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A Fifth-Order Combined Compact Difference Scheme for Stokes Flow on Polar Geometries

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Abstract. Incompressible flows with zero Reynolds number can be modeled by the Stokes equations. When numerically solving the Stokes flow in stream-vorticity formulation with high-order accuracy, it will be important to solve both the stream function and velocity components with the high-order accuracy simultaneously. In this work, we will develop a fifth-order spectral/combined compact difference (CCD) method for the Stokes equation in stream-vorticity formulation on the polar geometries, including a unit disk and an annular domain. We first use the truncated Fourier series to derive a coupled system of singular ordinary differential equations for the Fourier coefficients, then use a shifted grid to handle the coordinate singularity without pole condition. More importantly, a three-point CCD scheme is developed to solve the obtained system of differential equations. Numerical results are presented to show that the proposed spectral/CCD method can obtain all physical quantities in the Stokes flow, including the stream function and vorticity function as well as all velocity components, with fifth-order accuracy, which is much more accurate and efficient than low-order methods in the literature.

AMS subject classifications: 65N06

Key words: Stokes flow, combined compact difference (CCD) scheme, truncated Fourier series, shifted grid, coordinate singularity.

1. Introduction

The two-dimensional (2D) Stokes equations in the usual velocity-pressure formulation are given as follows,

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$$-\Delta u + \frac{\partial p}{\partial x} = f_1, \quad \text{in} \quad \Omega, \tag{1.1}$$

$$-\Delta v + \frac{\partial p}{\partial y} = f_2, \quad \text{in} \quad \Omega, \tag{1.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \text{in} \quad \Omega, \tag{1.3}$$

with boundary conditions

$$u = u_1(x, y), \quad v = v_1(x, y), \quad \text{on} \quad \Gamma = \partial\Omega, \tag{1.4}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the well-known Laplacian operator, (u, v) is the velocity field, p is the pressure, (f_1, f_2) is the external force, Ω is the 2D domain, and Γ is its boundary.

A flow governed by the Stokes equations is known as a creeping flow or Stokes flow. Such flow is used for the fluid with very low Reynolds number, that is the inertia is very small compared to the viscous force. When the inertial term is neglected, i.e., Reynolds number is zero, 2D Navier-Stokes equations reduce to Stokes equations (1.1)-(1.3). We refer [26] for a brief introduction to the life at low Reynolds number, and refer [13] for discussions on various methods for solving the viscous incompressible flow at low Reynolds numbers. The above 2D Stokes problem (1.1)-(1.4) in rectangular domain was numerically studied in [12] by a fourth-order compact MAC finite difference scheme. However, in many physical problems, one often needs to solve flow equations in a non-Cartesian domain, such as polar or cylindrical domains. For example, Navier-Stokes equations in the polar or cylindrical domains are discussed in [2, 10, 16, 19, 20, 24, 28, 29, 34, 36, 37]. This paper will focus on the 2D Stokes equations on polar geometries, including a unit disk and an annulus domain.

Now we first introduce the stream function ψ [27, 35], which satisfies

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}. \tag{1.5}$$

Through a simple calculation, it is easy to show that the above Stokes equations (1.1)-(1.3) is equivalent to the following boundary value problem of biharmonic equation for ψ,

$$\Delta^2 \psi = f(x, y), \quad \text{in } \Omega, \tag{1.6}$$

where

$$f = \frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial x},$$

and the boundary conditions (1.4) can be transformed as

$$\psi = g(x, y), \quad \frac{\partial \psi}{\partial \vec{n}} = h(x, y), \quad \text{on } \Gamma = \partial\Omega, \tag{1.7}$$

where

$$h(x, y) = v_1 n_1 - u_1 n_2, \quad g(x, y) = \int_\Gamma -u_1 dy + v_1 dx + g(x_0, y_0).$$

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Here $\vec{n} = (n_1, n_2)^T$ is the unit outer normal vector on the boundary, $\int_\Gamma -u_1 dy + v_1 dx$ is the curve integral on the boundary from one given point (x_0, y_0) to (x, y) and $g(x_0, y_0)$ can take arbitrary value. ψ, f, g, h are assumed to be sufficiently smooth functions.

Eqs. (1.6)-(1.7) are called the first boundary value problem of the biharmonic equation, which is proved to have a unique solution if the solution exists, see [38]. As mentioned in [4], direct discretisation of the biharmonic equation (1.6) leads to an ill-conditioned linear system of equations. Thus, classical iterative methods and Krylov subspace methods generally require a large number of iterations in order to get some satisfactory solutions. One simple approach to avoid solving such ill-conditioned linear system is to introduce the vorticity

$$w = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \Delta \psi$$

so that the biharmonic equation (1.6) can be split into two coupled Poisson equations [18],

$$\Delta w = f(x, y), \quad \Delta \psi = w, \quad \text{in } \Omega. \tag{1.8}$$

The above equations are actually the stream-vorticity formulation for the Stokes equations. One can easily see that under this formulation, there are two boundary conditions (1.7) for the stream function ψ but no boundary conditions for the vorticity w.

In this paper, we consider a high-order accurate numerical algorithm for Stokes equations in stream-vorticity formulation (1.8) on polar geometries, including a unit disk $\Omega_1 = \{(x, y) | 0 < \sqrt{x^2 + y^2} < 1\}$ and an annulus domain $\Omega_2 = \{(x, y) | 0.5 < \sqrt{x^2 + y^2} < 1\}$. It is natural to apply the polar coordinate transformation $x = r\cos\theta, y = r\sin\theta$ to the Eq. (1.8). The coupled system of $w(r, \theta), \psi(r, \theta)$ in the unit disk Ω_1 can be rewritten as follows [18]

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = f(r, \theta), \quad 0 < r < 1, \quad 0 \le \theta < 2\pi, \tag{1.9}$$

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = w(r, \theta), \quad 0 < r < 1, \quad 0 \le \theta < 2\pi, \tag{1.10}$$

with boundary conditions

$$\psi(1, \theta) = g(\theta), \quad \frac{\partial \psi}{\partial r}(1, \theta) = h(\theta). \tag{1.11}$$

In addition, the velocity field can be rewritten in radial and angular components by using the stream function as follows,

$$u_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad u_\theta = \frac{\partial \psi}{\partial r}, \tag{1.12}$$

where $u_r = u\cos\theta + v\sin\theta$ is the radial velocity and $u_\theta = v\cos\theta - u\sin\theta$ is the angular velocity.

Several approaches for the numerical solution of the boundary value problem (1.9)-(1.11), as well as the applications to the steady incompressible flow inside circular geometries have been developed in existing literatures. Those include the spectral method [14],

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the integral equation method [5], and the spectral/difference method [3, 11]. And more recently, Lai and Liu [18] used a spectral/finite difference method for solving the boundary value problem (1.9)-(1.11), however, the method is only second-order accurate.

When numerically solving the Stokes equations in stream-vorticity formulation (1.9)-(1.11) with high-order accuracy, it is also important to solve velocity field (1.12) with high-order accuracy simultaneously. Thus, one needs to solve both the stream function and its derivatives with the high-order accuracy simultaneously. The three-point combined compact difference (CCD) scheme is a method which can achieve this objective. The CCD scheme was first proposed by Chu and Fan [1] to solve one-dimensional convection-diffusion equation. When using the CCD method to solve the differential equations, the equation is assumed to be valid at the boundary, and the first and second derivatives together with the function values of unknowns at grid points are computed simultaneously [1]. Fourier analysis shows that the CCD scheme has better spectral resolution than many other existing compact or noncompact high-order schemes [22]. Due to the compact three-point stencils and high-order accuracy, the CCD scheme has been proved to be very useful and efficient for solving the convection-diffusion equation [30], the shallow water equations in spherical geometry [25], the time-fractional advection-diffusion equation [6, 7], the 2D elliptic equation with mixed derivatives [31], the 2D linear telegraph equation [8], the 2D lid-driven cavity problem [32, 33], and the three-dimensional nonlinear Schrödinger equation [9].

In this paper, we will develop a spectral/CCD method to solve the Stokes equations in stream-vorticity formulation (1.8) on the polar geometries, which includes a unit disk and an annular domain. In our approach, the truncated Fourier series and shifted grids are adopted as in [18], which give a coupled system of singular ordinary differential equations for the Fourier coefficients of stream function and vorticity function. The main contribution of this paper is to develop a CCD scheme for solving the Fourier coefficients of stream function and vorticity function as well as their derivatives from the coupled differential equations with high-order accuracy simultaneously. Numerical results show that the current method is the fifth-order accurate for all physical quantities, including the stream function and vorticity function as well as all velocity components, which is much more accurate and efficient than the second-order method in [18]. Our spectral/CCD method can be well used to simulate the Stokes flow on polar geometries with very high-order accuracy.

2. Numerical Method

2.1. CCD scheme for the second-order differential equation

In this subsection, we present the CCD method for solving the following ordinary differential equation, readers can also refer to [1],

$$p(x)\frac{d^2u}{dx^2} + q(x)\frac{du}{dx} + \gamma(x)u = r(x), \quad 0 \le x \le L, \tag{2.1}$$

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with boundary conditions

$$u(0)=\alpha,\quad u(L)=\beta, \tag{2.2}$$

where p,q,γ,r are given sufficiently smooth functions and the Eq. (2.1) is assumed to be valid on the boundary.

Discretize the interval $[0,L]$ into a uniform grid $0=x_0<x_1<\cdots<x_S=L$ with a spacing $h=L/S$. Denote $U_i=u(x_i),U_i'=\frac{du}{dx}(x_i),U_i''=\frac{d^2u}{dx^2}(x_i),i=0,1,\cdots,S$. For internal grid points, the CCD method relates the first and second derivatives as follows,

$$\frac{7}{16}(U_{i+1}'+U_{i-1}')+U_i'-\frac{h}{16}(U_{i+1}''-U_{i-1}'')-\frac{15}{16h}(U_{i+1}-U_{i-1})=0,\ \ 1\le i\le S-1, \tag{2.3}$$

$$\frac{9}{8h}(U_{i+1}'-U_{i-1}')+U_i''-\frac{1}{8}(U_{i+1}''+U_{i-1}'')-\frac{3}{h^2}(U_{i+1}-2U_i+U_{i-1})=0,\ 1\le i\le S-1. \tag{2.4}$$

The above two relations are valid up to $O(h^6)$ [1].

At the two boundary points, a pair of fifth-order one-sided CCD schemes are introduced as follows [1],

$$14U_0'+16U_1'+2hU_0''-4hU_1''+\frac{1}{h}(31U_0-32U_1+U_2)=0, \tag{2.5}$$

$$14U_S'+16U_{S-1}'-2hU_S''+4hU_{S-1}''-\frac{1}{h}(31U_S-32U_{S-1}+U_{S-2})=0. \tag{2.6}$$

In addition, the Eq. (2.1) and boundary conditions (2.2) are also used for the CCD scheme as follows,

$$p(x_i)U_i''+q(x_i)U_i'+\gamma(x_i)U_i=r(x_i),\quad 0\le i\le S, \tag{2.7}$$

and

$$U_0=\alpha,\quad U_S=\beta. \tag{2.8}$$

The CCD scheme for the second-order differential equation (2.1) with boundary condition (2.2) are given by Eqs. (2.3)-(2.8). There are $3(S+1)$ unknowns: U_i,U_i',U_i'' $(0\le i\le S)$, and there are totally $3(S+1)$ equations in (2.3)-(2.8). Thus, the system is closed.

We remark that although the fifth-order boundary conditions are exploited at boundaries, numerical experiments show that the sixth-order accuracy of the CCD method is guaranteed, see more details in [1, 30]. Furthermore, since the coefficient matrix of the CCD system possesses a block tridiagonal structure [33], it can be efficiently solved by the block Thomas algorithm.

2.2. CCD scheme for Stokes equations on polar geometries

Since a function in the unit disk is periodic in θ, we can approximate the solutions using the truncated Fourier series [15, 18],

$$w(r,\theta)=\sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1}\hat{w}^n(r)e^{in\theta},\quad \psi(r,\theta)=\sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1}\hat{\psi}^n(r)e^{in\theta}, \tag{2.9}$$

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and $\hat{w}^n(r), \hat{\psi}^n(r)$ are the complex Fourier coefficient given by

$$\hat{w}^n(r) = \frac{1}{N}\sum_{j=0}^{N-1} w(r,\theta_j)e^{-in\theta_j}, \quad \hat{\psi}^n(r) = \frac{1}{N}\sum_{j=0}^{N-1} \psi(r,\theta_j)e^{-in\theta_j}, \tag{2.10}$$

where $\theta_j = j\triangle\theta$ with $\triangle\theta = (2\pi)/N$ and N is the number of grid points along the circle. The above transformation between the physical space and Fourier space can be efficiently performed using the fast Fourier transform (FFT) with $O(N\log_2 N)$ arithmetic operations.

Substituting those expansions (2.9)-(2.10) into Eq. (1.12), we obtain that

$$u_r(r,\theta) = -\frac{i}{r}\sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} n\hat{\psi}^n(r)e^{in\theta}, \quad u_\theta(r,\theta) = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \frac{\partial\hat{\psi}^n(r)}{\partial r}e^{in\theta}. \tag{2.11}$$

Substituting those expansions (2.9)-(2.10) into Eq. (1.9)-(1.11) and equating the Fourier coefficients, the n-th Fourier mode $\hat{\psi}^n(r), \hat{w}^n(r)$ satisfy the following ordinary differential equations

$$\frac{d^2\hat{w}^n(r)}{dr^2} + \frac{1}{r}\frac{d\hat{w}^n(r)}{dr} - \frac{n^2}{r^2}\hat{w}^n(r) = \hat{f}^n(r), \quad 0 < r \le 1, \tag{2.12}$$

$$\frac{d^2\hat{\psi}^n(r)}{dr^2} + \frac{1}{r}\frac{d\hat{\psi}^n(r)}{dr} - \frac{n^2}{r^2}\hat{\psi}^n(r) = \hat{w}^n(r), \quad 0 < r \le 1, \tag{2.13}$$

with boundary conditions

$$\hat{\psi}^n(1) = \hat{g}^n, \quad \frac{d\hat{\psi}^n(1)}{dr} = \hat{h}^n, \tag{2.14}$$

where the complex Fourier coefficients $\hat{f}^n(r)$ and $\hat{g}^n$ are defined in a manner similar to that of (2.10), and the Eqs. (1.9)-(1.10) are assumed to be valid on the boundary $r = 1$ so that (2.12)-(2.13) are also valid on $r = 1$.

As we know, the above Eqs. (2.12)-(2.13) is singular at origin $r = 0$. A special choice of girds r_i to avoid the polar singularity [15, 17, 18, 21, 23] is given by

$$r_i = \left(i - \frac{1}{2}\right)\triangle r, \quad 0 \le i \le M+1, \tag{2.15}$$

with mesh size $\triangle r = 2/(2M+1)$. Under such grid, $r_0 = -\Delta r/2, r_1 = \Delta r/2, r_{M+1} = 1$. The advantage of this grid is that grid points are not placed directly at the origin. Thus, the boundary value on the origin is not needed.

Let $\Psi_i = \hat{\psi}^n(r_i), W_i = \hat{w}^n(r_i), i = 0, 1, \cdots, M+1$. For internal grid points r_i $(1 \le i \le M)$, the three-point sixth-order CCD scheme relates the first and second derivatives of $\hat{\psi}^n, \hat{w}^n$ as follows,

$$\frac{7}{16}(\Psi'_{i+1} + \Psi'_{i-1}) + \Psi'_i - \frac{\triangle r}{16}(\Psi''_{i+1} - \Psi''_{i-1}) - \frac{15}{16\triangle r}(\Psi_{i+1} - \Psi_{i-1}) = 0, \tag{2.16}$$

$$\frac{9}{8\triangle r}(\Psi'_{i+1} - \Psi'_{i-1}) + \Psi''_i - \frac{1}{8}(\Psi''_{i+1} + \Psi''_{i-1}) - \frac{3}{(\triangle r)^2}(\Psi_{i+1} - 2\Psi_i + \Psi_{i-1}) = 0, \tag{2.17}$$

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and

$$\frac{7}{16}(W'_{i+1}+W'_{i-1})+W'_i-\frac{\triangle r}{16}(W''_{i+1}-W''_{i-1})-\frac{15}{16\triangle r}(W_{i+1}-W_{i-1})=0, \tag{2.18}$$

$$\frac{9}{8\triangle r}(W'_{i+1}-W'_{i-1})+W''_i-\frac{1}{8}(W''_{i+1}+W''_{i-1})-\frac{3}{(\triangle r)^2}(W_{i+1}-2W_i+W_{i-1})=0, \tag{2.19}$$

where Ψ'_i and Ψ''_i denote the first and second derivatives of $\hat{\psi}^n$ with respect to r at r_i, while W'_i and W''_i denote the first and second derivatives of $\hat{w}^n$ with respect to r at r_i. The above four relations are valid upon to $O((\triangle r)^6)$, see above subsection and [1].

At the two end grid points (r_0, r_{M+1}), two pairs of fifth-order one-sided CCD schemes are introduced

$$14\Psi'_0+16\Psi'_1+2\triangle r\Psi''_0-4\triangle r\Psi''_1+\frac{1}{\triangle r}(31\Psi_0-32\Psi_1+\Psi_2)=0, \tag{2.20}$$

$$14\Psi'_{M+1}+16\Psi'_M-2\triangle r\Psi''_{M+1}+4\triangle r\Psi''_M-\frac{1}{\triangle r}(31\Psi_{M+1}-32\Psi_M+\Psi_{M-1})=0, \tag{2.21}$$

and

$$14W'_0+16W'_1+2\triangle rW''_0-4\triangle rW''_1+\frac{1}{\triangle r}(31W_0-32W_1+W_2)=0, \tag{2.22}$$

$$14W'_{M+1}+16W'_M-2\triangle rW''_{M+1}+4\triangle rW''_M-\frac{1}{\triangle r}(31W_{M+1}-32W_M+W_{M-1})=0. \tag{2.23}$$

And Eq. (2.14) gives the condition at $r=1$ as follows

$$\Psi_{M+1}=\hat{g}^n, \quad \Psi'_{M+1}=\hat{h}^n. \tag{2.24}$$

Moreover, Eqs. (2.12) and (2.13) are also used for the CCD scheme, i.e.,

$$W''_i+\frac{1}{r_i}W'_i-\frac{n^2}{r_i^2}W_i=\hat{f}^n(r_i), \quad 0\le i\le M+1, \tag{2.25}$$

and

$$\Psi''_i+\frac{1}{r_i}\Psi'_i-\frac{n^2}{r_i^2}\Psi_i=W_i, \quad 0\le i\le M+1. \tag{2.26}$$

Here we point out that Eqs. (2.12) and (2.13) are also valid for $r<0$. The following is a short illustration.

The transformation between Cartesian and polar coordinates can be written as $x=r\cos\theta$, $y=r\sin\theta$. When we replace r with $-r$, and θ with $\theta+\pi$, the Cartesian coordinates of a point remain the same. Therefore, any scalar function $w(r,\theta)$ satisfies $w(-r,\theta)=w(r,\theta+\pi)$. Thus we have,

$$\begin{aligned} w(-r,\theta)&=\sum_{n=-\infty}^{+\infty}\hat{w}^n(-r)e^{in\theta}=w(r,\theta+\pi)=\sum_{n=-\infty}^{+\infty}\hat{w}^n(r)e^{in(\theta+\pi)}\\ &=\sum_{n=-\infty}^{+\infty}(-1)^n\hat{w}^n(r)e^{in\theta}. \end{aligned} \tag{2.27}$$

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This yields

$$\hat{w}^n(-r) = (-1)^n \hat{w}^n(r). \tag{2.28}$$

Therefore, we have $\hat{\psi}^n(-r) = (-1)^n \hat{\psi}^n(r)$ and $\hat{w}^n(-r) = (-1)^n \hat{w}^n(r)$. Hence, the following conditions are imposed at r_0 in the CCD scheme,

$$\Psi_0 = (-1)^n \Psi_1, \qquad W_0 = (-1)^n W_1. \tag{2.29}$$

Taking the first and second derivative with respect to r for (2.28), we have

$$-\frac{d\hat{w}^n(-r)}{dr} = (-1)^n \frac{d\hat{w}^n(r)}{dr}, \tag{2.30}$$

and

$$\frac{d^2\hat{w}^n(-r)}{dr^2} = (-1)^n \frac{d^2\hat{w}^n(r)}{dr^2}. \tag{2.31}$$

Additionally, we have $\hat{f}^n(-r) = (-1)^n \hat{f}^n(r)$. Thus,

$$\frac{d^2\hat{w}^n(-r)}{dr^2} - \frac{1}{r}\frac{d\hat{w}^n(-r)}{dr} - \frac{n^2}{r^2}\hat{w}^n(-r) = \hat{f}^n(-r), \tag{2.32}$$

which means that Eq. (2.12) is still valid when extending to a negative value of r. Therefore, Eq. (2.25) is used for $r_0 = -\triangle r/2$. Similarly, Eq. (2.13) is still valid when extending to a negative value of r. Therefore, Eq. (2.26) is also used for $r_0 = -\triangle r/2$. Our three-point CCD scheme is Eqs. (2.16)-(2.26) and (2.29). There are $6(M+2)$ unknowns: $\Psi_i, \Psi_i', \Psi_i'', W_i, W_i', W_i''$ $(0 \le i \le M+1)$, and there are totally $6(M+2)$ equations in (2.16)-(2.26) and (2.29). Thus, the system is closed. Once Ψ_i, Ψ_i', W_i are obtained, one can obtain the numerical solution for ψ, w from (2.9) and u_r, u_θ from (2.11), these truncated Fourier series can be efficiently performed by using the fast Fourier transform (FFT) with $O(N\log_2 N)$ arithmetic operations.

For the case of Stokes flow in the annulus domain Ω_2, the situation is simpler. In this case, we just use the regular grids: $r_i = 0.5 + (i-1)\triangle r, \triangle r = 0.5/(M+1)$, where M is a positive integer. And the Eqs. (1.9)-(1.10) are assumed to be valid on both boundaries $r = 0.5$ and $r = 1$ while the boundary conditions on $r = 0.5$ are similar to (2.24). And Eqs. (2.16)-(2.26) are used. Since there are no boundary conditions for w, Eqs. (2.22)-(2.23) are actually approximations for the internal points of unknowns with fifth-order accuracy. This will reduce the CCD method to fifth-order accuracy, which will be confirmed by the numerical results in the next section. We point out this is different from the usual sixth-order accurate CCD method [1,30] where the fifth-order approximations are actually applied to the boundary points of unknowns where the values of the boundary points are given.

2.3. Summary of the spectral/CCD algorithm

We now close the section by summarizing the spectral/CCD algorithm for solving the Stokes equations of stream-vorticity formulation (1.9)-(1.11). The detailed algorithm is put in **Algorithm** 2.1.

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Algorithm 2.1 Spectral/CCD algorithm for the Stokes flow on the polar geometry.

1: Compute the Fourier coefficients $\hat{f}^n(r), \hat{w}^n(r), \hat{g}^n$ and $\hat{h}^n$ using FFT described in (2.10).
2: Solve $\Psi_i, \Psi_i', \Psi_i'', W_i, W_i', W_i''$ $(0 \le i \le M+1)$ from the CCD scheme (2.16)-(2.26) and (2.29) for $n = -N/2, \cdots, N/2$.
3: Convert the Fourier coefficients $\hat{\psi}^n(r_i)$ and $\hat{w}^n(r_i)$ to $\psi(r_i,\theta)$ and $w(r_i,\theta)$ using inverse FFT described in (2.9).
4: Compute the radial velocity u_r and the angular velocity u_θ using (2.11).

3. Numerical Results

In this section, numerical results of the spectral/CCD method presented in the previous section and Lai's second-order method [18] for two examples are given. All numerical experiments were carried out on a desktop with 4.00GHz Intel i7-4790K and 8GB RAM using double-precision arithmetics. Our code is written in Matlab, and the software Matlab 2014a is used in the computation.

The discrete L^∞ error of the stream function is defined by

$$||e_\psi||_\infty = \sup_{0\le i\le M, 0\le j\le N} |\psi(r_i,\theta_j) - \psi_{i,j}|, \tag{3.1}$$

where $\psi(r_i,\theta_j)$ and $\psi_{i,j}$ are the exact solution and numerical solution at (r_i,θ_j), respectively. Similar definitions are used for $||e_w||_\infty$, $||e_{u_r}||_\infty$ and $||e_{u_\theta}||_\infty$. The rate of convergence is calculated by the ratio of two consecutive errors.

Example 3.1. Consider the problem (1.9)-(1.10) with the following exact solution

$$\begin{aligned}
\psi &= e^{r\sin\theta + r\cos\theta},\\
w &= 2e^{r\sin\theta + r\cos\theta},\\
u_r &= (\sin(\theta) - \cos(\theta))e^{r\sin\theta + r\cos\theta},\\
u_\theta &= (\sin(\theta) + \cos(\theta))e^{r\sin\theta + r\cos\theta},
\end{aligned}$$

and $f = 4e^{r\sin\theta + r\cos\theta}$ is obtained by the exact solution. This example is taken from [18].

By using the current spectral/CCD method, we have run a series of tests by fixing the value M but varying the number N, or vise versa. The range of M and N are from 8 to 128. Table 1 shows the errors for ψ in the L^∞ norm for distinct values of M and N for Example 3.1 on a unit disk. As we can see that when the number $N = 8$, the accuracy does not improve when we refine the radial mesh. This is because too few Fourier modes causes the dominant error coming from the azimuthal discretisation. When the number N is greater than or equal to 32, the dominant error comes from the radial discretisation. Similar discussion can be found in [15].

Table 2-Table 4 give the convergence study for Example 3.1 on both the unit disk and the annulus domain by using the present spectral/CCD method and Lai's second-order

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Table 1: The L^∞ errors of ψ by using the spectral/CCD method for Example 3.1 on the unit disk $0 < r < 1$ with different N and M.

M \ N	8	16	32	64	128
8	1.08e−04	1.00e−06	1.00e−06	1.00e−06	1.00e−06
16	1.11e−04	4.08e−08	4.13e−08	4.13e−08	4.13e−08
32	1.11e−04	1.82e−09	1.49e−09	1.49e−09	1.49e−09
64	1.11e−04	1.12e−09	5.00e−11	4.98e−11	4.97e−11
128	1.11e−04	1.16e−09	1.51e−12	1.52e−12	1.52e−12

Table 2: The L^∞ errors and CPU times (in seconds) obtained by the spectral/CCD method with $N = 32$ for Example 3.1 on the unit disk $0 < r < 1$.

M	CPU	$\| e_\psi \|_\infty$	rate	$\| e_w \|_\infty$	rate	$\| e_{u_r} \|_\infty$	rate	$\| e_{u_\theta} \|_\infty$	rate
4	0.016	2.04E−05	-	8.78E−04	-	2.69E−05	-	1.02E−04	-
8	0.023	1.00E−06	4.73	4.51E−05	4.67	1.36E−06	4.70	6.57E−06	4.31
16	0.027	4.13E−08	4.81	1.85E−06	4.81	5.78E−08	4.76	3.18E−07	4.57
32	0.032	1.49E−09	4.90	6.65E−08	4.91	2.11E−09	4.89	1.25E−08	4.77
64	0.056	5.00E−11	4.95	2.23E−09	4.95	7.12E−11	4.94	4.42E−10	4.88

method [18]. For comparison purpose, Table 2 and Table 3 also list the computational costs by using the two methods on the unit disk. Here, we fix the number of grid points in the azimuthal direction as $N = 32$ and vary the number of grid points M in the radial direction.

As we can see that, for both the unit disk and the annulus domain, numerical solutions of ψ, w, u_r, u_θ obtained by our spectral/CCD method are all fifth-order accurate, which is much more accurate than Lai's second-order method. For instance, as shown in Table 2, in order to get the $O(10^{-5})$ error of ψ for the problem in the unit disk, current CCD method only needs to use 4 grid points in the radial direction, while the method in [18] needs to use 64 grid points.

Here, we point out that the radial and angular velocities u_r, u_θ obtained from Eq. (2.11) are fifth-order accurate since both $\hat{\psi}^n(r)$ and $\frac{\partial \hat{\psi}^n(r)}{\partial r}$ are solved simultaneously with fifth-order accuracy by our spectral/CCD method, no numerical differentiation is needed. This is actually another advantage of our spectral/CCD method.

Furthermore, as we can see that, to achieve the same accuracy, the present spectral/CCD method needs much less computational costs than Lai's method. For example, as shown in Table 3, current CCD method only needs 0.016 seconds and 4 grid points in the radial direction in order to get the absolute error 2.04×10^{-5} for ψ, while Lai's method in [18] needs 0.072 seconds and 128 grid points in the radial direction in order to get the absolute error 2.03×10^{-5} for ψ. For the same grid points in the radial direction, the computational costs by using the present spectral/CCD method are roughly 1.5 times of that by using Lai's method. Therefore, current spectral/CCD method is very efficient and accurate.

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Table 3: The L^∞ errors and CPU times (in seconds) obtained by Lai's second-order method [18] with $N = 32$ for Example 3.1 on the unit disk $0 < r < 1$.

M	CPU	$\| e_\psi \|_\infty$	rate	$\| e_w \|_\infty$	rate	$\| e_{u_r} \|_\infty$	rate	$\| e_{u_\theta} \|_\infty$	rate
4	0.014	1.59E−02	-	2.87E−01	-	3.08E−02	-	4.97E−02	-
8	0.015	4.59E−03	1.95	8.76E−02	1.86	1.01E−02	1.75	1.44E−02	1.95
16	0.017	1.23E−03	1.99	2.39E−02	1.96	3.11E−03	1.78	3.96E−03	1.95
32	0.023	3.18E−04	1.99	6.22E−03	1.99	9.13E−04	1.81	1.09E−03	1.91
64	0.037	8.07E−05	2.00	1.58E−03	2.00	2.61E−04	1.83	2.98E−04	1.89
128	0.072	2.03E−05	2.00	3.99E−04	2.00	7.32E−05	1.85	8.16E−05	1.88
256	0.139	5.11E−06	2.00	1.00E−04	2.00	2.02E−05	1.86	2.22E−05	1.88
512	0.280	1.28E−06	2.00	2.51E−05	2.00	5.53E−06	1.87	6.00E−06	1.89

Table 4: The L^∞ errors using $N = 32$ for Example 3.1 on the annulus domain $0.5 < r < 1$.

M	$\| e_\psi \|_\infty$	rate	$\| e_w \|_\infty$	rate	$\| e_{u_r} \|_\infty$	rate	$\| e_{u_\theta} \|_\infty$	rate
4	2.91E−07	-	2.59E−05	-	5.39E−07	-	2.86E−06	-
8	1.66E−08	4.87	1.50E−06	4.84	3.12E−08	4.85	1.95E−07	4.57
16	7.39E−10	4.89	6.68E−08	4.90	1.40E−09	4.88	9.68E−09	4.72
32	2.78E−11	4.94	2.51E−09	4.95	5.28E−11	4.94	3.87E−10	4.86
64	9.89E−13	4.92	9.37E−10	4.85	1.90E−12	4.91	1.47E−11	4.82

Example 3.2. Consider the problem (1.9)-(1.10) with the following exact solution

$$\psi = \frac{1}{2\pi}(1-\cos(2\pi r\cos\theta))(1-\cos(2\pi r\sin\theta)),$$
$$w = 2\pi\cos(2\pi r\cos\theta) + 2\pi\sin(2\pi r\sin\theta) - 4\pi\cos(2\pi r\cos\theta)\cos(2\pi r\sin\theta),$$
$$u_r = \sin\theta\sin(2\pi r\cos\theta)(1-\cos(2\pi r\sin\theta)) - \cos\theta\sin(2\pi r\sin\theta)(1-\cos(2\pi r\cos\theta)),$$
$$u_\theta = \cos\theta\sin(2\pi r\cos\theta)(1-\cos(2\pi r\sin\theta)) + \sin\theta\sin(2\pi r\sin\theta)(1-\cos(2\pi r\cos\theta)),$$

and f is obtained by this exact solution. This example is taken from [12] where the domain is set to be the unit disk.

For this example, we fix the number of grid points in the azimuthal direction as $N = 64$ and vary the number of grid points M in the radial direction.

Table 5 – Table 6 give the convergence study and CPU times (in seconds) by using the present spectral/CCD method and Lai's second-order method [18], respectively. Again, from Table 5 and Table 6, we can see that the current spectral/CCD method is fifth-order accurate for all the physical quantities, which is much more accurate than Lai's method. Moreover, up to the same accuracy, the computational costs of the current spectral/CCD method are much less than that of Lai's method.

4. Conclusions and Discussions

In this paper, we developed a spectral/CCD method for the Stokes equations in stream-vorticity formulation on polar geometries, including a unit disk and an annular domain. By

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Table 5: The L^∞ errors and CPU times (in seconds) obtained by the spectral/CCD method with $N = 64$ for Example 3.2 on the unit disk $0 < r < 1$.

M	CPU	$\| e_\psi \|_\infty$	rate	$\| e_w \|_\infty$	rate	$\| e_{u_r} \|_\infty$	rate	$\| e_{u_\theta} \|_\infty$	rate
32	0.10	1.41E−06	-	6.95E−05	-	1.68E−06	-	1.00E−05	-
64	0.16	6.47E−08	4.49	2.32E−06	4.96	6.30E−08	4.79	4.05E−07	4.68
128	0.30	2.37E−09	4.80	7.39E−08	5.00	2.15E−09	4.90	1.42E−08	4.87
256	0.60	7.97E−11	4.91	2.33E−09	5.00	6.99E−11	4.96	4.66E−10	4.94
512	1.19	2.56E−12	4.97	7.26E−11	5.01	2.21E−12	4.99	1.48E−11	4.98

Table 6: The L^∞ errors and CPU times (in seconds) obtained by Lai's second-order method [18] with $N = 64$ for Example 3.2 on the unit disk $0 < r < 1$.

M	CPU	$\| e_\psi \|_\infty$	rate	$\| e_w \|_\infty$	rate	$\| e_{u_r} \|_\infty$	rate	$\| e_{u_\theta} \|_\infty$	rate
32	0.08	2.69E−03	-	7.09E−02	-	4.34E−03	-	6.45E−03	-
64	0.12	6.78E−04	2.17	1.81E−02	2.14	1.10E−03	2.16	1.64E−03	2.16
128	0.23	1.70E−04	2.08	4.58E−03	2.07	2.78E−04	2.08	4.12E−04	2.08
256	0.47	4.28E−05	2.04	1.15E−03	2.04	6.97E−05	2.04	1.03E−04	2.04
512	0.95	1.07E−05	2.02	2.88E−04	2.02	1.75E−05	2.02	2.59E−05	2.02
1024	1.97	2.68E−06	2.01	7.21E−05	2.01	4.37E−06	2.01	6.48E−06	2.01
2048	4.21	6.70E−07	2.00	1.80E−05	2.00	1.09E−06	2.00	1.62E−06	2.00

using the truncated Fourier series and shifted grids as in [18], we apply the CCD method for the resulting coupled system of ordinary differential equations for the Fourier coefficients of the stream function and vorticity function. Since there is a lack of boundary conditions for vorticity function w and two fifth-order approximations for end index points of w are used, which are actually fifth-order approximations for the internal points of unknowns, our CCD method reduces to fifth-order accuracy. The fifth-order accuracy for all physical quantities, including the stream function and vorticity function as well as all velocity components, are well confirmed by our numerical tests. Numerical results show that the spectral/CCD method developed in this paper is not only much more accurate but also much more efficient than the second-order method in [18].

Finally, we point out that, although the CCD method has been successfully applied to many different differential equations, the theoretical analysis of the CCD scheme for variable coefficient and non-periodic problems is still open. To our best knowledge, very few theoretical results of the CCD scheme are available in the literature. These include the stability for the unsteady convection-diffusion equation [30] and the linear telegraph equation [8], and the stability and the convergence for the constant coefficient time-fractional advection-diffusion equation subject to periodic boundary conditions [6, 7].

In the future, we plan to use the CCD method to study the incompressible flow with nonzero Reynolds number on polar geometries [16, 34, 36]. In addition, the theoretical analysis for the proposed CCD scheme of Stokes problems will also be our future work.

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