eta' + \eta'\xi')^2}{2(1+(\eta')^2)} = f(\eta).\end{aligned} \tag{2.28b}$$

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We recall that (2.28*b*) is a quadratic in $\xi'(z)$ which can be solved to give

$$\xi'(z) = \gamma\eta + c - \sqrt{(1+(\eta')^2)(c^2 + f(\eta) - 2g\eta)}, \tag{2.29}$$

Replacing in (2.28*a*) ξ by the right-hand side of (2.29), equation (2.28*a*) becomes a single equation for the unknown η. In the case of zero vorticity and zero surface tension, the expression simplifies considerably and the global relation (2.28*a*) becomes

$$\int \mathrm{e}^{-\mathrm{i}kz}\left[\left(1-\sqrt{(1+(\eta')^2)\left(1-\frac{2g}{c^2}\eta\right)}\right)\sinh(k(\eta+h)) + \mathrm{i}\eta'\cosh(k(\eta+h))\right]\mathrm{d}z = 0. \tag{2.30}$$

3. Formal asymptotic results for the two-dimensional rotational case

In this section we non-dimensionalize the equations in (2.26) and approach the problem perturbatively. Throughout this section we make the assumption that each of $\{\eta, \eta_t, \eta_x, \xi_x\}$ are bounded and have sufficient decay at ∞ so that the results that follow remain valid. The rigorous justification of these results should be achieved using similar arguments to those in §3, but this is not pursued here. We suppose that ℓ is a typical length scale for the wavelengths and a is a typical amplitude of oscillation. Then we make the following substitutions:

$$x \mapsto \ell x, \quad k \mapsto \frac{k}{\ell}, \quad t \mapsto \frac{\ell}{\sqrt{gh}}t, \quad \xi \mapsto \frac{g\ell a}{\sqrt{gh}}\xi, \quad \eta \mapsto a\eta, \quad \gamma \mapsto \sqrt{\frac{g}{h}}\gamma. \tag{3.1}$$

We introduce the dimensionless parameters (ϵ, δ) defined by

$$\epsilon = \frac{a}{h}, \quad \delta = \frac{h}{\ell}, \tag{3.2}$$

which are assumed to be small. In this case (2.26*a*) becomes

$$\int \mathrm{e}^{-\mathrm{i}kx}\{\delta^{-1}\xi_x \sinh[\delta k(\epsilon\eta+1)] - \mathrm{i}[\eta_t - \epsilon\gamma\eta\eta_x]\cosh[\delta k(\epsilon\eta+1)]\}\,\mathrm{d}x = 0. \tag{3.3}$$

It is straightforward to individually dominate the terms appearing in the integrand, assuming appropriate bounds on $\|\xi_x\|_{L^1}$, $\|\eta\|_{L^\infty}$ and $\|\eta_t\|_{L^1}$. An application of the dominated convergence theorem allows us to expand the relevant expressions as a power series in (ϵ, δ), so (3.3) yields the following expression:

$$\sum_{n,m=0}^{\infty} \epsilon^n\delta^m \int \mathrm{e}^{-\mathrm{i}kx}A_{nm}(k,\eta,\eta_t,\xi)\,\mathrm{d}x = 0. \tag{3.4}$$

Using the correspondence between $\partial_x \mapsto \mathrm{i}k$ in the Fourier integral for a finite number of terms (so the relevant expression is well-defined in a classical sense) the same equation yields the following:

$$\sum_{n,m=0}^{finite} \epsilon^n\delta^m \int \mathrm{e}^{\mathrm{i}kx}A_{nm}(\mathrm{i}\partial,\eta,\eta_t,\xi)\,\mathrm{d}x + \sum_{n,m}^{\infty} \epsilon^n\delta^m \int \mathrm{e}^{\mathrm{i}kx}A_{nm}(k,\eta,\eta_t,\xi)\,\mathrm{d}x = 0. \tag{3.5}$$

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It is straightforward to bound the terms in the latter integral so the sum is $O(\epsilon^M \delta^N)$ for some specified M, N. Using the completeness of the Fourier transform we deduce

$$\sum_{n,m=0}^{finite} \epsilon^n \delta^m A_{nm}(\eta, \eta_t, \xi) \sim 0, \tag{3.6}$$

where for convenience of notation we have dropped the $\mathrm{i}\partial$ dependence. The first few A_{nm} can be easily computed:

$$A(\eta, \eta_t, \xi) = \begin{bmatrix} \eta_t + \xi_{xx} & 0 & -\frac{1}{2}\eta_{txx} - \frac{1}{6}\xi_{xxxx} & \cdots \\ (\eta\xi_x)_x - \gamma\eta\eta_x & 0 & -(\eta\eta_t)_{xx} - \frac{1}{2}(\eta\xi_x)_{xxx} + \frac{1}{2}\gamma(\eta\eta_x)_{xx} & \cdots \\ 0 & 0 & -\frac{1}{2}(\eta^2\eta_t)_{xx} - \frac{1}{2}(\eta^2\xi_x)_{xxx} + \gamma(\eta^2\eta_x)_{xx} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{3.7}$$

We note that each of the coefficients is real, and hence (3.3) is one equation. This is expected, otherwise equations (2.26) would constitute an overdetermined system of equations for the unknowns (η, ξ).

Now we look at the (2.26*b*) for (η, ξ). Using the non-dimensional parameters this equation becomes

$$\xi_t + \frac{1}{2}\epsilon\xi_x^2 - \gamma\partial_x^{-1}\eta_t + \eta + \frac{\epsilon\gamma\eta(2\epsilon\delta^2\eta\eta_x - 2\xi_x + \gamma\eta)}{2(1+\epsilon^2\delta^2\eta_x^2)} - \frac{\epsilon\delta^2(\eta_t + \epsilon\eta_x\xi_x)^2}{2(1+\epsilon^2\delta^2\eta_x^2)} - \delta^2\hat{\sigma}\left(\frac{\eta_x}{\sqrt{1+\epsilon^2\delta^2\eta_x^2}}\right)_x = 0, \tag{3.8}$$

where $\hat{\sigma} \equiv \sigma/gh^2$ is the reciprocal of the bond number. Again we expand the expression in terms the dimensionless parameters (ϵ, δ) to find a series of the form

$$\sum_{n,m=0}^{finite} \epsilon^n \delta^m B_{nm}(\eta, \eta_t, \xi, \xi_t) \sim 0. \tag{3.9}$$

Again, the computation of the coefficients B_{nm} is straightforward:

$$B(\eta, \eta_t, \xi, \xi_t) = \begin{bmatrix} \xi_t + \eta - \gamma\partial_x^{-1}\eta_t & 0 & -\hat{\sigma}\eta_{xx} & \cdots \\ \frac{1}{2}\xi_x^2 - \gamma\eta\xi_x + \frac{1}{2}\gamma^2\eta^2 & 0 & -\frac{1}{2}\eta_t^2 & \cdots \\ 0 & 0 & -\eta_t\eta_x\xi_x & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{3.10}$$

The above analysis yields a graded system of equations according to the perturbation parameters (ϵ, δ).

3.1. *Solitons and perturbed hamiltonian structure*

In this section we consider the governing equations at each order, using the expressions obtained in (3.7) and (3.10). We first investigate the Hamiltonian structure of these equations at each $O(\epsilon^N \delta^M)$, and then demonstrate the existence of solitons at an appropriate truncation.

Order $(\boldsymbol{\epsilon^0\delta^0})$. To lowest order the evolution equations are

$$\eta_t = -\xi_{xx}, \tag{3.11a}$$

$$\xi_t - \gamma\partial_x^{-1}\eta_t = -\eta. \tag{3.11b}$$

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The right-hand side of (3.11) are Hamiltonian with respect to (η, ξ) with the Hamiltonian density:

$$\mathscr{H}_{00}(\eta, \xi) = \frac{1}{2} \int \left(\eta^2 + \xi_x^2\right) \mathrm{d}x \tag{3.12}$$

and the standard symplectic structure. That is to say, (3.11) can be written in the form:

$$\partial_t \begin{pmatrix} \eta \\ \xi - \gamma \partial_x^{-1} \eta \end{pmatrix} = J \delta \mathscr{H}_{00}, \quad \text{where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \delta = \begin{pmatrix} \delta_\eta \\ \delta_\xi \end{pmatrix}. \tag{3.13}$$

Here δ_η represents the usual varitational derivative with respect to η.

Order $(\boldsymbol{\epsilon^1 \delta^0})$. To next order the equations are

$$\eta_t = -\xi_{xx} - \epsilon \left(\eta \xi_x\right)_x + \epsilon \gamma \eta \eta_x, \tag{3.14a}$$

$$\xi_t - \gamma \partial_x^{-1} \eta_t = -\eta - \tfrac{1}{2} \epsilon \xi_x^2 + \epsilon \gamma \eta \xi_x - \tfrac{1}{2} \epsilon \gamma^2 \eta^2. \tag{3.14b}$$

In the same sense, the right-hand side of this system is again Hamiltonian with the following Hamiltonian density:

$$\mathscr{H}_{10}(\eta, \xi) = \mathscr{H}_{00}(\eta, \xi) + \frac{1}{2} \epsilon \int \left(\eta \xi_x^2 + 2\gamma \eta \eta_x \xi + \frac{1}{3} \gamma^2 \eta^3\right) \mathrm{d}x. \tag{3.15}$$

Indeed (3.14) can be written in the form

$$\partial_t \begin{pmatrix} \eta \\ \xi - \gamma \partial_x^{-1} \eta \end{pmatrix} = J \delta \mathscr{H}_{10}. \tag{3.16}$$

Order $(\boldsymbol{\epsilon^1 \delta^1})$. There is no contribution at this order from the perturbation expansion, so the equations remain unchanged:

$$\partial_t \begin{pmatrix} \eta \\ \xi - \gamma \partial_x^{-1} \eta \end{pmatrix} = J \delta \mathscr{H}_{11}, \tag{3.17}$$

where we have set $\mathscr{H}_{11} = \mathscr{H}_{10} + 0$.

Order $(\boldsymbol{\epsilon^0 \delta^2})$. At this order these exists a slight complication, because the right-hand side of the evolution equations involves η_t:

$$\partial_t \begin{pmatrix} \eta \\ \xi - \gamma \partial_x^{-1} \eta \end{pmatrix} = J \delta \mathscr{H}_{11} + \delta^2 \begin{pmatrix} \frac{1}{2} \eta_{txx} + \frac{1}{6} \xi_{xxxx} \\ \hat{\sigma} \eta_{xx} \end{pmatrix}. \tag{3.18}$$

However, by using the expression for η_t recursively, we can express the right-hand side in terms of (η, ξ) and x-derivatives thereof. One must keep track of the order of the relevant terms in the recursive routine. Implementing this approach gives to $O(\delta^2)$ the following equations:

$$\partial_t \begin{pmatrix} \eta \\ \xi - \gamma \partial_x^{-1} \eta \end{pmatrix} = J \delta \mathscr{H}_{11} + \delta^2 \begin{pmatrix} -\frac{1}{3} \xi_{xxxx} \\ \hat{\sigma} \eta_{xx} \end{pmatrix}. \tag{3.19}$$

The right-hand side of this system is again Hamiltonian with respect to the Hamiltonian density given by

$$\mathscr{H}_{02} = \mathscr{H}_{11} + \frac{1}{2} \delta^2 \int \left(\hat{\sigma} \eta_x^2 - \frac{1}{3} \xi_{xx}^2\right) \mathrm{d}x. \tag{3.20}$$

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Order ($\epsilon^1\delta^2$). Computing the relevant $O(\epsilon\delta^2)$ terms through the recursion process, we find the following augmented Hamiltonian system:

$$\partial_t \begin{pmatrix} \eta \\ \xi - \gamma\partial_x^{-1}\eta \end{pmatrix} = J\delta\mathscr{H}_{02} + \epsilon\delta^2 \begin{pmatrix} -(\eta\xi_{xx})_{xx} \\ \frac{1}{2}\xi_{xx}^2 \end{pmatrix}. \tag{3.21}$$

The associated Hamiltonian density is

$$\mathscr{H}_{12} = \mathscr{H}_{02} - \tfrac{1}{2}\epsilon\delta^2 \int \xi_{xx}^2\eta \,\mathrm{d}x. \tag{3.22}$$

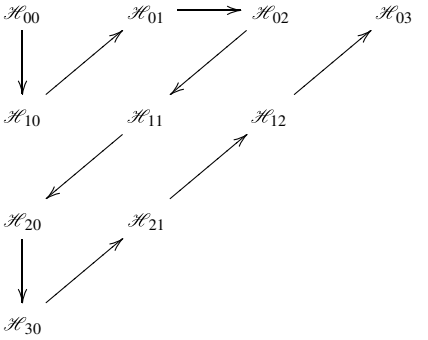
The above analysis suggests that at each order the system takes the form

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = J\delta\mathscr{H}_{ij} + \begin{pmatrix} 0 \\ \gamma\partial_x^{-1}\eta_t \end{pmatrix}. \tag{3.23}$$

Using the first of these equations, $\eta_t = \delta_\xi\mathscr{H}_{ij}$, this can be written as

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = J(\gamma)\delta\mathscr{H}_{ij}, \quad J(\gamma) = \begin{pmatrix} 0 & 1 \\ -1 & \gamma\partial_x^{-1} \end{pmatrix} \tag{3.24}$$

where $\mathscr{H}_{ij}$ is an element of the infinite chain of Hamiltonians



and $J(\gamma)$ represents the perturbed symplectic structure.

CONJECTURE 1. *The full system has a perturbed Hamiltonian structure, where the perturbation is defined by the perturbed symplectic form $J(\gamma)$ and the following Hamiltonian:*

$$\mathscr{H} = \bigoplus_{n=0}^{\infty}\bigoplus_{m=0}^{\infty}\mathscr{H}_{nm} \tag{3.25}$$

where the grading is with respect to the perturbation parameters (ϵ, δ) .

It is important to establish the context of the perturbation expansion in this conjecture. Our formulation suggests a perturbed symplectic Hamiltonian structure for the full rotational water wave problem in two dimensions, in the form:

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = J(\gamma) \begin{pmatrix} \delta_\eta \\ \delta_\xi \end{pmatrix} \left(\sum_{n,m=0}^{\infty} \epsilon^n\delta^m\mathscr{H}_{nm} \right). \tag{3.26}$$

That is to say, we claim there is a way to 'sum up' the series above. This is in contrast to the usual perturbative approach to Hamiltonian systems, in which one

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makes perturbative expansions for the Hamiltonian and the symplectic form (e.g. Olver 1984).

We note that our results are in agreement with the work of Constantin, Ivanov & Prodanov (2008), where the authors constructed a nearly Hamiltonian structure for water waves with constant vorticity using the Hilbert transform of the stream function on the free surface. It is well known that the Hilbert transform relates the boundary data for harmonic functions and their conjugates. In light of this, it is not surprising that our formulation is explicit, given that we have chosen to work with the harmonic conjugate of $\psi^h = \psi + (1/2)\gamma y^2$, rather than the stream function itself.

From this point onwards we truncate the equations at $O(\epsilon^N \delta^M)$, with $N + M > 2$. This gives us the perturbed Hamiltonian system:

$$\begin{aligned}\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} &= J(\gamma) \begin{pmatrix} \delta_\eta \\ \delta_\xi \end{pmatrix} \int \frac{1}{2} \left[(\eta^2 + \xi_x^2) + \epsilon \left(\eta \xi_x^2 + 2\gamma \eta \eta_x \xi + \frac{1}{3}\gamma^2 \eta^3 \right) \right. \\ &\quad \left. + \delta^2 \left(\hat{\sigma} \eta_x^2 - \frac{1}{3} \xi_{xx}^2 \right) \right] \mathrm{d}x \\ &= \begin{pmatrix} -\xi_{xx} - \epsilon (\eta \xi_x)_x + \epsilon \gamma \eta \eta_x - \dfrac{1}{3}\delta^2 \xi_{xxxx} \\ -\eta - \gamma \xi_x - \dfrac{1}{2}\epsilon \xi_x^2 + \delta^2 \hat{\sigma} \eta_{xx} - \dfrac{1}{3}\gamma \delta^2 \xi_{xxx} \end{pmatrix}. \end{aligned} \tag{3.27}$$

Eliminating η from (3.27) and discarding terms of order $O(\epsilon^N \delta^M)$ with $N + M > 2$ we find the equation

$$\begin{aligned} &\xi_{tt} - \xi_{xx} + \gamma \xi_{xt} + \epsilon((2 + \gamma^2)(\xi_{xt}\xi_x + \gamma \xi_x \xi_{xx}) + (1 + \gamma^2)\xi_{xx}\xi_t + \gamma \xi_t \xi_{xt}) \\ &\quad + \delta^2(\hat{\sigma} - \tfrac{1}{3})\xi_{xxxx} + \tfrac{1}{3}\gamma \delta^2 \xi_{xxxt} = 0. \end{aligned} \tag{3.28}$$

It is important to note that (3.28) is actually just one member of an equivalence class of equations that can be attained by truncating the asymptotic expansion. By using lower-order relations and substituting back in, different equations can be found. A formal analysis of the linearization of (3.28) using normal modes shows that the equation is ill-posed. To address this, we use

$$\xi_{xx} = \xi_{tt} + \gamma \xi_{xt} + O(\epsilon, \delta^2). \tag{3.29}$$

By differentiating this expression twice with respect to x and substituting back into (3.28) we find

$$\begin{aligned} &\xi_{tt} - \xi_{xx} + \gamma \xi_{xt} + \epsilon((2 + \gamma^2)(\xi_{xt}\xi_x + \gamma \xi_x \xi_{xx}) + (1 + \gamma^2)\xi_{xx}\xi_t + \gamma \xi_t \xi_{xt}) \\ &\quad + \delta^2 \hat{\sigma} \xi_{xxxx} - \tfrac{1}{3}\delta^2 \xi_{xxtt} = 0. \end{aligned} \tag{3.30}$$

Again through a formal analysis of normal modes at the linear level, we find this equation is well-posed so the corresponding solutions are physically meaningful. We now consider (3.30) in two classical scaling regimes and provide a reduction to a nonlinear evolution equation which admits solitons.

3.2. *Soliton equations*

In this section we consider solitary wave solutions to (3.30) in two physically important scaling regimes.

3.2.1. *Korteweg–de Vries scaling regime with small vorticity*

In this case the weak nonlinearity and the weak dispersion balance in the form $\epsilon = \delta^2$, valid for shallow water. We also assume the vorticity is $O(\epsilon)$, so we make the

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replacement $\gamma \mapsto \epsilon\gamma$. We use the slow time scale $T = \epsilon t$ and go to a moving frame $X = x - t$. Setting $\xi = \xi(X, T)$ and denoting $\xi_X = \zeta$, the (3.30) becomes

$$-2\zeta_T - \gamma\zeta_X - 3\zeta\zeta_X + \left(\hat{\sigma} - \tfrac{1}{3}\right)\zeta_{XXX} = 0. \tag{3.31}$$

This is the celebrated Korteweg–de Vries (KdV) equation. In this case the solitary wave solutions are

$$\xi'(z) = (2c - \gamma)\mathrm{sech}^2\left[\sqrt{\frac{\gamma - 2c}{\hat{\sigma} - \frac{1}{3}}}\frac{(z - z_0)}{2}\right]. \tag{3.32}$$

Setting $\eta = \eta(z)$ in (3.27) and integrating up we find the following quadratic in η:

$$\tfrac{1}{2}\gamma\,(\epsilon\eta)^2 + (c - \xi')\epsilon\eta - (\xi' + \tfrac{1}{3}\xi''') = 0. \tag{3.33}$$

In the case $\gamma \neq 0$, we find the following two solutions

$$\eta(z) = \frac{\xi' - c \pm \sqrt{(\xi' - c)^2 + 2\gamma(\xi' + \frac{1}{3}\xi''')}}{\epsilon\gamma} \tag{3.34}$$

of which only the $(+)$ solution obeys $\eta \to 0$ as $|z| \to \infty$. These solutions are valid when the term in the square root is positive and the steepness of the corresponding wave increases as this term approaches zero. It is clear from (3.34) that these solutions break down if

$$\left(\frac{\mathrm{d}\xi}{\mathrm{d}z} - c\right)^2 + 2\gamma\left(\frac{\mathrm{d}\xi}{\mathrm{d}z} + \frac{1}{3}\frac{\mathrm{d}^3\xi}{\mathrm{d}z^3}\right) \leqslant 0 \tag{3.35}$$

for some value of z. Using (3.32), this results in a transcendental equation for z. However, note that (3.31), after one integration, gives ξ''' in terms of ξ'. Consequently (3.35) becomes the following quadratic in ξ':

$$\left[1 + \frac{\gamma}{\hat{\sigma} - \frac{1}{3}}\right](\xi')^2 + 2\left[\gamma - c + \frac{\gamma\,(\gamma - 2c)}{3\left(\hat{\sigma} - \frac{1}{3}\right)}\right]\xi' + c^2 \leqslant 0. \tag{3.36}$$

By examining the discriminant of this quadratic, one can identify a *subset* of the parameter space $(\hat{\sigma}, c, \gamma)$ for which the solution (3.34) is well-defined.

3.2.2. *Weak nonlinearity–dispersion balance*

In the balance between nonlinearity (governed by $\epsilon \ll 1$) and dispersion ($\delta \ll 1$) one assumes the two fundamental length scales are comparable, so we set $\epsilon = \delta$. We then operate with the fast scales $\epsilon X = x$, $\epsilon T = t$. In this regime the (3.30) transforms to

$$\begin{aligned}&\xi_{TT} - \xi_{XX} + \gamma\xi_{XT} + (2 + \gamma^2)(\xi_{XT}\xi_X + \gamma\xi_X\xi_{XX}) + (1 + \gamma^2)\xi_{XX}\xi_T + \gamma\xi_T\xi_{XT}\\ &\quad + \hat{\sigma}\xi_{XXXX} - \tfrac{1}{3}c^2\xi_{XXTT} = 0.\end{aligned} \tag{3.37}$$

For solitons, we look for travelling wave solutions of the form $\xi = \xi(X - cT)$. Using this ansatz in (3.37) we obtain

$$(c^2 - 1 - \gamma c)\xi'' + ((2 + \gamma^2)(\gamma - c) - c(1 + \gamma^2) + c^2\gamma)\xi'\xi'' + \left(\hat{\sigma} - \tfrac{1}{3}c^2\right)\xi'''' = 0 \tag{3.38}$$

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where the prime denotes differentiation with respect to $z = X - cT$. Integrating up the equation in (3.38) we find the classical sech^2 soliton solution:

$$\xi'(z) = A(\gamma, c)\,\mathrm{sech}^2\left[\sqrt{\frac{c^2-(1+\gamma c)}{\frac{1}{3}c^2-\hat{\sigma}}}\,\frac{(z-z_0)}{2}\right], \tag{3.39a}$$

where the amplitude is given by

$$A(\gamma, c) = 3\left[\frac{1+\gamma c-c^2}{(2+\gamma^2)(\gamma-c)-c(1+\gamma^2)+c^2\gamma}\right]. \tag{3.39b}$$

The corresponding expression for the wave amplitude $\eta(z)$ is given by (3.34). As with the KdV scaling regime, we can identify a subset of the parameter space for which these solutions are well-defined. The condition in (3.35) is now supplemented with (3.38) which, after one integration, gives ξ''' in terms of ξ' and $(\xi')^2$. Examining the discriminant of the resulting quadratic gives sufficient conditions for the amplitude to be well-defined.

4. Conclusions

We have presented the following generalizations of the results of Ablowitz *et al.* (2006): (*a*) we have identified appropriate function spaces for the new formulation to be set in; (*b*) we have considered the case of constant vorticity; and (*c*) we have incorporated the effect of a multivalued free surface. Also, following Deconinck & Oliveras (2011), we have derived an upper bound for the wave height in the case of constant vorticity and non-zero surface tension.

We recall that Craig & Sulem (1993) introduced an elegant Dirichlet to Neumann operator $G(\eta)$ associated with the velocity potential and also obtained a series for the operator $G(\eta)$, valid for small η. The integral equations presented in Ablowitz *et al.* (2006) and here, can be considered as the summation of the above series, i.e. the series of Craig & Sulem (1993) is the Neumann series associated with the integral equations derived by Ablowitz *et al.* (2006) and those presented here.

It appears that the new formulation provides an alternative, perhaps simpler approach, for: (*a*) the numerical investigation of water waves; (*b*) the derivation of various asymptotic limits; and (*c*) the rigorous analysis of water waves.

Regarding (*a*), we recall that two-dimensional lumps were computed by Ablowitz *et al.* (2006) in the case of sufficient surface tension.

Regarding (*b*) we recall that various asymptotic equations, including the Boussinesq, Benney–Luke and the nonlinear Schrödinger equations, were derived by Ablowitz *et al.* (2006). Similarly, an appropriate Boussinesq-type equation in the case of constant vorticity was derived in § 3. We observe that these equations have been derived by several other authors, however, it appears that the new formulation provides a straightforward way of deriving these equations. For example, regarding our results in § 3, we note that Chow (1989) uses a perturbative approach which involves solving many PDEs arising from the consistency of the perturbation expansion with the boundary conditions. To solve these PDEs, the author confines attention to separable solutions. The entire paper is devoted to computing the first few coefficients of the perturbation expansion. The author comments that this method is unwieldy at anything beyond second order. Our paper gives a more direct approach, and can easily be extended to higher orders.

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Regarding (*c*) the new formulation suggests a rigorous methodology which differs drastically from that employed in the important works of Amick, Fraenkel & Toland (1982), Constantin & Strauss (2002) and Wu (1997). The rigorous implementation of the new formulation is currently a work in progress.

Finally, we note that in the classical works of Benjamin (1962), the author does not include the effect of surface tension. In contrast, our formulation also includes the effect of surface tension. The main advantages of the new approach presented by Ablowitz *et al.* (2006) and in this paper are a consequence of the *explicit* nature of the equations in 1, 2 and 3.

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