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Modified Vlasov equation and dispersion relations for a relativistic radiating electron plasma

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A modified Vlasov equation is obtained by developing a covariant statistical mechanics for a system of electrons without considering the effects of the ions and including the Landau–Lifshitz equation of motion. General dispersion relations for the transverse and longitudinal modes for any temperature are expressed. The results are similar to those found by Hakim & Mangeney (*Phys. Fluids*, vol. 14, 1971, pp. 2751–2781) for both the modified Vlasov equation and the dispersion relations. However, for the longitudinal mode, unlike the development of Hakim and Mangeney, correct expansions are done in order to give a numerical approach to obtain the longitudinal relativistic dispersion relations for any value of the wavenumber. Accordingly, new loop solutions, with turning points, crossing the super-luminous region and the super-thermal region are found. Although the expressions for the Landau damping and the damping due to the radiation reaction force coincide with the Hakim and Mangeney results for some particular cases, in general they are different. A Landau anti-damping appears in the second branch of the loop in a small region between the cutoff point and the intersection with the super-thermal line. The analysis of this effect leads us to a kind of wave pulse. We will call them bipolar waves. The treatment contains the relativistic interactions between all the electrons in the system with retarded effects. This explain the differences with Zhang's recent work (*Phys. Plasmas*, vol. 20, 2013, 092112–092132). It is shown that for low densities, the cutoff of the wave is due to the dispersion relations and not due to the radiation reaction force damping. While for both high densities and temperatures, the damping due to the radiation reaction force is important.

Key words: plasma waves

1. Introduction

In the last century, Hakim developed a manifestly covariant statistical mechanics (Hakim I 1967; Hakim II 1967). A rigorous hierarchy of equations for suitable reduced densities is described by studying the electromagnetic interactions. In order to calculate the dispersion relations for a plasma, Prentice (1968) presented an

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Some characteristic plasmas

Plasma	Density (n in particles m^{-3})	Electron's temperature T in K	$\alpha = \frac{mc^2}{kT}$	$ne^2\tau_o$
Ionosphere	10^{12}	10^3	6 000 000	1.6×10^{-58}
Glow discharge	10^{16}	10^4	600 000	1.6×10^{-45}
Chromosphere	10^{15}	10^4	600 000	1.6×10^{-46}
Solar wind	10^6	10^5	60 000	1.6×10^{-55}
Sun's crown	5.5×10^{14}	10^7	600	8.8×10^{-47}
Solar core	10^{32}	10^7	600	1.6×10^{-29}
Tokamak	10^{20}	10^8	60	1.6×10^{-41}
Mirror reactor	10^{20}	3.5×10^9	1.7	1.6×10^{-41}

TABLE 1. All these different plasmas are such that $ne^2\tau_o \ll 1$. Mirror reactor data are from Riviere (1971).

interesting work obtaining transverse and longitudinal modes including the relativistic Landau damping but without including the radiation reaction force and the internal magnetic field produced by the relativistic electrons. Later, by including the radiation reaction force calculated by Dirac (1938) and all the interactions between the electrons of the plasma (retarded electric and magnetic fields), Hakim & Mangeney (1968) obtained a relativistic Vlasov equation which permits us to obtain wave dispersion relations, the conductivity tensor and other properties. However, the use of the Lorentz–Dirac equation obligates us to consider the hyper-acceleration as an independent variable since it plays the same role as the position and the velocity. The phase space results in a $10N$-dimensional space, where N is the number of particles ($12N - 2N$, due to the kinematic relations). In an approximation process, by using the Lorentz–Dirac equation at first order in the characteristic time of the charges τ_o and by considering absence of any correlation between particles, they obtained a modified Vlasov equation at first order in $ne^2\tau_o$ where n, e and $\tau_o = 2e^2/3mc^3$ represent the density of particles, the charge of the electron and the characteristic time of the electron, respectively. Among other properties that can be derived, as we mentioned above, it has to be highlighted that this modified Vlasov equation can be linearized in order to obtain dispersion relations for electromechanical waves. They obtained general expressions for the transverse and longitudinal collective modes. In a later article, Hakim & Mangeney (1971) used the continuity equation to obtain dispersion relations eliminating non-physical solutions. They give a series of approximations for the zero-temperature case, for low temperatures just for the neighbourhood of $k \simeq w_p$, for supra-luminous and supra-thermal cases (non-relativistic and relativistic) but limited by some constraints. Interesting results in the relativistic case such as the shift of the plasma frequency are shown. In all these approximations they described the Landau damping (LD) or the damping due to the radiation reaction force (RRFD). Hakim & Mangeney (1968) claimed that in an extreme relativistic plasma, their own expressions are no longer valid since the first-order approximation in $ne^2\tau_o$ becomes meaningless if higher-order terms are considered and, consequently, the decay of the wave is accentuated. However, the order of the approximation is related to the value of $ne^2\tau_o$ and many high-temperature systems can be considered within their model as long as the parameter $ne^2\tau_o$ stays small (see table 1 and Riviere (1971)).

On the other hand, although the Landau–Lifshitz equation (Landau & Lifshitz 1962) is considered as a first-order approximation in τ_o of the Lorentz–Dirac equation

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(Dirac 1938), many authors refer to it as the exact equation of motion for a charge particle. Indeed, for Spohn (2000) and Rohrlich (2000, 2001), the Lorentz–Dirac equation must be restricted to its critical surface yielding the Landau–Lifshitz equation and consequently this last one represents the correct equation of motion of a spinless classical point charge. Moreover, by considering a generalized quantum Langevin equation and by giving a structure to the electron with a form factor and a finite cutoff parameter, Ford & O'Connell (1991) derived a non-relativistic equation of motion for charged particles. Generalizing this equation to special relativity (Ford & O'Connell 1993), they obtained an equation that Eliezer (1948) derived fifty years before using distinct arguments. Although, the Landau–Lifshitz equation and the Eliezer–Ford equation are mathematically different, their physical implications are equivalent (Di Piazza 2008; Di Piazza *et al.* 2012; Kravets, Noble & Jaroszynski 2013; García-Camacho *et al.* 2015) within Shen's zone (Shen 1972) where quantum effects are negligible. Albeit that the Landau–Lifshitz equation can be considered or not as an approximation at first order in τ_o of the Lorentz–Dirac equation (Dirac 1938), the reality is that the differences with the Lorentz–Dirac equation or with the Eliezer–Ford–O'Connell equation (Eliezer 1948; Ford & O'Connell 1993), are negligible except in the cases of self-accelerations and pre-accelerations (Di Piazza 2008; Di Piazza *et al.* 2012; Kravets *et al.* 2013; García-Camacho *et al.* 2015).

Once the Landau–Lifshitz equation is chosen to describe the motion of the charges, it is necessary to eliminate the degree of freedom represented by considering the hyper-acceleration as an independent variable in the deduction of the density function. Indeed, the hyper-acceleration does not appear anymore. Therefore, we can follow Hakim & Mangeney (1968) in the deduction of the density just by considering the positions and the velocities of the particles as independent variables. The phase space results in a $7N$-dimensional space ($8N - N$, due to the kinematic relations). The final result coincides with Hakim's generalization of the Vlasov equation in special relativity (Hakim & Mangeney 1968) giving, by a linearization process, the same dispersion relations. Moreover, in appendix A, a more general equation is deduced by including a constant external weak electric field. The dispersion relations are obtained for this case and the effects of the coupling between the constant external weak electric field and the interaction of the particle do not play an important role. The dispersion relations are not affected despite having very large Debye length at relativistic temperatures. Moreover, the LD can be deduced and compared with the RRFD. By giving an expansion in terms of $\mu = w/ck$ ($\mu \geqslant 1$ or $\mu < 1$) in the integrals involved in the dispersion relations, a general expression for the wave dispersion relations for the longitudinal mode can be expressed in order to give numerical solutions.

It has to be highlighted that in many articles these dispersion relations have been studied for different relativistic cases; such as for examples, hot relativistic beam–plasma systems (Magneville 1990), the one-dimensional tearing mode equation in tokamaks (Jovanović & Dimitrijević 2013), the nonlinear case but at moderate plasma temperature (Pesch & Kull 2010). the nonlinear case of large amplitude electromagnetic waves coupled with electrostatic waves (Connor *et al.* 2018) and the one-dimensional linear longitudinal case (Zhang *et al.* 2013). Particularly, the work done by Zhang *et al.* (2013) describes the relativistic dispersion relations for longitudinal modes in a one-dimensional model and it is an excellent opportunity to compare the latter with the dispersion relations obtained in this covariant approach where the magnetic field generated by the relativistic motions of the electrons is taken into account. Indeed, this procedure must give different results due to two reasons:

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the first one, as we noticed before, is related to the fact that the magnetic field generated by high velocity electrons must be considered; the second consists of using the covariant 4-dimension Jüttner distribution function (Jüttner 1911; Synge 1956; Cubero *et al.* 2007; López-Carrera & Ares de Parga 2008). It has to be pointed out that in this paper a correct expansion is done in terms of μ ($\mu \geqslant 1$ or $\mu < 1$) in order to be able to calculate some integrals which appear in the dispersion relations, giving new loop types of solutions crossing the super-luminous region and the super-thermal region (two branches). The interesting result is that a cutoff for the wave appears for any value of the temperature. The LD vanishes in the super-luminous region as Hakim and Mangeney, and Zhang predicted (Hakim & Mangeney 1971; Zhang *et al.* 2013). However, for the super-thermal region after the cutoff, that is in the second branch, the LD turns to be an anti-damping and possible resonance may appear. However, those possible anti-damping solutions are near to the damping solutions with different close temperatures. Indeed, the perturbation heats locally the plasma in such a way that the temperature cannot be considered as exact data and the wave suffers an anti-damping and a damping during a short period or disappears due to the cutoff. Therefore, those solutions will be called bipolar waves. This represents a novel result which is due to the existence of the second branch. For the RRFD, our general expression differs from the one obtained by Hakim & Mangeney (1968, 1971) giving distinct behaviour of the damping. Indeed, the RRFD of Hakim and Mangeney is proportional to w/k which permits us to obtain strong damping for high temperatures. However, our result neglects such important damping unless the plasma frequency is so high that the density and the temperature are close to the degenerate case and the Hagedorn temperature. The result to highlight is to show that the wave has a cutoff due to the dispersion relations and not due to the RRFD when the densities are low. The RRFD is important when high densities and temperatures are present.

The purpose of this paper is threefold: first, a deduction of a modified covariant relativistic Vlasov equation which includes the effect of the radiation reaction force in the plasma by using the Landau–Lifshitz equation. General dispersion relations are obtained for transverse and longitudinal modes; second, by making an exact expansion in terms of μ, numerically relations are described showing the shift of the plasma frequency, the existence of loop solutions (two branches with a turning point) crossing the super-luminous region and the super-thermal region, the difference with the Zhang model and the cutoff of the wave for a certain value of the wavenumber k which depends on the parameter α (α is related to the inverse of the temperature); third, for the longitudinal mode, general expressions for the LD and the RRFD are obtained and the decay or growth of the waves due to these effects is analysed for different cases.

The article is organized as follows: in § 2, the modified Vlasov equation including the reaction force and an external weak constant electric field is derived. In § 3, the dispersion relations are derived for both transverse and longitudinal modes. The case of an external constant weak electric field is analysed in appendix A given the same dispersion relations. In § 4, the relativistic dispersion relations are expressed by using an expansion in terms of μ inside some integrals that appear in the dispersion relations and this permits us to describe numerically the dispersion relations and to compare them with the results obtained by Zhang *et al.* (2013). Interesting cutoffs of the waves due to the existence of loop solutions with two branches crossing the super-luminous region and the super-thermal region are described. The decay of the waves due to the LD and the RRFD is taken into account to describe the final waves. In § 5, we provide a summary. Some concluding remarks are given in § 6.

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2. The modified Vlasov equation including the radiation reaction force and an external weak constant electric field

In order to derive a relativistic modified Vlasov equation which includes the radiation reaction force and an external weak constant electric field, we will use the Landau–Lifshitz equation of motion. It is necessary to describe it for a system of N charges. We limit our attention only to a system of electrons because the action of the ions are neglected. This implies that we deal with the constraint in the paper that the thermal velocity V_{Te} is smaller than the phase velocity $V_\phi = w/ck = \mu$, $V_{Ti} \leqslant V_{Te} < V_\phi$.

2.1. *Landau–Lifshitz equation*

For a system of N charged particles with the same mass m and equal charge e, for each particle the Lorentz–Dirac equation (Dirac 1938) is

$$ma_i^\mu = \frac{e}{c}\boldsymbol{F}_{(i)}^{\mu\nu} v_{\nu i} + m\tau_o \left[\dot{a}_i^\mu + \frac{a_i^2}{c^2} v_i^\mu\right], \quad i = 1, \ldots, N, \tag{2.1}$$

where c, $\boldsymbol{F}_{(i)}^{\mu\nu}$ and $\tau_o = 2e^2/3mc^3$ are the speed of light, the field strength tensor due to the action of the other particles, the external field on the ith particle and the characteristic time of the charge e, respectively. The subscript 'i' represents the ith particle and v_i^μ, a_i^μ and $\dot{a}_i^\mu = \mathrm{d}a_i^\mu/\mathrm{d}\tau_i$ are the 4-velocity, the 4-acceleration and the 4-hyper-acceleration of the ith particle, respectively, with τ_i the proper time of the ith particle. The equation can be expressed as

$$ma_i^\mu = \frac{e}{c}\boldsymbol{F}_{(i)}^{\mu\nu} v_{\nu i} + m\tau_o \Delta^{\mu\nu}(v_i)\dot{a}_{\nu i}, \quad i = 1, \ldots, N, \tag{2.2}$$

where

$$\Delta^{\mu\nu}(v_i) = g^{\mu\nu} - \frac{v_i^\mu v_i^\nu}{c^2}, \tag{2.3}$$

with $g^{\mu\nu}$ representing the Minkowski metric and the kinematic relation $v^\mu \dot{a}_\mu + a^\mu a_\mu = 0$ has been used. In order to incorporate the Landau–Lifshitz proposal in (2.2), the hyper-acceleration must be substituted by

$$\dot{a}_{\nu i} \to \frac{e}{mc}\frac{\mathrm{d}}{\mathrm{d}\tau_i}\boldsymbol{F}_{\nu\alpha}^{(i)} v_i^\alpha = \frac{e}{mc}\left[\boldsymbol{F}_{\nu\alpha}^{(i)} a_i^\alpha + v_i^\rho v_i^\alpha \frac{\partial \boldsymbol{F}_{\nu\alpha}^{(i)}}{\partial x_i^\rho}\right], \tag{2.4}$$

and in a second step, in the first term of the right-hand side of (2.4), the acceleration a_i^α, must be substituted by the Lorentz force; that is:

$$\dot{a}_{\nu i} \to \frac{e}{mc}\left[\frac{e}{mc}\boldsymbol{F}_{\nu\alpha}^{(i)}\boldsymbol{F}_{(i)}^{\alpha\beta} v_{\beta i} + v_i^\rho v_i^\alpha \frac{\partial \boldsymbol{F}_{\nu\alpha}^{(i)}}{\partial x_i^\rho}\right]. \tag{2.5}$$

Therefore, the Landau–Lifshitz equation of motion for each particle is (Ares de Parga 2006) $i = 1, \ldots, N$,

$$ma_i^\mu = \frac{e}{c}\boldsymbol{F}_{(i)}^{\mu\nu} v_{\nu i} + m\tau_o \Delta^{\mu\nu}(v_i)\frac{e}{mc}\left[\frac{e}{mc}\boldsymbol{F}_{\nu\alpha}^{(i)}\boldsymbol{F}_{(i)}^{\alpha\beta} v_{\beta i} + v_i^\rho v_i^\alpha \frac{\partial \boldsymbol{F}_{\nu\alpha}^{(i)}}{\partial x_i^\rho}\right]. \tag{2.6}$$

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On the other hand, the force applied on each particle must be decomposed into two parts: the external field and the field produced by the other particles on the ith particle; that is

$$\boldsymbol{F}^{\mu\nu}_{(i)} = \boldsymbol{F}^{\mu\nu}_{\text{ext}} + \sum_{j\neq i} \boldsymbol{F}^{\mu\nu}_{ji}(x_\rho), \tag{2.7}$$

where

$$\begin{aligned}\boldsymbol{F}^{\mu\nu}_{ji}(x_\rho) \;=\; & \frac{4\pi e}{c}\iiint \mathrm{d}^4x'\,\mathrm{d}^4v'\,\mathrm{d}\tau'[v'^{\nu}\partial^{\mu} - v'^{\mu}\partial^{\nu}]D_{\text{ret}}(x_\rho - x'_\rho) \\ & \times \delta(x'_\rho - x_{\rho j}(\tau'))\delta(v'_\rho - v_{\rho j}(\tau')),\end{aligned} \tag{2.8}$$

where D_{ret} represents the retarded Green function (Jackson 1998).

Finally, it is important to note that the equation of motion described by (2.6) does not depend on the hyper-acceleration and consequently the phase space is described by a $8N$-dimensional space with the regular N constraint, $v^\mu v_\mu = c^2$. This point represents one of the different approaches to obtaining the relativistic Vlasov equation compared with the work done by Hakim & Mangeney (1968).

2.2. *Relativistic Vlasov equation including radiation reaction force*

Different derivations of a hierarchy of statistical equations were developed during the last century, independently, by Bogolyubov, Born and Green, Kirkwood and Yvon (the so-called BBGKY hierarchy). Particularly, we can cite the work done by Bogolyubov (1946) in the early sixties. An important approach was done by Hakim I (1967), Hakim II (1967) which lead to a hierarchy of manifestly covariant equations for a system of charge particles including the radiation reaction force (Hakim & Mangeney 1968, 1971) with the constraint of the absence of any correlation between the particles and equations to first order in $ne^2\tau_o$. We will follow Hakim and Mangeney in order to obtain a relativistic covariant modified Vlasov equation but with the difference that we will start from a phase space without the hyper-acceleration as an independent variable and including a weak constant electric field. The fact of not considering the hyper-acceleration as an independent variable is because we consider the Landau–Lifshitz equation of motion, which does not contain such dependence.

Let us make a summary of the derivation of the modified Vlasov equation just to place emphasis on the differences between our method and the Hakim and Mangeney one. The phase space in special relativity must be constructed by following the dynamics of the particles in the system. By looking at the equation of motion, equation (2.6), it is clear that the phase space is composed of

$$\Gamma = \{x_\mu\}^N \times \{v_\mu\}^N, \tag{2.9}$$

accompanied by the constraint

$$v_\mu v^\mu = c^2, \tag{2.10}$$

which can be incorporated directly in the densities by using a delta function. Notice that unlike Hakim & Mangeney (1968) the hyper-acceleration is not taken as part of the phase space since the equation of motion to be considered is the Landau–Lifshitz equation which is a second-order differential equation. Using the regular assumptions

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of the Gibbs ensemble (Hakim I 1967; Hakim II 1967), it is possible to define a random one-particle density as

$$R_1(x_\mu, v_\mu; \tau) = \sum_{i=1}^{N} \delta(x_\mu - x_{\mu i}(\tau))\delta(v_\mu - v_{\mu i}(\tau)), \tag{2.11}$$

where τ represents a parameter which will describe the trajectory of each particle by means of $v_{\mu i}(\tau) = \mathrm{d}x_{\mu i}(\tau)/\mathrm{d}\tau$. Also, following Hakim & Mangeney (1968), we define the proper time-dependent density,

$$D_1(x_\mu, v_\mu; \tau) = N\langle R_1(x_\mu, v_\mu; \tau)\rangle, \tag{2.12}$$

where the brackets $\langle R_1\rangle$ represent the regular average weighted by random initial conditions (Hakim I 1967; Hakim II 1967). Also, we will need

$$\begin{aligned} R_2(x_\mu, v_\mu, \tau; x'_\mu, v'_\mu, \tau') \;=\; & \sum_{i\neq j} \delta(x_\mu - x_{\mu i}(\tau)) \\ & \times \delta(v_\mu - v_{\mu i}(\tau))\delta(x'_\mu - x_{\mu i}(\tau'))\delta(v'_\mu - v_{\mu i}(\tau')), \end{aligned} \tag{2.13}$$

and

$$D_2 = (x_\mu, v_\mu, \tau; x'_\mu, v'_\mu, \tau') = N(N-1)\langle R_2(x_\mu, v_\mu, \tau; x'_\mu, v'_\mu, \tau')\rangle. \tag{2.14}$$

Finally, the relativistic one-particle density is

$$\mathcal{N}_1(x_\mu, v_\mu) = \int_{-\infty}^{\infty} D_1(x_\mu, v_\mu; \tau)\,\mathrm{d}\tau. \tag{2.15}$$

In order to obtain the continuity equation, let us follow the Klimontovich method (Klimontovich 1960), but with a difference since the force on each particle will not be the Lorentz force but the Lorentz force plus the radiation reaction force due to Landau and Lifshitz represented by (2.6). Starting from equation (2.11) in the equivalent following form

$$R_1(x^\mu, v^\mu; \tau) = \sum_{i=1}^{N} \delta(x^\mu - x_i^\mu(\tau))\delta(v^\mu - v_i^\mu(\tau)), \tag{2.16}$$

and partially differentiating it with respect to τ, we have,

$$\begin{aligned} \frac{\partial R_1(x^\mu, v^\mu; \tau)}{\partial \tau} \;=\; & \sum_{i=1}^{N} \left\{ \frac{\partial}{\partial \tau}\delta(x^\mu - x_i^\mu(\tau)) \right\} \delta(v^\mu - v_i^\mu(\tau)) \\ & + \delta(x^\mu - x_i^\mu(\tau)) \left\{ \frac{\partial}{\partial \tau}\delta(v^\mu - v_i^\mu(\tau)) \right\} \\ = \; & \sum_{i=1}^{N} \left\{ -\frac{\mathrm{d}x_i^\mu(\tau)}{\mathrm{d}\tau}\frac{\partial}{\partial x^\mu}\delta(x^\mu - x_i^\mu(\tau)) \right\} \delta(v^\mu - v_i^\mu(\tau)) \\ & + \sum_{i=1}^{N} \delta(x^\mu - x_i^\mu(\tau)) \left\{ -\frac{\mathrm{d}v_i^\mu(\tau)}{\mathrm{d}\tau}\frac{\partial}{\partial v^\mu}\delta(v^\mu - v_i^\mu(\tau)) \right\}. \end{aligned} \tag{2.17}$$

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Let us analyse the term in the second line of (2.17); first; it is necessary to take into account the property of the δ-function,

$$f(a)\delta(a-b) = f(b)\delta(a-b). \tag{2.18}$$

Then, since $\mathrm{d}x_i^\mu(\tau)/\mathrm{d}\tau = v_i^\mu(\tau)$, we obtain

$$\begin{aligned}
&\left\{-\frac{\mathrm{d}x_i^\mu(\tau)}{\mathrm{d}\tau}\frac{\partial}{\partial x^\mu}\delta(x^\mu - x_i^\mu(\tau))\right\}\delta(v^\mu - v_i^\mu(\tau)) \\
&\quad = -v^\mu\delta(v^\mu - v_i^\mu(\tau))\left\{\frac{\partial}{\partial x^\mu}\delta(x^\mu - x_i^\mu(\tau))\right\} \\
&\quad = -v_\mu\left\{\frac{\partial}{\partial x^\mu}[\delta(x^\mu - x_i^\mu(\tau))\delta(v^\mu - v_i^\mu(\tau))]\right\}.
\end{aligned} \tag{2.19}$$

Consequently,

$$\begin{aligned}
&\sum_{i=1}^{N}\left\{-\frac{\mathrm{d}x_i^\mu(\tau)}{\mathrm{d}\tau}\frac{\partial}{\partial x^\mu}\delta(x^\mu - x_i^\mu(\tau))\right\}\delta(v^\mu - v_i^\mu(\tau)) \\
&\quad = \sum_{i=1}^{N} -v^\mu\left\{\frac{\partial}{\partial x^\mu}[\delta(x^\mu - x_i^\mu(\tau))\delta(v^\mu - v_i^\mu(\tau))]\right\} \\
&\quad = -v^\mu\left\{\frac{\partial}{\partial x^\mu}\sum_{i=1}^{N}[\delta(x^\mu - x_i^\mu(\tau))\delta(v^\mu - v_i^\mu(\tau))]\right\} \\
&\quad = -v^\mu\frac{\partial R_1(x^\mu, v^\mu; \tau)}{\partial x^\mu}.
\end{aligned} \tag{2.20}$$

We must now analyse the term on the third line of (2.17),

$$\begin{aligned}
&\delta(x^\mu - x_i^\mu(\tau))\left\{-\frac{\mathrm{d}v_i^\mu(\tau)}{\mathrm{d}\tau}\frac{\partial}{\partial v^\mu}\delta(v^\mu - v_i^\mu(\tau))\right\} \\
&\quad = \delta(x^\mu - x_i^\mu(\tau))\left\{-a_i^\mu(\tau)\frac{\partial}{\partial v^\mu}\delta(v^\mu - v_i^\mu(\tau))\right\} \\
&\quad = \frac{\partial}{\partial v^\mu}\{-a_i^\mu(\tau)\delta(x^\mu - x_i^\mu(\tau))\delta(v^\mu - v_i^\mu(\tau))\},
\end{aligned} \tag{2.21}$$

where $a_i^\mu(\tau)$ enters inside the partial derivative since it depends only on τ. Therefore,

$$\begin{aligned}
&\sum_{i=1}^{N}\delta(x^\mu - x_i^\mu(\tau))\left\{-\frac{\mathrm{d}v_i^\mu(\tau)}{\mathrm{d}\tau}\frac{\partial}{\partial v^\mu}\delta(v^\mu - v_i^\mu(\tau))\right\} \\
&\quad = \sum_{i=1}^{N}\frac{\partial}{\partial v^\mu}\{-a_i^\mu(\tau)\delta(x^\mu - x_i^\mu(\tau))\delta(v^\mu - v_i^\mu(\tau))\}.
\end{aligned} \tag{2.22}$$

Noticing that $a_i^\mu(\tau) = a_i^\mu(x_i^\rho(\tau), v_i^\rho(\tau))$, we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\delta(x^\mu - x_i^\mu(\tau))\left\{-\frac{\mathrm{d}v_i^\mu(\tau)}{\mathrm{d}\tau}\frac{\partial}{\partial v^\mu}\delta(v^\mu - v_i^\mu(\tau))\right\} \\
&\quad = \sum_{i=1}^{N}\frac{\partial}{\partial v^\mu}\left\{\begin{array}{c}-a_i^\mu(x_i^\rho(\tau), v_i^\rho(\tau)) \\ \times\,\delta(x^\mu - x_i^\mu(\tau))\delta(v^\mu - v_i^\mu(\tau))\end{array}\right\}.
\end{aligned} \tag{2.23}$$

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By using (2.18), we arrive at

$$\sum_{i=1}^{N}\delta(x^{\mu}-x_i^{\mu}(\tau))\left\{-\frac{\mathrm{d}v_i^{\mu}(\tau)}{\mathrm{d}\tau}\frac{\partial}{\partial v^{\mu}}\delta(v^{\mu}-v_i^{\mu}(\tau))\right\}$$
$$=\frac{\partial}{\partial v^{\mu}}\left\{-a^{\mu}(x^{\rho},v^{\rho})\left[\sum_{i=1}^{N}\delta(x^{\mu}-x_i^{\mu}(\tau))\times\delta(v^{\mu}-v_i^{\mu}(\tau))\right]\right\}$$
$$=-\frac{\partial}{\partial v^{\mu}}[a^{\mu}(x^{\rho},v^{\rho})R]. \tag{2.24}$$

Here, it is important to analyse the following cases. First, let us consider that the force is the Lorentz force; that is,

$$a^{\mu}(x^{\rho},v^{\rho})=\frac{e}{mc}\boldsymbol{F}^{\mu\nu}(x^{\rho})v_{\nu}. \tag{2.25}$$

We can calculate the partial derivative of a^{μ} with respect to v^{μ},

$$\frac{\partial}{\partial v^{\mu}}a^{\mu}(x^{\rho},v^{\rho})=\frac{e}{mc}\boldsymbol{F}^{\mu\nu}(x^{\rho})\frac{\partial v_{\nu}}{\partial v^{\mu}}=\frac{e}{mc}\boldsymbol{F}^{\mu\nu}(x_{\rho})\delta_{\mu\nu}=\frac{e}{mc}\boldsymbol{F}^{\mu\mu}(x_{\rho})=0. \tag{2.26}$$

Therefore, in this case

$$\frac{\partial}{\partial v^{\mu}}[a^{\mu}(x^{\rho},v^{\rho})R]=a^{\mu}(x^{\rho},v^{\rho})\frac{\partial}{\partial v^{\mu}}R. \tag{2.27}$$

Finally for the Lorentz force case and by using (2.17), (2.20), (2.24) and (2.27), we obtain the classical Klimontovich result (Klimontovich 1960) for the continuity equation written as

$$\frac{\partial R_1(x_{\mu},v_{\mu};\tau)}{\partial\tau}+v^{\mu}\frac{\partial R_1(x_{\mu},v_{\mu};\tau)}{\partial x^{\mu}}+a^{\mu}\frac{\partial R_1(x_{\mu},v_{\mu};\tau)}{\partial v^{\mu}}=0. \tag{2.28}$$

However, when the equation of motion is governed by the Landau–Lifshitz equation, equation (2.6), (2.27) cannot be used due to the reaction radiation term. Therefore, by using (2.17), (2.20) and (2.24), in the case where the motion of the particles is governed by the Landau–Lifshitz equation, the classical Klimontovich result must be modified and the continuity equation can be expressed as

$$\frac{\partial R_1(x_{\mu},v_{\mu};\tau)}{\partial\tau}+v^{\mu}\frac{\partial R_1(x_{\mu},v_{\mu};\tau)}{\partial x^{\mu}}+\frac{\partial a^{\mu}R_1(x_{\mu},v_{\mu};\tau)}{\partial v^{\mu}}=0. \tag{2.29}$$

Notice that a difference appears with the Hakim & Mangeney (1968) dependence of R_1 since they obtained $R_1(x_i^{\mu},v_i^{\mu},\dot{a}_{vi})$ and our result is $R_1=R_1(x^{\mu},v^{\mu})$. This is because the equation of motion is taken from the beginning as equation (2.6) which does not depend on $\dot{a}_{vi}$. Therefore, by using (2.6) and (2.7), we arrive at,

$$\frac{\partial R_1(x_{\mu},v_{\mu};\tau)}{\partial\tau}+v^{\mu}\frac{\partial R_1(x_{\mu},v_{\mu};\tau)}{\partial x^{\mu}}$$
$$+\frac{e}{mc}\frac{\partial}{\partial v^{\mu}}\left[\begin{array}{c}\boldsymbol{F}^{\mu\nu}v_{\nu}\\+\tau_o\Delta^{\mu\nu}(v_{\rho})\left[\dfrac{e}{mc}\boldsymbol{F}_{\nu\alpha}\boldsymbol{F}^{\alpha\beta}v_{\beta}+v^{\rho}v^{\alpha}\dfrac{\partial\boldsymbol{F}_{\nu\alpha}}{\partial x^{\rho}}\right]R_1\end{array}\right]=0. \tag{2.30}$$

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Let us analyse the second line of (2.30). First, the strength tensor $\boldsymbol{F}^{\mu\nu}$ must be decomposed into two terms corresponding to the external force and the internal force, see (2.7) and (2.8); second, the non-radiation terms are represented in the second line of (2.31). Finally, the radiation terms are represented in the third line of (2.31).

$$\begin{aligned}&\frac{\partial R_1(x_\mu, v_\mu; \tau)}{\partial \tau} + v^\mu \frac{\partial R_1(x_\mu, v_\mu; \tau)}{\partial x^\mu} \\ &+ \frac{e}{mc}\frac{\partial}{\partial v^\mu}[\boldsymbol{F}^{\mu\nu}_{\text{ext}} v_\nu R_1] + \frac{e}{mc}\frac{\partial}{\partial v^\mu}[\boldsymbol{F}^{\mu\nu}_{\text{int}} v_\nu R_1] \\ &+ \frac{e}{mc}\tau_o \frac{\partial}{\partial v^\mu}\left[\Delta^{\mu\nu}(v_\rho)\left[\begin{array}{c}\dfrac{e}{mc}[\boldsymbol{F}^{\text{ext}}_{\nu\alpha} + \boldsymbol{F}^{\text{int}}_{\nu\alpha}] \\ \times [\boldsymbol{F}^{\alpha\beta}_{\text{ext}} + \boldsymbol{F}^{\alpha\beta}_{\text{int}}] v_\beta \\ + v^\rho v^\alpha \dfrac{\partial [\boldsymbol{F}^{\text{ext}}_{\nu\alpha} + \boldsymbol{F}^{\text{int}}_{\nu\alpha}]}{\partial x^\rho}\end{array}\right] R_1\right] = 0. \end{aligned} \tag{2.31}$$

By using (2.8), the first two lines of (2.31) are expressed in the first two lines of (2.32). The radiation term must be analysed as follows. Terms depending on a higher order of e^2 will be neglected. In this order of ideas, as we can notice in (2.8), the internal $\boldsymbol{F}^{\alpha\beta}_{\text{int}}$ depends on e and consequently the product of the external $\boldsymbol{F}^{\alpha\beta}_{\text{ext}}$ with the internal $\boldsymbol{F}^{\alpha\beta}_{\text{int}}$ as well as the product of the internal $\boldsymbol{F}^{\alpha\beta}_{\text{ext}}$ with the internal $\boldsymbol{F}^{\alpha\beta}_{\text{ext}}$ will be neglected since there is a factor $e\tau_o/mc$ in the radiation reaction term. Finally, the conserved terms are described in the third and fourth lines of (2.32).

$$\begin{aligned}&\frac{\partial R_1(x_\mu, v_\mu; \tau)}{\partial \tau} + v^\mu \frac{\partial R_1(x_\mu, v_\mu; \tau)}{\partial x^\mu} + \frac{e}{mc}\frac{\partial}{\partial v^\mu}[\boldsymbol{F}^{\mu\nu}_{\text{ext}} v_\nu R_1] \\ &+ \frac{4\pi e^2}{mc^2} v_\nu \frac{\partial}{\partial v_\mu}\int \mathrm{d}\tau'\, \mathrm{d}^4x'\, \mathrm{d}^4v' [\{v^{\nu'}\partial^\mu - v^{\mu'}\partial^\nu\} D_{\text{ret}}(x_\rho - x'_\rho)] \\ &\times R_2(x_\mu, v_\mu, \tau; x'_\mu, v'_\mu, \tau') \\ &+ \frac{e}{mc}\tau_o \frac{\partial}{\partial v^\mu}\left[\Delta^{\mu\nu}(v_\rho)\left[\frac{e}{mc}[\boldsymbol{F}^{\text{ext}}_{\nu\alpha}][\boldsymbol{F}^{\alpha\beta}_{\text{ext}}] v_\beta + v^\rho v^\alpha \frac{\partial \boldsymbol{F}^{\text{ext}}_{\nu\alpha}}{\partial x^\rho}\right] R_1\right] \\ &+ \tau_o \frac{\partial}{\partial v^\mu}\left\{\begin{array}{c}\Delta^{\mu\nu}(v_\rho)\dfrac{4\pi e^2}{mc^2} v^\rho v^\alpha \displaystyle\int \mathrm{d}\tau'\, \mathrm{d}^4x'\, \mathrm{d}^4v' \\ \left[\{v'_\alpha \partial_\nu - v'_\nu \partial_\alpha\}\dfrac{\partial}{\partial x^\rho} D_{\text{ret}}(x_\rho - x'_\rho)\right] R_2(x_\mu, v_\mu, \tau; x'_\mu, v'_\mu, \tau')\end{array}\right\} \\ &= 0. \end{aligned} \tag{2.32}$$

In the following, we abandon the inclusion of an external weak constant electric field since the dispersion relations are not changed as long as the Debye length is sufficiently small compared with the external weak constant field (see appendix A). Therefore, in the following,

$$\boldsymbol{F}^{\mu\nu}_{\text{ext}} = 0. \tag{2.33}$$

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We arrive at

$$\frac{\partial R_1(x_\mu, v_\mu; \tau)}{\partial \tau} + v^\mu \frac{\partial R_1(x_\mu, v_\mu; \tau)}{\partial x^\mu} + \frac{4\pi e^2}{mc^2} v_\nu \frac{\partial}{\partial v_\mu} \int \mathrm{d}\tau' \, \mathrm{d}^4x' \, \mathrm{d}^4v' [\{v^{\nu'}\partial^\mu - v^{\mu'}\partial^\nu\} D_{\mathrm{ret}}(x_\rho - x'_\rho)] \times R_2(x_\mu, v_\mu, \tau; x'_\mu, v'_\mu, \tau') + \tau_o \frac{\partial}{\partial v^\mu} \left\{ \begin{array}{c} \Delta^{\mu\nu}(v_\rho) \dfrac{4\pi e^2}{mc^2} v^\rho v^\alpha \displaystyle\int \mathrm{d}\tau' \, \mathrm{d}^4x' \, \mathrm{d}^4v' \\ \times \left[\{v'_\alpha \partial_\nu - v'_\nu \partial_\alpha\} \dfrac{\partial}{\partial x^\rho} D_{\mathrm{ret}}(x_\rho - x'_\rho) \right] R_2(x_\mu, v_\mu, \tau; x'_\mu, v'_\mu, \tau') \end{array} \right\} = 0. \tag{2.34}$$

Considering that the correlation function vanishes or is of order $(ne^2\tau_o)^2$; that is,

$$D_2(x_\mu, v_\mu, \tau; x'_\mu, v'_\mu, \tau') = D_1(x_\mu, v_\mu; \tau) D_1(x'_\mu, v'_\mu; \tau'). \tag{2.35}$$

By calculating the average of (2.12) and taking $N - 1 \simeq N$, we arrive at

$$\frac{\partial D_1(x_\mu, v_\mu; \tau)}{\partial \tau} + v^\mu \frac{\partial D_1(x_\mu, v_\mu; \tau)}{\partial x^\mu} + \frac{4\pi e^2}{mc^2} N v_\nu \frac{\partial}{\partial v_\mu} \int \mathrm{d}\tau' \, \mathrm{d}^4x' \, \mathrm{d}^4v' [\{v^{\nu'}\partial^\mu - v^{\mu'}\partial^\nu\} D_{\mathrm{ret}}(x_\rho - x'_\rho)] \times D_1(x_\mu, v_\mu; \tau) D_1(x'_\mu, v'_\mu; \tau') + \tau_o \frac{4\pi e^2}{mc^2} N \frac{\partial}{\partial v^\mu} \left\{ \begin{array}{c} \Delta^{\mu\nu}(v_\rho) v^\rho v^\alpha \displaystyle\int \mathrm{d}\tau' \, \mathrm{d}^4x' \, \mathrm{d}^4v' \\ \left[\{v'_\alpha \partial_\nu - v'_\nu \partial_\alpha\} \dfrac{\partial}{\partial x^\rho} D_{\mathrm{ret}}(x_\rho - x'_\rho) \right] \\ \times D_1(x_\mu, v_\mu; \tau) D_1(x'_\mu, v'_\mu; \tau') \end{array} \right\} = 0. \tag{2.36}$$

Considering that

$$\lim_{\tau \to \mp\infty} D_1(x_\mu, v_\mu; \tau) = 0, \tag{2.37}$$

and integrating over the proper time τ, we obtain

$$v^\mu \frac{\partial \mathcal{N}_1(x_\mu, v_\mu)}{\partial x^\mu} + \frac{4\pi e^2}{mc^2} N v_\nu \frac{\partial}{\partial v_\mu} \mathcal{N}_1(x_\mu, v_\mu) \times \int \mathrm{d}^4x' \, \mathrm{d}^4v' [\{v'^\nu \partial^\mu - v'^\mu \partial^\nu\} D_{\mathrm{ret}}(x_\rho - x'_\rho)] \mathcal{N}_1(x'_\mu, v'_\mu) + \tau_o \frac{4\pi e^2}{mc^2} N \frac{\partial}{\partial v^\mu} \left\{ \begin{array}{c} \Delta^{\mu\nu}(v_\rho) v^\rho v^\alpha \displaystyle\int \mathrm{d}^4x' \, \mathrm{d}^4v' \\ \left[\{v'_\alpha \partial_\nu - v'_\nu \partial_\alpha\} \dfrac{\partial}{\partial x^\rho} D_{\mathrm{ret}}(x_\rho - x'_\rho) \right] \\ \times \mathcal{N}_1(x_\mu, v_\mu) \mathcal{N}_1(x'_\mu, v'_\mu) \end{array} \right\} = 0, \tag{2.38}$$

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which represents a modified relativistic Vlasov equation up to the $ne^2\tau_o$ order. It has to be highlighted that no collision term is obtained because all of the interactions are electromagnetic and there are no collisions or friction forces. Nevertheless, the radiation reaction term can be taken as a collision term because without it we recover the usual Vlasov equation with no collisions.

3. Dispersion relations

Once we obtain the modified relativistic Vlasov equation which includes the effect of the radiation reaction force, it is necessary to consider that the system is in equilibrium and our goal is to obtain the 4-current perturbation vector in order to calculate the dispersion relations.

3.1. *First-order perturbation*

We assume now that the plasma is in an equilibrium state described by the Jüttner distribution function (Jüttner 1911; Synge 1956; Cubero *et al.* 2007; López-Carrera & Ares de Parga 2008). Accordingly, a small perturbation can be described

$$\mathcal{N} = \mathcal{N}_{eq} + \mathcal{N}_p \simeq \mathcal{N}_J + \mathcal{N}_p, \tag{3.1}$$

where $\mathcal{N}_{eq}$, $\mathcal{N}_p$ and $\mathcal{N}_J$ represent the distribution function in equilibrium, the perturbed distribution function and the Jüttner distribution function (see appendix B). Therefore, applying this to (2.38), considering that $\mathcal{N}_{eq} \simeq \mathcal{N}_J$ is the solution of the unperturbed case which does not depend on x_μ, that is $\mathcal{N}_{eq} \neq \mathcal{N}_{eq}(x_\mu)$ and taking into account that the derivatives of the perturbation and the product of the perturbations must both be neglected, we arrive at

$$\begin{aligned} v^\mu \frac{\partial \mathcal{N}_p(x_\mu, v_\mu)}{\partial x^\mu} + \frac{4\pi e^2}{mc^2} N v_\nu \frac{\partial}{\partial v_\mu} \mathcal{N}_{eq}(v_\mu) \int \mathrm{d}^4x' \, \mathrm{d}^4v' \\ \times [\{v'^\nu \partial^\mu - v'^\mu \partial^\nu\} D_{\mathrm{ret}}(x_\rho - x'_\rho)] \mathcal{N}_p(x'_\lambda, v'_\lambda) \\ = -\tau_o \frac{4\pi e^2}{mc^2} \frac{\partial}{\partial v^\mu} \left\{ \begin{array}{c} \Delta^{\mu\nu}(v_\rho) v^\rho v^\alpha \int \mathrm{d}^4x' \, \mathrm{d}^4v' \\ \left[\{v'_\alpha \partial_\nu - v'_\nu \partial_\alpha\} \dfrac{\partial}{\partial x^\rho} D_{\mathrm{ret}}(x_\rho - x'_\rho) \right] \\ \times \mathcal{N}_{eq}(v_\lambda) \mathcal{N}_p(x'_\sigma, v'_\sigma) \end{array} \right\}. \end{aligned} \tag{3.2}$$

Let us use the Fourier transformation in order to obtain the 4-current. We have

$$\tilde{\mathcal{N}}_p(k_\mu, v_\mu) = \frac{1}{(2\pi)^2} \int \mathrm{d}^4x \exp[-\mathrm{i}k^\mu x_\mu] \mathcal{N}_p(x_\rho, v'_\mu). \tag{3.3}$$

By using (3.2) and (3.3), we obtain

$$\begin{aligned} \mathrm{i}k^\mu v_\mu \tilde{\mathcal{N}}_p(k_\mu, v_\mu) + \mathrm{i} \left(\frac{4\pi e^2}{mc^2} \right) v_\nu \frac{\partial}{\partial v_\mu} \mathcal{N}_{eq}(v_\mu) \int \mathrm{d}^4v' \frac{k^\mu v'^\nu - k^\nu v'^\mu}{-k^\lambda k_\lambda} \mathcal{N}_p(k_\rho, v'_\rho) \\ = \left(\frac{4\pi e^2}{mc^2} \tau_o \right) \frac{\partial}{\partial v^\mu} \{\Delta^{\mu\nu}(v_\rho) v^\rho v^\alpha \mathcal{N}_{eq}(v_\lambda)\} \\ \times \int \mathrm{d}^4v' \frac{v'_\alpha k_\nu - v'_\nu k_\alpha}{-k^\lambda k_\lambda} k_\rho \mathcal{N}_p(k_\sigma, v'_\sigma). \end{aligned} \tag{3.4}$$

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Finally, we arrive at

$$\begin{aligned}\tilde{\mathcal{N}}_p(k_\mu, v_\mu) \;=\; & \frac{\omega_p^2}{n_o} v_\nu \frac{\partial}{\partial v_\mu} \mathcal{N}_{eq}(v_\mu) \\ & \times \frac{1}{k^\mu v_\mu} \int \mathrm{d}^4 v' \frac{k^\mu v'^\nu - k^\nu v'^\mu}{k^\lambda k_\lambda} \mathcal{N}_p(x_\rho, v'_\rho) \\ & + \mathrm{i} \left(\frac{\omega_p^2}{n_o} \tau_o \right) \frac{\partial}{\partial v^\mu} \{ \Delta^{\mu\nu}(v_\rho) v^\rho v^\alpha \mathcal{N}_{eq}(v_\lambda) \} \\ & \times \frac{1}{k^\mu v_\mu} \int \mathrm{d}^4 v' \frac{v'_\alpha k_\nu - v'_\nu k_\alpha}{k^\lambda k_\lambda} k_\rho \mathcal{N}_p(k_\sigma, v'_\sigma), \end{aligned} \tag{3.5}$$

where ω_p represents the regular plasma frequency. The Fourier component of the 4-current vector $J^\beta(k_v)$ is

$$J^\beta(k_v) = \int \mathrm{d}^4 v v^\beta \mathcal{N}_p(k_\nu, v_\sigma). \tag{3.6}$$

By using (3.5), we arrive at

$$J^\beta(k_v) = \frac{w_p^2}{k^\alpha k_\alpha} \left\{ \begin{array}{c} -I_\lambda k^\beta - k^\alpha k_\alpha I_\lambda^\beta + k^\alpha I_\alpha g_\lambda^\beta \\ + \mathrm{i}\tau_o (k^\alpha k_\alpha J_\lambda^\beta + \xi^\alpha k_\alpha \xi^{-1} g_\lambda^\beta - k^\beta \xi_\lambda \xi^{-1}) \end{array} \right\} J^\lambda(k_v), \tag{3.7}$$

where

$$I_\nu(k_\rho, \xi_\rho) = -\frac{1}{n_o} \int \mathrm{d}v^4 \frac{v_v}{k^\lambda v_\lambda} \mathcal{N}_{eq}(v_\mu), \tag{3.8}$$

and

$$I_\mu^\nu(k_\rho, \xi_\rho) = -\frac{1}{n_o} \int \mathrm{d}v^4 \frac{v_\mu v^\nu}{(k^\lambda v_\lambda)^2} \mathcal{N}_{eq}(v_\rho), \tag{3.9}$$

and

$$J_\mu^\nu(k_\rho, \xi_\rho) = -\frac{1}{n_o} \int \mathrm{d}v^4 \frac{v_\mu v^\nu}{k^\lambda v_\lambda} \mathcal{N}_{eq}(v_\rho), \tag{3.10}$$

where (see appendix B)

$$n_o = \frac{K_2(m\xi)}{K_1(m\xi)}, \tag{3.11}$$

where K_n are is Kelvin function of order n (Synge 1956) and

$$\xi = \frac{1}{\kappa T_o} \quad \text{and} \quad \xi^\mu = \xi \bar{v}^\mu. \tag{3.12a,b}$$

It has to be noted that other works constrain the problem to one dimension and, consequently, they use the one-dimensional Jüttner distribution function (Zhang *et al.* 2013).

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3.2. *General dispersion relations*

Once we obtain the expression for the 4-vector current, equation (3.7), we need just to calculate the determinant and equal it to zero; that is

$$\det G^{\beta}_{\lambda} = \det \left\{ g^{\beta}_{\lambda} \left(1 - w_p^2 \frac{k^{\alpha} I_{\alpha} + \mathrm{i}\tau_o \xi^{\alpha} k_{\alpha} \xi^{-1}}{k^{\alpha} k_{\alpha}} \right) + w_p^2 (I^{\beta}_{\lambda} - \mathrm{i}\tau_o J^{\beta}_{\lambda}) + w_p^2 k^{\beta} \frac{I_{\lambda} + \mathrm{i}\tau_o \xi_{\lambda} \xi^{-1}}{k^{\alpha} k_{\alpha}} \right\} = 0. \tag{3.13}$$

In a reference frame (the laboratory frame) where

$$\xi^{\mu} = (\xi, \mathbf{0}), \tag{3.14}$$

and choosing the z-axis as the direction of propagation of the perturbation mode, we have

$$k^{\mu} = (k^0, 0, 0, k^3) = (w, 0, 0, k). \tag{3.15}$$

For the longitudinal mode, we have

$$\begin{aligned} &\left\{ \left[1 - \frac{w_p^2}{k^{\alpha} k_{\alpha}} G \right] + w_p^2 \{ I_0^0 - \mathrm{i}\tau_o J_0^0 \} + \frac{w_p^2 k^0}{k^{\alpha} k_{\alpha}} (I_0 + \mathrm{i}\tau_o) \right\} \\ &\quad \times \left\{ \left[1 - \frac{w_p^2}{k^{\alpha} k_{\alpha}} G \right] + w_p^2 \{ I_3^3 - \mathrm{i}\tau_o J_3^3 \} + \frac{w_p^2 k^3}{k^{\alpha} k_{\alpha}} (I_3) \right\} \\ &= w_p^4 \left\{ \frac{k^3}{k^{\alpha} k_{\alpha}} (I_0 + \mathrm{i}\tau_o) + I_0^3 - \mathrm{i}\tau_o J_0^3 \right\} \times \left\{ I_3^0 - \mathrm{i}\tau_o J_3^0 + \frac{k^0 I_3}{k^{\alpha} k_{\alpha}} \right\}, \end{aligned} \tag{3.16}$$

where

$$G = k^{\nu} I_{\nu} + \mathrm{i}\tau_o k^{\nu} \xi_{\nu} \xi^{-1}. \tag{3.17}$$

For the transverse mode, we have

$$\left\{ 1 - \frac{w_p^2}{k^{\alpha} k_{\alpha}} G \right\} + w_p^2 \{ I_1^1 + \mathrm{i}\tau_o J_1^1 \} = 0. \tag{3.18}$$

We will now incorporate the continuity equation into the calculation of the dispersion relations. However, when the density of point particles is transformed to a continuous density, some properties can be lost and we must impose the continuity equation (see appendix C). Then, in (3.6) the continuity equation must be considered as a constraint embedded in the theory; that is,

$$k^{\mu} J_{\mu} = w J^0(k_v) - \boldsymbol{k} \cdot \boldsymbol{J}(k_v) = 0. \tag{3.19}$$

This is equivalent to the procedure used in (2.9) and (2.10). Indeed, if we consider (3.7), (3.13) and (3.19), we have a system of equations

$$\left\{ \begin{array}{c} G^{\beta}_{\lambda} J^{\lambda} = 0 \\ \det G^{\beta}_{\lambda} = 0 \\ k^{\mu} J_{\mu} = w J^0 - k J^3 = 0 \end{array} \right\}, \tag{3.20}$$

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and using (3.13), (3.14) and (3.15)), and noticing the parity of the integrals I_λ, I_λ^β, J_λ^β, we arrive at

$$G_0^1 = G_0^2 = G_1^0 = G_2^0 = G_0^1 = G_2^1 = G_3^1 = G_0^2 = G_1^2 = G_3^2 = G_1^3 = G_2^3 = 0 \quad \text{and} \quad G_1^1 = G_2^2. \tag{3.21}$$

Then,

$$G_\nu^\mu = \begin{pmatrix} G_0^0 & 0 & 0 & G_3^0 \\ 0 & G_1^1 & 0 & 0 \\ 0 & 0 & G_1^1 & 0 \\ G_0^3 & 0 & 0 & G_3^3 \end{pmatrix}. \tag{3.22}$$

Finally, we obtain,

$$[G_1^1]^2[G_0^0 G_3^3 - G_3^0 G_0^3] = 0. \tag{3.23}$$

For the transverse mode, we have

$$G_1^1 G_1^1 = 0, \tag{3.24}$$

and for the longitudinal mode, by using (3.20), we arrive at

$$G_0^0 G_3^3 - G_3^0 G_0^3 = 0, \quad wJ^0 - kJ^3 = 0, \quad \text{and} \quad G_0^0 w + G_3^0 J = 0. \tag{3.25a–c}$$

Therefore, by combining the last equations, we obtain

$$wG_3^0 + kG_0^0 = 0 \quad \text{and} \quad wG_3^3 + kG_0^3 = 0. \tag{3.26a,b}$$

However, these two equations are equivalent as we can see easily. Therefore, we will just use the first of the two equations. In this case, and after a long calculation and using the relations obtained in the appendix of Hakim & Mangeney (1971), we arrive at

$$G_0^0 = \left(1 - w_p^2 \frac{k^\alpha I_\alpha + \mathrm{i}\tau_o \xi^\alpha k_\alpha \xi^{-1}}{k^\alpha k_\alpha}\right) + w_p^2 (I_0^0 - \mathrm{i}\tau_o J_0^0) + w_p^2 k^0 \frac{I_0 + \mathrm{i}\tau_o \xi_0 \xi^{-1}}{k^\alpha k_\alpha}. \tag{3.27}$$

Since we consider the system at rest,

$$\xi_0 = \xi\, \bar{v}_0 = \xi, \tag{3.28}$$

we have

$$M = k^\alpha I_\alpha + \mathrm{i}\tau_o \xi^\alpha k_\alpha \xi^{-1} = k^\alpha I_\alpha + \mathrm{i}\tau_o w. \tag{3.29}$$

Therefore,

$$G_0^0 = \left(1 - \frac{w_p^2}{k^\alpha k_\alpha} M\right) + w_p^2 (I_0^0 - \mathrm{i}\tau_o J_0^0) + w_p^2 k^0 \frac{I_0 + \mathrm{i}\tau_o}{k^\alpha k_\alpha}. \tag{3.30}$$

Noticing that for $\mu = 1, 2, 3$, we have

$$\xi^\mu = \xi\, \bar{v}^\mu = 0 \quad \text{and} \quad I_3^0 = J_3^0 = 0. \tag{3.31a,b}$$

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Then,

$$G_3^0 = w_p^2(I_3^0 - \mathrm{i}\tau_o J_3^0) + w_p^2 k^0 \frac{I_3}{k^\alpha k_\alpha}. \tag{3.32}$$

Therefore equation (3.26) can be written as,

$$k\left(1 - \frac{w_p^2}{k^\alpha k_\alpha}M + w_p^2(I_0^0 - \mathrm{i}\tau_o J_0^0) + w_p^2 k^0 \frac{I_0 + \mathrm{i}\tau_o}{k^\alpha k_\alpha}\right) + w w_p^2\left(I_3^0 - \mathrm{i}\tau_o J_3^0 + k^0 \frac{I_3}{k^\alpha k_\alpha}\right) = 0, \tag{3.33}$$

where

$$\left.\begin{aligned}
I_3^0 &= -\frac{1}{k}\left\{w\frac{\partial I_0}{\partial w} + I_0\right\},\\
I_3 &= \frac{1}{k}\left\{\frac{K_1(\alpha)}{K_2(\alpha)} - wI_0\right\} = \frac{1}{k}\{k^\alpha I_\alpha - wI_0\},\\
I_0^0 &= \frac{\partial I_0}{\partial w},\\
J_3^0 &= -K\frac{\partial}{\partial \alpha}(K^{-1}I_3) = -K\frac{\partial}{\partial \alpha}\left(\frac{4\pi K_2(\alpha)}{\alpha k}\left\{\frac{K_1(\alpha)}{K_2(\alpha)} - wI_0\right\}\right)\\
&= -K\frac{\partial}{\partial \alpha}\left(\frac{1}{Kk}\{k^\mu I_\mu - wI_0\}\right),\\
J_0^0 &= -K\frac{\partial K^{-1}I_0}{\partial \alpha},
\end{aligned}\right\} \tag{3.34}$$

with

$$K = \frac{m\xi}{4\pi K_2(m\xi)} = \frac{\alpha}{4\pi K_2(\alpha)}. \tag{3.35}$$

Finally, the general dispersion relation for the longitudinal mode can be expressed as

$$\begin{aligned}
\frac{k^2 + w_p^2 k^\alpha I_\alpha}{k^\alpha k_\alpha} &= \frac{2w w_p^2 I_o}{k^\alpha k_\alpha} + w_p^2\frac{\partial I_o}{\partial w}\\
&\quad + \mathrm{i}\tau_o\frac{w_p^2}{k^\alpha k_\alpha}\left(w + k^\alpha k_\alpha\frac{\partial I_o}{\partial \alpha} - k^\alpha k_\alpha\left(k^\alpha I_\alpha + \frac{3}{\alpha}\right)I_o\right),
\end{aligned} \tag{3.36}$$

where

$$\alpha = m\xi = \frac{m}{\kappa T_o}, \tag{3.37}$$

with α the so-called inverse temperature parameter. This result differs from Hakim & Mangeney (1971) by an extra factor w/k in the imaginary part which appears in their expression. However, Hakim and Mangeney always calculated the dispersion relations for $w \simeq k$ and the results are equivalent in this neighbourhood. Nevertheless, this implies a difference when the damping due to the radiation reaction force is calculated in the general case. This result permits us to understand some important aspects of the longitudinal modes.

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4. Relativistic dispersion relation for the longitudinal case

4.1. *Real part of the dispersion relation*

Let us restrict our attention to the longitudinal case. First it has to be noted that in general I_0 is a complex function which can be written as:

$$I_0 = \mathrm{Re}\, I_0 + \mathrm{i}\, \mathrm{Im}\, I_0, \tag{4.1}$$

which can be verified by making an inspection of (3.8) and noticing that the term

$$\frac{1}{k^\lambda u_\lambda} = \mathrm{i}\pi \left\{ \delta(k^\lambda u_\lambda) - \frac{\mathrm{i}}{\pi} P.V. \left(\frac{1}{k^\lambda u_\lambda} \right) \right\}. \tag{4.2}$$

This is inherited from the expression for I_0, equation (3.8). Therefore, taking into account that

$$k^\lambda I_\lambda = \frac{K_2(\alpha)}{K_1(\alpha)}, \tag{4.3}$$

which is real, equation (3.36) can be written as:

$$\begin{aligned} \frac{k^2 + w_p^2 k^\alpha I_\alpha}{k^\alpha k_\alpha} &= \frac{2w w_p^2 (\mathrm{Re}\, I_0 + \mathrm{i}\, \mathrm{Im}\, I_0)}{k^\alpha k_\alpha} + w_p^2 \frac{\partial (\mathrm{Re}\, I_0 + \mathrm{i}\, \mathrm{Im}\, I_0)}{\partial w} + \mathrm{i}\tau_o \frac{w_p^2}{k^\alpha k_\alpha} \\ &\times \left(w + k^\alpha k_\alpha \frac{\partial (\mathrm{Re}\, I_0 + \mathrm{i}\, \mathrm{Im}\, I_0)}{\partial \alpha} - k^\alpha k_\alpha \left(k^\alpha I_\alpha + \frac{3}{\alpha} \right) (\mathrm{Re}\, I_0 + \mathrm{i}\, \mathrm{Im}\, I_0) \right). \end{aligned} \tag{4.4}$$

Therefore, we must consider that the frequency as a function of the wavelength is complex. However, we can consider that the complex part is smaller with respect the real part ($w = w_o + \mathrm{i}\gamma$, $\gamma \ll w_o$). Accordingly, let us first calculate the dispersion relation without considering the imaginary parts; that is, to obtain a relation which can be calculated by using a numerical approach without making any approximation but only taking into consideration at first glance that $\mathrm{Im}\, I_o$ is not big enough to change the fundamental relation. Notice that the term multiplied by τ_o is also neglected. Then,

$$\frac{k^2 + w_p^2 k^\alpha I_\alpha}{k^\alpha k_\alpha} = \frac{2w w_p^2 \mathrm{Re}\, I_o}{k^\alpha k_\alpha} + w_p^2 \frac{\partial \mathrm{Re}\, I_o}{\partial w}. \tag{4.5}$$

It has to be noticed that for $\mu > 1$, as will be demonstrated later in the paper, $\mathrm{Im}\, I_o = 0$ and in this region this approximation is exact without taking into account the terms multiplied by τ_o. We have to analyse the function $\mathrm{Re}\, I_o$ and use the expression deduced by Hakim & Mangeney (1968)

$$\alpha = m\xi = \frac{m}{\kappa T_o}, \quad \mathrm{Re}\, \psi = \xi \mathrm{Re}\, I_o = -\frac{\xi \mu \alpha}{k K_2(\alpha)} \frac{\mathrm{d}}{\mathrm{d}\alpha} \left[\frac{1}{\alpha} \int_0^\infty \frac{\exp[-\alpha \cosh \chi]}{(\mu^2 - 1)\cosh^2 \chi + 1} \mathrm{d}\chi \right]. \tag{4.6a,b}$$

From which

$$\begin{aligned} \mathrm{Re}\, I_o &= -\frac{\mu \alpha}{k K_2(\alpha)} \frac{\mathrm{d}}{\mathrm{d}\alpha} \left[\frac{1}{\alpha} \int_0^\infty \frac{\exp[-\alpha \cosh \chi]}{(\mu^2 - 1)\cosh^2 \chi + 1} \mathrm{d}\chi \right] \\ &= \frac{\mu}{k K_2(\alpha)} \int_0^\infty \frac{[\alpha^{-1} + \cosh \chi] \exp[-\alpha \cosh \chi]}{(\mu^2 - 1)\cosh^2 \chi + 1} \mathrm{d}\chi. \end{aligned} \tag{4.7}$$

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It has to be pointed out that this is an improper integral if $\mu < 1$ and in this case it has to be calculated by using its principal value (the imaginary part $\mathrm{Im}\, I_o$ is neglected as we mentioned above and $\mathrm{Im}\, I_o = 0$ for $\mu > 1$),

$$\mathrm{Re}\, I_o = \left\{ \begin{array}{ll} \dfrac{\mu}{kK_2(\alpha)} \displaystyle\int_0^\infty \frac{[\alpha^{-1} + \cosh\chi]\exp[-\alpha\cosh\chi]}{(\mu^2-1)\cosh^2\chi + 1}\,\mathrm{d}\chi & \text{for } \mu \geqslant 1 \\ P.V\dfrac{\mu}{kK_2(\alpha)} \displaystyle\int_0^\infty \frac{[\alpha^{-1} + \cosh\chi]\exp[-\alpha\cosh\chi]}{(\mu^2-1)\cosh^2\chi + 1}\,\mathrm{d}\chi & \text{for } \mu < 1 \end{array} \right\}. \quad (4.8)$$

A fundamental difference compared to the work done by Hakim & Mangeney (1971) can be pointed out. Let us analyse the second line, that is for $\mu < 1$

$$\mathrm{Re}\, I_o = \frac{\mu}{kK_2(\alpha)\alpha} \left\{ \begin{array}{l} \displaystyle\lim_{\varepsilon\to 0} \int_0^{\chi_c-\varepsilon} \frac{[\alpha^{-1} + \cosh\chi]\exp[-\alpha\cosh\chi]}{1-\sigma^2\cosh^2\chi}\,\mathrm{d}\chi \\ + \displaystyle\lim_{\varepsilon\to 0} \int_{\chi_c+\varepsilon}^\infty \frac{[\alpha^{-1} + \cosh\chi]\exp[-\alpha\cosh\chi]}{1-\sigma^2\cosh^2\chi}\,\mathrm{d}\chi \end{array} \right\} \quad \text{for } \mu < 1, \quad (4.9)$$

where

$$\sigma^2 = (1-\mu^2) < 1 \quad \text{and} \quad \chi_c = \mathrm{arccosh}\sqrt{\frac{1}{(1-\mu^2)}}. \quad (4.10a,b)$$

Equation (4.9) can be expressed as

$$\mathrm{Re}\, I_o = \frac{\mu}{kK_2(\alpha)\alpha}[H_1 + H_2], \quad \mu < 1, \quad (4.11)$$

where H_1 and H_2 represent the first and second integrals in (4.9), respectively. Now, in H_1, we can make an expansion around $y = 0$, because

$$\chi_c = \mathrm{arccosh}\sqrt{\frac{1}{(1-\mu^2)}} > \chi \quad \Rightarrow \quad y = \sigma^2\cosh^2\chi = (1-\mu^2)\cosh^2\chi < 1. \quad (4.12a,b)$$

Then, we can use

$$\frac{1}{1-y} = \sum_{j=0}^{\infty} y^j, \quad (4.13)$$

to obtain

$$H_1 = \lim_{\varepsilon\to 0} \int_0^{\chi_c-\varepsilon} \sum_{j=0}^{\infty} (1-\mu^2)^j \cosh^{2j}\chi\,[\alpha^{-1} + \cosh\chi]\exp[-\alpha\cosh\chi]\,\mathrm{d}\chi. \quad (4.14)$$

For H_2, we have

$$\chi_c = \mathrm{arccosh}\sqrt{\frac{1}{(1-\mu^2)}} < \chi \quad \Rightarrow \quad y = \sigma^2\cosh^2\chi = (1-\mu^2)\cosh^2\chi > 1. \quad (4.15a,b)$$

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Then, we can use

$$\begin{aligned}\frac{1}{1-\sigma^2\cosh^2\chi} &= -\frac{1}{\sigma^2\cosh^2\chi-1}\\ &= -\frac{1}{\sigma^2\cosh^2\chi\left(1-\dfrac{1}{\sigma^2\cosh^2\chi}\right)}\\ &= -\frac{1}{\sigma^2\cosh^2\chi}\left(\frac{1}{1-\dfrac{1}{\sigma^2\cosh^2\chi}}\right),\end{aligned} \tag{4.16}$$

and

$$-\frac{1}{y\left(1-\dfrac{1}{y}\right)} = -\frac{1}{y}\sum_{j=0}^{\infty}\left(\frac{1}{y}\right)^j. \tag{4.17}$$

Therefore, we express H_2 as

$$\begin{aligned}H_2 &= -\lim_{\varepsilon\to 0}\int_{\chi_c-\varepsilon}^{\infty}\frac{1}{(1-\mu^2)\cosh^2\chi}\sum_{j=0}^{\infty}\left(\frac{1}{(1-\mu^2)\cosh^2\chi}\right)^j\\ &\quad\times[\alpha^{-1}+\cosh\chi]\exp[-\alpha\cosh\chi]\,\mathrm{d}\chi\\ &= -\lim_{\varepsilon\to 0}\int_{\chi_c-\varepsilon}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{(1-\mu^2)\cosh^2\chi}\right)^j[\alpha^{-1}+\cosh\chi]\exp[-\alpha\cosh\chi].\end{aligned} \tag{4.18}$$

There is no way to identify this term with Bessel functions as Hakim and Mangeney did (Hakim & Mangeney 1971). It has to be noted that the summatory cannot be put out of the integral unless the series converges uniformly and this depends on the value of μ. Given the complexity of (4.14) and (4.18) that include integrals of infinite series, and our inability to find an analytical result, we must use numerical calculation. In this case we need to cut off the series at some point and to evaluate the error induced by using a finite series. However, this procedure does not provide additional value compared to the numerical calculation of the whole integral in (4.8). For the evaluation of these integrals we used NIntegrate built-in symbol of Wolfram Mathematic, and particularly, for $\mu < 1$ we used the 'Principal Value' method of NIntegrate. Since we are interested in high temperatures, let us present graphically the results for different values of α. In figures 1, 2, 3, 4 and 13, we present such dispersion relations for $\alpha = 100$, 10, 1, 0.5 and the extreme case of the Hagedorn temperature $\alpha = 0.0059$ respectively.

In figure 1, for $\alpha = 100$, which is very close to a tokamak temperature ($\alpha \simeq 59$), we can notice that the dispersion relation gives a loop which represents one of the important results of the paper. The first part of the wave, $0 < k/w_p < 1$, is equal to the classical result $w \simeq w_p$. However, due to the correct expansions done in equations, (4.8), (4.9), (4.10), (4.14) and (4.18), after crossing the super-luminous region, the turning point appears and the loop is obtained contrary to other works. The main point is that, due to the existence of the turning point, it is not possible to find waves

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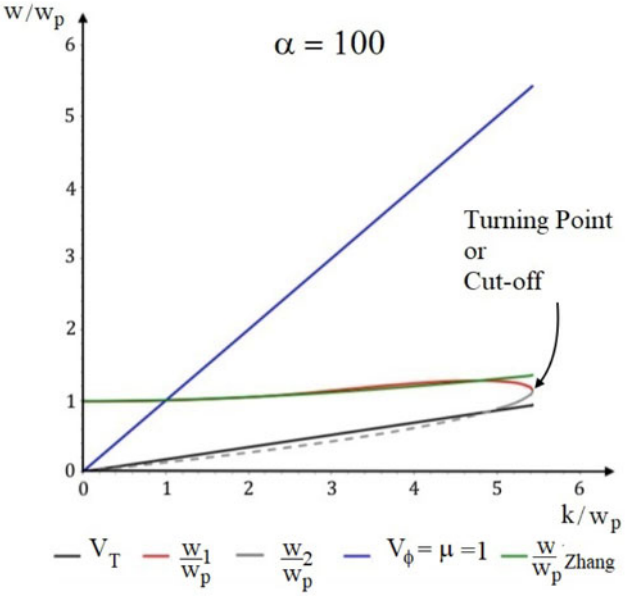


FIGURE 1. Value of w/w_p plotted against k/w_p (red for the first branch w_1/w_p and grey line for w_2/w_p), Zhang curve (green line for w_{Zhang}/w_p), $\mu = 1$ (blue line) and V_{Te} (black line). For $\alpha = 100$, the Zhang curve coincides with the first branch w_1/w_p until the cutoff. It has to be highlighted that the second branch w_2/w_p crosses the line which represents the thermal velocity (V_T) and consequently below this point the second branch is represented by a grey dashed line. In this region the dispersion relation is not valid anymore.

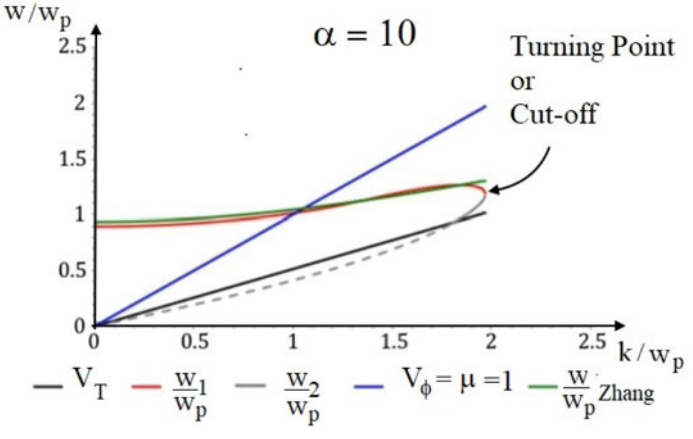


FIGURE 2. Same annotations than in figure 1. For $\alpha = 10$, Zhang curve is close to the first branch w_1/w_p until the cut-off. It has to be highlighted that the second branch w_2/w_p crosses the line which represents the thermal velocity (V_T) and consequently below this point the second branch is represented by a grey dashed line. In this region the dispersion relation is not anymore valid.

above a critical wavenumber and this is described by two branches: the first one is the red line in the figure 1 and it finishes at the turning point (or cutoff) where the second one begins which is represented by the grey line. It has to be highlighted

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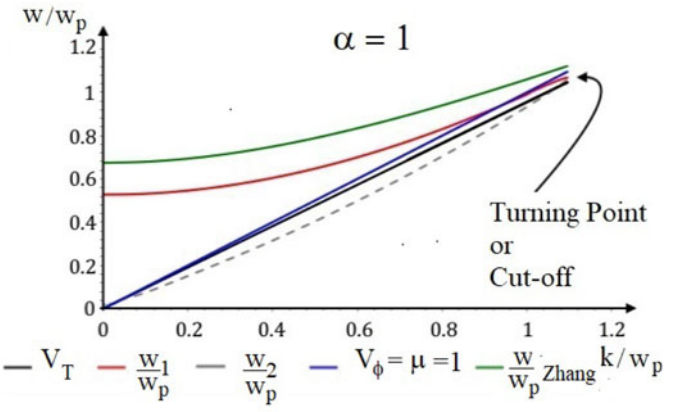


FIGURE 3. Same annotations as figure 1. For $\alpha = 1$, the Zhang curve differs from the first branch w_1/w_p. It has to be highlighted that the second branch w_2/w_p crosses the line which represents the thermal velocity (V_T) close to the turning point (cutoff) and therefore, just a very small part of the second branch w_2/w_p subsists. The other part of the second branch is represented by a grey dashed line. In this region the dispersion relation is not anymore valid.

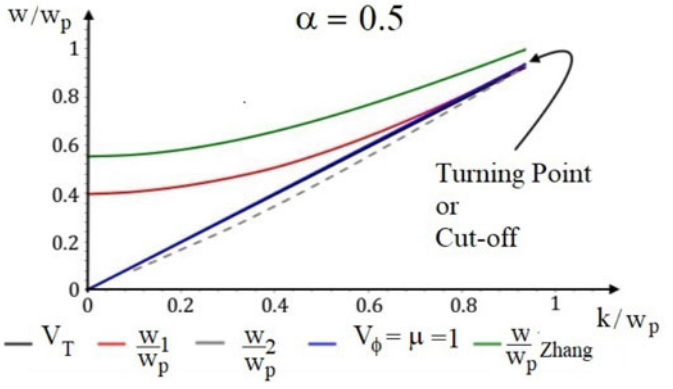


FIGURE 4. Same annotations as figure 1. For $\alpha = 0.5$, the Zhang curve differs from the first branch w_1/w_p. It has to be highlighted that the second branch w_2/w_p crosses the line which represents the thermal velocity (V_T) close to the turning point (cutoff) and therefore, just a very small part of the second branch w_2/w_p subsists. The other part of the second branch is represented by a grey dashed line. In this region the dispersion relation is not anymore valid.

that the Zhang dispersion relation (Zhang *et al.* 2013) described by the green line coincides with the red line until the effects of the real dispersion relations, which include the effects of the magnetic field and the 3- dimension case, are present. The Zhang dispersion relation is considered for a maximum wavenumber $k/w_p = 3$, arguing that it is sufficient for typical relativistic plasma oscillations and not because a cutoff is predicted as happens in our approach. It has to be noticed that part of the two branches belongs to the super-thermal wave region ($V_{Te} < \mu < 1$). It should also be noted that the black line, which represents the identity between the phase velocity and

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the thermal velocity, indicates that below ($\mu < V_{Te}$) this line the dispersion relation is not valid anymore and consequently the grey line, which corresponds to the second branch of the loop, is dashed when it crosses the boundary of the thermal velocity. Indeed, for $\mu < V_{Te}$ or $\mu < V_{Ti}$ different physical arguments have to be included in the dispersion relation, obtaining distinct types of waves. It is also important that the thermal velocity has to be expressed by using the Synge relation and not the regular non-relativistic one; that is

$$V_{Te}^2 = 1 - \left[\frac{3}{\alpha} + \mathrm{Re}\, G(\alpha)\right]^{-2}, \tag{4.19}$$

where

$$G = k^\nu I_\nu + \mathrm{i}\tau_o k^\nu \xi_\nu \xi^{-1} \quad \Rightarrow \quad \mathrm{Re}\, G(\alpha) = k^\nu I_\nu = \frac{K_2(\alpha)}{K_1(\alpha)}. \tag{4.20a,b}$$

Notice that when α tends to infinity,

$$V_{Te}^2 = \frac{3}{\alpha} = \frac{3\kappa T}{m}, \tag{4.21}$$

which corresponds to the non-relativistic case. Finally, it has to be pointed out that the wave has a cutoff (or turning point) at $k_{\max} \simeq 5.4 w_p$.

In figure 2, for $\alpha = 10$, the loop still exists but a difference with the Zhang result begins to appear. The wave has a cutoff at $k_{\max} \simeq 1.9 w_p$. Moreover, there is a shift of the plasma frequency when the wavenumber tends to zero; that is

$$w(k \to 0) \neq w_p \quad \text{and} \quad w(k \to 0) = \chi w_p. \tag{4.22a,b}$$

Zhang obtained a similar result

$$w(k \to 0) \neq w_p \quad \text{and} \quad w(k \to 0) = \chi_z w_p, \tag{4.23a,b}$$

but

$$1 > \chi_z > \chi. \tag{4.24}$$

This effect is increased when α drops (temperature grows) as we can see for $\alpha = 1$, $\alpha = 0.5$ and the extreme case of the Hagedorn temperature ($\alpha = 0.0059$). We call this effect the frequency plasma shift.

In figure 3, for $\alpha = 1$, the shift effect is more important and the difference from the Zhang result is increased. However, since the thermal velocity line ($\mu = V_{Te}$) approaches the line of the plane wave ($\mu = 1$), the second branch exists just in a small neighbourhood around the turning point. This effect is more important for decreasing α, as can be seen in the cases for $\alpha = 0.5$ and the Hagedorn temperature. The wave has a cutoff at $k_{\max} \simeq 1.1 w_p$.

In figure 4, for $\alpha = 0.5$, as is expected due to the relativistic effect, the difference from the Zhang result is accentuated. However, the second branch disappears completely due to coincidence between the thermal velocity line and the supra-luminous edge wave line ($V_T \simeq \mu = 1$). The wave has a cutoff at $k_{\max} \simeq 0.95 w_p$.

In figure 13, for $\alpha = 0.0059$, the Hagedorn temperature, which represents the boundary where hadronic matter is no longer stable and which is the limit of

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the theory, the second branch cannot be considered due to the thermal velocity line equalling the super-luminous region wave line. However the plasma frequency continues to decrease and the cutoff wavenumber also decreases; that is

$$w(k \to 0) \simeq 0.044 \quad \text{and} \quad k_{\max} \simeq 0.1 w_p. \tag{4.25a,b}$$

Although for $w > k$ and for not very low α our results coincide with Hakim & Mangeney (1968, 1971) and Zhang *et al.* (2013), for $w < k$ the existence of the loop was not predicted by Hakim and Mangeney, neither by Zhang. In the case of Hakim and Mangeney the differences came from the expansion done in an incorrect form and for Zhang due to the 1-dimension character of his model which cannot consider the effects of the magnetic field generated by high temperatures. Moreover, as the temperature grows, the turning or cutoff point tends to the point (0, 0) and the wave does not exist. Although, the Hagedorn temperature represents the lower limit for α, it has to be noticed that when dealing with high temperatures care must be taken since the density must be such that there is no degeneracy. This point is considered later in the article when the RRFD is analysed.

4.2. *Landau damping*

Let us now consider the imaginary parts of (3.36) or (4.4). First, we will not take into account the terms containing τ_o. We need now to consider that

$$w = w_o + \mathrm{i}\gamma \quad k^\alpha k_\alpha = (w_o^2 + 2\mathrm{i}\gamma w - k^2), \tag{4.26a,b}$$

where w_o and γ are real. It can be supposed that $w_o \gg \gamma$. Therefore, by neglecting terms of second power in γ ($\gamma^2 \simeq 0$), and making an expansion in $i\gamma$, we have

$$\left.\begin{aligned} \operatorname{Re} I_0(w_o + \mathrm{i}\gamma) &= [\operatorname{Re} I_0]_{w=w_o} + \mathrm{i}\gamma \left[\frac{\partial \operatorname{Re} I_0}{\partial w_o}\right]_{w=w_o} \quad \text{and} \\ \operatorname{Im} I_0(w_o + \mathrm{i}\gamma) &= [\operatorname{Im} I_0]_{w=w_o} + \mathrm{i}\gamma \left[\frac{\partial \operatorname{Im} I_0}{\partial w_o}\right]_{w=w_o}. \end{aligned}\right\} \tag{4.27}$$

In the following we will drop the parenthesis notation; that is for example $[\operatorname{Re} I_0]_{w=w_o} = \operatorname{Re} I_0$, etc. We arrive at

$$\begin{aligned} k^2 + w_p^2 k^\alpha I_\alpha \;=\; & 2w_o w_p^2 \left[\left(\operatorname{Re} I_0 + \mathrm{i}\gamma \frac{\partial \operatorname{Re} I_0}{\partial w_o}\right) + \mathrm{i}\left[\operatorname{Im} I_0 + \mathrm{i}\gamma \frac{\partial \operatorname{Im} I_0}{\partial w_o}\right]\right] + w_p^2 k^\alpha k_\alpha \\ & \times \left(\left(\frac{\partial \operatorname{Re} I_o}{\partial w_o} + \mathrm{i}\gamma \frac{\partial^2 \operatorname{Re} I_0}{\partial w_o^2}\right) + \mathrm{i}\left(\frac{\partial \operatorname{Im} I_0}{\partial w_o} + \mathrm{i}\gamma \frac{\partial^2 \operatorname{Im} I_0}{\partial w_o^2}\right)\right). \end{aligned} \tag{4.28}$$

Therefore, we have

$$\begin{aligned} k^2 + w_p^2 k^\alpha I_\alpha \;=\; & 2(w_o + \mathrm{i}\gamma) w_p^2 \left[\operatorname{Re} I_0 - \gamma \frac{\partial \operatorname{Im} I_0}{\partial w_o} + \mathrm{i}\left[\operatorname{Im} I_0 + \gamma \frac{\partial \operatorname{Re} I_0}{\partial w_o}\right]\right] \\ & + w_p^2 [w_o^2 + 2\mathrm{i}\gamma w_o - k^2] \\ & \times \left(\frac{\partial \operatorname{Re} I_0}{\partial w_o} - \gamma \frac{\partial^2 \operatorname{Im} I_0}{\partial w_o^2} + \mathrm{i}\left[\frac{\partial \operatorname{Im} I_0}{\partial w_o} + \gamma \frac{\partial^2 \operatorname{Re} I_0}{\partial w_o^2}\right]\right). \end{aligned} \tag{4.29}$$

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The real part of this equation corrects the dispersion relation and it is:

$$k^2 + w_p^2 k^\alpha I_\alpha = 2w_p^2 \left(w_o \mathrm{Re}\, I_0 - \gamma w_o \frac{\partial \mathrm{Im}\, I_0}{\partial w_o} - \gamma \mathrm{Im}\, I_0 \right) + w_p^2 [w_o^2 - k^2] \left[\frac{\partial \mathrm{Re}\, I_0}{\partial w_o} - \gamma \frac{\partial^2 \mathrm{Im}\, I_0}{\partial w_o^2} \right] + w_p^2 \left[-2w_o \gamma \frac{\partial \mathrm{Im}\, I_0}{\partial w_o} \right], \quad (4.30)$$

which can be solved numerically by firstly analysing the behaviour of both $\mathrm{Re}\, I_o$ and $\mathrm{Im}\, I_0$, and knowing the value of γ. However, since $\gamma \ll w_o$ the differences between (4.30) and (4.5) are negligible. This fact can be shown by solving (4.30) for different values of α and comparing the results with those already obtained in § 4.1. However, for the imaginary part, the result is important as it leads to a damping.

For the imaginary part, we have

$$0 = 2 \left(w_o \left[\mathrm{Im}\, I_0 + \gamma \frac{\partial \mathrm{Re}\, I_0}{\partial w_o} \right] + \gamma \mathrm{Re}\, I_0 \right) + [w_o^2 - k^2] \left[\frac{\partial \mathrm{Im}\, I_0}{\partial w_o} + \gamma \frac{\partial^2 \mathrm{Re}\, I_0}{\partial w_o^2} \right] + 2\gamma w_o \frac{\partial (\mathrm{Re}\, I_0)}{\partial w_o}. \quad (4.31)$$

We obtain

$$2w_o \mathrm{Im}\, I_0 + [w_o^2 - k^2] \frac{\partial \mathrm{Im}\, I_0}{\partial w_o} = -2w_o \gamma \frac{\partial \mathrm{Re}\, I_0}{\partial w_o} - 2\gamma \mathrm{Re}\, I_0 - [w_o^2 - k^2] \gamma \frac{\partial^2 \mathrm{Re}\, I_0}{\partial w_o^2} - 2\gamma w_o \frac{\partial (\mathrm{Re}\, I_0)}{\partial w_o}. \quad (4.32)$$

Finally, The LD can be represented by the Landau factor γ_L,

$$\gamma_L = - \frac{2w_o \mathrm{Im}\, I_0 + [w_o^2 - k^2] \frac{\partial \mathrm{Im}\, I_0}{\partial w_o}}{2 \left[2w_o \frac{\partial \mathrm{Re}\, I_0}{\partial w_o} + \mathrm{Re}\, I_0 + \frac{1}{2} [w_o^2 - k^2] \frac{\partial^2 \mathrm{Re}\, I_0}{\partial w_o^2} \right]}. \quad (4.33)$$

Before continuing, we need to analyse the value of $\mathrm{Im}\, I_o$ (we have already analysed $\mathrm{Re}\, I_o$ and see Hakim & Mangeney (1968, 1971) for $\mathrm{Im}\, I_o$),

$$\mathrm{Im}\, I_o = \left\{ \begin{array}{cc} 0 & \text{for } \mu > 1 \\ \dfrac{\pi}{2kK_2(\alpha)} \exp \left\{ \dfrac{-\alpha}{(1-\mu^2)^{1/2}} \right\} \left[\dfrac{1}{\alpha} + \dfrac{1}{(1-\mu^2)^{1/2}} \right] & \text{for } \mu < 1 \end{array} \right\}. \quad (4.34)$$

Therefore,

$$\gamma_L = \left\{ \begin{array}{cc} 0 & \text{for } \mu > 1 \\ - \dfrac{2w_o \mathrm{Im}\, I_0 + [w_o^2 - k^2] \frac{\partial \mathrm{Im}\, I_0}{\partial w_o}}{2 \left[2w_o \frac{\partial \mathrm{Re}\, I_0}{\partial w_o} + \mathrm{Re}\, I_0 + \frac{1}{2} [w_o^2 - k^2] \frac{\partial^2 \mathrm{Re}\, I_0}{\partial w_o^2} \right]} & \text{for } \mu < 1 \end{array} \right\}. \quad (4.35)$$

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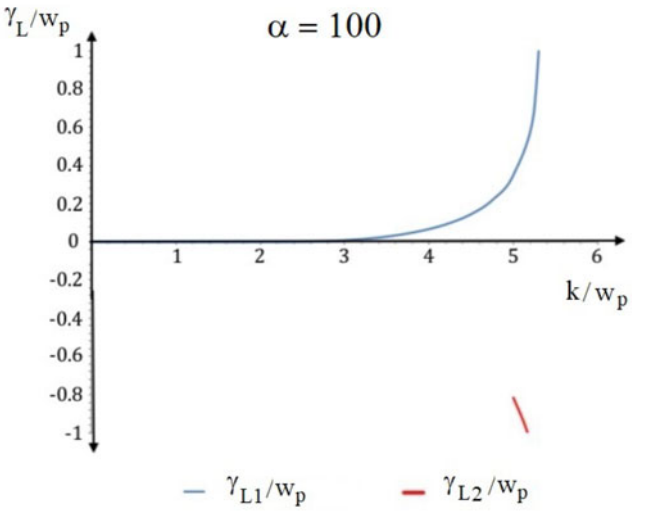


FIGURE 5. For $\alpha = 100$, the blue line represents the Landau damping in the first branch (γ_L/w_p). For $k < 2.5w_p$, $\gamma_L/w_p \simeq 0$ and for $k > 2.5w_p$, $\gamma_L/w_p > 0$. The red line represents the Landau damping in the second branch (γ_L/w_p). For $k > 5w_p$, $\gamma_L/w_p < 0$.

This means that Landau damping does not exist for $\mu > 1$; that is: there is no LD in the super-luminous region (see figures 5, 7, 9 and 11).

To understand the effect of the LD, it is necessary to compare the Landau decay time T_d with the period $T = 2\pi/w_o$ obtained from (4.5). This is

$$T = \frac{2\pi}{w_o} \quad \text{and} \quad T_d = \frac{1}{\gamma}, \tag{4.36a,b}$$

where the wave is damped in the cases where the ratio

$$R_L = \frac{T_d}{T} < 1. \tag{4.37}$$

Do not confuse R_L with the logarithmic damping decrement used in astrophysics, it is related to the inverse of it.

Let us describe the properties of the waves for the different values of R_L. It has to be noted that the definition of the Fourier transform implies that a positive γ corresponds to damped oscillations while a negative value corresponds to unstable oscillations (resonance). This means that for:

$R_L > 1$ the wave is practically not damped due to the LD and we will call it a stable wave;

$0 < R_L < 1$ the wave is severely damped and it decays in a time less than its period and we will call it a damped wave;

$0 > R_L > -1$ the wave suffers a resonance and it is unstable and we will call it an unstable wave (below in the paper it will be called a bipolar wave due to the change of local temperature generated by itself).

4.2.1. *Analysis of the stability of the first branch*

Let us first analyse the first branches of the loops for different temperatures.

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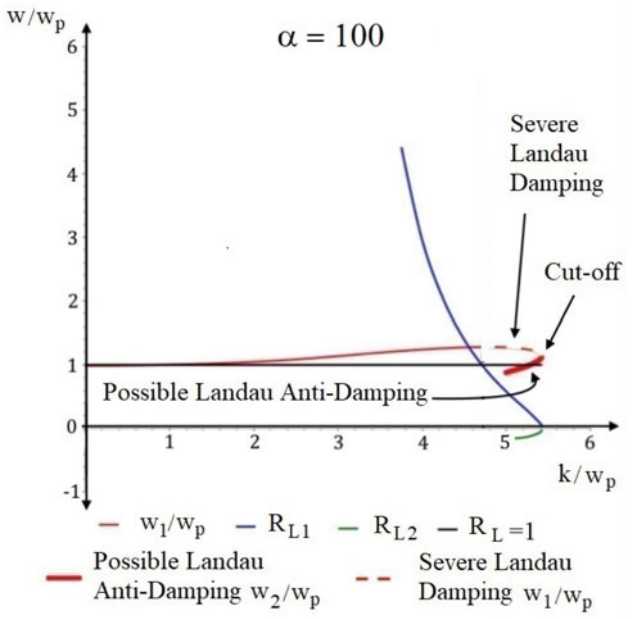


FIGURE 6. For $\alpha = 100$, the red line represents a stable first branch since $R_{L1} > 1$ when $k < 4.7w_p$. The dashed red line represents a damped first branch (severe Landau damping) since $0 < R_{L1} < 1$ when $k > 4.7w_p$. The thick red line represents an unstable second branch (possible anti-damping) since $0 > R_{L2} > -1$ when $k > 5w_p$.

The LD ratio R_L is represented in figures 6, 8, 10 and 12 by the blue lines and the wave is severely damped when (4.37) is satisfied. Therefore, below the horizontal black line ($R_L = 1$) the wave is damped and for this reason it is dashed (see figures 6, 8, 10 and 12). For each value of α, we have:

(i) $\alpha = 100$ and $k < 4.7w_p$, $R_L > 1$ and the wave is stable;

$\alpha = 100$ and $k > 4.7w_p$, $0 < R_L < 1$ and the wave is damped;

(ii) $\alpha = 10$ and $k < 1.8w_p$, $R_L > 1$ and the wave is stable;

$\alpha = 10$ and $k > 1.8w_p$, $0 < R_L < 1$ and the wave is damped;

(iii) $\alpha = 1$ and $k > 1.04w_p$, $R_L > 1$; and the wave is stable;

$\alpha = 1$ and $k > 1.04w_p$, $0 < R_L < 1$ and the wave is damped but just in a very small interval $1.05w_p > k > k_T$, where k_T is the wavenumber corresponding to the turning point;

(iv) $\alpha = 0.5$ and $k > 0.9w_p$, $R_L > 1$; and the wave is stable;

$\alpha = 0.5$ and $k > 0.9w_p$, $0 < R_L < 1$ and the wave is damped but in a very small interval $0.9w_p > k > k_T$, where k_T is the wavenumber corresponding to the turning point.

For some values of k, the γ values are close to w_o which contradicts our supposition in (4.26). However, the idea consists of describing the tendency of the damping.

4.2.2. *Analysis of the stability of the second branch*

It has to be noted that the second branch is not completely present in the dispersion relation curve for values of $\alpha = 100$ and $\alpha = 10$ (see figures 6 and 8). Indeed, the second branches cross the thermal velocity line ($\mu = V_\phi = V_{Te}$) and from this point the dispersion relation is not valid. For each value of α, we have

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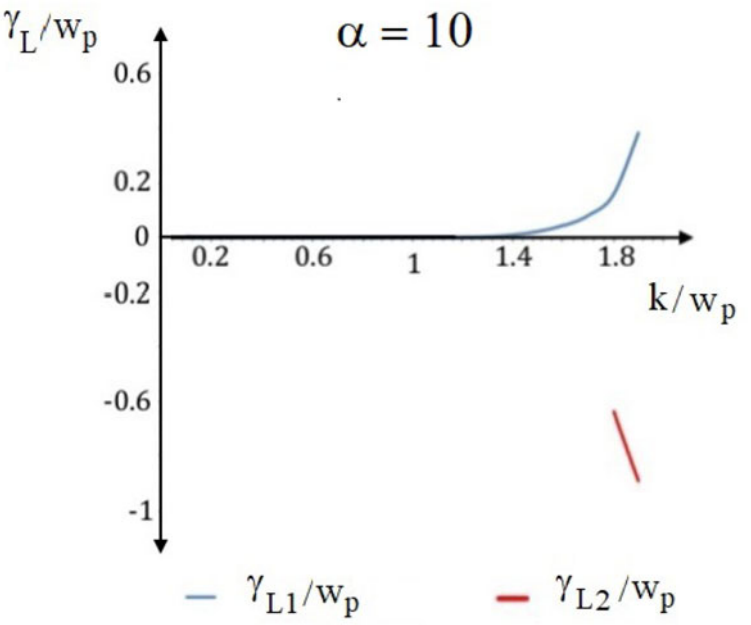


FIGURE 7. For $\alpha = 10$, the blue line represents the Landau damping in the first branch (γ_L/w_p). For $k < 1.25w_p$, $\gamma_L/w_p \simeq 0$ and for $k > 1.25w_p$, $\gamma_L/w_p > 0$. The red line represents the Landau damping in the second branch (γ_L/w_p). For $k > 1.8w_p$, $\gamma_L/w_p < 0$.

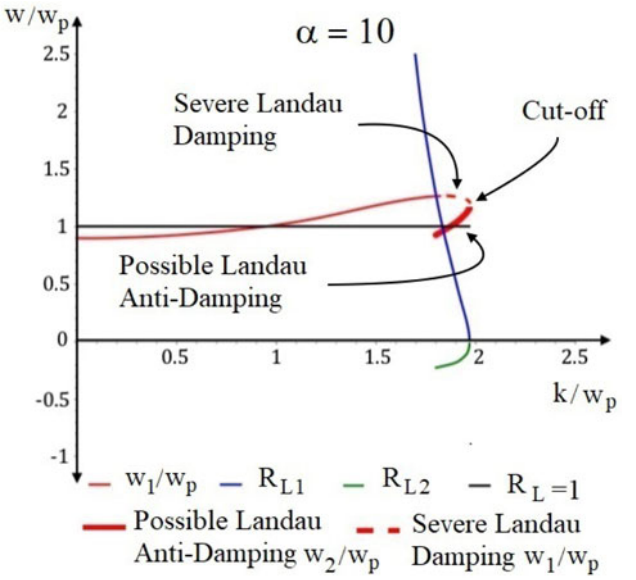


FIGURE 8. For $\alpha = 10$, the red line represents a stable first branch since $R_{L1} > 1$ when $k < 1.8w_p$. The dashed red line represents a damped first branch (severe Landau damping) since $0 < R_{L1} < 1$ when $k > 1.8w_p$. The thick red line represents an unstable second branch (possible anti-damping) since $0 > R_{L2} > -1$ when $k > 1.8w_p$.

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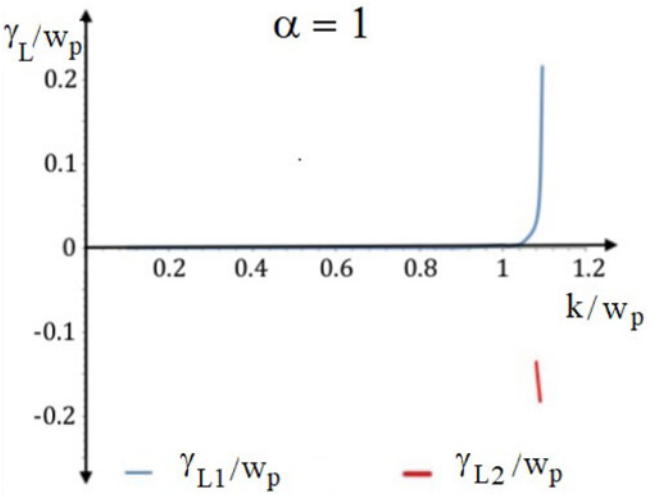


FIGURE 9. For $\alpha = 1$, the blue line represents the Landau damping in the first branch (γ_L/w_p). For $k < 1.04w_p$, $\gamma_L/w_p \simeq 0$ and for $k > 1.04w_p$, $\gamma_L/w_p > 0$. The red line represents the Landau damping in the second branch (γ_{L2}/w_p). For $k > 1.04w_p$, $\gamma_L/w_p < 0$.

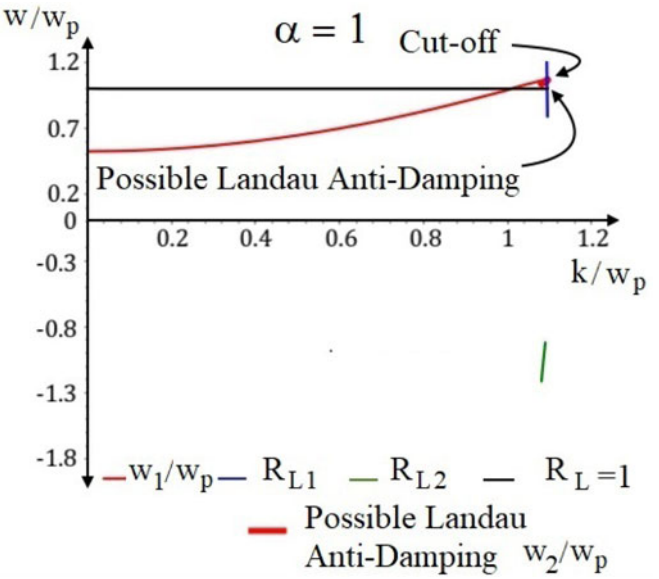


FIGURE 10. For $\alpha = 1$, the red line represents a stable first branch since $R_{L1} > 1$ when $k < 1.04w_p$. There is not a dashed red line representing a damped wave since $0 < R_{L1} < 1$ practically coincides with the turning point at $k \simeq 1.04w_p$. The thick red line represents an unstable second branch, $0 > R_{L2} > -1$ (possible anti-damping), but it exists in a small neighbourhood around $k \simeq 1.04w_p$.

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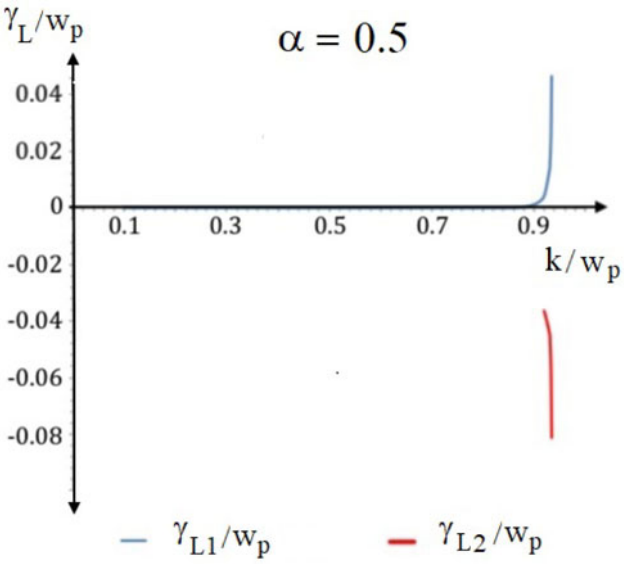


FIGURE 11. For $\alpha = 0.5$, the blue line represents the Landau damping in the first branch (γ_L/w_p). For $k < 0.9w_p$, $\gamma_L/w_p \simeq 0$ and for $k > 0.9w_p$, $\gamma_L/w_p > 0$. The red line represents the Landau damping in the second branch (γ_L/w_p). For $k > 0.9w_p$, $\gamma_L/w_p < 0$.

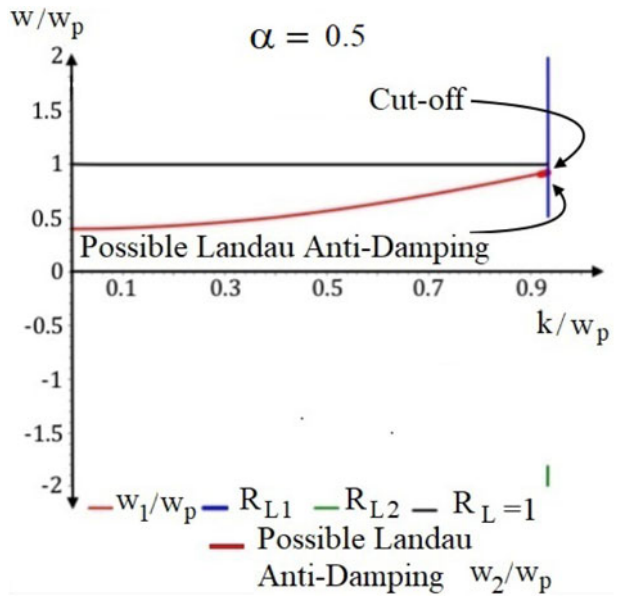


FIGURE 12. For $\alpha = 0.5$, the red line represents a stable first branch since $R_{L1} > 1$ when $k < 0.9w_p$. There is not a dashed red line representing a damped wave since $0 < R_{L1} < 1$ practically coincides with the turning point at $k \simeq 0.9w_p$. The thick red line represents an unstable second branch, $0 > R_{L2} > -1$ (possible anti-damping), but it exists in a small neighbourhood around $k \simeq 0.9w_p$.

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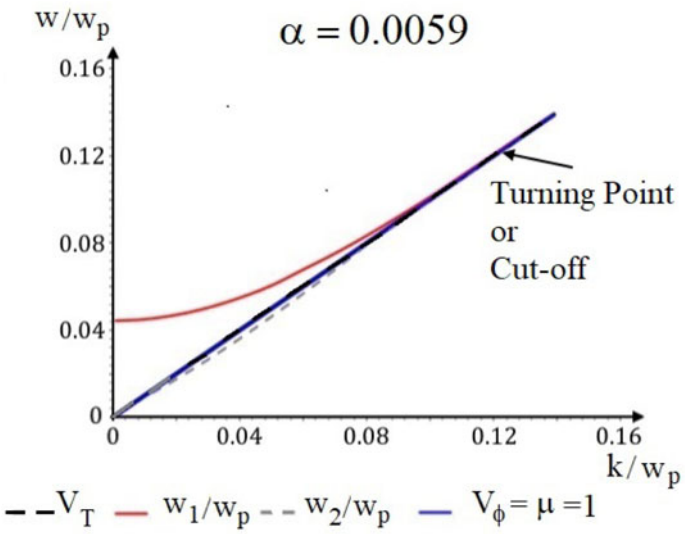


FIGURE 13. The wave at the Hagedorn temperature $\alpha = 0.0059$ and $w_p \simeq 5.085 \times 10^{22}$ rad sec^{-1}. The red line represents the first branch w_1/w_p. The second branch does not exist since it is below the line $\mu = V_T$. The turning point corresponds to a wavenumber $k = 0.12 w_p$.

(i) $\alpha = 100$ and $5w_p > k > k_T$, where k_T is the wavenumber corresponding to the turning point, $0 > R_L > -1$ and the wave is unstable (possible anti-damping);

(ii) $\alpha = 10$ and $1.8w_p > k > k_T$, where k_T is the wavenumber corresponding to the turning point, $0 > R_L > -1$ and the wave is unstable (possible anti-damping);

(iii) $\alpha = 1$, the second branch exists in a small neighbourhood around $k \simeq 1.04 w_p$ where $0 > R_L > -1$ and the wave is unstable (possible anti-damping, see figure 10);

(iv) $\alpha = 0.5$, the second branch exists in a small neighbourhood around $k \simeq 0.9 w_p$ where $0 > R_L > -1$ and the wave is unstable (possible anti-damping, see figure 12).

The interesting point consists of noticing that in the second branch there is a possible anti-damping. However, the existence of this apparent resonance can be objected to for the following reasons: when a perturbation is made and a resonance may appear, the motion of the particles increases the temperature slightly. Therefore, we can consider that the wave exists in the plasma with a new α which has decayed a little. Finally, the wavenumber at this increased temperature may give a frequency corresponding to the first branch which is damped by the LD effect in this region or disappears due to the cutoff. That is, the wave is alternatively anti-damped and damped or disappears, and a pulse can be detected. Therefore, those waves can be called bipolar waves.

4.3. *Radiation reaction force damping*

In order to calculate the RRFD, let us now consider the term which only contains τ_o multiplied by $k^\alpha k_\alpha$ in order to calculate the imaginary part due to the radiation reaction force; that is,

$$\Gamma = \mathrm{i}\tau_o w_p^2 \left(w + k^\alpha k_\alpha \frac{\partial (\mathrm{Re}\, I_0 + \mathrm{i}\, \mathrm{Im}\, I_0)}{\partial \alpha} - k^\alpha k_\alpha \left(k^\alpha I_\alpha + \frac{3}{\alpha} \right) (\mathrm{Re}\, I_0 + \mathrm{i}\, \mathrm{Im}\, I_0) \right). \quad (4.38)$$

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By using (4.26) and (4.27), we have

$$\Gamma = \mathrm{i}\tau_o w_p^2 \begin{pmatrix} w_o + \mathrm{i}\gamma + (w_o^2 + 2\mathrm{i}\gamma w_o - k^2) \\ \left(\dfrac{\partial \mathrm{Re}\, I_0}{\partial \alpha} + \mathrm{i}\gamma \dfrac{\partial^2 \mathrm{Re}\, I_0}{\partial \alpha \partial w} + \mathrm{i}\left(\dfrac{\partial \mathrm{Im}\, I_0}{\partial \alpha} + \mathrm{i}\gamma \dfrac{\partial^2 \mathrm{Im}\, I_0}{\partial \alpha \partial w}\right)\right) \\ -(w_o^2 + 2\mathrm{i}\gamma w_o - k^2)\left(k^\alpha I_\alpha + \dfrac{3}{\alpha}\right) \\ \left(\mathrm{Re}\, I_0 + \mathrm{i}\gamma \dfrac{\partial \mathrm{Re}\, I_0}{\partial w_o} + \mathrm{i}\left[\mathrm{Im}\, I_0 + \mathrm{i}\gamma \dfrac{\partial \mathrm{Im}\, I_0}{\partial w_o}\right]\right) \end{pmatrix}. \tag{4.39}$$

The real part generated by this term can be neglected due to the value of τ_o. Therefore, let us put our attention just in the imaginary part.

$$\mathrm{i}\,\mathrm{Im}\,\Gamma = \mathrm{i}\tau_o w_p^2 \begin{bmatrix} w_o + (w_o^2 - k^2)\left(\dfrac{\partial \mathrm{Re}\, I_0}{\partial \alpha} - \gamma \dfrac{\partial^2 \mathrm{Im}\, I_0}{\partial \alpha \partial w}\right) \\ -2\gamma w_o \left(\dfrac{\partial \mathrm{Im}\, I_0}{\partial \alpha}\right) \\ -(w_o^2 - k^2)\left(k^\alpha I_\alpha + \dfrac{3}{\alpha}\right)\left(\mathrm{Re}\, I_0 - \gamma \dfrac{\partial \mathrm{Im}\, I_0}{\partial w_o}\right) \\ -2\gamma w_o \left(k^\alpha I_\alpha + \dfrac{3}{\alpha}\right) \mathrm{Im}\, I_0 \end{bmatrix}. \tag{4.40}$$

This must equal the denominator of the Landau damping, equation (4.33); that is

$$\begin{aligned} &\tau_o w_p^2 \begin{bmatrix} w_o + (w_o^2 - k^2)\left(\dfrac{\partial \mathrm{Re}\, I_0}{\partial \alpha} - \gamma \dfrac{\partial^2 \mathrm{Im}\, I_0}{\partial \alpha \partial w}\right) - 2\gamma w_o \left(\dfrac{\partial \mathrm{Im}\, I_0}{\partial \alpha}\right) \\ -(w_o^2 - k^2)\left(k^\alpha I_\alpha + \dfrac{3}{\alpha}\right)\left(\mathrm{Re}\, I_0 - \gamma \dfrac{\partial \mathrm{Im}\, I_0}{\partial w_o}\right) - 2\gamma w_o \left[\left(k^\alpha I_\alpha + \dfrac{3}{\alpha}\right) \mathrm{Im}\, I_0\right] \end{bmatrix} \\ &= -2w_p^2 \gamma \left[2w_o \frac{\partial \mathrm{Re}\, I_0}{\partial w_o} + \mathrm{Re}\, I_0 + \frac{1}{2}[w_o^2 - k^2]\frac{\partial^2 \mathrm{Re}\, I_0}{\partial w_o^2}\right], \end{aligned} \tag{4.41}$$

which can be solved and the RRFD can be represented by the RRFD factor γ_R

$$\gamma_R = -\frac{\tau_o \left[w_o + (w_o^2 - k^2)\dfrac{\partial \mathrm{Re}\, I_0}{\partial \alpha} - (w_o^2 - k^2)\left(k^\alpha I_\alpha + \dfrac{3}{\alpha}\right)\mathrm{Re}\, I_0\right]}{\begin{bmatrix} 2\left[2w_o \dfrac{\partial \mathrm{Re}\, I_0}{\partial w_o} + \mathrm{Re}\, I_0 + \dfrac{1}{2}[w_o^2 - k^2]\dfrac{\partial^2 \mathrm{Re}\, I_0}{\partial w_o^2}\right] \\ -\tau_o \begin{bmatrix} (w_o^2 - k^2)\left(\dfrac{\partial^2 \mathrm{Im}\, I_0}{\partial \alpha \partial w}\right) + 2w_o \left(k^\alpha I_\alpha + \dfrac{3}{\alpha}\right)\left(\mathrm{Im}\, I_0 + \dfrac{\partial \mathrm{Im}\, I_0}{\partial \alpha}\right) \\ -\dfrac{\partial \mathrm{Im}\, I_0}{\partial w_o}(w_o^2 - k^2)\left(k^\alpha I_\alpha + \dfrac{3}{\alpha}\right) \end{bmatrix} \end{bmatrix}}. \tag{4.42}$$

In the denominator the term with τ_o may be neglected for some particular values of w_o leading to a simple expression, see equation (4.43).

However, this damping due to the radiation is in general very small, unlike other works (Hakim & Mangeney 1968, 1971), unless the plasma frequency is very large.

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Indeed, for many authors the RRFD is increased for high temperatures ($\alpha \ll 1$) but, in these cases, the cutoff appears for $k \leqslant w_p$. That is, for high temperatures ($\alpha \ll 1$), the wave exists just in the first branch where the LD vanishes. Moreover, for $k \leqslant w_p$ the RRFD is very small in general unless the value of w_p is such that $\tau_o w_p \simeq 1$ and not just for $\alpha \ll 1$. Let us see now why it is necessary that $\tau_o w_p \simeq 1$ in order that the RRFD is important. To show this fact, let us consider $\alpha \gg 1$ (low temperatures). For our purposes we can put $\alpha = \infty$. In this case $\mathrm{Im}\, I_o \simeq 0$ and (4.42) turns out to be

$$\gamma_R = -\frac{\tau_o \left[w_o + (w_o^2 - k^2) \left(\frac{\partial \mathrm{Re}\, I_0}{\partial \alpha} \right) - (w_o^2 - k^2) \left(k^\alpha I_\alpha + \frac{3}{\alpha} \right) (\mathrm{Re}\, I_0) \right]}{2 \left[2 w_o \frac{\partial \mathrm{Re}\, I_0}{\partial w_o} + \mathrm{Re}\, I_0 + \frac{1}{2} [w_o^2 - k^2] \frac{\partial^2 \mathrm{Re}\, I_0}{\partial w_o^2} \right]}. \tag{4.43}$$

Additionally, for $\alpha = \infty$, we have

$$I_o = \frac{1}{w_o}, \quad \frac{\partial \mathrm{Re}\, I_0}{\partial w_o} = -\frac{1}{w^2}, \quad \frac{K_1(\infty)}{K_2(\infty)} = \frac{1}{n_o} = 1 \quad \text{and} \quad w_o = w_p. \tag{4.44a–d}$$

Using these approximations, we arrive at

$$\gamma_R = \frac{\tau_o w_p^2}{2}. \tag{4.45}$$

Therefore, since $\tau_o \simeq 6.26 \times 10^{-24}$ sec, the plasma frequency must be big enough in order that the damping can be appreciable. Since the RRFD is important only when its decay time $T_R = 1/\gamma_R$ is smaller than the corresponding period of the wave; then

$$R_R = \frac{T_R}{T} < 1. \tag{4.46}$$

In this case

$$R_R = \frac{1}{\pi \tau_o w_p} < 1. \tag{4.47}$$

Therefore, in order that the RRFD is important we need

$$w_p > \frac{1}{\pi \tau_o}. \tag{4.48}$$

Consequently, it is necessary that for a system of electrons,

$$w_p \simeq 5.085 \times 10^{22} \text{ rad sec}^{-1}. \tag{4.49}$$

However, with this kind of frequency we risk having a degenerate plasma and the applied theory is meaningless. Indeed (in international system of units (SI)),

$$\tau_o w_p = \tau_o \sqrt{\frac{ne^2}{m_e \varepsilon_o}} \simeq \frac{1}{\pi}, \tag{4.50}$$

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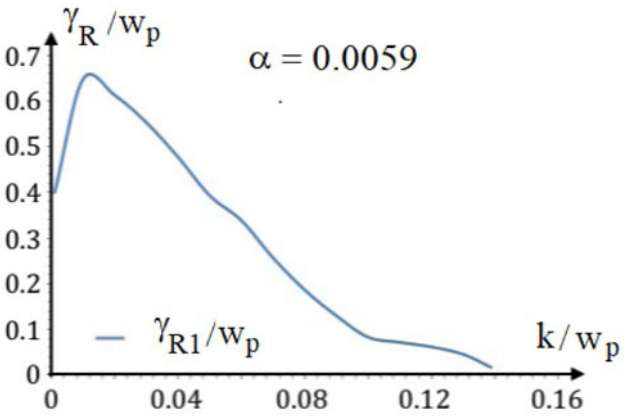


FIGURE 14. For $\alpha = 0.0059$ and $w_p \simeq 5.085 \times 10^{22}$ rad sec^{-1}, $\gamma_{L1}/w_p = 0$ and γ_{R1}/w_p is represented by the blue line.

where n, e, m_e and ε_o are the density of electrons, the charge of the electron, the electron mass and the dielectric constant in a vacuum, respectively. Accordingly,

$$n = \frac{m_e \varepsilon_o}{\pi^2 \tau_o^2 e^2} \simeq 0.8124 \times 10^{42}. \tag{4.51}$$

However, following Phillips (1998), Brown (2005) the limit between the degenerate and the non-degenerate cases is represented by equation

$$\frac{1}{\alpha} = \frac{\kappa T}{mc^2} > \left[\left(\frac{n}{n_o} \right)^{2/3} + 1 \right]^{1/2}, \tag{4.52}$$

where $n_o = 5.88 \times 10^{35}$ particles per cubic metre. This means that for the limit case of the Hagedorn temperature ($T = 1.005 \times 10^{12}$ or $\alpha = 0.0059$), the permitted limit for the density of particles per cubic metre is such that

$$\left[\left(\frac{n_{\max}}{5.88 \times 10^{35}} \right)^{2/3} + 1 \right]^{1/2} = 169.5, \tag{4.53}$$

which corresponds to $n_{\max} = 2.8633 \times 10^{42}$ particles per cubic metre, Therefore,

$$n = \frac{m_e \varepsilon_o}{\pi^2 \tau_o^2 e^2} \simeq 0.8124 \times 10^{42} < n_{\max}. \tag{4.54}$$

In astrophysical systems it may be possible to find such conditions.

We can now evaluate the RRFD by using $w_p \simeq 5.085 \times 10^{22}$ rad sec^{-1}, and $\bar{\gamma}_R = \gamma_R/w_p$ and the value of R_R (see figures 14 and 15). In these conditions, the RRFD cancels the wave for $\alpha = 0.0059$ (see figure 15) from $k = 0$ until $k = 0.12 w_p$ and permits the existence of the wave from $k = 0.12 w_p$ until the cutoff point (see figure 16). However, since in these circumstances the thermal velocity V_{Te} is very close to the phase velocity $V_\phi = w/ck = \mu = 1$, we cannot ensure the existence of this part of the wave.

It has to be noticed that for $\alpha = 100$, 10, 1 and 0.5, by using (4.52) and (4.53), it is necessary to have very small $n_{\max}$ and consequently $\tau_o w_p \ll 1$ and the RRFD is neglected.

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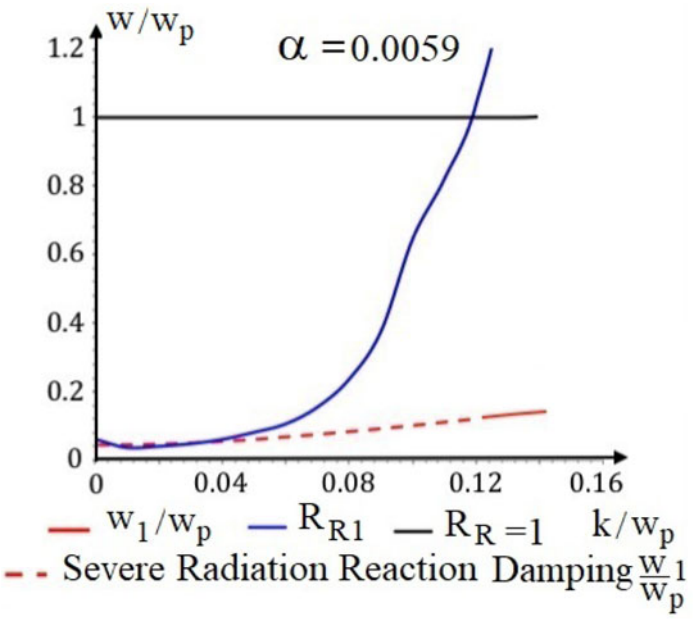


FIGURE 15. For $\alpha = 0.0059$ and $w_p \simeq 5.085 \times 10^{22}$ rad sec^{-1}. The dashed red line represents the damped first branch since $0 < R_{R1} < 1$ for $k < 0.12w_p$. The red line corresponds to a stable wave in the first branch since $R_{R1} > 1$ when $k > 0.12w_p$.

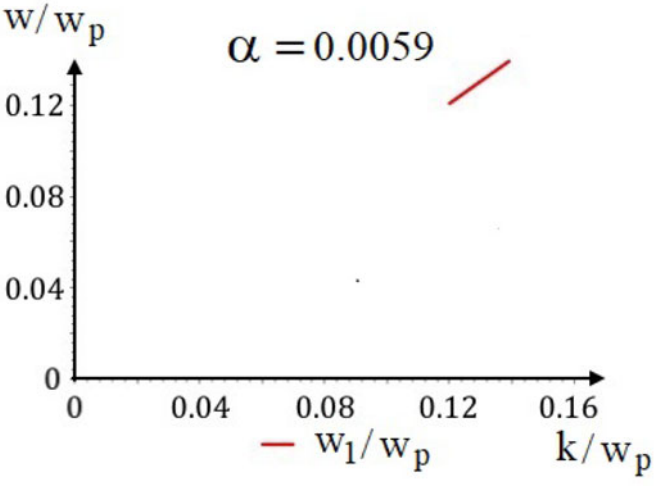


FIGURE 16. Final wave for $\alpha = 0.0059$ and $w_p \simeq 5.085 \times 10^{22}$ rad sec^{-1}. The wave is stable just when $0.12w_p > k > 0.14w_p$.

5. Summary

The dispersion relations have been found for the different temperatures accompanied by the distinct damping. The waves are bounded by the existence of the cutoffs obtained from the dispersion relationships themselves, from the different damping and from the restriction with respect to the thermal velocity. We have also seen

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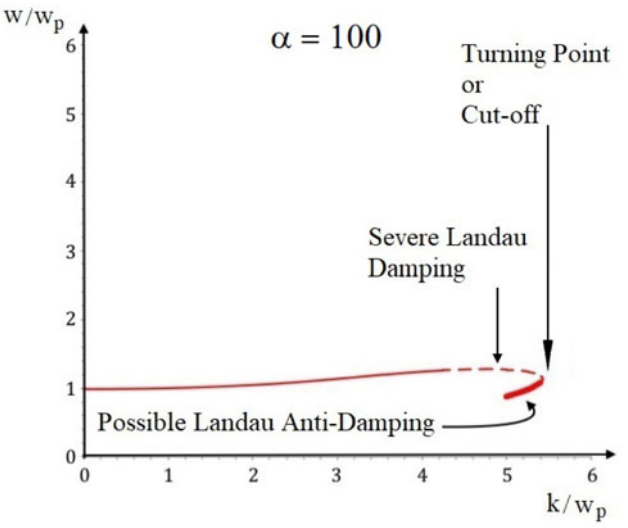


FIGURE 17. Final wave for $\alpha = 100$. The red line represents the stable first branch; the dashed red line represents the damped first branch wave (severe Landau damping); the thick red line represents the unstable second branch (possible anti-damping).

the existence of special waves that due to an analysis of the local increase of the temperature cannot be considered as resonances. In figures 17, 18, 19, 20 and 16, the final resulting waves for $\alpha = 100, 10, 1, 0.5, 0.0059$, are shown, respectively.

In figure 21, the surface generated by the waves as a function of w, k and α for low densities, is described. We can see the regular waves, the damping region of the waves due to the LD, the bipolar wave region, the plasma frequency shifts and how the wave disappears for very high values of temperature.

6. Concluding remarks

In this paper we have proved many properties of a relativistic plasma under the considerations that the thermal velocities of the electrons and ions are smaller than the phase velocity of the electrons and that $ne^2\tau_o \ll 1$; they are the followings:

(i) A modified covariant Vlasov equation was obtained by means of the Landau–Lifshitz equation including a weak constant electric field.

(ii) General dispersion relations are deduced for transverse and longitudinal modes, yielding a difference from the literature (Hakim & Mangeney 1968, 1971) in the RRFD term.

(iii) For each value of the temperature, a turning point (loop) is described as the solution of the dispersion relation for the longitudinal mode due to correct expansions in the integrals involved (two branches).

(iv) A general expression for the LD is exposed and its behaviour is computed for different temperatures. Although it vanishes for $\mu > 1$ as other authors predict, it is shown that it practically continues to vanish for an important interval of k in the $\mu < 1$ region; then it grows, not permitting the wave to attain the turning point of the loop for values of α close to 1. For values of α much lower than 1, the LD vanishes since the wave only exists in the first branch of the loop.

(v) In the second branch of the loop, the LD turns out to be an anti-damping; however, this effect just exists in a neighbourhood between the region of the turning

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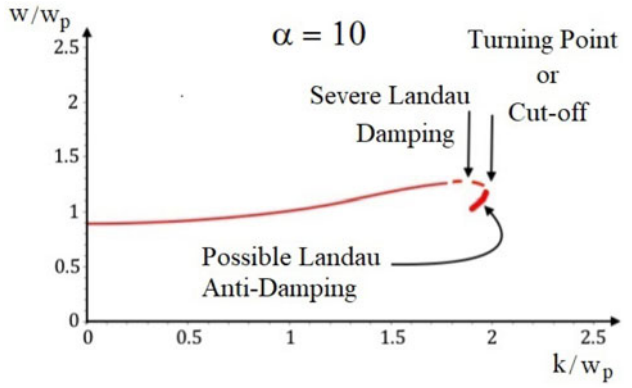


FIGURE 18. Final wave for $\alpha = 10$. The red line represents the stable first branch; the dashed red line represents the damped first branch wave (severe Landau damping); the thick red line represents the unstable second branch (possible anti-damping).

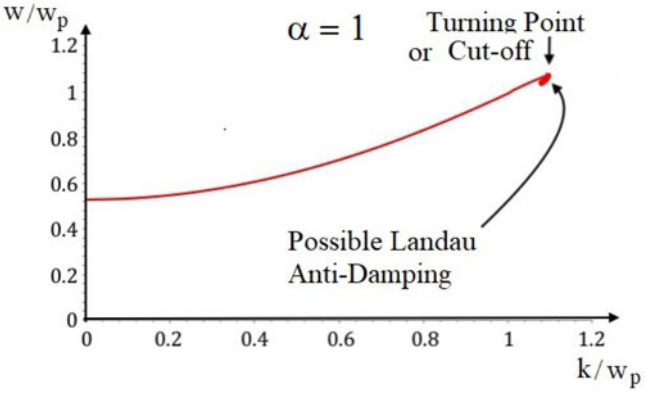


FIGURE 19. Final wave for $\alpha = 1$. The red line represents the stable first branch; the thick red line represents the unstable second branch (possible anti-damping).

point and the crossing point of the second branch with the thermal velocity line; the apparent resonance locally produces a few increments in the temperature giving a frequency corresponding to the first branch which is damped by the LD effect in this region or sending the frequency above the cutoff where a wave does not exist. These waves that can be detected as pulses, we called them bipolar waves.

(vi) Regularly, it is thought that for very low α (high temperatures), when the LD decays, the RRFD begins to be important; nevertheless, the RRFD is negligible unless $w_p > (1/\pi\tau_o)$ which implies high densities. The Hagedorn temperature is taken as a limit for finding the necessary conditions for the applicability of our model and the predominant role of the RRFD. For low densities and high temperatures, the disappearance of the wave is due to the existence of the cutoff and not due to the RRFD. Only for high densities and temperatures is the RRFD important.

(vii) A general surface in a α, w, k diagram is presented showing the edges and the critical regions of the surface including the LD and the bipolar wave regions.

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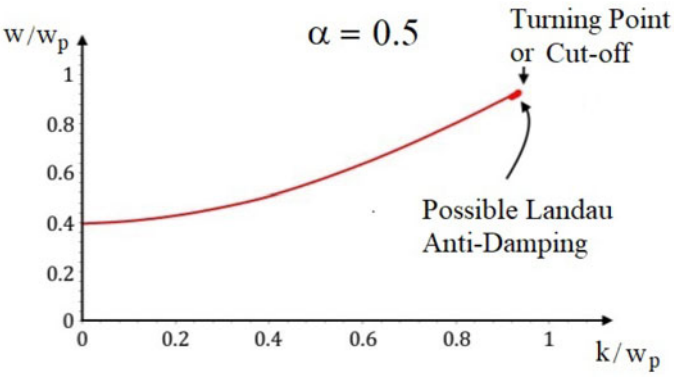


FIGURE 20. Final wave for $\alpha = 0.5$. The red line represents the stable first branch; the thick red line represents the unstable second branch (possible anti-damping).

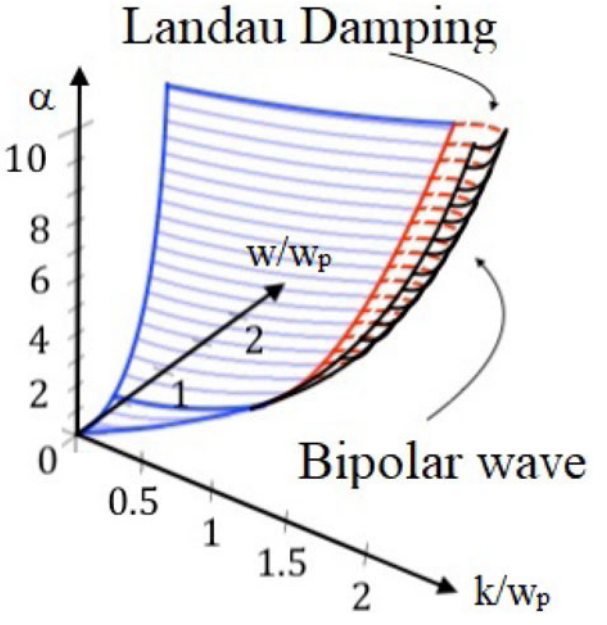


FIGURE 21. The surface of the waves.

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Appendix A

In the following, we will consider the case of an external constant electric field in the x-axis. Therefore, due to the antisymmetry of $\boldsymbol{F}^{\mu\nu}_{\text{ext}}$ and remembering that the 4-current which produces the external field is outside of the volume and is constant

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with respect the velocity, we have

$$\frac{e}{mc}\frac{\partial}{\partial v^{\mu}}[\boldsymbol{F}^{\mu\nu}_{\rm ext}v_{\nu}R_1]=\frac{e}{mc}\boldsymbol{F}^{\mu\nu}_{\rm ext}v_{\nu}\frac{\partial}{\partial v^{\mu}}R_1. \tag{A 1}$$

On the other hand, for a constant electric field it is a well-known fact that the radiation term vanishes, as has been proved by many authors (Ares de Parga, Mares & Domínguez-Hernández 2005*b*). Indeed, consider the part of the external radiation reaction term, since the electric field is constant, we have

$$\Delta^{\mu\nu}(v_{\rho})\left[\frac{e}{mc}[\boldsymbol{F}^{\rm ext}_{\nu\alpha}][\boldsymbol{F}^{\alpha\beta}_{\rm ext}]v_{\beta}+v^{\rho}v^{\alpha}\frac{\partial \boldsymbol{F}^{\rm ext}_{\nu\alpha}}{\partial x^{\alpha}}\right]=\Delta^{\mu\nu}(v_{\rho})\left[\frac{e}{mc}[\boldsymbol{F}^{\rm ext}_{\nu\alpha}][\boldsymbol{F}^{\alpha\beta}_{\rm ext}]v_{\beta}\right], \tag{A 2}$$

then,

$$\begin{aligned}\Delta^{\mu\nu}(v_{\rho})\left[\frac{e}{mc}[\boldsymbol{F}^{\rm ext}_{\nu\alpha}][\boldsymbol{F}^{\alpha\beta}_{\rm ext}]v_{\beta}+v^{\rho}v^{\alpha}\frac{\partial \boldsymbol{F}^{\rm ext}_{\nu\alpha}}{\partial x^{\alpha}}\right] &= \left(n^{\mu\nu}-\frac{v^{\mu}v^{\nu}}{c^2}\right)\left[\frac{e}{mc}[\boldsymbol{F}^{\rm ext}_{\nu\alpha}][\boldsymbol{F}^{\alpha\beta}_{\rm ext}]v_{\beta}\right]\\ &= E^2v^{\mu}\left(1-\frac{c^2}{c^2}\right)=0,\end{aligned} \tag{A 3}$$

where E represents the constant electric field. Notice that within Hammond theory (Hammond 2010*a*,*b*) the radiation reaction force does not vanish for a constant electric field and this will predict different dispersion relations. Starting from (2.32), we arrive at

$$\begin{aligned}&\frac{\partial R_1(x_{\mu},v_{\mu};\tau)}{\partial\tau}+v^{\mu}\frac{\partial R_1(x_{\mu},v_{\mu};\tau)}{\partial x^{\mu}}+\frac{e}{mc}\boldsymbol{F}^{\mu\nu}_{\rm ext}v_{\nu}\frac{\partial}{\partial v^{\mu}}R_1\\ &\quad+\frac{4\pi e^2}{mc^2}v_{\nu}\frac{\partial}{\partial v_{\mu}}\int \mathrm{d}\tau'\,\mathrm{d}^4x\prime\,\mathrm{d}^4v'[\{v'^{\nu}\partial^{\mu}-v'^{\mu}\partial^{\nu}\}D_{\rm ret}(x_{\rho}-x'_{\rho})]\\ &\quad\times R_2(x_{\mu},v_{\mu},\tau;x'_{\mu},v'_{\mu},\tau')\\ &\quad+\tau_o\frac{\partial}{\partial v^{\mu}}\left\{\begin{array}{c}\Delta^{\mu\nu}(v_{\rho})\dfrac{4\pi e^2}{mc^2}v^{\rho}v^{\alpha}\displaystyle\int \mathrm{d}\tau'\,\mathrm{d}^4x'\,\mathrm{d}^4v'\\ \times\left[\{v'_{\alpha}\partial_{\nu}-v'_{\nu}\partial_{\alpha}\}\dfrac{\partial}{\partial x^{\rho}}D_{\rm ret}(x_{\rho}-x'_{\rho})\right]R_2(x_{\mu},v_{\mu},\tau;x'_{\mu},v'_{\mu},\tau')\end{array}\right\}\\ &=O((ne^2\tau_o)^2).\end{aligned} \tag{A 4}$$

Considering that the correlation function vanishes or is of order $(ne^2\tau_o)^2$; that is,

$$D_2(x_{\mu},v_{\mu},\tau;x'_{\mu},v'_{\mu},\tau')=D_1(x_{\mu},v_{\mu};\tau)D_1(x'_{\mu},v'_{\mu};\tau'). \tag{A 5}$$

By calculating the average of (2.12) and taking $N-1\simeq N$, we arrive at

$$\begin{aligned}&\frac{\partial D_1(x_{\mu},v_{\mu};\tau)}{\partial\tau}+v^{\mu}\frac{\partial D_1(x_{\mu},v_{\mu};\tau)}{\partial x^{\mu}}+\frac{e}{mc}\boldsymbol{F}^{\mu\nu}_{\rm ext}v_{\nu}\frac{\partial}{\partial v^{\mu}}D_1\\ &\quad+\frac{4\pi e^2}{mc^2}Nv_{\nu}\frac{\partial}{\partial v_{\mu}}\int \mathrm{d}\tau'\,\mathrm{d}^4x'\,\mathrm{d}^4v'[\{v'^{\nu}\partial^{\mu}-v'^{\mu}\partial^{\nu}\}D_{\rm ret}(x_{\rho}-x'_{\rho})]\\ &\quad\times D_1(x_{\mu},v_{\mu};\tau)D_1(x'_{\mu},v'_{\mu};\tau')\end{aligned}$$

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$$+\tau_o \frac{4\pi e^2}{mc^2} N \frac{\partial}{\partial v^\mu} \left\{ \begin{array}{c} \Delta^{\mu\nu}(v_\rho) v^\rho v^\alpha \int \mathrm{d}\tau' \, \mathrm{d}^4 x' \, \mathrm{d}^4 v^{4\prime} \\ \left[\{ v'_\alpha \partial_\nu - v'_\nu \partial_\alpha \} \frac{\partial}{\partial x^\rho} D_{\mathrm{ret}}(x_\rho - x'_\rho) \right] \\ \times D_1(x_\mu, v_\mu; \tau) D_1(x'_\mu, v'_\mu; \tau') \end{array} \right\}$$

$$= 0. \tag{A 6}$$

Considering that

$$\lim_{\tau \to \mp\infty} D_1(x_\mu, v_\mu; \tau) = 0, \tag{A 7}$$

and integrating over the proper time τ, we obtain

$$\begin{aligned} & v^\mu \frac{\partial \mathcal{N}_1(x_\mu, v_\mu)}{\partial x^\mu} + \frac{e}{mc} \boldsymbol{F}^{\mu\nu}_{\mathrm{ext}} v_\nu \frac{\partial}{\partial v^\mu} \mathcal{N}_1(x_\mu, v_\mu) \\ & \quad + \frac{4\pi e^2}{mc^2} N v_\nu \frac{\partial}{\partial v_\mu} \mathcal{N}_1(x_\mu, v_\mu) \\ & \quad \times \int \mathrm{d}^4 x' \, \mathrm{d}^4 v' [\{ v'^\nu \partial^\mu - v'^\mu \partial^\nu \} D_{\mathrm{ret}}(x_\rho - x'_\rho)] \mathcal{N}_1(x'_\mu, v'_\mu) \\ & \quad + \tau_o \frac{4\pi e^2}{mc^2} N \frac{\partial}{\partial v^\mu} \left\{ \begin{array}{c} \Delta^{\mu\nu}(v_\rho) v^\rho v^\alpha \int \mathrm{d}^4 x' \, \mathrm{d}^4 v' \\ \left[\{ v'_\alpha \partial_\nu - v'_\nu \partial_\alpha \} \frac{\partial}{\partial x^\rho} D_{\mathrm{ret}}(x_\rho - x'_\rho) \right] \\ \times \mathcal{N}_1(x_\mu, v_\mu) \mathcal{N}_1(x'_\mu, v'_\mu) \end{array} \right\} \\ & = 0, \end{aligned} \tag{A 8}$$

which represents a modified Vlasov equation with an external constant electric field up to the $ne^2\tau_o$ order.

On the other hand, when a constant electric field is present the distribution function may be weighted by an exponential factor which takes into account the potential due to the electric constant field in the x-axis (Krall & Trivelpiece 1973); that is,

$$\phi = \phi_o \frac{\exp \frac{x}{\lambda} - \exp -\frac{x}{\lambda}}{\exp \frac{R}{\lambda} - \exp -\frac{R}{\lambda}}, \tag{A 9}$$

where ϕ_o, $\lambda = (8\pi e^2 n/\kappa T)^{-1/2}$ and R correspond to the potential in one of the plates (κ the Boltzmann constant), to the Debye length of the plasma in equilibrium and the length between the plates of the capacitor which generates the electric constant field, respectively. For $R \ll \lambda$, the potential becomes

$$\phi \simeq \phi_o \frac{x}{\lambda}, \tag{A 10}$$

and there is no shielding. It cannot be considered that we have a plasma in equilibrium. Even if we have a weak constant electric field but a high temperature

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such that $R \ll \lambda$, the electric field cannot be neglected. However, when $R \gg \lambda$, the potential is

$$\phi \simeq \phi_o \frac{\exp \frac{x}{\lambda} - \exp -\frac{x}{\lambda}}{\exp \frac{R}{\lambda}}. \tag{A 11}$$

Accordingly, the electric field is very small and the current generated by it can be neglected. Therefore, the equilibrium distribution function may be considered as

$$\mathcal{N}_{eq} \simeq \mathcal{N}_J \exp -\frac{\phi}{\kappa T} \simeq \mathcal{N}_J, \tag{A 12}$$

where $\mathcal{N}_{eq}$, $\mathcal{N}_J$ and T correspond to the equilibrium distribution function, the Jüttner distribution function and the temperature, respectively. We assume now that the plasma is in an equilibrium state described by the Jüttner distribution function which may be considered good enough to describe an equilibrium plasma with an external weak constant electric field.

The external constant electric field represented by $\boldsymbol{F}^{\mu\nu}_{\text{ext}} v_\nu$ is much smaller than it would (3.2) be if there were no plasma due to the shielding. Accordingly, when the perturbed distribution function is considered following the same procedure as in (3.1) and (3.2), the term

$$\frac{e}{mc} \boldsymbol{F}^{\mu\nu}_{\text{ext}} v_\nu \frac{\partial}{\partial v^\mu} \mathcal{N}_p(x_\mu, v_\mu), \tag{A 13}$$

can be withdrawn because for high volume and external weak constant electric field with $R \gg \lambda$, it represents the product of two small quantities. This means that the perturbed plasma will not be affected by an external weak constant electric field and (A 8) leads after the perturbation process to (3.2) as in the case without a weak constant electric field.

Appendix B

The Jüttner distribution function is expressed by Hakim & Mangeney (1968) as

$$N_J = \frac{n_o m \xi}{4\pi K_2(m\xi)} [\exp -m\xi^\mu v_\mu] 2\Theta(v^0) \delta(v^\mu v_\mu - 1), \tag{B 1}$$

with

$$\xi = \frac{1}{\kappa T_o}, \quad \text{and} \quad \xi^\mu = \xi \bar{v}^\mu. \tag{B 2a,b}$$

K_2 represents the Kelvin function (normally, the Hankel function of second order is used in order to describes the Jüttner function (Jüttner 1911; Synge 1956; Cubero *et al.* 2007; López-Carrera & Ares de Parga 2008)), $\gamma(\boldsymbol{u}) = (1 - u^2)^{-1/2}$ and Θ represents the Heaviside function. Notice that T_o is defined in the comoving frame of the space where the particles are moving with a velocity $\bar{\boldsymbol{u}}$ where $\boldsymbol{u} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$ and $\boldsymbol{v} = \gamma(\boldsymbol{u})\boldsymbol{u}$ with $\gamma(\boldsymbol{u}) = (1 - u^2)^{-1/2}$; that is, the rest frame K_o where the system is at rest and the temperature is T_o. The existence of such a rest frame is assured because in the free particle system there is no interaction and a rest frame can be defined. However,

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when there is interaction between the particles it is necessary to define a volume in a particular frame and in such a frame the instantaneity is considered (Gamba 1967; Ares de Parga, López-Carrera & Angulo-Brown 2005*a*; Nakamura 2006; López-Carrera & Ares de Parga 2008; Ares de Parga & López-Carrera 2011). This means that the Jüttner distribution function can be used even if there is interaction as long as the temperature is defined in a particular frame. In our case, the temperature is in the rest frame K_o where $\bar{\boldsymbol{u}} = \boldsymbol{0}$. Therefore, the Jüttner distribution function takes into account the effects of relativistic thermodynamics.

Appendix C

Before continuing with the development, it is necessary to note some properties of the 4-vector current. Actually, we define the 4-vector point particle current for a system of point particles,

$$J^{\alpha}(x_{\rho}) = \int \mathrm{d}\tau \left\{ \sum_{i=1}^{n} e_i \delta^4(x_{\rho} - x_{i\rho}(\tau)) \frac{\mathrm{d}x_i^{\alpha}(\tau)}{\mathrm{d}\tau} \right\}, \tag{C 1}$$

which represents a well-defined 4-vector. It can be decomposed and written as:

$$\boldsymbol{J}(x, t) = \sum_{i=1}^{n} e_i \delta^3(\boldsymbol{x} - \boldsymbol{x}_i(t)) \frac{\mathrm{d}\boldsymbol{x}_i(t)}{\mathrm{d}t} \quad \text{and} \quad \rho(x, t) = \sum_{i=1}^{n} e_i \delta^3(\boldsymbol{x} - \boldsymbol{x}_i(t)). \tag{C 2}$$

It is easy to show that

$$\frac{\partial \rho(x, t)}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J}(x, t) = 0. \tag{C 3}$$

That is, the continuity equation is satisfied. If we take the Fourier transform of the 4-vector point particle current, the continuity equation must be satisfied. If we want to pass to a continuous system of particles with the same charge e, we put

$$\frac{\partial e\mathcal{N}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J}(x, t) = 0, \tag{C 4}$$

where $\mathcal{N}$ represents a continuous particle density such as the one introduced in (2.38). Of course, the continuity equation must be accomplished if we use the Maxwell equations. However, when the density of point particles is transformed to a continuous density, some properties can be lost and we must impose the continuity equation. Then, in (3.6) the continuity equation must be considered as a constraint embedded in the theory; that is,

$$k^{\mu} J_{\mu} = w J^0(k_v) - \boldsymbol{k} \cdot \boldsymbol{J}(k_v) = 0. \tag{C 5}$$

This is equivalent to the procedure used in (2.9) and (2.10).

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