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CUT ELIMINATION IN HYPERSEQUENT CALCULUS FOR SOME LOGICS OF LINEAR TIME

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Abstract. This is a sequel article to [10] where a hypersequent calculus (HC) for some temporal logics of linear frames including **Kt4.3** and its extensions for dense and serial flow of time was investigated in detail. A distinctive feature of this approach is that hypersequents are noncommutative, i.e., they are finite lists of sequents in contrast to other hypersequent approaches using sets or multisets. Such a system in [10] was proved to be cut-free HC formalization of respective logics by means of semantical argument. In this article we present an equivalent variant of this calculus for which a constructive syntactical proof of cut elimination is provided.

§1. Introduction. Hypersequent calculi (HC) were developed by A. Avron [1] as a generalised form of Gentzen's sequent calculus (SC), although they were introduced first (under different name) by Mints [18] and then by G. Pottinger [19]. In particular, HC proved to be very useful for construction of cut-free formalizations of many nonclassical logics, including modal, many-valued, relevant, paraconsistent and fuzzy logics.[1]

Hypersequents, the basic items of HC, are usually defined as sets or multisets of ordinary sequents. Such a notion of hypersequent works well for many logics, including some monomodal logics of linear frames (see, e.g., Indrzejczak [8, 9], Kurokawa [12] or Lahav [14]) but it is not flexible enough to cover the symmetry of future and past operators. To provide a cut-free HC system for temporal logics of linear frames we applied in [10] hypersequents defined as finite lists of sequents. In the context of temporal logics such a solution naturally corresponds to linear order of time points. As a result we got a syntactic modelling of both forward and backward transmission of data on the time axis by means of suitably defined rules for temporal operators.

We have provided in [10] a cut-free calculus with subformula-property for the minimal temporal logic of linear frames **Kt4.3** and all its extensions with additional structural rules covering features of density and past or future seriality. The problem of adequacy of this HC was investigated there by means of semantical techniques. In particular, we have shown the completeness using Schütte/Hintikka's style argument to construct a countermodel for any unprovable hypersequent out of saturated hypersequents. This semantical approach allows for developing well-behaved proof-search strategies. Moreover, it was significantly

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[1] See for example Avron [2, 3] Baaz, Ciabattoni and Fermüller [4], Metcalfe, Olivetti and Gabbay [17], Bednarska and Indrzejczak [6]. A more comprehensive treatment of the origins and applications of HC and a discussion of different strategies applied in formalization of logics of linear frames may be found in Indrzejczak [10].

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improved by Baelde, Lick and Schmitz [5] by inclusion of clusters representation yielding decision procedures of optimal complexity.

However, from the standpoint of proof theory, it is profitable to have purely syntactical, constructive proof of admissibility/elimination of cut rule. The problem of finding such a proof for noncommutative hypersequents is harder than for hypersequents being multisets or sets of sequents. Most of the techniques applied in such proofs for hypersequent calculi are of no use in the present setting. A strategy based on Schütte [20] proof of admissibility of cut for the one-sided sequent calculus was applied in [11]. It works for sequents built from finite sets but is troublesome for multisets and primitive contraction rule. Hence this approach cannot be treated as fully satisfying. Fortunately, a combination of a technique applied by Avron [3] with general strategy proposed in Metcalfe, Olivetti and Gabbay [17] and modified in Ciabattoni, Metcalfe and Montagna [7] works for suitably defined cut rules in noncommutative hypersequents. Below we provide a fully syntactical, constructive proof of cut elimination along these lines for a slightly modified variant of the HC from [10].

§2. Temporal logic. For the sake of reference we briefly recall some standard characterization of respective temporal logics, more fully described in [10]. In what follows a standard bimodal language with countable set of propositional variables *VAR*, ordinary boolean constants and two Priorean unary temporal operators: $\Box^F$ (always in the future) and $\Box^P$ (always in the past) is used. Dual operators $\Diamond^F$ and $\Diamond^P$ are treated as definitional shortcuts. The weakest linear temporal logic **Kt4.3** may be obtained by addition to any Hilbert system for classical propositional logic closed under MP (modus ponens) the following schemata:

KF $\Box^F(\varphi \to \psi) \to (\Box^F\varphi \to \Box^F\psi)$
KP $\Box^P(\varphi \to \psi) \to (\Box^P\varphi \to \Box^P\psi)$
PF $\varphi \to \Box^P\neg\Box^F\neg\varphi$
FP $\varphi \to \Box^F\neg\Box^P\neg\varphi$
4 $\Box^F\varphi \to \Box^F\Box^F\varphi$
LF $\Box^P\varphi \wedge \Box^F\varphi \wedge \varphi \to \Box^F\Box^P\varphi$
LP $\Box^P\varphi \wedge \Box^F\varphi \wedge \varphi \to \Box^P\Box^F\varphi$

and two rules of necessitation:
$\vdash \varphi \ / \ \vdash \Box^F\varphi$
$\vdash \varphi \ / \ \vdash \Box^P\varphi$

Kt4.3 may be proved adequate with respect to the class of relational frames $\langle T, R\rangle$ where acessibility relation R on the nonempty set of time points T is transitive and satisfies the condition of trichotomy:

$$\forall t, t'(Rtt' \vee Rt't \vee t = t').$$

Models $\mathfrak{M}$ are built on frames by the addition of valuation function $v : VAR \longrightarrow \mathcal{P}(T)$. Formulae are evaluated at the points of the model in the standard way by means of recursively defined satisfaction relation $\vDash$. In particular, the clauses for temporal operators are the following:

$\mathfrak{M}, t \vDash \Box^F\varphi$ iff $\mathfrak{M}, t' \vDash \varphi$ for every t' such that Rtt'
$\mathfrak{M}, t \vDash \Box^P\varphi$ iff $\mathfrak{M}, t' \vDash \varphi$ for every t' such that $Rt't$
$\mathfrak{M}, t \vDash \Diamond^F\varphi$ iff $\mathfrak{M}, t' \vDash \varphi$ for some t' such that Rtt'
$\mathfrak{M}, t \vDash \Diamond^P\varphi$ iff $\mathfrak{M}, t' \vDash \varphi$ for some t' such that $Rt't$

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A formula φ is **Kt4.3**-valid ($\models \varphi$) iff it is satisfied at every point of every **Kt4.3**-Model. A formula is falsifiable if it is nonvalid, i.e., $\mathfrak{M}, t \not\models \varphi$ for some $\mathfrak{M}$ and t.

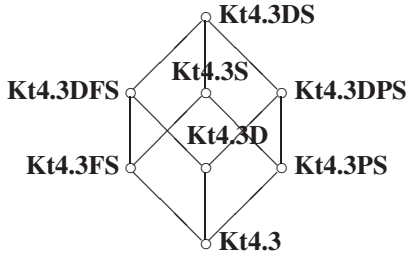
We consider extensions of **Kt4.3** which are obtained by the addition of any of the following axioms:

FS $\Box^F\varphi \rightarrow \Diamond^F\varphi$
PS $\Box^P\varphi \rightarrow \Diamond^P\varphi$
D $\Box^F\Box^F\varphi \rightarrow \Box^F\varphi$

These axioms correspond to conditions of future seriality (no ending point), past seriality (no starting point) and density of R, i.e.:

FS $\forall t, \exists t' Rtt'$
PS $\forall t, \exists t' Rt't$
D $\forall t, t'(Rtt' \rightarrow \exists t''(Rtt'' \wedge Rt''t'))$

Since the conditions are independent we obtain 7 different extensions of **Kt4.3** determined by classes of frames with R satisfying at least one of the above conditions. We will call them **Kt4.3FS**, **Kt4.3PS**, **Kt4.3D**, **Kt4.3S** (=**Kt4.3FSPS**), **Kt4.3DFS**, **Kt4.3DPS**, **Kt4.3DS** (=**Kt4.3DFSPS**). They form the following lattice:



§3. Hypersequent calculus. A formalization of all these logics in HC setting provided below is similar (and in fact, equivalent) to the system from [10]. We define hypersequents as finite lists of Gentzen's sequents and the following notation is useful in further considerations:

- $\Gamma \Rightarrow \Delta$ or s (usually with subscripts) for sequents; note that Γ, Δ are finite, possibly empty, multisets of formulae.
- $s_1 \mid \cdots \mid s_n$, G, H, G_i, H_i stand for hypersequents; in particular G, G_i will always denote nonempty hypersequent, whereas H, H_i will be used if we admit that it is empty.
- $H_1 \mid s \mid H_2$ (or $H_1 \mid \Gamma \Rightarrow \Delta \mid H_2$) stand for hypersequents with displayed sequent s (or $\Gamma \Rightarrow \Delta$). Such displayed sequents in schemata of rules applications will be called active.
- If G is $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$ and G' is $\Gamma'_1 \Rightarrow \Delta'_1 \mid \cdots \mid \Gamma'_n \Rightarrow \Delta'_n$ for $n \geq 1$, then $G \otimes G'$ denotes $\Gamma_1, \Gamma'_1 \Rightarrow \Delta_1, \Delta'_1 \mid \cdots \mid \Gamma_n, \Gamma'_n \Rightarrow \Delta_n, \Delta'_n$. Note that this operation is defined only for hypersequents having the same number of members.

The calculus for **Kt4.3** consists of the following rules:

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$$(AX)\ \ H_1 \mid p, \Gamma \Rightarrow \Delta, p \mid H_2 \qquad (Cut)\ \ \frac{H_1 \mid \Gamma \Rightarrow \Delta, \varphi \mid H_2 \qquad H_1' \mid \varphi, \Gamma' \Rightarrow \Delta' \mid H_2'}{H_1 \otimes H_1' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid H_2 \otimes H_2'}$$

$$(C\Rightarrow)\ \ \frac{H_1 \mid \varphi, \varphi, \Gamma \Rightarrow \Delta \mid H_2}{H_1 \mid \varphi, \Gamma \Rightarrow \Delta \mid H_2} \qquad (\Rightarrow C)\ \ \frac{H_1 \mid \Gamma \Rightarrow \Delta, \varphi, \varphi \mid H_2}{H_1 \mid \Gamma \Rightarrow \Delta, \varphi \mid H_2}$$

$$(\neg\Rightarrow)\ \ \frac{H_1 \mid \Gamma \Rightarrow \Delta, \varphi \mid H_2}{H_1 \mid \neg\varphi, \Gamma \Rightarrow \Delta \mid H_2} \qquad (\Rightarrow\neg)\ \ \frac{H_1 \mid \varphi, \Gamma \Rightarrow \Delta \mid H_2}{H_1 \mid \Gamma \Rightarrow \Delta, \neg\varphi \mid H_2}$$

$$(\wedge\Rightarrow)\ \ \frac{H_1 \mid \varphi, \psi, \Gamma \Rightarrow \Delta \mid H_2}{H_1 \mid \varphi \wedge \psi, \Gamma \Rightarrow \Delta \mid H_2} \qquad (\Rightarrow\wedge)\ \ \frac{H_1 \mid \Gamma \Rightarrow \Delta, \varphi \mid H_2 \qquad H_1 \mid \Gamma \Rightarrow \Delta, \psi \mid H_2}{H_1 \mid \Gamma \Rightarrow \Delta, \varphi \wedge \psi \mid H_2}$$

$$(\Rightarrow\vee)\ \ \frac{H_1 \mid \Gamma \Rightarrow \Delta, \varphi, \psi \mid H_2}{H_1 \mid \Gamma \Rightarrow \Delta, \varphi \vee \psi \mid H_2} \qquad (\vee\Rightarrow)\ \ \frac{H_1 \mid \varphi, \Gamma \Rightarrow \Delta \mid H_2 \qquad H_1 \mid \psi, \Gamma \Rightarrow \Delta \mid H_2}{H_1 \mid \varphi \vee \psi, \Gamma \Rightarrow \Delta \mid H_2}$$

$$(\Rightarrow\rightarrow)\ \ \frac{H_1 \mid \varphi, \Gamma \Rightarrow \Delta, \psi \mid H_2}{H_1 \mid \Gamma \Rightarrow \Delta, \varphi \rightarrow \psi \mid H_2} \qquad (\rightarrow\Rightarrow)\ \ \frac{H_1 \mid \Gamma \Rightarrow \Delta, \varphi \mid H_2 \qquad H_1 \mid \psi, \Gamma \Rightarrow \Delta \mid H_2}{H_1 \mid \varphi \rightarrow \psi, \Gamma \Rightarrow \Delta \mid H_2}$$

$$(\Rightarrow\Box^F)\ \ \frac{H \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \varphi}{H \mid \Gamma \Rightarrow \Delta, \Box^F\varphi} \qquad (\Box^F\Rightarrow)\ \ \frac{H_1 \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \Box^F\varphi, \varphi, \Pi \Rightarrow \Sigma \mid H_2}{H_1 \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma \mid H_2}$$

$$(\Rightarrow\Box^P)\ \ \frac{\Rightarrow \varphi \mid \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta, \Box^P\varphi \mid H} \qquad (\Box^P\Rightarrow)\ \ \frac{H_1 \mid \Box^P\varphi, \varphi, \Gamma \Rightarrow \Delta \mid \Box^P\varphi, \Pi \Rightarrow \Sigma \mid H_2}{H_1 \mid \Gamma \Rightarrow \Delta \mid \Box^P\varphi, \Pi \Rightarrow \Sigma \mid H_2}$$

$$(\Rightarrow\Box^{F'})\ \ \frac{H_1 \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \varphi \mid \Pi \Rightarrow \Sigma \mid H_2 \qquad H_1 \mid \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \varphi \mid H_2 \qquad H_1 \mid \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \Box^F\varphi \mid H_2}{H_1 \mid \Gamma \Rightarrow \Delta, \Box^F\varphi \mid \Pi \Rightarrow \Sigma \mid H_2}$$

$$(\Rightarrow\Box^{P'})\ \ \frac{H_1 \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \varphi \mid \Pi \Rightarrow \Sigma \mid H_2 \qquad H_1 \mid \Gamma \Rightarrow \Delta, \varphi \mid \Pi \Rightarrow \Sigma \mid H_2 \qquad H_1 \mid \Gamma \Rightarrow \Delta, \Box^P\varphi \mid \Pi \Rightarrow \Sigma \mid H_2}{H_1 \mid \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \Box^P\varphi \mid H_2}$$

The following differences with the system from [10] may be noted:

1. (*Cut*) is primitive in the present formulation.
2. In the present system axioms are restricted to atomic formulae as active formulae, whereas in the system from [10] this restriction is absent. This solution simplifies the proof of cut elimination.
3. $(\Box^F\Rightarrow)$ and $(\Box^P\Rightarrow)$ are slightly different in both systems. In the system from [10] they have the following form:

$$\frac{H_1 \mid \Gamma \Rightarrow \Delta \mid \cdots \mid \varphi, \Pi \Rightarrow \Sigma \mid H_2}{H_1 \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \cdots \mid \Pi \Rightarrow \Sigma \mid H_2} \qquad \frac{H_1 \mid \varphi, \Gamma \Rightarrow \Delta \mid \cdots \mid \Pi \Rightarrow \Sigma \mid H_2}{H_1 \mid \Gamma \Rightarrow \Delta \mid \cdots \mid \Box^P\varphi, \Pi \Rightarrow \Sigma \mid H_2}.$$

It may be easily noticed that these rules are nonlocal in the sense that side formula φ may be put in any sequent to the right or to the left of the active sequent. Such form of these rules is very convenient from the point of view of proof search. However, for keeping control on inferences required in the proof of cut elimination such rules are very bad. By contrast, both rules in the present system are local in the sense that φ occurs in the immediate neighbour-sequent to the right or to the left which is essential for the proof of cut elimination to be carried. However, we will show in the conclusion that both systems are in fact equivalent.

In [10] we provided a detailed analysis of the rules and rationale behind their shape so now we only briefly point out two important things concerning both systems and one connected with the shape of (*Cut*):

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1. All rules (except (*Cut*)) satisfy the subformula property.
2. Definition of hypersequents as finite lists of sequents allows for direct syntactical encoding of linear time. Informally, every sequent corresponds to some point on the time axis and if we consider a hypersequent $H_1 \mid s_i \mid \cdots \mid s_j \mid H_2$, then the time point represented by s_i is earlier than that represented by s_j. This explains why, with exception of four rules introducing temporal operators to succedents, we require that rules must have the same number of sequents in all involved hypersequents. Semantically speaking this provides an encoding of the same partial interpretation. In case of abovementioned temporal rules the introduction of additional sequent in (one) premiss corresponds to the situation where a new time point extends an interpretation encoded by the conclusion.
3. In the light of the previous remark one may wonder why (*Cut*) is different from other two-premiss rules. In fact, the operation $\otimes$ is used only in the schema of (*Cut*) but still allowed only if the number of sequents occurring in both premisses is the same, in accordance with aforementioned semantical motivation. One may of course use a simpler form of (*Cut*), similar to other two-premiss rules, i.e., having both premisses identical to the conclusion (i.e., with only addition of cut formula in one of sequent displayed as active in the schema). Such additive form of cut was considered in [11] where essentially different strategy of proof of cut elimination was examined, strongly based on invertibility of rules. The present proof does not appeal to invertibility of rules but requires greater flexibility of combining hypersequents. Still we must have the same number of sequents in two hypersequents which are to be combined by means of (*Cut*). So the choice of such multiplicative form of (*Cut*) is a reasonable compromise.

The proof of a hypersequent G is defined in the usual way as a tree of hypersequents with G as the root, axioms as leaves and all other nodes regulated by the application of rules. The height of a proof of a hypersequent is defined as the number of nodes in the longest branch of this proof.

Example proof of the instance of *LF* looks like that.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box^P p, \Box^F p, p \Rightarrow \mid \Box^F p, p \Rightarrow p \mid \Rightarrow}{\Box^P p, \Box^F p, p \Rightarrow \mid \Rightarrow p \mid \Rightarrow}\,(\Box^F \Rightarrow) \qquad \Box^P p, \Box^F p, p \Rightarrow p \mid \Rightarrow \qquad D}{\Box^P p, \Box^F p, p \Rightarrow \mid \Rightarrow \Box^P p}\,(\Rightarrow \Box^{P'})}{\Box^P p, \Box^F p, p \Rightarrow \Box^F \Box^P p}\,(\Rightarrow \Box^F)}{\Box^P p \wedge \Box^F p \wedge p \Rightarrow \Box^F \Box^P p}\,(\wedge \Rightarrow) \text{ twice}}{\Rightarrow \Box^P p \wedge \Box^F p \wedge p \rightarrow \Box^F \Box^P p}\,(\Rightarrow \rightarrow)$$

where D represents the proof of the rightmost premiss:

$$\dfrac{\dfrac{\dfrac{\Box^P p, p \Rightarrow p \mid \Box^P p, \Box^F p, p \Rightarrow \mid \Rightarrow}{\Rightarrow p \mid \Box^P p, \Box^F p, p \Rightarrow \mid \Rightarrow}\,(\Box^P \Rightarrow)}{\Box^P p, \Box^F p, p \Rightarrow \Box^P p \mid \Rightarrow.}\,(\Rightarrow \Box^P)$$

By the addition of any of the following structural rules we obtain suitable extensions of our calculus:

$$(PS)\ \dfrac{\Rightarrow \mid G}{G} \text{ or } (FS)\ \dfrac{G \mid \Rightarrow}{G} \text{ or } (D)\ \dfrac{G_1 \mid \Rightarrow \mid G_2}{G_1 \mid G_2}.$$

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These rules cover past seriality (i.e., no starting point), future seriality (or no ending point) and density. Cut-free deductions of suitable axioms by means of these rules is a routine matter as the reader may easily check.

In what follows we will denote with **HCL** $\vdash G$, or with $\vdash G$ simply, the provability of G in **HCL**, where **L** is either **Kt4.3.** or any of its seven extensions obtained by the addition of at least one of the (*PS*), (*FS*), (*D*).

§4. Soundness. We recall briefly the principles of interpretation of noncommutative hypersequents after [10].

Satisfaction of a sequent at a state in a model is defined in a standard way:

$\mathfrak{M}, t \vDash \Gamma \Rightarrow \Delta$ iff $\mathfrak{M}, t \vDash \wedge\Gamma \rightarrow \vee\Delta$,

and

$\mathfrak{M}, t \nvDash \Gamma \Rightarrow \Delta$ iff $\mathfrak{M}, t \nvDash \wedge\Gamma \rightarrow \vee\Delta$,

These notions extend to noncommutative hypersequents in the following way:

DEFINITION 4.1. *For any* **Kt4.3***-model* $\mathfrak{M}$ *and hypersequent* $G = s_1 \mid \cdots \mid s_n$:

- $\mathfrak{M} \models G$ *iff for all states* $t_1, \ldots t_n$ *of* $\mathfrak{M}$:
  if $t_1 R t_2 R \ldots R t_n$, *then for some* $i \leq n$, $\mathfrak{M}, t_i \vDash s_i$;
- $\models G$ (*G is* **Kt4.3***-valid*) *iff* $\mathfrak{M} \models G$ *for every* $\mathfrak{M}$.

Note that in consequence: $\not\models G$ iff there is $\mathfrak{M}$ such that $\mathfrak{M} \not\models G$ and this means that there are $t_1, \ldots t_n$ such that $t_1 R t_2 R \ldots R t_n$ and $t_1 \nvDash s_1, \ldots t_n \nvDash s_n$. The same applies to all considered extensions of **Kt4.3**.

Since HC calculus introduced in this article is a variant of the calculus from [10] there is no problem with showing its soundness. One should only show that the present rules ($\Box^F\Rightarrow$) and ($\Box^P\Rightarrow$) and the version of cut are validity-preserving under the interpretation of hypersequent specified above. Hence we state without a proof:

LEMMA 4.2 (Validity-preservation). *All rules are validity-preserving* (*in suitable logics*)

As a consequence we obtain:

THEOREM 4.3 (Soundness). *If* **HCL** $\vdash G$, *then* $\models G$ *in any logic* **L**.

§5. Preliminary results. Before proving cut elimination theorem we must provide proofs of some auxiliary results. In particular, we must provide proofs of admissibility of structural rules of weakening:

$$(IW)\ \frac{H_1 \mid \Gamma \Rightarrow \Delta \mid H_2}{H_1 \mid \Gamma' \Rightarrow \Delta' \mid H_2} \qquad (EW)\ \frac{H_1 \mid H_2}{H_1 \mid \Rightarrow \mid H_2}$$

where in (*IW*) (internal weakening) $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$ (at least one strictly subsumed in the sense of multiset inclusion) and in (*EW*) (external weakening) at least one of H_1, H_2 is nonempty. Notice that by (*EW*) and (*IW*) we may introduce any sequent s as an additional element of a given hypersequent

In what follows we provide as examples only proofs for the cases of $\Box^F$ leaving "mirror" proofs for $\Box^P$ to the reader. Also we will use an abbreviation (*IC*) for both (internal) contraction rules.

The following result holds for **HCL**:

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LEMMA 5.1. 1. $\vdash H_1 \mid \varphi, \Gamma \Rightarrow \Delta, \varphi \mid H_2$;

2. (*IW*) *is height-preserving admissible*;

3. (*EW*) *is admissible.*

Proof. ad. 1: By induction on the complexity of φ. In case of boolean formulae it is standard. For $\Box^F\varphi$ we need subsidiary induction on the number of sequents to the right of the active sequent (in case of $\Box^P\varphi$ we perform induction on the number of sequents to the left). The basis is simple:

$$\dfrac{\dfrac{G \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \Box^F\varphi, \varphi \Rightarrow \varphi}{G \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \Rightarrow \varphi}\,(\Box^{\mathrm{F}} \Rightarrow)}{G \mid \Box^F\varphi, \Gamma \Rightarrow \Delta, \Box^F\varphi}\,(\Rightarrow \Box^{\mathrm{F}})$$

where the top-hypersequent is derivable by the induction hypothesis. For induction step consider the following application of $(\Rightarrow \Box^{\mathrm{F}'})$:

$$\dfrac{G \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \Rightarrow \varphi \mid \Theta \Rightarrow \Lambda \mid H \qquad G \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda, \varphi \mid H \qquad G \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda, \Box^F\varphi \mid H}{G \mid \Box^F\varphi, \Gamma \Rightarrow \Delta, \Box^F\varphi \mid \Theta \Rightarrow \Lambda \mid H}$$

To each of the premisses we may apply $(\Box^{\mathrm{F}} \Rightarrow)$ in the following way:

$$\dfrac{G \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \Box^F\varphi, \varphi \Rightarrow \varphi \mid \Theta \Rightarrow \Lambda \mid H}{G \mid \Box^F\varphi, \Gamma \Rightarrow \Delta, \mid \Rightarrow \varphi \mid \Theta \Rightarrow \Lambda \mid H}\,(\Box^{\mathrm{F}} \Rightarrow)$$

$$\dfrac{G \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \Box^F\varphi, \varphi, \Theta \Rightarrow \Lambda, \varphi \mid H}{G \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda, \varphi \mid H}\,(\Box^{\mathrm{F}} \Rightarrow)$$

and

$$\dfrac{G \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \Box^F\varphi, \varphi, \Theta \Rightarrow \Lambda, \Box^F\varphi \mid H}{G \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda, \Box^F\varphi \mid H}\,(\Box^{\mathrm{F}} \Rightarrow).$$

In the first and the second case the starting hypersequent is provable by the induction hypothesis of the main induction. In the third case the number of the sequents to the right was decreased in the premiss so this hypersequent is also provable by the induction hypothesis of the subsidiary induction.

ad. 2. This is trivial since all rules, including temporal and the additional structural ones, are context insensitive and the addition of new formulae does not harm their correctness. Note also that no new applications of cut are required.

ad. 3. The proof is by induction on the height of the proof of $H_1 \mid H_2$. The problem is only with temporal rules since all of them assume dependence of active sequents on their neighbours. There are three cases for $\Box^F\varphi$ and three mirror cases for $\Box^P\varphi$. In [10] (Lemma 6.1.) we provided a proof of admissibility of a similar form of (*EW*) in the presence of $(\Rightarrow\Box^F)$ and $(\Rightarrow\Box^{F'})$ and these rules are the same in the present system. So we must consider as new only the case of $(\Box^F\Rightarrow)$ which is different (suitable rules of [10] were unproblematic in this respect). However, for the reader's convenience, we also repeat below the elaborate arguments for $(\Rightarrow\Box^F)$ and $(\Rightarrow\Box^{F'})$.

Let us start with $(\Box^F\Rightarrow)$. In fact, the only troublesome situation is when we want to add a new sequent immediately to the right of the active sequent. Let H_1 denote $G \mid \Box^F\varphi, \Gamma \Rightarrow \Delta$ and H_2 be $\Theta \Rightarrow \Lambda \mid H$. We obtain the result as follows:

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$$\dfrac{\dfrac{\dfrac{G \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \Rightarrow \mid \Box^F\varphi, \varphi, \Theta \Rightarrow \Lambda \mid H}{G \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \Box^F\varphi, \varphi \Rightarrow \mid \Box^F\varphi, \varphi, \Theta \Rightarrow \Lambda \mid H}\,(IW)}{G \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \Box^F\varphi, \varphi \Rightarrow \mid \Theta \Rightarrow \Lambda \mid H}\,(\Box^{\mathrm{F}} \Rightarrow)}{G \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \Rightarrow \mid \Theta \Rightarrow \Lambda \mid H}\,(\Box^{\mathrm{F}} \Rightarrow)$$

The first line is by the induction hypothesis ($\Rightarrow$ inserted). This example (as well as cases of $(\Rightarrow\Box^F)$ and $(\Rightarrow\Box^{F'})$) shows why (EW) is not height-preserving admissible; the additional application of $(\Box^{\mathrm{F}} \Rightarrow)$ increases the height of this proof.

Now consider the case of $(\Rightarrow\Box^F)$. Let our H_2 be empty and $H_1 = H_1' \mid \Gamma \Rightarrow \Delta, \Box^F\varphi$ obtained by $(\Rightarrow\Box^F)$ from $H_1' \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \varphi$. We must show that the addition of $\Rightarrow$ to the right is also provable. This may be provable only via $(\Rightarrow\Box^{F'})$ in the following way:

$$\dfrac{H_1' \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \varphi \mid \Rightarrow \qquad H_1' \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \varphi \qquad H_1' \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \Box^F\varphi}{H_1' \mid \Gamma \Rightarrow \Delta, \Box^F\varphi \mid \Rightarrow}$$

Hence we must provide proofs of the three required premisses from the original one. But the central premiss is just the original premiss whereas the leftmost one is obtained from it by the induction hypothesis. Finally, from the original premiss we get $H_1' \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \mid \Rightarrow \varphi$ by the induction hypothesis, and then by $(\Rightarrow\Box^F)$ we get the rightmost premiss; again this last step destroys preservation of the height.

The case of $(\Rightarrow\Box^{F'})$ is also problematic when we want to insert an additional sequent immediately after the active sequent, i.e., if $H_1 = H_1' \mid \Gamma \Rightarrow \Delta, \Box^F\varphi$ and $H_2 = \Pi \Rightarrow \Sigma \mid H_2'$. Therefore what we have is:

$$\dfrac{H_1' \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \varphi \mid \Pi \Rightarrow \Sigma \mid H_2' \qquad H_1' \mid \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \varphi \mid H_2' \qquad H_1' \mid \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \Box^F\varphi \mid H_2'}{H_1' \mid \Gamma \Rightarrow \Delta, \Box^F\varphi \mid \Pi \Rightarrow \Sigma \mid H_2'}$$

and we want to prove $H_1' \mid \Gamma \Rightarrow \Delta, \Box^F\varphi \mid \Rightarrow \mid \Pi \Rightarrow \Sigma \mid H_2'$. This may be done by $(\Rightarrow \Box^{\mathrm{F}'})$ from the following premisses:

(1) $H_1' \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \varphi \mid \Rightarrow \mid \Pi \Rightarrow \Sigma \mid H_2'$,
(2) $H_1' \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \varphi \mid \Pi \Rightarrow \Sigma \mid H_2'$ and
(3) $H_1' \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \Box^F\varphi \mid \Pi \Rightarrow \Sigma \mid H_2'$.

By the induction hypothesis we get (1) from the leftmost premiss which is also just (2). It remains to prove (3). We can do this again by $(\Rightarrow \Box^{\mathrm{F}'})$ on the basis of the following premisses:

(1') $H_1' \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \mid \Rightarrow \varphi \mid \Pi \Rightarrow \Sigma \mid H_2'$
(2') $H_1' \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \mid \Pi \Rightarrow \Sigma, \varphi \mid H_2'$ and
(3') $H_1' \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \mid \Pi \Rightarrow \Sigma, \Box^F\varphi \mid H_2'$.

It suffices to notice that all follow from the original premisses by the induction hypothesis.

Note that the addition of extra structural rules (P), (FS) or (D) does not spoil the proof of admissibility of these structural rules. Also in case of the proof of (EW) no extra applications of cut are needed. □

REMARK. *One may wonder what is the situation with external contraction rule which is often considered in hypersequent calculi. In this case we cannot prove its admissibility,*

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rather the contrary holds due to the way we semantically interpret noncommutative hypersequents. We may prove that it is not sound in our calculi since it is valid in reflexive frames.[2] *One may easily check it semantically applying the interpretation of hypersequents described in §4. Here we will show that its addition to any of our* **HCL** *leads to provability of axiom T which is nonvalid in considered temporal logics.*

$$\frac{\Box^F\varphi \Rightarrow \mid \Box^F\varphi, \varphi \Rightarrow \varphi}{\Box^F\varphi \Rightarrow \mid \Rightarrow \varphi}\ (\Box^{\mathrm{F}} \Rightarrow) \qquad \frac{\Box^F\varphi \Rightarrow \varphi \mid \Box^F\varphi \Rightarrow \varphi}{\Box^F\varphi \Rightarrow \varphi}\ (EC)$$

$$\frac{\Box^F\varphi \Rightarrow \mid \Rightarrow \varphi}{\Box^F\varphi \Rightarrow \varphi}\ (IW)$$

In fact (*EW*) (*at least in the variant with both H nonempty*) *also expresses some semantic property, namely transitivity. We could simplify both* $(\Box^F \Rightarrow)$ *and* $(\Box^P \Rightarrow)$ *by deleting the copies of principal formulae in the premisses and add instead* (*EW*) *as a primitive rule. However such a solution destroys the possibility of proving cut elimination, at least by the method presented in the next section.*

Since (*Cut*) is primitive in the present version and we have proved also the general form of axiomatic hypersequents in Lemma 5.1, the completeness is provable in a routine way. One should only prove the remaining axioms. Both rules of necessitation are easily simulated with the help of (*EW*). Hence we can conclude:

THEOREM 5.2. **HCL** *is adequate with respect to all considered logics.*

§6. Cut elimination. Now we provide a constructive syntactical proof that (*Cut*) is eliminable for all 8 considered logics.

In the following proof we will combine a technique of Avron [3] with the strategy applied in Ciabattoni, Metcalfe, Montagna proof in [7] which is based on the general approach described by Metcalfe, Olivetti and Gabbay [17]. Avron [3] introduced a technique of separating modal cut formulae as a special case of multiplicative cut covering external contraction. Although there is no EC in our system (see the remark in the preceding section), still the construction of rules requires cutting on more than one sequent at once. This is only needed in one lemma but the crucial one. The whole proof is based on the strategy from [7] which already has shown its usefulness for cut elimination proofs in hypersequent calculi for modal logics (see: Kurokawa [12], Indrzejczak [9], Lellmann [13, 15]), so it seems to offer the simplest solution for cut elimination in our calculus as well.

Let $|\varphi|$ denote the *complexity* of any formula φ which is counted as the number of occurrences of its connectives and $[\varphi]^i$ denote a multiset consisting of i occurrences of φ. We define the cut rank $\rho(D)$ of a derivation D as the maximal complexity of cut formulae in D plus 1. Any cut-free derivation D has $\rho(D) = 0$. Also let H be an abbreviation for $[\varphi]^{i_1}, \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid [\varphi]^{i_n}, \Gamma_n \Rightarrow \Delta_n$, $n \geq 1$ and for every $k \leq n$ let $i_k \geq 0$ but with φ not included in any Γ_k, then $H - \varphi$ denotes $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$. Similarly $\Gamma - \varphi$ denotes the result of deleting all occurrences of φ from Γ if there are any.

We need to prove two lemmata.

LEMMA 6.1. *Let* D_l *and* D_r *be derivations in* **HCL** *such that*:

[2] Similar observation was made by Lellmann in [16].

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1. *D_r is a derivation of $H_1' \mid [\varphi]^i, \Gamma' \Rightarrow \Delta' \mid H_2'$;*
2. *D_l is a derivation of $H_1 \mid \Gamma \Rightarrow \Delta, \varphi \mid H_2$;*
3. *$\rho(D_l) \leq |\varphi|$ and $\rho(D_r) \leq |\varphi|$;*
4. *φ is the principal formula of the last applied logical rule in D_l.*

Then a derivation D with $\rho(D) \leq |\varphi|$ in respective **HCL** *can be constructed of $H_1 \otimes H_1^* \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid H_2 \otimes H_2^*$ where $H_1^* = H_1' - \varphi$ if $\varphi = \Box^P \psi$ and $H_2^* = H_2' - \varphi$ if $\varphi = \Box^F \psi$; otherwise $H_1^* = H_1'$ and $H_2^* = H_2'$.*

Proof. The proof runs by induction on the height of D_r. If it is an axiom then we are done trivially, since φ being compound may be only parametric. Otherwise we must analyse the last rule applied in D_r. We will consider in part A. some cases which take place when all occurrences of cut formula are parametric and in part B. when one occurrence of cut formula is principal in the right premiss. Note that admissible structural rules of weakening preserve cut rank of transformed proofs since no new cuts were required in the proof of their admissibility.

A. In general if φ is parametric, then we apply the induction hypothesis to premisses and then a respective rule since all rules are context insensitive. However we must take care of three possible complications which additionally deserve extra applications of structural rules.

1. The last rule by which $H_1' \mid [\varphi]^i, \Gamma' \Rightarrow \Delta' \mid H_2'$ was derived is cut. By assumption $\rho(D_l) \leq |\varphi|$ so cut formula is not φ and by the induction hypothesis we obtain two premisses of this application of cut with all occurrences of φ deleted either in the antecedent of active sequent or, in addition, in antecedents of all sequents to the left or to the right of $\Gamma' \Rightarrow \Delta'$. After application of cut to new premisses we obtain $H_1 \otimes H_1 \otimes H_1^* \mid \Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta' \mid H_2 \otimes H_2 \otimes H_2^*$ and we get a result after several applications of (IC).

2. φ may be a side formula of the applied rule in the active sequent and is deleted by the induction hypothesis. Now we must apply (IW) to reintroduce φ in the proper place before we can apply respective rule. We will illustrate this point with the application of $(\Rightarrow\rightarrow)$ using φ as a side formula in the last step of the proof of the right premiss. The situation looks like this:

$$(\Rightarrow\rightarrow)\ \frac{H_1' \mid [\varphi]^{i+1}, \Gamma' \Rightarrow \Delta', \psi \mid H_2'}{H_1' \mid [\varphi]^i, \Gamma' \Rightarrow \Delta', \varphi \rightarrow \psi \mid H_2'}$$

after the application of the induction hypothesis we proceed as follows:

$$(\Rightarrow\rightarrow)\ \frac{(IW)\ \dfrac{H_1 \otimes H_1^* \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta', \psi \mid H_2 \otimes H_2^*}{H_1 \otimes H_1^* \mid \varphi, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', \psi \mid H_2 \otimes H_2^*}}{H_1 \otimes H_1^* \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta', \varphi \rightarrow \psi \mid H_2 \otimes H_2^*}$$

3. The last rule was one of the $(\Rightarrow \Box^{\mathrm{F}})$, $(\Rightarrow \Box^{\mathrm{F}'})$, $(\Rightarrow \Box^{\mathrm{P}})$, $(\Rightarrow \Box^{\mathrm{P}'})$. In this case (one of) the premiss(es) is one sequent longer hence to apply the induction hypothesis we must first apply (EW) to $H_1 \mid \Gamma \Rightarrow \Delta, \varphi \mid H_2$ and add $\Rightarrow$ in suitable place. Then we apply the induction hypothesis and then the respective rule to new premisses and obtain $H_1 \otimes H_1^* \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid H_2 \otimes H_2^*$. The same remark applies to extensions of **HCKt4.3** if the last rule was one of the (D), (FS), (PS); we must also apply (EW) to the left premiss. The addition of $\Rightarrow$ as an additional element of the hypersequent ending the left premiss is required to have the same number of sequents in both premisses. However we must take care for the assumption 4. of our lemma; φ must be still principal in the new left premiss.

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It is not problematic if φ is boolean or even if it is a temporal formula but an additional sequent is added in a place which is not in the neighbourhood of the active sequent. In such case we add $\Rightarrow$ in the proper place of the premiss(es) of $H_1 \mid \Gamma \Rightarrow \Delta, \varphi \mid H_2$ and apply the rule introducing φ to the succedent of $\Gamma \Rightarrow \Delta$.

More problematic are all cases where $\Rightarrow$ must be added as a neighbour of the active sequent of some temporal rule introducing φ in such a way that $H_1 \mid \Gamma \Rightarrow \Delta, \varphi \mid H_2$ cannot be properly deduced from modified premisses with φ as principal formula. To repair the situation some adjustments are needed which we illustrate with two examples; the remaining ones are similar.

3.1. Consider a case of $\varphi = \Box^F \psi$, then D_l ends with the following derivation:

$$\frac{H_1 \mid \Pi \Rightarrow \Sigma \mid \Rightarrow \psi \mid \Gamma \Rightarrow \Delta \mid H_2 \qquad H_1 \mid \Pi \Rightarrow \Sigma \mid \Gamma \Rightarrow \Delta, \psi \mid H_2 \qquad H_1 \mid \Pi \Rightarrow \Sigma \mid \Gamma \Rightarrow \Delta, \Box^F \psi \mid H_2}{H_1 \mid \Pi \Rightarrow \Sigma, \Box^F \psi \mid \Gamma \Rightarrow \Delta \mid H_2}$$

and let D_r ends with the application of structural rule (D) in such a way:

$$\frac{H_1' \mid [\Box^F \psi]^i, \Pi' \Rightarrow \Sigma' \mid \Rightarrow \mid \Gamma' \Rightarrow \Delta' \mid H_2'}{H_1' \mid [\Box^F \psi]^i, \Pi' \Rightarrow \Sigma' \mid \Gamma' \Rightarrow \Delta' \mid H_2'}$$

In order to satisfy assumption 4. of the lemma we first apply (EW) to three premisses of the original application of $(\Rightarrow \Box^{\mathrm{F}'})$ and obtain:

$G_1 = H_1 \mid \Pi \Rightarrow \Sigma \mid \Rightarrow \mid \Rightarrow \psi \mid \Gamma \Rightarrow \Delta \mid H_2$
$G_2 = H_1 \mid \Pi \Rightarrow \Sigma \mid \Rightarrow \mid \Gamma \Rightarrow \Delta, \psi \mid H_2$
$G_3 = H_1 \mid \Pi \Rightarrow \Sigma \mid \Rightarrow \mid \Gamma \Rightarrow \Delta, \Box^F \psi \mid H_2$

Moreover from the leftmost premiss by (EW) we also obtain:

$G_4 = H_1 \mid \Pi \Rightarrow \Sigma \mid \Rightarrow \psi \mid \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H_2$

Finally we can combine them in the following way:

$$\dfrac{G_4 \qquad H_1 \mid \Pi \Rightarrow \Sigma \mid \Rightarrow \psi \mid \Gamma \Rightarrow \Delta \mid H_2 \qquad \dfrac{G_1 \qquad G_2 \qquad G_3}{H_1 \mid \Pi \Rightarrow \Sigma \mid \Rightarrow \Box^F \psi \mid \Gamma \Rightarrow \Delta \mid H_2}\,(\Rightarrow \Box^{\mathrm{F}'})}{H_1 \mid \Pi \Rightarrow \Sigma, \Box^F \psi \mid \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H_2}\,(\Rightarrow \Box^{\mathrm{F}'})$$

where the central premiss is just the leftmost premiss of the input inference and the last hypersequent satisfies all requirements of the lemma. Hence we may apply the induction hypothesis to $H_1' \mid [\Box^F \psi]^i, \Pi' \Rightarrow \Sigma' \mid \Rightarrow \mid \Gamma' \Rightarrow \Delta' \mid H_2'$ and then by (D) the required result.

3.2. Moreover, in some cases, we must change the application of $(\Rightarrow \Box^{\mathrm{F}})$ or $(\Rightarrow \Box^{\mathrm{P}})$ into $(\Rightarrow \Box^{\mathrm{F}'})$ or $(\Rightarrow \Box^{\mathrm{P}'})$. However, φ is still principal after transformation. Let us illustrate this with the case where the right premiss comes by $(\Rightarrow \Box^{\mathrm{F}})$ on the active sequent, thus we have:

$$(\Rightarrow \Box^{\mathrm{F}})\ \frac{H_1' \mid [\Box^F \psi]^i, \Gamma' \Rightarrow \Delta' \mid \Rightarrow \chi}{H_1' \mid [\Box^F \psi]^i, \Gamma' \Rightarrow \Delta', \Box^F \chi}$$

and the left premiss is also by $(\Rightarrow \Box^{\mathrm{F}})$:

$$(\Rightarrow \Box^{\mathrm{F}})\ \frac{H_1 \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \psi}{H_1 \mid \Gamma \Rightarrow \Delta, \Box^F \psi}$$

Now, to apply the induction hypothesis we must have $H_1 \mid \Gamma \Rightarrow \Delta, \Box^F \psi \mid \Rightarrow$ but to satisfy assumption 4 of the lemma we must have $\Box^F \psi$ still principal. This is only possible by $(\Rightarrow \Box^{\mathrm{F}'})$. To obtain suitable proof we use not only the original premiss but also apply

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to it (EW) in two different ways and then $(\Rightarrow \Box^{\mathrm{F}})$ to one of them in order to get three premisses for the application of $(\Rightarrow \Box^{\mathrm{F}'})$ introducing $H_1 \mid \Gamma \Rightarrow \Delta, \Box^F \psi \mid \Rightarrow$. As we noted cut rank is preserved despite the application of admissible structural rule. The new proof of thus transformed left premiss looks like this:

$$\dfrac{(EW)\dfrac{H_1 \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \psi}{H_1 \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \psi \mid \Rightarrow} \qquad H_1 \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \psi \qquad \dfrac{\dfrac{\dfrac{H_1 \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \psi}{H_1 \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \mid \Rightarrow \psi}(EW)}{H_1 \mid \Gamma \Rightarrow \Delta \mid \Rightarrow \Box^F \psi}(\Rightarrow \Box^{\mathrm{F}})}{H_1 \mid \Gamma \Rightarrow \Delta, \Box^F \psi \mid \Rightarrow}(\Rightarrow \Box^{\mathrm{F}'})$$

We apply this hypersequent and the induction hypothesis to the right premiss and obtain the required result:

$$(\Rightarrow \Box^{\mathrm{F}}) \dfrac{H_1 \otimes H_1' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid \Rightarrow \chi}{H_1 \otimes H_1' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta', \Box^F \chi.}$$

B. In case of one of the occurrences of φ principal, in general we must refer to the fact that φ is principal in $H_1 \mid \Gamma \Rightarrow \Delta, \varphi \mid H_2$ and use additionally one or two cuts with premisse(s) of it. We will illustrate the problem with one boolean and one temporal case. The other cases can be handled in a similar way.

1. Let us consider as an example the case of $\varphi = \psi \vee \chi$. We have the following situation – D_l ends with:

$$(\Rightarrow \vee) \dfrac{H_1 \mid \Gamma \Rightarrow \Delta, \psi, \chi \mid H_2}{H_1 \mid \Gamma \Rightarrow \Delta, \psi \vee \chi \mid H_2}$$

and D_r ends with:

$$\dfrac{H_1' \mid \psi, [\psi \vee \chi]^i, \Gamma' \Rightarrow \Delta' \mid H_2' \qquad H_1' \mid \chi, [\psi \vee \chi]^i, \Gamma' \Rightarrow \Delta' \mid H_2'}{H_1' \mid [\psi \vee \chi]^{i+1}, \Gamma' \Rightarrow \Delta' \mid H_2'.}(\vee \Rightarrow)$$

By the induction hypothesis we obtain derivations of:

G_1 of the form $H_1 \otimes H_1^* \mid \psi, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid H_2 \otimes H_2^*$ and
G_2 of the form $H_1 \otimes H_1^* \mid \chi, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid H_2 \otimes H_2^*$.

The following proof provides the expected result:

$$\dfrac{\dfrac{(Cut)\dfrac{H_1 \mid \Gamma \Rightarrow \Delta, \psi, \chi \mid H_2 \qquad G_2.}{H_1 \otimes H_1 \otimes H_1^* \mid \Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta', \psi \mid H_2 \otimes H_2 \otimes H_2^*.} \qquad G_1}{H_1 \otimes H_1 \otimes H_1 \otimes H_1^* \otimes H_1^* \mid \Gamma, \Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta, \Delta', \Delta' \mid H_2 \otimes H_2 \otimes H_2 \otimes H_2^* \otimes H_2^*.}(Cut)(IC)$$

Note that ψ and χ being subformulae of φ guarantee that the cut rank of this new derivation is $\leq |\varphi|$.

2. Let us consider the case of $\varphi = \Box^P \psi$ principal on the right (and on the left by assumption of the lemma). Then we have the following last steps of the two derivations:

$$\dfrac{H_1 \mid \Pi \Rightarrow \Sigma \mid \Rightarrow \psi \mid \Gamma \Rightarrow \Delta \mid H_2 \qquad H_1 \mid \Pi \Rightarrow \Sigma, \psi \mid \Gamma \Rightarrow \Delta \mid H_2 \qquad H_1 \mid \Pi \Rightarrow \Sigma, \Box^P \psi \mid \Gamma \Rightarrow \Delta \mid H_2}{H_1 \mid \Pi \Rightarrow \Sigma \mid \Gamma \Rightarrow \Delta, \Box^P \psi \mid H_2}$$

and

$$\dfrac{H_1' \mid \psi, \Box^P \psi, \Pi' \Rightarrow \Sigma' \mid [\Box^P \psi]^i, \Gamma' \Rightarrow \Delta' \mid H_2'}{H_1' \mid \Pi' \Rightarrow \Sigma' \mid [\Box^P \psi]^i, \Gamma' \Rightarrow \Delta' \mid H_2'.}(\Box^{\mathrm{P}} \Rightarrow)$$

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By the induction hypothesis we get:
$H_1 \otimes H_1^* \mid \psi, \Pi, \Pi' - \Box^P\psi \Rightarrow \Sigma, \Sigma' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid H_2 \otimes H_2^*$
where H_1^* is $H_1' - \Box^P\psi$ and H_2^* is just H_2', and we construct the following derivation:

$$\dfrac{\dfrac{H_1 \mid \Pi \Rightarrow \Sigma, \psi \mid \Gamma \Rightarrow \Delta \mid H_2 \qquad H_1 \otimes H_1^* \mid \psi, \Pi, \Pi' - \Box^P\psi \Rightarrow \Sigma, \Sigma' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid H_2 \otimes H_2^*}{H_1 \otimes H_1 \otimes H_1^* \mid \Pi, \Pi, \Pi' - \Box^P\psi \Rightarrow \Sigma, \Sigma, \Sigma' \mid \Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta' \mid H_2 \otimes H_2 \otimes H_2^*}\,(Cut)}{H_1 \otimes H_1^* \mid \Pi, \Pi, \Pi' - \Box^P\psi \Rightarrow \Sigma, \Sigma' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid H_2 \otimes H_2^*}\,(IC)$$

which yields our expected result by derivation with cut rank $\leq |\varphi|$. □

REMARK. *Before we go further with the proof perhaps some comments on the form of* $H_1 \otimes H_1^* \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid H_2 \otimes H_2^*$ *postulated in our lemma will be helpful. In case of* φ *not being of the form* $\Box^F\psi$ *or* $\Box^P\psi$ *it is just cut with deletion of all occurrences of* φ *in the antecedent of the active sequent only. This is just to take care of* (*IC*) *on* φ *as the last step of* D_r. *The clauses for* φ *being* $\Box^P\psi$ *or* $\Box^F\psi$ *require additional deletion of all occurrences of* φ *in sequents to the left of the active sequent in the former, and to the right in the latter case. This is based on the solution of Avron* [3] *but specified here for different semantic interpretation of the notion of hypersequent. In Avron's, as well as in other proofs recorded in introductory remarks, such a solution was dictated by troubles with* (*EC*). *Here, a limited version follows from the construction of temporal rules only. We can in fact extract from the proof two specific rules of temporal multicut of the form*:

$$(Cut\Box^F)\ \dfrac{H_1 \mid \Gamma \Rightarrow \Delta, \Box^F\varphi \mid H_2 \qquad H_1' \mid [\Box^F\varphi]^i, \Gamma' \Rightarrow \Delta' \mid H_2'}{H_1 \otimes H_1' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid H_2 \otimes H_2^*}$$

$$(Cut\Box^P)\ \dfrac{H_1 \mid \Gamma \Rightarrow \Delta, \Box^P\varphi \mid H_2 \qquad H_1' \mid [\Box^P\varphi]^i, \Gamma' \Rightarrow \Delta' \mid H_2'}{H_1 \otimes H_1^* \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid H_2 \otimes H_2'}$$

Although a proof of the previous lemma demonstrates syntactically their admissibility it is also easy to show that they are validity-preserving. Let us sketch this for the case of $(Cut\Box^F)$. *If we assume that the conclusion is invalid, then, for the left premiss to be valid, we must have, under the same interpretation,* $\Box^F\psi$ *true in the succedent of the active sequent. But then the right premiss is falsified under this interpretation since* $\Box^F\psi$ *is true in the antecedent of the active sequent and, by transitivity, in the antecedents of all sequents on the right. So both premisses cannot be valid, if the conclusion is invalid.*

LEMMA 6.2. *Let* D_l *and* D_r *be derivations in* **HCL** *such that*:

1. D_r *is a derivation of* $H_1' \mid \varphi, \Gamma' \Rightarrow \Delta' \mid H_2'$;
2. D_l *is a derivation of* $H_1 \mid \Gamma \Rightarrow \Delta, [\varphi]^i \mid H_2$;
3. $\rho(D_l) \leq |\varphi|$ *and* $\rho(D_r) \leq |\varphi|$.

Then a derivation D *with* $\rho(D) \leq |\varphi|$ *in respective* **HCL** *can be constructed of* $H_1 \otimes H_1' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid H_2 \otimes H_2'$.

Proof. The proof is by induction on the height of D_l and is very similar to the proof of the preceding lemma but not so restrictive. In particular, despite the form of φ we delete only all its occurrences in the active sequent and we do not require φ to be compound and principal in the right premiss. The last difference makes some change in the proof; when

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the last rule is logical with φ principal we must refer not only to the induction hypothesis but also to Lemma 6.1.

In the basis we must consider additionally the case of φ being active formula of the active sequent. In such a case φ occurs in Γ and we obtain a result by (*IW*) on the right premiss.

In case of φ parametric we may encounter similar situations as those analysed in the proof of Lemma 6.1. In particular, we may be also forced to apply additionally (*EW*). Let us illustrate the point with $(\Rightarrow \Box^{\mathbb{P}'})$ applied on the next sequent to the right (but note that the place where cut formula actually occurs is inessential – transformations are the same if it is in any sequent, active or not). Then we have the following last steps of the two derivations:

$H_1 \mid \Pi \Rightarrow \Sigma, [\varphi]^i \mid \Gamma \Rightarrow \Delta, \Box^P \psi \mid H_2$ and
$H_1' \mid \varphi, \Pi' \Rightarrow \Sigma' \mid \Gamma' \Rightarrow \Delta' \mid H_2'$

The former derived by $(\Rightarrow \Box^{\mathbb{P}'})$ from:

$G_1 = H_1 \mid \Pi \Rightarrow \Sigma, [\varphi]^i \mid \Rightarrow \psi \mid \Gamma \Rightarrow \Delta \mid H_2$
$G_2 = H_1 \mid \Pi \Rightarrow \Sigma, [\varphi]^i, \psi \mid \Gamma \Rightarrow \Delta \mid H_2$
$G_3 = H_1 \mid \Pi \Rightarrow \Sigma, [\varphi]^i, \Box^P \psi \mid \Gamma \Rightarrow \Delta \mid H_2$

By the induction hypothesis we get:

$G_1' = H_1 \otimes H_1' \mid \Pi, \Pi' \Rightarrow \Sigma, \Sigma' \mid \Rightarrow \psi \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid H_2 \otimes H_2'$
$G_2' = H_1 \otimes H_1' \mid \Pi, \Pi' \Rightarrow \Sigma, \Sigma', \psi \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid H_2 \otimes H_2'$
$G_3' = H_1 \otimes H_1' \mid \Pi, \Pi' \Rightarrow \Sigma, \Sigma', \Box^P \psi \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid H_2 \otimes H_2'$

Notice that to obtain G_1' we must apply also (*EW*) to the right premiss to obtain $H_1' \mid \varphi, \Pi' \Rightarrow \Sigma' \mid \Rightarrow \mid \Gamma' \Rightarrow \Delta' \mid H_2'$

From G_1', G_2' and G_3' by $(\Rightarrow \Box^{\mathbb{P}'})$ we obtain:

$H_1 \otimes H_1' \mid \Pi, \Pi' \Rightarrow \Sigma, \Sigma' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta', \Box^P \psi \mid H_2 \otimes H_2'$.

As an example of the principal temporal cut formula consider $\varphi = \Box^F \psi$. Then D_l ends with:

$H_1 \mid \Pi \Rightarrow \Sigma, [\Box^F \psi]^{i+1} \mid \Gamma \Rightarrow \Delta \mid H_2$

derived from:

$G_1 = H_1 \mid \Pi \Rightarrow \Sigma, [\Box^F \psi]^i \mid \Rightarrow \psi \mid \Gamma \Rightarrow \Delta \mid H_2$
$G_2 = H_1 \mid \Pi \Rightarrow \Sigma, [\Box^F \psi]^i \mid \Gamma \Rightarrow \Delta, \psi \mid H_2$
$G_3 = H_1 \mid \Pi \Rightarrow \Sigma, [\Box^F \psi]^i \mid \Gamma \Rightarrow \Delta, \Box^F \psi \mid H_2$

and D_r ends with:

$H_1' \mid \Box^F \psi, \Pi' \Rightarrow \Sigma' \mid \Gamma' \Rightarrow \Delta' \mid H_2'$

We apply the induction hypothesis thrice to get:

$G_1' = H_1 \otimes H_1' \mid \Pi, \Pi' \Rightarrow \Sigma, \Sigma' \mid \Rightarrow \psi \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid H_2 \otimes H_2'$
$G_2' = H_1 \otimes H_1' \mid \Pi, \Pi' \Rightarrow \Sigma, \Sigma' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta', \psi \mid H_2 \otimes H_2'$
$G_3' = H_1 \otimes H_1' \mid \Pi, \Pi' \Rightarrow \Sigma, \Sigma' \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta', \Box^F \psi \mid H_2 \otimes H_2'$

Notice again that to obtain G_1 we must apply also (*EW*) to the right premiss.
Now we can derive from G_1', G_2' and G_3' by $(\Rightarrow \Box^{\mathbb{F}'})$:

$H_1 \otimes H_1' \mid \Pi, \Pi' \Rightarrow \Sigma, \Sigma', \Box^F \psi \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid H_2 \otimes H_2'$.

where $\Box^F \psi$ is principal. Hence we can apply Lemma 6.1 to this hypersequent and the right premiss and obtain:

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$H_1 \otimes H_1' \otimes H_1' \mid \Pi, \Pi', \Pi' - \Box^F\psi \Rightarrow \Sigma, \Sigma', \Sigma' \mid \Gamma, \Gamma', \Gamma' - \Box^F\psi \Rightarrow \Delta, \Delta', \Delta' \mid H_2 \otimes H_2' \otimes H_2^*$.

which by (*IC*) yields a desired result. □

These two lemmata yield:

THEOREM 6.3. (*Cut*) *is eliminable.*

Proof. Consider any derivation D with $\rho(D) > 0$, The proof proceeds by a double induction on $\rho(D)$ and $n\rho(D)$ – the number of applications of (*Cut*) with cut rank $\rho(D)$. We start with an uppermost application of (*Cut*) with cut rank $\rho(D)$ and apply Lemma 6.2 to its premisses and possibly (*IW*) to restore some extra occurrences of cut formula which were also deleted. Again note that cut rank is preserved despite invoking admissible rules. As a result we decrease either $\rho(D)$ or $n\rho(D)$. By repeating systematically this procedure we finish with cut-free derivation. □

§7. Conclusion. As we noted HC calculus introduced in this article is a variant of the calculus presented in [10]. However both were proved complete in different ways and not without a reason. It may be instructive to close the article with demonstration of their equivalence. For easier reference we will use the names **1-HCL** and **2-HCL** to denote hypersequent systems from [10] and from the present article respectively (where **L** is any of the 8 logics depicted in §2).

We already noticed that there are three differences between the two systems: in **2-HCL** (*Cut*) is primitive, axioms are restricted to atomic formulae as active formulae, and the rules for introduction of temporal operators to antecedents are different.

For the first two points it is enough to notice that in [10] we obtained the admissibility of additive variant of (*Cut*) as a byproduct of semantical completeness of cut-free **1-HCL**, and that we proved in Lemma 5.1 that axiomatic hypersequents without restriction to atoms are provable in **2-HCL**. Hence in order to prove that both calculi are equivalent we need only to show that respective rules of antecedent introduction for temporal operators may be derivable or admissible in the other calculus.

Let us recall that in **1-HCL** both rules had the following form:

$$\frac{H_1 \mid \Gamma \Rightarrow \Delta \mid \cdots \mid \varphi, \Pi \Rightarrow \Sigma \mid H_2}{H_1 \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \cdots \mid \Pi \Rightarrow \Sigma \mid H_2} \qquad \frac{H_1 \mid \varphi, \Gamma \Rightarrow \Delta \mid \cdots \mid \Pi \Rightarrow \Sigma \mid H_2}{H_1 \mid \Gamma \Rightarrow \Delta \mid \cdots \mid \Box^P\varphi, \Pi \Rightarrow \Sigma \mid H_2}$$

and will be called subsequently as ($\Box^{F'}\Rightarrow$) and ($\Box^{P'}\Rightarrow$).

Derivability of these rules in **2-HCL** may be shown by induction on the number of sequents between two active sequents in the premiss. For this we need only respective rule of **2-HCL** and (*IW*). In the basis we have:

$$\dfrac{\dfrac{H_1 \mid \Gamma \Rightarrow \Delta \mid \varphi, \Pi \Rightarrow \Sigma \mid H_2}{H_1 \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \Box^F\varphi, \varphi, \Pi \Rightarrow \Sigma \mid H_2}\,(IW)}{H_1 \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma \mid H_2}\,(\Box^F\Rightarrow)$$

In the inductive step we have:

$$\dfrac{\dfrac{\dfrac{H_1 \mid \Gamma \Rightarrow \Delta \mid \cdots \mid \Theta \Rightarrow \Lambda \mid \varphi, \Pi \Rightarrow \Sigma \mid H_2}{H_1 \mid \Gamma \Rightarrow \Delta \mid \cdots \mid \Box^F\varphi, \Theta \Rightarrow \Lambda \mid \Box^F\varphi, \varphi, \Pi \Rightarrow \Sigma \mid H_2}\,(IW)}{H_1 \mid \Gamma \Rightarrow \Delta \mid \cdots \mid \Box^F\varphi, \Theta \Rightarrow \Lambda \mid \Pi \Rightarrow \Sigma \mid H_2}\,(\Box^F\Rightarrow)}{H_1 \mid \Gamma \Rightarrow \Delta \mid \cdots \mid \Box^F\varphi, \varphi, \Theta \Rightarrow \Lambda \mid \Pi \Rightarrow \Sigma \mid H_2}\,(IW)$$

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From the last line a goal hypersequent $H_1 \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \cdots \mid \Theta \Rightarrow \Lambda \mid \Pi \Rightarrow \Sigma \mid H_2$ is provable by the induction hypothesis.

To show that respective rules of **2-HCL** are derivable in **1-HCL** we must use admissible (additive) cut however.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow\mid\Rightarrow\mid\varphi\Rightarrow\varphi}{\Box^F\varphi\Rightarrow\mid\Rightarrow\mid\Rightarrow\varphi}(\Rightarrow\Box^{\mathrm{F}})}{\Box^F\varphi\Rightarrow\mid\Rightarrow\Box^F\varphi}(\Rightarrow\Box^{\mathrm{F}})}{\Box^F\varphi\Rightarrow\Box^F\Box^F\varphi}(\Box^{\mathrm{F}'}\Rightarrow)}{H_1 \mid \Box^F\varphi, \Gamma \Rightarrow \Delta, \Box^F\Box^F\varphi \mid \Pi \Rightarrow \Sigma \mid H_2}(IW, EW) \qquad \dfrac{\dfrac{H_1 \mid \Gamma \Rightarrow \Delta \mid \Box^F\varphi, \varphi, \Pi \Rightarrow \Sigma \mid H_2}{H_1 \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \Box^F\varphi, \Pi \Rightarrow \Sigma \mid H_2}(\Box^{\mathrm{F}'}\Rightarrow)}{H_1 \mid \Box^F\Box^F\varphi, \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma \mid H_2}(\Box^{\mathrm{F}'}\Rightarrow)$$

$$\dfrac{H_1 \mid \Box^F\varphi, \Gamma \Rightarrow \Delta, \Box^F\Box^F\varphi \mid \Pi \Rightarrow \Sigma \mid H_2 \qquad H_1 \mid \Box^F\Box^F\varphi, \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma \mid H_2}{H_1 \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma \mid H_2}(Cut)$$

REMARK. *We could prove the left premiss of* (*Cut*) *without admissible rules of external and internal weakening*[3] *but the proof would be more complicated since instead of* ($\Rightarrow \Box^{\mathrm{F}}$) *we had to use* ($\Rightarrow \Box^{\mathrm{F}'}$). *On the other hand, the application of* (*admissible for* **1-HCL**) (*Cut*) *is indispensable in this proof.*

Thus we are entitled to claim:

LEMMA 7.1 (Equivalence). **1-HCL** $\vdash G$ *iff* **2-HCL** $\vdash G$

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