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The projective bundle theorem for $\mathbf{I}^j$-cohomology

by

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Abstract

We compute the total $\mathbf{I}^j$-cohomology of a projective bundle over a smooth scheme.

Key Words: Projective bundles, I^j-cohomology, quadratic forms, Gersten complex.

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Introduction

Let X be a smooth integral scheme of dimension d over a field k and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. One of the basic tools to study the Witt groups of X with value in $\mathcal{L}$ is the so-called Gersten-Witt complex $C(X,\mathbf{W},\mathcal{L})$ of X constructed in [6]

$$W(k(X),\omega_{x_0}^{\mathcal{L}}) \xrightarrow{d_0} \bigoplus_{x\in X^{(1)}} W(k(x),\omega_x^{\mathcal{L}}) \xrightarrow{d_1} \cdots \longrightarrow \bigoplus_{x\in X^{(d)}} W(k(x),\omega_x^{\mathcal{L}})$$

where $\omega_x^{\mathcal{L}}$ denotes for any point $x \in X^{(j)}$ the vector space $\mathrm{Ext}^j_{\mathcal{O}_{X,x}}(k(x),\mathcal{L}_x)$, with $\mathcal{L}_x$ the free $\mathcal{O}_{X,x}$-module obtained from $\mathcal{L}$. For any $x \in X^{(j)}$, the group $W(k(x),\omega_x^{\mathcal{L}})$ is endowed with a structure of a $W(k(x))$-module induced by left multiplication. One can therefore define for any $n \in \mathbb{N}$ the analogues $I^n(k(x),\omega_x^{\mathcal{L}})$ of the powers of the fundamental ideal $I(k(x))$ by $I^n(k(x),\omega_x^{\mathcal{L}}) := I^n(k(x)) \cdot W(k(x),\omega_x^{\mathcal{L}})$. We also set $I^n(k(x),\omega_x^{\mathcal{L}}) = W(k(x),\omega_x^{\mathcal{L}})$ if $n \leq 0$. The differentials of the Gersten complex respect the subgroups $I^n(k(x),\omega_x^{\mathcal{L}})$ ([12, Corollary 7.2], or [9, Théorème 9.2.4]) and therefore one obtains a filtered Gersten-Witt complex $C(X,\mathbf{I}^j,\mathcal{L})$:

$$I^j(k(X),\omega_{x_0}^{\mathcal{L}}) \xrightarrow{d_0^{\mathcal{L}}} \bigoplus_{x\in X^{(1)}} I^{j-1}(k(x),\omega_x^{\mathcal{L}}) \xrightarrow{d_1^{\mathcal{L}}} \cdots \longrightarrow \bigoplus_{x\in X^{(d)}} I^{j-d}(k(x),\omega_x^{\mathcal{L}}).$$

for any $j \in \mathbb{Z}$ and any invertible $\mathcal{O}_X$-module $\mathcal{L}$ over X. This complex carries in some sense more K-theoretic information than the usual Gersten complex $C(X,\mathbf{W},\mathcal{L})$. Indeed, for any $j \in \mathbb{Z}$ let $C(X,\mathbf{I}^{j+1},\mathcal{L}) \to C(X,\mathbf{I}^j,\mathcal{L})$ be the morphism of complexes given by the term-wise inclusions $I^{n+1}(k(x),\omega_x^{\mathcal{L}}) \subset I^n(k(x),\omega_x^{\mathcal{L}})$. The cokernel of this map is the complex $C(X,\bar{\mathbf{I}}^j)$ as described below:

$$\overline{I}^j(k(X)) \longrightarrow \bigoplus_{x\in X^{(1)}} \overline{I}^{j-1}(k(x)) \longrightarrow \cdots \longrightarrow \bigoplus_{x\in X^{(d)}} \overline{I}^{j-d}(k(x)).$$

Its groups and differentials are independent of $\mathcal{L}$ and its cohomology groups $H^i(X,\bar{\mathbf{I}}^j)$ are isomorphic to the groups $H^i(X,\mathbf{H}^j_{et}(_,\mu_2))$ obtained from the sheaf associated to the presheaf $V \mapsto H^j_{et}(V,\mu_2)$ ([25, Theorem 7.4] and [21,

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Theorem 4.1]). In particular, one sees that $H^j(X,\bar{\mathbf{I}}^j) = Ch^j(X)$ for any $j \in \mathbb{Z}$, where $Ch^j(X)$ denotes the Chow group of codimension j-cycles modulo 2. By construction, there is a long exact sequence in cohomology

$$\cdots \to H^i(X,\mathbf{I}^{j+1},\mathcal{L}) \to H^i(X,\mathbf{I}^j,\mathcal{L}) \to H^i(X,\bar{\mathbf{I}}^j) \to H^{i+1}(X,\mathbf{I}^{j+1},\mathcal{L}) \to \cdots$$

linking the cohomology groups $H^i(X,\mathbf{I}^j,\mathcal{L})$, or the $\mathbf{I}^j$*-cohomology* of X, with the groups $H^i(X,\bar{\mathbf{I}}^j)$.

The cohomology of the filtered Gersten-Witt complex is also involved in the definition of the Chow-Witt groups (see [9, Chapitre 10]). In fact there is an exact sequence (where X is of dimension d):

$$CH^d(X) \longrightarrow \widetilde{CH^d}(X,\mathcal{L}) \longrightarrow H^d(X,\mathbf{I}^d,\mathcal{L}) \longrightarrow 0$$

for any invertible $\mathcal{O}_X$-module $\mathcal{L}$. Thus the group $H^d(X,\mathbf{I}^d,\mathcal{L})$ can be seen as the quadratic part of the top Chow-Witt group $\widetilde{CH^d}(X,\mathcal{L})$.

All in all, the cohomology of $C(X,\mathbf{I}^j,\mathcal{L})$ is an object of study at least as interesting as the cohomology of the usual Gersten-Witt complex. An important question that one may ask is the following: What kind of characteristic classes do we have in this cohomology theory?

In the usual Chow theory, or more generally for oriented cohomology theories in the sense of [17] or [22], one way to get characteristic classes is the computation of the total cohomology of a projective bundle $\mathbb{P}(E)$ over X. This computation is also interesting because it gives a (vague) idea on how the cohomology groups of projective varieties look like. In this paper, we compute the total $\mathbf{I}^j$-cohomology of a projective bundle over a smooth base and we construct some characteristic classes.

One of the first observations we make is that part of the total I^j-cohomology actually comes from $\bar{I}^j$-cohomology. Namely, we construct for any $i,j \in \mathbb{Z}$ and any $m \in \mathbb{N}$ homomorphisms

$$\mu^m_{\mathcal{L}} : H^i(X,\bar{\mathbf{I}}^j) \to H^{i+m}(\mathbb{P}(E),\mathbf{I}^{j+m},\mathcal{O}(-m)\otimes p^*\mathcal{L})$$

which we prove to be split injective provided $1 \leq m \leq \operatorname{rank}(E)-1$ (Corollary 5.8). Since the Gersten-Witt complexes twisted by an invertible module $\mathcal{L}$ or by an invertible module $\mathcal{L}\otimes\mathcal{N}^{\otimes 2}$ are canonically isomorphic, we see that the homomorphisms $\mu^m_{\mathcal{L}}$ have images in different groups depending on the parity of m. We therefore define split injective homomorphisms

$$\Theta^{\mathcal{L}}_{\text{even}} : \bigoplus_{m \text{ even}}^{2\leq m\leq \operatorname{rank}(E)-1} H^{i-m}(X,\bar{\mathbf{I}}^{j-m}) \to H^i(\mathbb{P}(E),\mathbf{I}^j,p^*\mathcal{L})$$

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and

$$\Theta^{\mathcal{L}}_{\text{odd}} : \bigoplus_{\substack{1\leq m\leq \text{rank}(E)-1\\ m \text{ odd}}} H^{i-m}(X,\bar{\mathbf{I}}^{j-m}) \to H^i(\mathbb{P}(E),\mathbf{I}^j,p^*\mathcal{L}\otimes\mathcal{O}(-1))$$

as the sum of the corresponding $\mu^m_{\mathcal{L}}$. We then define the *reduced cohomology groups* $\tilde{H}^i(\mathbb{P}(E),\mathbf{I}^j,p^*\mathcal{L})$ and $\tilde{H}^i(\mathbb{P}(E),\mathbf{I}^j,p^*\mathcal{L}\otimes\mathcal{O}(-1))$ to be the cokernels of $\Theta^{\mathcal{L}}_{\text{even}}$ and $\Theta^{\mathcal{L}}_{\text{odd}}$. The real challenge is to compute these reduced groups. In case the bundle is of odd rank, we find the following result (Theorem 9.1):

Theorem *Let X be a smooth scheme and let E be a vector bundle over X of odd rank r. Let $p:\mathbb{P}(E)\to X$ be the projective bundle associated to E and*

$$0 \longrightarrow \mathcal{G}_E \longrightarrow p^*\mathcal{E}^\vee \longrightarrow \mathcal{O}(1) \longrightarrow 0$$

the canonical sequence. Then

$$p^* : H^i(X,\mathbf{I}^j,\mathcal{L}) \to \tilde{H}^i(\mathbb{P}(E),\mathbf{I}^j,p^*\mathcal{L})$$

and

$$c(G_E^\vee) : H^{i+1-r}(X,\mathbf{I}^{j+1-r},\mathcal{L}) \to \tilde{H}^i(\mathbb{P}(E),\mathbf{I}^j,p^*\mathcal{L}\otimes\omega_{\mathbb{P}(E)/X})$$

are isomorphisms for any $i,j\in\mathbb{Z}$ and any line bundle L over X.

The class $c(G_E^\vee)$ appearing in the statement is the Euler class (defined for instance in [9]) of the total space $G_E^\vee$ of $\mathcal{G}_E^\vee$. When the bundle E is of even rank, then there is a simple case and a more difficult case depending on the twist involved (Theorems 9.2 and 9.4).

Theorem *Let X be a smooth scheme and let E be a vector bundle over X of even rank r. Let $p:\mathbb{P}(E)\to X$ be the projective bundle associated to E. Then*

$$\tilde{H}^i(\mathbb{P}(E),\mathbf{I}^j,\mathcal{O}(-1)) = 0$$

for any $i,j\in\mathbb{Z}$, and we have a long exact sequence

$$\cdots \to H^{i-r}(X,\mathbf{I}^{j-r+1}) \xrightarrow{\epsilon c(E)} H^i(X,\mathbf{I}^j,\det(\mathcal{E})^\vee) \xrightarrow{p^*} \tilde{H}^i(\mathbb{P}(E),\mathbf{I}^j,\omega_{\mathbb{P}(E)/X}) \xrightarrow{p_*} H^{i-r+1}(X,\mathbf{I}^{j-r+1}) \to \cdots$$

The sequence splits if and only if $\epsilon c(E)=0$. The same results hold for groups twisted by a line bundle L over X.

Here $\epsilon : H^{i-r}(X,\mathbf{I}^{j-r+1}) \to H^{i-r}(X,\mathbf{I}^{j-r})$ is the homomorphism induced by the morphism of complexes $C(X,\mathbf{I}^{j-r+1}) \to C(X,\mathbf{I}^{j-r})$ and $c(E)$ is the Euler class of E. We provide an example where $\epsilon c(E)$ is non zero and therefore the long exact sequence is not split in general. As a consequence, p^* is not injective and thus

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the useful splitting principle valid for Chow groups doesn't hold in the context of $\mathbf{I}^j$-cohomology.

For the special case of the cohomology of the (non-filtered) Gersten-Witt complex, we re-obtain the results of Walter [27] by a different method. The fact that there is no clean answer in the even rank case forces us to restrict to the case of odd rank for the definition of the integral Stiefel-Whitney classes of E. Interestingly, we only get classes $c_m(E)$ with m odd and they satisfy the usual properties of the characteristic classes (Proposition 10.4):

Theorem *Let X be a smooth scheme and let E be a vector bundle of odd rank r over X. Then the integral Stiefel-Whitney classes of E satisfy the following properties:*

1. *If $f : X \to Y$ is a flat morphism, then $f^* c_m(E) = c_m(f^* E) \circ f^*$.*
2. *If $f : X \to Y$ is a proper morphism, then $f_*(c_m(f^* E)) = c_m(E) \circ f_*$.*
3. *$c_1(E) = c(\det(E))$ and $c_r(E) = c(E)$.*
4. *If $[E] = [E']$ in $K_0(X)$, then $c_m(E) = c_m(E')$ for any odd m.*

The name of these classes comes from the fact that they lift the (classical) Stiefel-Whitney classes defined in Section 4 and therefore potentially carry more information than the latter.

Finally, we compute the cohomology of $\mathbb{P}^n_k$ with coefficients in the Milnor-Witt sheaf $\mathbf{K}^{MW}_j$ as defined in [20, §3]. We obtain the following theorem (Theorem 11.7):

Theorem *We have*

$$H^i(\mathbb{P}^n_k, \mathbf{K}^{MW}_j) = \begin{cases} K^{MW}_j(k) & \text{if } i = 0. \\ K^M_{j-i}(k) & \text{if } i \text{ is even and } i \neq 0. \\ 2K^M_{j-i}(k) & \text{if } i \text{ is odd and } i \neq n. \\ K^{MW}_{j-n}(k) & \text{if } i = n \text{ and } n \text{ is odd.} \end{cases}$$

$$H^i(\mathbb{P}^n_k, \mathbf{K}^{MW}_j, \mathcal{O}(-1)) = \begin{cases} 2K^M_{j-i}(k) & \text{if } i \text{ is even and } i \neq n. \\ K^M_{j-i}(k) & \text{if } i \text{ is odd.} \\ K^{MW}_{j-n}(k) & \text{if } i = n \text{ and } n \text{ is even.} \end{cases}$$

A philosophical (and unfortunately imprecise) explanation on why we have to restrict to this case is the following.

Let X be a smooth scheme over $\mathbb{R}$. Then, the $\mathbf{I}^j$-cohomology of X is (in a sense that we hope to make more precise in the future) the analogue of

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the singular cohomology groups $H^*(X(\mathbb{R}),\mathbb{Z})$, while the $\bar{\mathbf{I}}^j$-cohomology is the analogue of the singular cohomology groups $H^*(X(\mathbb{R}),\mathbb{Z}/2)$. The comparison between $\mathbf{I}^{j+1}$-cohomology and $\mathbf{I}^j$-cohomology corresponds to the homomorphism $H^*(X(\mathbb{R}),\mathbb{Z}) \to H^*(X(\mathbb{R}),\mathbb{Z})$ induced by the multiplication by 2 on the coefficients, and the connecting homomorphism comparing $\bar{\mathbf{I}}^j$-cohomology and $\mathbf{I}^{j+1}$-cohomology is analogous to the Bockstein homomorphism $H^*(X(\mathbb{R}),\mathbb{Z}) \to H^{*+1}(X(\mathbb{R}),\mathbb{Z}/2)$.

Consider the sheaf $\mathbf{K}^M_j$ associated to Milnor K-theory and its modulo 2 version $\mathbf{K}^M_j/2$. The cohomology groups $H^*(X,\mathbf{K}^M_j)$ represent an analogue of the singular cohomology groups $H^*(X(\mathbb{C}),\mathbb{Z})$ and the cohomology groups $H^*(X,\mathbf{K}^M_j/2)$ correspond to $H^*(X(\mathbb{C}),\mathbb{Z}/2)$.

By definition, the Milnor-Witt sheaves $\mathbf{K}^{MW}_j$ is a way to put together the information obtained from both $\mathbf{I}^j$-cohomology and $\mathbf{K}^M_j$-cohomology. Thus, for a scheme over $\mathbb{R}$, the analogies of the previous paragraphs show that $H^*(X,\mathbf{K}^{MW}_j)$ glue together the singular cohomology groups $H^*(X(\mathbb{R}),\mathbb{Z})$ and $H^*(X(\mathbb{C}),\mathbb{Z})$. We can summarize this in the following motto: *Milnor-Witt K-theory is an analogue of the singular cohomology of both the real and complex points, and shows how these glue together.*

The sad truth is that the information of both these theories doesn't glue together well in the case of a projective bundle (and there is no reason on why it should be the case!), unless the base scheme is a point.

The article is organized as follows: In Section 1, we quickly review the basic facts about projective bundles that we will need throughout the article. We also fix some notation used in the sequel. Section 2 recalls the construction of the filtered Gersten complex and its functorial behaviour. There is also a discussion of the product on the cohomology groups. The next section deals with Euler classes, and in particular with Euler classes associated to line bundles. We prove a crucial lemma (Lemma 3.1) describing the Euler class of a line bundle in a new way.

In Section 4, we recall the well-known computation of the projective bundle theorem for the Chow groups modulo 2. The technical part of the article begins in Section 5. We construct the already mentioned homomorphisms

$$\mu^m_{\mathcal{L}} : H^i(X,\bar{\mathbf{I}}^j) \to H^{i+m}(\mathbb{P}(E),\mathbf{I}^{j+m},\mathcal{O}(-m)\otimes p^*\mathcal{L})$$

which will generate the $\bar{\mathbf{I}}^j$-cohomology part in the total $\mathbf{I}^j$-cohomology of $\mathbb{P}(E)$, and then we prove that they are split injective provided m is smaller than the rank of E. We then attack the computation of the reduced groups by constructing some interesting classes $\xi_{e_1,\dots,e_n}$ when $\mathbb{P}(E) = \mathbb{P}^{n-1}_X$. These classes already appear in Balmer's paper [2], where they are used to give explicit generators of the Witt groups

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of $\mathbb{P}^{n-1}_X$. It turns out that these classes come from global ones when n is odd, and this is where the vector bundles of even and odd rank behave differently. Having these tools in the pocket, we spend Section 7 to explain the links between the I^j-cohomology of $\mathbb{P}(E)$ and the I^j-cohomology of $\mathbb{P}(E \oplus \mathbf{1}_X)$. Finally, we prove the projective bundle theorem in the split case in Section 8, before passing to the general case in Section 9.

We end up the article with a discussion on integral Stiefel-Whitney classes of odd rank vector bundles in Section 10 and with the computation of the projective bundle theorem for Milnor-Witt K-theory in Section 11.

Conventions

All schemes considered in this article are assumed to be connected, noetherian, separated of finite type over a field k, whose characteristic is different from 2. We also assume that the schemes we consider are smooth. If X is a scheme, we write $\Omega_{X/k}$ for the sheaf of differentials of X over k, which in our case is a locally free $\mathcal{O}_X$-module of rank equal to the dimension of X. We denote by $\omega_{X/k}$ its highest exterior power.

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1. Projective bundles

In this section, we recall some well-known facts about projective bundles. The reader is referred to [15] for more details.

Let X be a scheme. For any $\mathcal{O}_X$-module $\mathcal{M}$, we denote by $\mathcal{M}^\vee$ the $\mathcal{O}_X$-module $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X)$. If $\mathcal{E}$ is a locally free coherent $\mathcal{O}_X$-module of rank r, we denote by $\det(\mathcal{E})$ the invertible module $\wedge^r \mathcal{E}$. Observe that there is a canonical isomorphism $\det(\mathcal{E})^\vee \simeq \det(\mathcal{E}^\vee)$.

Let $\mathcal{E}$ be a locally free coherent $\mathcal{O}_X$-module over X. The vector bundle E associated to $\mathcal{E}$ is the scheme $\mathrm{Spec}(Sym(\mathcal{E}^\vee))$ together with the natural projection $E \to X$. The sheaf of sections of this morphism is naturally isomorphic to $\mathcal{E}$. In particular, we have the zero section $z : X \to E$. In this article, we will always

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use capital calligraphic letters for $\mathcal{O}_X$-modules and capital roman letters for the associated vector bundles.

The projective bundle $\mathbb{P}(E)$ associated to E is the scheme $\mathrm{Proj}(Sym(\mathcal{E}^\vee))$. We usually denote by $p : \mathbb{P}(E) \to X$ the projection. The scheme $\mathbb{P}(E)$ comes with an exact sequence of locally free $\mathcal{O}_{\mathbb{P}(E)}$-modules

$$0 \longrightarrow \mathcal{G}_E \longrightarrow p^*\mathcal{E}^\vee \longrightarrow \mathcal{O}(1) \longrightarrow 0.$$

The invertible sheaf $\mathcal{O}(1)$ is called the canonical quotient. Its dual is $\mathcal{O}(-1)$, the canonical subbundle. As usual, we denote by G_E the total space of $\mathcal{G}_E$.

It follows from [15, II,Theorem 8.13] that the sheaf of differentials $\Omega_{\mathbb{P}(E)/X}$ of $\mathbb{P}(E)$ over X can be computed with the exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}(E)/X} \longrightarrow p^*\mathcal{E}^\vee \otimes \mathcal{O}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}(E)} \longrightarrow 0.$$

Tensoring with $\mathcal{O}(1)$, we see that $\Omega_{\mathbb{P}(E)/X} \otimes \mathcal{O}(1) \simeq \mathcal{G}_E$.

Let $\mathbf{1}_X$ denote the module $\mathcal{O}_X$. Consider the projective bundle $\mathbb{P}(E \oplus \mathbf{1}_X)$ with projection $p' : \mathbb{P}(E \oplus \mathbf{1}_X) \to X$. Since $X = \mathbb{P}(\mathbf{1}_X)$, there is a canonical embedding $s : X \to \mathbb{P}(E \oplus \mathbf{1}_X)$. Let U be the open complement of X in $\mathbb{P}(E \oplus \mathbf{1}_X)$, with inclusion $\eta : U \to \mathbb{P}(E \oplus \mathbf{1}_X)$. If $X = \mathrm{Spec}(A)$ and $E = A^r$, then the restriction of $\mathcal{O}_{\mathbb{P}(E\oplus\mathbf{1}_X)}(1)$ to U is globally generated by r elements and therefore there is a unique morphism $q : U \to \mathbb{P}(E)$ such that $\mathcal{O}_{\mathbb{P}(E\oplus\mathbf{1}_X)}(1)_{|U} = q^*\mathcal{O}_{\mathbb{P}(E)}(1)$ ([15, II, Theorem 7.1]). Glueing these morphisms, we get a morphism $q : U \to \mathbb{P}(E)$ for any vector bundle E. It turns out that U is a vector bundle over $\mathbb{P}(E)$ and can be identified with the total space of $\mathcal{O}_{\mathbb{P}(E)}(1)$ over $\mathbb{P}(E)$. Under this identification, the morphism $q : U \to \mathbb{P}(E)$ can be identified with the canonical projection (see [14, Corollaire 8.6.4] for instance).

The embedding $\iota : \mathbb{P}(E) \to \mathbb{P}(E \oplus \mathbf{1}_X)$ factors through U and gives the zero section of the vector bundle $q : U \to \mathbb{P}(E)$. The complement of $\mathbb{P}(E)$ in $\mathbb{P}(E \oplus \mathbf{1}_X)$ is E, with inclusion $\tilde{\iota} : E \to \mathbb{P}(E \oplus \mathbf{1}_X)$, and the zero section $z : X \to E$ is just the restriction to E of the embedding $s : X \to \mathbb{P}(E \oplus \mathbf{1}_X)$.

Let $Bl_X(\mathbb{P}(E \oplus \mathbf{1}_X))$ be the blow-up of $\mathbb{P}(E \oplus \mathbf{1}_X)$ along $X = \mathbb{P}(\mathbf{1}_X)$ with projection $\tilde{p} : Bl_X(\mathbb{P}(E \oplus \mathbf{1}_X)) \to \mathbb{P}(E \oplus \mathbf{1}_X)$. Observe that the exceptional fibre of $Bl_X(\mathbb{P}(E \oplus \mathbf{1}_X))$ is $\mathbb{P}(E)$. Consider the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}(E)}(1) \oplus \mathbf{1}_{\mathbb{P}(E)})$ over $\mathbb{P}(E)$ with projection $\tilde{p}' : \mathbb{P}(\mathcal{O}_{\mathbb{P}(E)}(1) \oplus \mathbf{1}_{\mathbb{P}(E)}) \to \mathbb{P}(E)$. Again, we can identify $\mathbb{P}(E)$ with $\mathbb{P}(\mathbf{1}_{\mathbb{P}(E)})$ and its complement in $\mathbb{P}(\mathcal{O}_{\mathbb{P}(E)}(1) \oplus \mathbf{1}_{\mathbb{P}(E)})$ is the total space of $\mathcal{O}_{\mathbb{P}(E)}(1)$ over $\mathbb{P}(E)$. This identifies U with the open complement of $\mathbb{P}(E)$ in $\mathbb{P}(\mathcal{O}_{\mathbb{P}(E)}(1) \oplus \mathbf{1}_{\mathbb{P}(E)})$ and $Bl_X(\mathbb{P}(E \oplus \mathbf{1}_X))$ with $\mathbb{P}(\mathcal{O}_{\mathbb{P}(E)}(1) \oplus \mathbf{1}_{\mathbb{P}(E)})$ (see [14, §8.7]).

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1.1. Canonical modules

Recall from our conventions that if X is a (smooth) scheme we denote by $\omega_{X/k}$ the highest exterior power of the locally free sheaf $\Omega_{X/k}$. If $f : X \to Y$ is a morphism of schemes, we denote by $\omega_{X/Y}$ the invertible sheaf $\omega_{X/k} \otimes f^*\omega_{Y/k}^\vee$.

If $E \to X$ is vector bundle of rank r and $p : \mathbb{P}(E) \to X$ is the associated projection, the exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}(E)/X} \longrightarrow p^*\mathcal{E}^\vee \otimes \mathcal{O}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}(E)} \longrightarrow 0.$$

yields a canonical isomorphism $\omega_{\mathbb{P}(E)/X} := \det(\Omega_{\mathbb{P}(E)/X}) \simeq p^*\det(\mathcal{E})^\vee \otimes \mathcal{O}(-r)$. It follows immediately that $\omega_{\mathbb{P}(E\oplus\mathbf{1}_X)/X} = (p')^*\det(\mathcal{E})^\vee \otimes \mathcal{O}(-r-1)$.

Let $\eta : U \subset \mathbb{P}(E \oplus \mathbf{1}_X)$ be the open complement of $X = \mathbb{P}(\mathbf{1}_X)$ and $q : U \to \mathbb{P}(E)$ be the morphism defined in the previous section. We have

$$\omega_{U/X} = \rho^*\omega_{\mathbb{P}(E\oplus\mathbf{1}_X)/X} = \rho^*(p')^*\det(\mathcal{E})^\vee \otimes \mathcal{O}(-r) = \rho^*(p')^*\det(\mathcal{E})^\vee \otimes \rho^*\mathcal{O}(-r).$$

Since $p'\rho = pq$ and $\rho^*\mathcal{O}(-1) = q^*\mathcal{O}(-1)$, we find $\omega_{U/X} = q^*(\omega_{\mathbb{P}(E)/X} \otimes \mathcal{O}(-1))$.

For the convenience of the reader, we summarize the computations of the canonical twists up to squares in the following table.

	r even	r odd
$\omega_{\mathbb{P}(E)/X}$	$p^*\det(\mathcal{E})^\vee$	$p^*\det(\mathcal{E})^\vee \otimes \mathcal{O}(-1)$
$\omega_{\mathbb{P}(E\oplus\mathbf{1}_X)/X}$	$(p')^*\det(\mathcal{E})^\vee \otimes \mathcal{O}(-1)$	$(p')^*\det(\mathcal{E})^\vee$

1.2. Notation

We set the following notation:

- $p : \mathbb{P}(E) \to X$ the projection.
- $\mathbf{1}_X := \mathcal{O}_X$
- $\mathcal{G}_E := \Omega_{\mathbb{P}(E)/X} \otimes \mathcal{O}(1)$.
- $p' : \mathbb{P}(E \oplus \mathbf{1}_X) \to X$ the projection.
- $s : X \to \mathbb{P}(E \oplus \mathbf{1}_X)$ the inclusion given by the identification $X = \mathbb{P}(\mathbf{1}_X)$.
- $\tilde{\imath} : E \to \mathbb{P}(E \oplus \mathbf{1}_X)$ the inclusion.
- $p'' : E \to X$ the projection.
- $z : X \to E$ the restriction of $s : X \to \mathbb{P}(E \oplus \mathbf{1}_X)$ to E.

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- $U := \mathbb{P}(E \oplus \mathbf{1}_X) \setminus X$.
- $\eta : U \to \mathbb{P}(E \oplus \mathbf{1}_X)$ the inclusion.
- $q : U \to \mathbb{P}(E)$ the natural projection when U is identified to the total space of $\mathcal{O}_{\mathbb{P}(E)}(1)$ over $\mathbb{P}(E)$.
- $\iota : \mathbb{P}(E) \to \mathbb{P}(E \oplus \mathbf{1}_X)$ the inclusion.
- $Bl_X(\mathbb{P}(E \oplus \mathbf{1}_X)$ the blow-up of $\mathbb{P}(E \oplus \mathbf{1}_X)$ along X.
- $\tilde{p} : Bl_X(\mathbb{P}(E \oplus \mathbf{1}_X) \to \mathbb{P}(E \oplus \mathbf{1}_X)$ the projection.
- $\omega_{\mathbb{P}(E)/X}$ the canonical bundle of $\mathbb{P}(E)$ over X.
- $\omega_{\mathbb{P}(E)/X}(i) := \omega_{\mathbb{P}(E)/X} \otimes \mathcal{O}(i)$ for any $i \in \mathbb{Z}$.
- For any line bundle L over X and any $i \in \mathbb{Z}$, $L(i) := p^*L \otimes \mathcal{O}(i)$.

2. Witt groups and Gersten complexes

In this section, we refer to [6] for the construction of the Gersten-Witt complex, to [9] for its properties and to [3] for more information on the theory of derived Witt groups.

2.1. *The complex*

Let X be a connected smooth scheme of dimension d and let L be an invertible $\mathcal{O}_X$-module. Recall from the introduction that we denote by $\mathcal{L}_x$ the free $\mathcal{O}_{X,x}$-module $\mathcal{L} \otimes \mathcal{O}_{X,x}$ and by $\omega_x^{\mathcal{L}}$ the $k(x)$-vector space $\mathrm{Ext}^n_{\mathcal{O}_{X,x}}(k(x), \mathcal{L}_x)$ for any point $x \in X^{(n)}$.

For any $j \in \mathbb{Z}$, we denote by $I^j(k(x), \omega_x^{\mathcal{L}})$ the group $I^j(k(x)) \cdot W(k(x), \omega_x^{\mathcal{L}})$, where $I^j(k(x))$ stands for the j-th power of the fundamental ideal $I(k(x)) \subset W(k(x))$, with the convention that $I^j(k(x), \omega_x^{\mathcal{L}}) = W(k(x), \omega_x^{\mathcal{L}})$ if $j \leq 0$. The quotient $I^j(k(x), \omega_x^{\mathcal{L}})/I^{j+1}(k(x), \omega_x^{\mathcal{L}})$ is independent of $\mathcal{L}$ by [9, Lemme E.1.3] and 2-torsion. We denote it by $\overline{I}^j(k(x))$. By definition, there is an exact sequence of groups

$$0 \longrightarrow I^{j+1}(k(x), \omega_x^{\mathcal{L}}) \longrightarrow I^j(k(x), \omega_x^{\mathcal{L}}) \longrightarrow \overline{I}^j(k(x)) \longrightarrow 0. \qquad (1)$$

Recall from [6] that there is a Gersten-Witt complex $C(X, \mathbf{W}, \mathcal{L})$

$$W(k(X), \omega_{x_0}^{\mathcal{L}}) \xrightarrow{d_0^{\mathcal{L}}} \bigoplus_{x \in X^{(1)}} W(k(x), \omega_x^{\mathcal{L}}) \xrightarrow{d_1^{\mathcal{L}}} \cdots \longrightarrow \bigoplus_{x \in X^{(d)}} W(k(x), \omega_x^{\mathcal{L}})$$

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which is the first page of a coniveau spectral sequence. Since the differentials of the Gersten-Witt complex respect the fundamental ideals ([12, Theorem 6.4]), we get for any $j \in \mathbb{Z}$ a Gersten-Witt complex $C(X,\mathbf{I}^j,\mathcal{L})$

$$I^j(k(X),\omega_{x_0}^{\mathcal{L}}) \xrightarrow{d_0^{\mathcal{L}}} \bigoplus_{x\in X^{(1)}} I^{j-1}(k(x),\omega_x^{\mathcal{L}}) \xrightarrow{d_1^{\mathcal{L}}} \cdots \longrightarrow \bigoplus_{x\in X^{(d)}} I^{j-d}(k(x),\omega_x^{\mathcal{L}}).$$

Now let $\mathbf{I}^j_{\mathcal{L}}$ be the sheaf on X associated to the presheaf defined on $U \subset X$ by the exact sequence

$$0 \longrightarrow \mathbf{I}^j_{\mathcal{L}}(U) \longrightarrow I^j(k(U),\omega_{x_0}^{\mathcal{L}}) \xrightarrow{d_0^{\mathcal{L}}} \bigoplus_{x\in U^{(1)}} I^{j-1}(k(x),\omega_x^{\mathcal{L}}).$$

It turns out that $C(X,\mathbf{I}^j,\mathcal{L})$ is a flasque resolution of $\mathbf{I}^j_{\mathcal{L}}$ ([12, Corollary 7.7]). Thus the homology groups $H^i(X,\mathbf{I}^j,\mathcal{L})$ of the complex $C(X,\mathbf{I}^j,\mathcal{L})$ compute the (Zariski) cohomology of the sheaf $\mathbf{I}^j_{\mathcal{L}}$. When $\mathcal{L}$ is trivial, we simply drop it in the above notation. By convention, we have $H^i(X,\mathbf{I}^j,\mathcal{L}) = H^i(X,\mathbf{W},\mathcal{L})$ if $i > j$ and in particular $H^i(X,\mathbf{I}^j,\mathcal{L}) = H^i(X,\mathbf{W},\mathcal{L})$ if $j \leq 0$.

Another consequence of the fact that the differentials preserve the fundamental ideals is the existence for any $j \in \mathbb{Z}$ of a complex $C(X,\overline{\mathbf{I}}^j)$:

$$\overline{I}^j(k(X)) \xrightarrow{\overline{d}_0} \bigoplus_{x\in X^{(1)}} \overline{I}^{j-1}(k(x)) \xrightarrow{\overline{d}_1} \cdots \longrightarrow \bigoplus_{x\in X^{(d)}} \overline{I}^{j-d}(k(x)).$$

Since all groups appearing in this sequence are 2-torsion, its homology groups $H^i(X,\overline{\mathbf{I}}^j)$ are also 2-torsion and we will not bother about signs in the computations involving these groups. As in the above case, this complex is a flasque resolution of the sheaf $\overline{\mathbf{I}}^j$ associated to the presheaf defined on U by the exact sequence

$$0 \longrightarrow \overline{\mathbf{I}}^j(U) \longrightarrow \overline{I}^j(k(U)) \xrightarrow{\overline{d}_0} \bigoplus_{x\in U^{(1)}} \overline{I}^{j-1}(k(x)).$$

Observe that $\overline{I}^j(k(x))$ coincides with $H^j_{et}(k(x),\mu_2)$ for any point $x \in X$ and any $j \in \mathbb{Z}$ by [21] and that both groups are trivial when $j < 0$.

For any $x \in X$, the exact sequence (1) shows that for any $j \in \mathbb{Z}$ and any invertible $\mathcal{O}_X$-module $\mathcal{L}$, there is an exact sequence of sheaves

$$0 \longrightarrow \mathbf{I}^{j+1}_{\mathcal{L}} \xrightarrow{\epsilon_{\mathcal{L}}} \mathbf{I}^j_{\mathcal{L}} \xrightarrow{\pi_{\mathcal{L}}} \overline{\mathbf{I}}^j \longrightarrow 0.$$

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We also denote by $\epsilon_{\mathcal{L}}$ the homomorphism $H^i(X,\mathbf{I}^{j+1},\mathcal{L}) \to H^i(X,\mathbf{I}^j,\mathcal{L})$ induced by the morphism $\epsilon_{\mathcal{L}} : \mathbf{I}^{j+1}_{\mathcal{L}} \to \mathbf{I}^j_{\mathcal{L}}$, by $\pi_{\mathcal{L}} : H^i(X,\mathbf{I}^j,\mathcal{L}) \to H^i(X,\bar{\mathbf{I}}^j)$ the homomorphism induced by $\pi_{\mathcal{L}} : \mathbf{I}^j_{\mathcal{L}} \to \bar{\mathbf{I}}^j$ and by $\partial_{\mathcal{L}}$ the connecting homomorphism $H^i(X,\bar{\mathbf{I}}^j) \to H^{i+1}(X,\mathbf{I}^{j+1},\mathcal{L})$. By definition $\partial_{\mathcal{L}}(\overline{\alpha}) = d_i^{\mathcal{L}}(\alpha)$, where $\alpha \in C^i(X,\mathbf{I}^j,\mathcal{L})$ is any lift of $\overline{\alpha} \in C^i(X,\bar{\mathbf{I}}^j)$.

2.2. *Functorial properties*

The Gersten complex constructed in the previous section satisfies good functorial properties, which we recall now. More details can be found in [9].

If $f : X \to Y$ is a flat morphism, then there is a morphism of complexes

$$f^* : C(Y,\mathbf{I}^j,\mathcal{L}) \to C(X,\mathbf{I}^j,f^*\mathcal{L})$$

of degree 0, and hence homomorphisms $f^* : H^i(Y,\mathbf{I}^j,\mathcal{L}) \to H^i(X,\mathbf{I}^j,f^*\mathcal{L})$ for any $i \in \mathbb{N}$. In the particular case where X is a vector bundle over Y and f is the projection, then f^* is an isomorphism ([9, Théorème 11.2.9]). We call this phenomenon *homotopy invariance*. More generally, there is a pull-back homomorphism $f^* : H^i(Y,\mathbf{I}^j,\mathcal{L}) \to H^i(X,\mathbf{I}^j,f^*\mathcal{L})$ associated to any morphism $f : X \to Y$ which coincides with the above homomorphism when f is flat, but its construction is a little bit more delicate and does not come from a morphism of complexes in general (see [8]). This generalized pull-back allowed us to define a ring structure ([8, Corollary 4.6, Lemma 4.20]) on the total cohomology group

$$H^*(X,\mathbf{I}^{**},_) := \bigoplus_{i\in\mathbb{N}}\bigoplus_{j\in\mathbb{Z}}\bigoplus_{\mathcal{L}\in \mathrm{Pic}(X)/2} H^i(X,\mathbf{I}^j,\mathcal{L}).$$

in which the unit is the class of $\langle 1\rangle$ in $W(k(X))$ provided X is connected. The other ingredient is the exterior product on complexes ([8, Corollary 4.6])

$$\star : C^i(X,\mathbf{I}^j,\mathcal{L}) \times C^m(X,\mathbf{I}^n,\mathcal{N}) \to C^{i+j}(X\times X,\mathbf{I}^{j+n},\mathcal{L}\otimes\mathcal{N})$$

which induces an exterior product on cohomology groups using a kind of Leibnitz formula ([8, Corollary 4.8]).

If $f : X \to Y$ is a proper morphism between smooth schemes, then there is a morphism of complexes

$$f_* : C(X,\mathbf{I}^j,\omega_{X/k}\otimes f^*\mathcal{L}) \to C(Y,\mathbf{I}^{j-d},\omega_{Y/k}\otimes\mathcal{L})$$

of degree d, where $d = \dim X - \dim Y$, and therefore we get push-forward homomorphisms $f_* : H^i(X,\mathbf{I}^j,\omega_{X/k}\otimes f^*\mathcal{L}) \to H^{i-d}(Y,\mathbf{I}^{j-d},\omega_{Y/k}\otimes\mathcal{L})$ for any

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$i \in \mathbb{N}$. In particular, if $\mathcal{L} = \omega_{Y/k}^{\vee}$, we get homomorphisms

$$f_* : H^i(X, \mathbf{I}^j, \omega_{X/Y}) \to H^{i-d}(Y, \mathbf{I}^{j-d})$$

where $\omega_{X/Y} = \omega_{X/k} \otimes f^* \omega_{Y/k}^{\vee}$. For instance, if $X = \mathbb{P}(E)$ for some rank r vector bundle E over Y then $\omega_{X/Y} = p^* \det(\mathcal{E})^{\vee} \otimes \mathcal{O}(-r)$ as already seen in Section 1.

The compatibility between pull-backs and push-forwards is as good as one can hope in some particular cases. If

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array}$$

is a Cartesian square of smooth schemes with f proper and u flat, then $u^* f_* = g_* v^*$ by [9, Théorème 12.3.6]. More generally, suppose that in the fibre product

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array}$$

the morphism f is a regular embedding (u is not necessarily flat). Let $N_Y X$ be the normal cone to X in Y and $N_{Y'} X'$ be the normal cone to X' in Y' with respective sheaves $\mathcal{N}_Y X$ and $\mathcal{N}_{Y'} X'$. Then we have an exact sequence of $\mathcal{O}_{X'}$-modules

$$0 \longrightarrow \mathcal{N}_{Y'} X' \longrightarrow v^* \mathcal{N}_Y X \longrightarrow \mathcal{E} \longrightarrow 0$$

and $\mathcal{E}$ is locally free. In this situation, we have $u^* f_*(_) = g_*(c(E) \cdot v^*(_))$ where $c(E)$ is the Euler class of E defined in Section 3 (see [10]). This generalized base change allows us to prove the following projection formula, that we will need later in this paper. Let $f : X \to Y$ be a proper morphism, then

$$f_*(\alpha) \cdot \beta = f_*(\alpha \cdot f^* \beta)$$

for any $\alpha \in H^i(X, \mathbf{I}^j, \omega_{Y/X} \otimes f^* \mathcal{L})$ and $\beta \in H^r(Y, \mathbf{I}^s, \mathcal{N})$. This is easily deduced from the Cartesian square

$$\begin{array}{ccc} X & \xrightarrow{\Gamma_f} & X \times Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \times 1} \\ Y & \xrightarrow[\Delta]{} & Y \times Y \end{array}$$

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where Γ_f is the graph of f and Δ is the diagonal embedding.

To conclude this section, we state some obvious properties of the connecting homomorphism $\partial_L : H^i(X,\bar{\mathbf{I}}^j) \to H^{i+1}(X,\mathbf{I}^{j+1},\mathcal{L})$ defined in the previous section. These results express the fact that the connecting homomorphism is natural.

Proposition 2.1 *Let X,Y be a smooth schemes and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.*

1. *If $p : Y \to X$ is flat, then $p^*\partial_{\mathcal{L}} = \partial_{p^*\mathcal{L}}p^*$.*

2. *If $p : Y \to X$ is proper, then $p_*\partial_{p^*\mathcal{L}\otimes\omega_{Y/X}} = \partial_{\mathcal{L}}p_*$.*

3. Euler classes

Let $\rho : V \to X$ be a vector bundle of rank r and let $v : X \to V$ be the zero section. Recall that the *Euler class of V* is the homomorphism (see [9] for instance)

$$H^i(X,\mathbf{I}^j,\mathcal{L}) \xrightarrow{v_*} H^{i+r}(V,\mathbf{I}^{j+r},\rho^*(\det(\mathcal{V})^\vee \otimes \mathcal{L})) \xrightarrow{(\rho^*)^{-1}} H^{i+r}(X,\mathbf{I}^{j+r},\det(\mathcal{V})^\vee \otimes \mathcal{L}).$$

It follows immediately from the projection formula discussed in the above section that for any $\alpha \in H^i(X,\mathbf{I}^j,\mathcal{N})$ we have

$$(\rho^*)^{-1}v_*(\alpha) = (\rho^*)^{-1}v_*(1\cdot v^*\rho^*\alpha) = [(\rho^*)^{-1}v_*(1)]\cdot\alpha.$$

Therefore this homomorphism is entirely determined by the image of $1 \in H^0(X,\mathbf{W})$ which we denote by $c(V)$. By abuse of language, we also call this element the Euler class of V. The Euler class satisfies the expected functorial behaviour, as proved in [9, Chapitre 13]. Explicitly,

1. If $f : X \to Y$ is proper and V is a vector bundle over Y, then

$$f_*(c(f^*V)\cdot_) = c(V)\cdot f_*(_).$$

2. If $f : X \to Y$ is flat and V is a vector bundle over Y, then

$$f^*(c(V)\cdot_) = c(f^*V)\cdot f^*(_).$$

3. If $0 \to V_1 \to V_2 \to V_3 \to 0$ is an exact sequence of vector bundles over X, then $c(V_2) = c(V_1)\cdot c(V_2)$.

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Of course, we can replace I^j by $\overline{I}^j$ in the definition to get a homomorphism $\overline{c}(E)$:

$$H^i(X,\bar{\mathbf{I}}^j) \xrightarrow{v_*} H^{i+r}(E,\bar{\mathbf{I}}^{j+r}) \xrightarrow{(\rho^*)^{-1}} H^{i+r}(X,\bar{\mathbf{I}}^{j+r}).$$

This homomorphism is also completely determined by the image of $1 \in H^0(X,\overline{\mathbf{W}})$ which we denote by $\overline{c}(E)$. We will call $\overline{c}(E)$ the *top Stiefel-Whitney class* of E. This denomination will be made clearer in the next section, where Stiefel-Whitney classes of vector bundles will be defined.

The following Lemma gives a useful computation of the Euler class of a line bundle:

Lemma 3.1 *Let L and N be line bundles over X. The following diagram commutes:*

$$\begin{array}{ccc} H^i(X,\mathbf{I}^j,\mathcal{N}) & \xrightarrow{c(L)} & H^{i+1}(X,\mathbf{I}^{j+1},\mathcal{N}\otimes\mathcal{L}^\vee) \\ \pi_{\mathcal{N}}\downarrow & \nearrow_{\partial_{\mathcal{N}\otimes\mathcal{L}^\vee}} & \\ H^i(X,\bar{\mathbf{I}}^j). & & \end{array}$$

Proof: Let $\rho : L \to X$ be the projection and let $v : X \to L$ be the zero section, with associated morphism of sheaves $\mathcal{O}_L \to v_*\mathcal{O}_X$. As $L = \mathrm{Spec}(Sym(\mathcal{L}^\vee))$, the zero section is given by a morphism of $\mathcal{O}_L$-modules $s : \rho^*\mathcal{L}^\vee \to \mathcal{O}_L$ such that the sequence

$$0 \longrightarrow \rho^*\mathcal{L}^\vee \xrightarrow{s} \mathcal{O}_L \longrightarrow v_*\mathcal{O}_X \longrightarrow 0$$

is exact. Since $\mathcal{O}_L \simeq \mathrm{Hom}_{\mathcal{O}_L}(\rho^*\mathcal{L}^\vee,\rho^*\mathcal{L}^\vee)$, we can see s as a symmetric morphism for the duality $\mathrm{Hom}_{\mathcal{O}_L}(_,\rho^*\mathcal{L}^\vee)$.

Let $k(L)$ be the residue field at the generic point ξ of L. Localizing s at ξ, it becomes an isomorphism and therefore defines an element of $W(k(L),\omega_\xi^{\rho^*\mathcal{L}^\vee})$ whose class in $\overline{W}(k(L))$ is the unit $\overline{1}$ (since the underlying $k(L)$-vector space is of dimension 1). Now $d_0^{\rho^*\mathcal{L}^\vee}(s)$ is the class of the symmetric isomorphism

$$\begin{array}{ccc} \rho^*\mathcal{L}^\vee & \xrightarrow{s} & \mathcal{O}_L \\ -1\downarrow & & \| \\ \rho^*\mathcal{L}^\vee & \xrightarrow[-s]{} & \mathcal{O}_L \end{array}$$

in $H^1(L,\mathbf{I},\rho^*\mathcal{L}^\vee)$ (see for instance [6, Proposition 8.5]). By definition, this is precisely $\partial_{\rho^*\mathcal{L}^\vee}(\overline{1})$ in $H^1(L,\mathbf{I},\rho^*\mathcal{L}^\vee)$. Using [13, §2.4], we see that this element is $v_*(1)$, where $1 \in H^0(X,\mathbf{W})$ is the unit of $W(k(X))$. Since the Euler class commutes with pull-backs,

$$v_*(1) = \rho^*c(L) = c(\rho^*L)$$

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thus showing that $\partial_{\rho^*\mathcal{L}^\vee}\pi(1) = c(\rho^*L)$. We can apply Proposition 2.1 and homotopy invariance to get $\partial_{\mathcal{L}^\vee}\pi(1) = c(L)$.

Let $\alpha \in H^i(X,\mathbf{I}^j,\mathcal{N})$. Then $\rho^*\alpha \in H^i(L,\mathbf{I}^j,\rho^*\mathcal{N})$ is the class of an element $\theta \in C^i(L,\mathbf{I}^j,\rho^*\mathcal{N})$. We can consider the exterior product $s \star \theta \in C^i(L\times L,\mathbf{I}^j,\rho^*\mathcal{L}^\vee \otimes \mathcal{N})$. Applying [8, Corollary 4.8], we get

$$d_0^{\mathcal{L}^\vee}(s) \star \theta = (-1)^i d_i^{\mathcal{L}^\vee\otimes\mathcal{N}}(s \star \theta).$$

Now $s \star \theta$ is a lift of $\rho^*\pi_{\mathcal{N}}(\alpha) = \pi_{\rho^*\mathcal{N}}(\rho^*\alpha)$ in $C^i(L,\mathbf{I}^j,\rho^*\mathcal{L}^\vee \otimes \mathcal{N})$ and we conclude that

$$\partial_{\rho^*(\mathcal{N}\otimes\mathcal{L}^\vee)}(\pi_{\rho^*\mathcal{N}}(\rho^*\alpha)) = (-1)^i c(L)\cdot\rho^*\alpha.$$

Now $c(L) = \partial_{\mathcal{L}^\vee}\pi(1)$ is 2-torsion, and we can use homotopy invariance to get the result. □

Next we prove a compatibility lemma between the Euler class of a vector bundle and the connecting homomorphism.

Lemma 3.2 *Let X be a smooth scheme, V be a rank r vector bundle over X and L be a line bundle over X. Then $c(V)\partial_{\mathcal{L}} = \partial_{\mathcal{L}\otimes\det(\mathcal{V})^\vee}\overline{c}(E)$.*

Proof: Let $\rho : V \to X$ be the projection, and $\nu : X \to V$ be the zero section. Then $c(V)\partial_{\mathcal{L}} = (\rho^*)^{-1}\nu_*\partial_{\mathcal{L}}$. By Proposition 2.1 (applied twice), we get

$$(\rho^*)^{-1}\nu_*\partial_{\mathcal{L}} = \partial_{\mathcal{N}}(\rho^*)^{-1}\nu_* = \partial_{\mathcal{L}\otimes\det(\mathcal{V})^\vee}\overline{c}(E).$$

□

4. $\bar{\mathbf{I}}^j$-cohomology of $\mathbb{P}(E)$

Let E be a rank r vector bundle over X and let $p : \mathbb{P}(E) \to X$ be the projection. In this section, we recall quickly the computation of the $\bar{\mathbf{I}}^j$-cohomology of $\mathbb{P}(E)$. We set $\xi := \overline{c}(\mathcal{O}(1))$ and for any $m \in \mathbb{N}$ we denote by

$$\xi^m : H^i(X,\bar{\mathbf{I}}^j) \to H^{i+m}(\mathbb{P}(E),\bar{\mathbf{I}}^{j+m})$$

the homomorphism $\alpha \mapsto \xi^m \cdot p^*\alpha$, with the convention that $\xi^0 = Id$. When no confusion can arise we also denote $\xi^m\cdot\alpha$ the product $\xi^m \cdot p^*\alpha$.

Recall from Section 1 that the identification of $\mathbb{P}(\mathbf{1}_X)$ with X gives an inclusion $s : X \to \mathbb{P}(E\oplus\mathbf{1}_X)$, and that the complement U of X in $\mathbb{P}(E\oplus\mathbf{1}_X)$ is a vector bundle over $\mathbb{P}(E)$ with projection $\rho : U \to \mathbb{P}(E)$. This gives a long exact sequence of localization

$$\cdots \longrightarrow H^i_X(\mathbb{P}(E\oplus\mathbf{1}_X),\bar{\mathbf{I}}^j) \longrightarrow H^i(\mathbb{P}(E\oplus\mathbf{1}_X),\bar{\mathbf{I}}^j) \longrightarrow H^i(U,\bar{\mathbf{I}}^j) \longrightarrow \cdots$$

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for any $i, j \in \mathbb{N}$. Using the push-forward and homotopy invariance, this yields an exact sequence

$$\cdots \longrightarrow H^{i-r}(X,\bar{\mathbf{I}}^{j-r}) \xrightarrow{s_*} H^i(\mathbb{P}(E \oplus \mathbf{1}_X),\bar{\mathbf{I}}^j) \longrightarrow H^i(\mathbb{P}(E),\bar{\mathbf{I}}^j) \longrightarrow \cdots$$

Because $p' : \mathbb{P}(E \oplus \mathbf{1}_X) \to X$ is proper and satisfies $p's = Id_X$, $(p')_*$ is a retraction of s_* and the long sequence reduces to split short exact sequences

$$0 \longrightarrow H^{i-r}(X,\bar{\mathbf{I}}^{j-r}) \xrightarrow{s_*} H^i(\mathbb{P}(E \oplus \mathbf{1}_X),\bar{\mathbf{I}}^j) \longrightarrow H^i(\mathbb{P}(E),\bar{\mathbf{I}}^j) \longrightarrow 0.$$

We can now prove the following theorem:

Theorem 4.1 *Let X be a smooth scheme and let E be a rank r vector bundle over X. Then the homomorphism*

$$\chi_E : \bigoplus_{m=0}^{r-1} H^{i-m}(X,\bar{\mathbf{I}}^{j-m}) \to H^i(\mathbb{P}(E),\bar{\mathbf{I}}^j)$$

defined by $\chi_E(\alpha_i,\ldots,\alpha_{i-r+1}) = \sum_{m=0}^{r-1} \xi^m \cdot \alpha_{i-m}$ is an isomorphism for any $i \in \mathbb{N}$.

Proof: Assume that $E = \mathcal{O}_X^{r+1}$. Since everything is obvious for $r = 0$, we can prove the result by induction. The short split exact sequence above reads as

$$0 \longrightarrow H^{i-r}(X,\bar{\mathbf{I}}^{j-r}) \xrightarrow{s_*} H^i(\mathbb{P}^r_X,\bar{\mathbf{I}}^j) \longrightarrow H^i(\mathbb{P}^{r-1}_X,\bar{\mathbf{I}}^j) \longrightarrow 0$$

and the classes $\xi^m \in H^m(\mathbb{P}^r_X,\bar{\mathbf{I}}^j)$ restrict to the classes $\xi^m \in H^m(\mathbb{P}^{r-1}_X,\bar{\mathbf{I}}^j)$ under the homomorphism $H^i(\mathbb{P}^r_X,\bar{\mathbf{I}}^j) \to H^i(\mathbb{P}^{r-1}_X,\bar{\mathbf{I}}^j)$ for any $m \in \mathbb{N}$. We therefore have an isomorphism

$$s_* \oplus \chi_{\mathcal{O}_X^r} : H^{i-r}(X,\bar{\mathbf{I}}^{j-r}) \oplus \bigoplus_{m=0}^{r-1} H^{i-m}(X,\bar{\mathbf{I}}^{j-m}) \to H^i(\mathbb{P}^r_X,\bar{\mathbf{I}}^j).$$

We now prove that if $\alpha \in H^{i-r}(X,\bar{\mathbf{I}}^{j-r})$, then the restriction of $\xi^r \cdot \alpha$ to the group $H^i(\mathbb{P}^{r-1}_X,\bar{\mathbf{I}}^j)$ is zero.

Let $\psi : \mathbb{P}^{r-1}_X \to \mathbb{P}^{r-1}_k$ be the projection induced by $X \to \mathrm{Spec}(k)$. Pulling back, we get $\xi = \overline{c}(\mathcal{O}(1)) = \psi^*(\overline{c}(\mathcal{O}(1))$ and $\xi^r \cdot \alpha = \psi^*\overline{c}(\mathcal{O}(1))^r \cdot \alpha$. Since $C^r(\mathbb{P}^{r-1}_k,\bar{\mathbf{I}}^r)$ is trivial because there are no points of codimension r in $\mathbb{P}^{r-1}_k$, we find $\overline{c}(\mathcal{O}(1))^r = 0$ and therefore $\xi^r = 0$ when restricted to $\mathbb{P}^{r-1}_X$.

This shows that for any $\alpha \in H^{i-r}(X,\bar{\mathbf{I}}^{j-r})$, we have $\xi^r \cdot \alpha = s_*(\beta)$ for some β in $H^{i-r}(X,\bar{\mathbf{I}}^{j-r})$. Applying p_*, we get $\alpha = \beta$ by [11, Proposition 3.1 (a) (ii)]. Therefore the result is proven if $E = \mathcal{O}_X^{r+1}$. For a general E, we can use Mayer-Vietoris to conclude. □

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This theorem allows us to define the Stiefel-Whitney classes of a vector bundle E of rank r. It shows that

$$\xi^r = \sum_{m=0}^{r-1} \xi^m \alpha_{r-m}$$

for some uniquely determined $\alpha_{r-m} \in H^{r-m}(X,\bar{\mathbf{I}}^{r-m})$. We set $\bar{c}_0(E) = 1$ and $\bar{c}_{r-m}(E) = \alpha_{r-m}$ for any $0 \leq m \leq r-1$.

Definition 4.2 For any $0 \leq n \leq r$, we call the element $\bar{c}_n(E) \in H^n(X,\bar{\mathbf{I}}^n)$ the *n-th Stiefel-Whitney class of E*.

The Stiefel-Whitney classes satisfy the functorial properties listed in [11, Theorem 3.2]. Observe that the Stiefel-Whitney class $\bar{c}_r(E)$ coincides with the top Stiefel-Whitney class defined in Section 3 by [11, Example 3.3.2].

To finish the section, we shall compute the homomorphism

$$s_* : H^{i-r}(X,\bar{\mathbf{I}}^{j-r}) \to H^i(\mathbb{P}(E \oplus \mathbf{1}_X),\bar{\mathbf{I}}^j).$$

This result will be needed in the sequel. We recall first the computation of the top Stiefel-Whitney class of the vector bundle associated to the dual of $\mathcal{G}_E = \Omega_{\mathbb{P}(E)/X} \otimes \mathcal{O}(1)$.

Lemma 4.3 *We have* $\bar{c}(G_E^\vee) = \sum_{m=0}^{r-1} \xi^m \bar{c}_{r-1-m}(E)$.

Proof: We have an exact sequence of sheaves on $\mathbb{P}(E)$:

$$0 \longrightarrow G_E \longrightarrow p^* E^\vee \longrightarrow \mathcal{O}(1) \longrightarrow 0.$$

The result then follows from the Whitney formula ([11, Theorem 3.2 (e)]) and [11, Remark 3.2.3 (a)]. □

Lemma 4.4 *For any* $\alpha \in H^{i-r}(X,\bar{\mathbf{I}}^{j-r})$, *we have* $s_*(\alpha) = \bar{c}(G^\vee_{E\oplus \mathbf{1}_X}) \cdot \alpha$.

Proof: Using the projection formula, we see that it is sufficient to prove the result for $\alpha = 1$.

As seen at the beginning of the section, the exact sequence of localization associated to the closed embedding $s : X \to \mathbb{P}(E \oplus \mathbf{1}_X)$ yields a split exact sequence

$$0 \longrightarrow H^0(X,\overline{\mathbf{W}}) \xrightarrow{s_*} H^r(\mathbb{P}(E \oplus \mathbf{1}_X),\bar{\mathbf{I}}^r) \longrightarrow H^r(\mathbb{P}(E),\bar{\mathbf{I}}^r) \longrightarrow 0.$$

Since $s_*(1) \in H^r(\mathbb{P}(E \oplus \mathbf{1}_X),\bar{\mathbf{I}}^r)$ and any element in this group is a linear combination of $1,\xi,\ldots,\xi^r$ by Theorem 4.1,

$$s_*(1) = \sum_{m=0}^{r} \xi^m \alpha_{r-m}$$

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with $\alpha_{r-m} \in H^{r-m}(X,\bar{\mathbf{I}}^{r-m})$. Restricting to $\mathbb{P}(E)$, we get $\sum_{m=0}^{r}\xi^m\alpha_{r-m} = 0$. By definition of the Stiefel-Whitney classes of E, we have

$$\xi^r = \sum_{m=0}^{r-1}\xi^m\overline{c}_{r-m}(E).$$

Using Theorem 4.1 again, we find $\alpha_{r-m} = \overline{c}_{r-m}(E)\cdot\alpha_0$ for any $0 \leq m \leq r-1$. We therefore have $s_*(1) = \sum_{m=0}^{r}\xi^m(\overline{c}_{r-m}(E)\cdot\alpha_0)$. Applying $(p')_*$ and [11, Proposition 3.1 (a)], we obtain $\alpha_0 = 1$. Now for any $0 \leq r-m \leq r$, the Whitney formula shows that $\overline{c}_{r-m}(E) = \overline{c}_{r-m}(E\oplus\mathbf{1}_X)$. The result then follows from Lemma 4.3. □

5. Twisting homomorphisms

Definition 5.1 Let X be a scheme and E be a vector bundle over X. For any $m \geq 1$ and any line bundle L over X, let

$$\mu_{\mathcal{L}}^m : H^i(X,\bar{\mathbf{I}}^j) \to H^{i+m}(\mathbb{P}(E),\mathbf{I}^{j+m},\mathcal{L}(-m))$$

denote the composite

$$H^i(X,\bar{\mathbf{I}}^j) \xrightarrow{p^*} H^i(\mathbb{P}(E),\bar{\mathbf{I}}^j) \xrightarrow{\partial_{\mathcal{L}(-1)}} H^{i+1}(\mathbb{P}(E),\mathbf{I}^{j+1},\mathcal{L}(-1)) \xrightarrow{c(\mathcal{O}(1))^{m-1}} H^{i+m}(\mathbb{P}(E),\mathbf{I}^{j+m},\mathcal{L}(-m)).$$

These homomorphisms are a crucial tool to understand the $\mathbf{I}^j$-cohomology of a projective bundle. Our concern in this section is to understand their behaviour after composition with the projection $\pi_{\mathcal{L}(-m)}$:

$$\begin{array}{ccc} H^i(X,\bar{\mathbf{I}}^j) & \xrightarrow{\mu_{\mathcal{L}}^m} & H^{i+m}(\mathbb{P}(E),\mathbf{I}^{j+m},L(-m)) \\ & & \Big\downarrow{\scriptstyle \pi_{\mathcal{L}(-m)}} \\ & & H^{i+m}(\mathbb{P}(E),\bar{\mathbf{I}}^{j+m}). \end{array}$$

Lemma 5.2 *Suppose that* $E = E' \oplus \mathbf{1}_X$. *Then for any* $\alpha \in H^i(X,\bar{\mathbf{I}}^j)$ *we have*

$$\pi_{\mathcal{L}(-m)}\mu_{\mathcal{L}}^m(\alpha) = \xi^m\alpha + \xi^{m-1}\pi_{\mathcal{L}}\partial_{\mathcal{L}}(\alpha).$$

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Proof: Let $\beta \in C^i(X,\mathbf{I}^j,\mathcal{L})$ be a lift of α. The element $1 \in \mathbf{1}_X$ gives a global section of $\mathcal{O}(1)$ which we denote by x and we get a symmetric morphism

$$\mathcal{O}(-1) \xrightarrow{x} \mathcal{O}_{\mathbb{P}(E)}$$

for the duality $\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}(E)}}(_,\mathcal{O}(-1))$.

Letting ξ be the generic point of $\mathbb{P}(E)$, we can see x as an element in the group $W(k(\mathbb{P}(E)),\omega_\xi^{\mathcal{O}(-1)}) = C^0(\mathbb{P}(E),\mathbf{W},\mathcal{O}(-1))$ (see also the proof of Lemma 3.1). We consider the exterior product $p^*(\beta) \star x \in C^i(\mathbb{P}(E)\times\mathbb{P}(E),\mathbf{I}^j,\mathcal{L}(-1))$ and we can use the Leibnitz rule [8, proof of Proposition 4.7] to compute $d_i^{L(-1)}(p^*(\beta)\star x)$. We get (up to signs):

$$\begin{aligned} d_i^{\mathcal{L}(-1)}(p^*(\beta)\star x) &= d_i^{p^*\mathcal{L}}(p^*\beta)\star x + p^*(\beta)\star d_0^{\mathcal{O}(-1)}(x) \\ &= p^* d_i^{\mathcal{L}}(\beta)\star x + p^*(\beta)\star d_0^{\mathcal{O}(-1)}(x). \end{aligned}$$

Observe that $d_i^{\mathcal{L}(-1)}(p^*(\beta)\star x) = \mu_{\mathcal{L}}^1(\alpha)$ by definition and therefore

$$\pi_{\mathcal{L}(-1)}\mu_{\mathcal{L}}^1(\alpha) = \pi_{\mathcal{L}(-1)}(p^* d_i^{\mathcal{L}}(\beta)\star x + p^*(\beta)\star d_0^{\mathcal{O}(-1)}(x)).$$

Now $d_0^{\mathcal{O}(-1)}(x) = c(\mathcal{O}(1))$ by Lemma 3.1 and thus $\pi_{\mathcal{O}(-1)} d_0^{\mathcal{O}(-1)}(x) = \xi$. Then

$$\pi_{\mathcal{L}(-1)}\mu_{\mathcal{L}}^1(\alpha) = p^*\pi_{\mathcal{L}}\partial_{\mathcal{L}}(\alpha) + p^*(\alpha)\xi.$$

and the case $m = 1$ is settled. In general,

$$\pi_{\mathcal{L}(-m)}\mu_{\mathcal{L}}^m(\alpha) = \xi^{m-1}(\pi_{\mathcal{L}(-1)}\mu_{\mathcal{L}}^1(\alpha))$$

and we deduce the result from the case $m = 1$. □

Corollary 5.3 *Suppose that $E = E' \oplus \mathbf{1}_X$. Then for any $\alpha \in H^i(X,\mathbf{I}^j)$ we have*

$$\pi_{\mathcal{L}(-m)}\mu_{\mathcal{L}}^m\pi(\alpha) = \xi^m\pi(\alpha) + \xi^{m-1}\overline{c}(L)\pi(\alpha).$$

Proof: The above lemma yields

$$\pi_{\mathcal{L}(-m)}\mu_{\mathcal{L}}^m\pi(\alpha) = \xi^m\alpha + \xi^{m-1}\pi_{\mathcal{L}}\partial_{\mathcal{L}}\pi(\alpha)$$

and Lemma 3.1 gives a commutative diagram

$$\begin{array}{ccc} H^i(X,\mathbf{I}^j) & \xrightarrow{c(L^\vee)} & H^{i+1}(X,\mathbf{I}^{j+1},\mathcal{L}) \\ \pi\downarrow & \nearrow_{\partial_{\mathcal{L}}} & \\ H^i(X,\overline{\mathbf{I}}^j). & & \end{array}$$

It follows that $\pi_{\mathcal{L}}\partial_{\mathcal{L}}\pi(\alpha) = \overline{c}(L^\vee)\pi(\alpha)$. Now $\overline{c}(L) = \overline{c}(L^\vee)$ and the result is proved. □

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Next we prove some results which will be used to understand the composition $\pi_{\mathcal{L}(-m)}\mu_{\mathcal{L}}^m$ for a general bundle E. They will also be useful for the computation of the total cohomology of a projective bundle. Using the notations of Section 1, we have:

Lemma 5.4 *The following diagram commutes:*

$$\begin{array}{ccc} H^i(\mathbb{P}(E\oplus\mathbf{1}_X),\mathbf{I}^j,\det(\mathcal{E})^\vee(-m)) & \xrightarrow{\eta^*} & H^i(U,\mathbf{I}^j,q^*(\det(\mathcal{E})^\vee(-m))) \\ \uparrow{\scriptstyle \mu^m_{\det(\mathcal{E}\oplus\mathbf{1}_X)^\vee}} & & \uparrow{\scriptstyle q^*} \\ H^{i-m}(X,\bar{\mathbf{I}}^{j-m}) & \xrightarrow[\mu^m_{\det(\mathcal{E})^\vee}]{} & H^i(\mathbb{P}(E),\mathbf{I}^j,\det(\mathcal{E})^\vee(-m)). \end{array}$$

Proof: This is a straightforward consequence of Proposition 2.1 and of the fact that the Euler classes commute with flat pull-backs. □

We now consider, for any line bundle L over $\mathbb{P}(E\oplus\mathbf{1}_X)$, the push-forward

$$\iota_*: H^{i-1}(\mathbb{P}(E),\mathbf{I}^{j-1},\omega_{\mathbb{P}(E)/X}\otimes\iota^*\mathcal{L}) \to H^i_{\mathbb{P}(E)}(\mathbb{P}(E\oplus\mathbf{1}_X),\mathbf{I}^j,\omega_{\mathbb{P}(E\oplus\mathbf{1}_X)/X}\otimes\mathcal{L})$$

induced by the closed immersion $\iota:\mathbb{P}(E)\to\mathbb{P}(E\oplus\mathbf{1}_X)$. We start by computing ι_* for cycles coming from X.

Lemma 5.5 *Let L be a line bundle over X. For any $\alpha\in H^i(X,\mathbf{I}^j,\mathcal{L})$, we have*

$$\iota_* p^*(\alpha) = c(\mathcal{O}(1))(p')^*(\alpha)$$

in $H^{i+1}(\mathbb{P}(E\oplus\mathbf{1}_X),\mathbf{I}^{j+1},\mathcal{L}(-1))$.

Proof: Let $x:\mathcal{O}(-1)\to\mathcal{O}_{\mathbb{P}(E\oplus\mathbf{1}_X)}$ be the morphism given by the global section corresponding to $1\in\mathbf{1}_X$. Then the class in $H^1(\mathbb{P}(E\oplus\mathbf{1}_X),\mathbf{I},\mathcal{O}(-1))$ of the symmetric isomorphism $d_0^{\mathcal{O}(-1)}(x)$ given by the diagram

$$\begin{array}{ccc} \mathcal{O}(-1) & \xrightarrow{x} & \mathcal{O}_{\mathbb{P}(E\oplus\mathbf{1}_X)} \\ {\scriptstyle -1}\downarrow & & \| \\ \mathcal{O}(-1) & \xrightarrow[-x]{} & \mathcal{O}_{\mathbb{P}(E\oplus\mathbf{1}_X)} \end{array}$$

represents $c(\mathcal{O}(1))$. This is precisely $\iota_*(1)$ by [8, Remark 3.33]. The result follows on noting that $\iota_* p^*(\alpha) = \iota_*(1)\cdot p^*(\alpha)$. □

Lemma 5.6 *For any $m\geq 1$ and any line bundle L over X, we have $\iota_*\mu_{\mathcal{L}}^m = \mu_{\mathcal{L}}^{m+1}$.*

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Proof: First observe that $\iota^*\mathcal{O}(1) = \mathcal{O}(1)$. Using [9, Théorème 13.3.1], we get for any m

$$\iota_* c(\mathcal{O}(1))^{m-1} = \iota_* c(\iota^*\mathcal{O}(1))^{m-1} = c(\mathcal{O}(1))^{m-1}\iota_*$$

and we can then suppose that $m = 1$. By Proposition 2.1, we get

$$\iota_*\mu^1_{\mathcal{L}} = \iota_*\partial_{\mathcal{L}(-1)}p^* = \partial_{(p')^*\mathcal{L}}\iota_* p^*.$$

Now $\iota_* p^* = \overline{c}(\mathcal{O}(1))(p')^*$ by [11, Proposition 2.6 (b)] and Lemma 3.2 shows that

$$\partial_{(p')^*\mathcal{L}}\iota_* p^* = \partial_{(p')^*\mathcal{L}}\overline{c}(\mathcal{O}(1))(p')^* = c(\mathcal{O}(1))\partial_{\mathcal{L}(-1)}(p')^*.$$

The latter is $\mu^2_{\mathcal{L}}$ and we are done. □

We thus get the description we wanted, as stated in the next result.

Proposition 5.7 *For any $m \geq 1$ and any $\alpha \in H^i(X, \bar{\mathbf{I}}^j)$, we have*

$$\pi_{\mathcal{L}(-m)}\mu^m_{\mathcal{L}}(\alpha) = \xi^m\alpha + \xi^{m-1}\pi_{\mathcal{L}}\partial_{\mathcal{L}}(\alpha).$$

Proof: We have a commutative diagram:

$$\begin{array}{ccc} H^{i+m}(\mathbb{P}(E), \mathbf{I}^{j+m}, \mathcal{L}(-m)) & \xrightarrow{\pi_{\mathcal{L}(-m)}} & H^{i+m}(\mathbb{P}(E), \bar{\mathbf{I}}^{j+m}) \\ \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} \\ H^{i+m+1}(\mathbb{P}(E \oplus \mathbf{1}_X), \mathbf{I}^{j+m+1}, \mathcal{L}(-m-1)) & \xrightarrow[\pi_{\mathcal{L}(-m-1)}]{} & H^{i+m+1}(\mathbb{P}(E \oplus \mathbf{1}_X), \bar{\mathbf{I}}^{j+m+1}). \end{array}$$

The right-hand ι_* is injective because the long exact sequence of localization associated to the embedding ι reduces to (split) short exact sequences. In order to prove the result, it suffices therefore to prove that

$$\iota_*\pi_{\mathcal{L}(-m)}\mu^m_{\mathcal{L}}(\alpha) = \iota_*(\xi^m\alpha + \xi^{m-1}\pi_{\mathcal{L}}\partial_{\mathcal{L}}(\alpha)).$$

Using the commutativity of the above diagram, we see that the left-hand side is equal to $\pi_{\mathcal{L}(-m-1)}\iota_*\mu^m_{\mathcal{L}}(\alpha)$. By Lemma 5.6, we have $\iota_*\mu^m_{\mathcal{L}} = \mu^{m+1}_{\mathcal{L}}$, and we are thus reduced to prove that

$$\pi_{\mathcal{L}(-m-1)}\mu^{m+1}_{\mathcal{L}}(\alpha) = \iota_*(\xi^m\alpha + \xi^{m-1}\pi_{\mathcal{L}}\partial_{\mathcal{L}}(\alpha)).$$

Using once again that $\iota_* p^* = \overline{c}(\mathcal{O}(1))(p')^* = \xi \cdot (p')^*$ and the fact that Stiefel-Whitney classes commute with proper push-forward, we see that the right-hand side is equal to $\xi^{m+1}\alpha + \xi^m\pi_{\mathcal{L}}\partial_{\mathcal{L}}(\alpha)$. The result follows then from Lemma 5.2. □

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Corollary 5.8 *Let E be a bundle of rank r on a smooth scheme X, and let L be a line bundle over X. Then the homomorphisms*

$$\Theta^{\mathcal{L}}_{\text{even}} : \bigoplus_{m \text{ even}}^{2\le m\le r-1} H^{i-m}(X,\bar{\mathbf{I}}^{j-m}) \xrightarrow{\sum \mu^m_{\mathcal{L}}} H^i(\mathbb{P}(E),\mathbf{I}^j,\mathcal{L})$$

$$\Theta^{\mathcal{L}}_{\text{odd}} : \bigoplus_{m \text{ odd}}^{1\le m\le r-1} H^{i-m}(X,\bar{\mathbf{I}}^{j-m}) \xrightarrow{\sum \mu^m_{\mathcal{L}}} H^i(\mathbb{P}(E),\mathbf{I}^j,\mathcal{L}\otimes\mathcal{O}(-1))$$

are split injective.

Proof: We prove that the first homomorphism is split injective when $\mathcal{L}$ is trivial, the arguments to prove the full statement being similar. By Proposition 5.7, the composition

$$\bigoplus_{m \text{ even}}^{2\le m\le r-1} H^{i-m}(X,\bar{\mathbf{I}}^{j-m}) \xrightarrow{\Theta_{\text{even}}} H^i(\mathbb{P}(E),\mathbf{I}^j) \xrightarrow{\pi} H^i(\mathbb{P}(E),\bar{\mathbf{I}}^j)$$

maps $\sum_i \alpha_i$ to $\sum_i (\xi^i \alpha_i + \xi^{i-1}\pi\partial(\alpha_i))$. Now Theorem 4.1 shows that we have an isomorphism

$$\chi_E : \bigoplus_{m=0}^{r-1} H^{i-m}(X,\bar{\mathbf{I}}^{j-m}) \to H^i(\mathbb{P}(E),\bar{\mathbf{I}}^j).$$

Composing π, the inverse of this isomorphism and the projection on the even factors, we get a retraction of Θ_{even}. □

This corollary leads to the following definition.

Definition 5.9 Let X be a smooth scheme, E be a vector bundle of rank r over X and L be a line bundle over X. For any $i, j \in \mathbb{Z}$, we denote by $\tilde{H}^i(\mathbb{P}(E),\mathbf{I}^j,\mathcal{L})$ the cokernel of $\Theta^{\mathcal{L}}_{\text{even}}$ and by $\tilde{H}^i(\mathbb{P}(E),\mathbf{I}^j,\mathcal{L}\otimes\mathcal{O}(-1))$ the cokernel of $\Theta^{\mathcal{L}}_{\text{odd}}$. We call these groups *reduced cohomology groups*.

The proof of the projective bundle theorem will consist in studying these reduced cohomology groups. The next section is devoted to the definition and the study of a useful class.

6. The orientation class

Let X be a smooth scheme and let $\mathcal{E} = \mathcal{O}^r_X$; thus $\mathbb{P}(E) = \mathbb{P}^{r-1}_X$. As usual, we denote by $p : \mathbb{P}^{r-1}_X \to X$ the projection. Given any basis $e_1,\dots,e_r$ of $\mathcal{O}^r_X$, we denote by $e_1^\vee,\dots,e_r^\vee$ its dual basis and by $\tilde{e}_1,\dots,\tilde{e}_r$ the images of $p^*e_1^\vee,\dots,p^*e_r^\vee$ under the

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canonical morphism $p^*((\mathcal{O}_X^r)^\vee) \to \mathcal{O}(1)$. We get symmetric morphisms $\tilde{e}_i$ of the form

$$\mathcal{O}(-1) \xrightarrow{\tilde{e}_i} \mathcal{O}_{\mathbb{P}_X^{r-1}}$$

whose localizations at the generic point of $\mathbb{P}_X^{r-1}$ define elements in $C^0(\mathbb{P}_X^{r-1}, \mathbf{W}, \mathcal{O}(-1))$. Their images under $d_0^{\mathcal{O}(-1)}$ are symmetric forms $d_0^{\mathcal{O}(-1)}(\tilde{e}_i)$:

$$\begin{array}{ccc} \mathcal{O}(-1) & \xrightarrow{\tilde{e}_i} & \mathcal{O}_{\mathbb{P}_X^{r-1}} \\ {\scriptstyle -1}\downarrow & & \| \\ \mathcal{O}(-1) & \xrightarrow[-\tilde{e}_i]{} & \mathcal{O}_{\mathbb{P}_X^{r-1}}. \end{array}$$

Observe that $d_0^{\mathcal{O}(-1)}(\tilde{e}_i)$ is supported on $V(\tilde{e}_i)$, which is of codimension 1, and that it represents $c(\mathcal{O}(1)) = \partial_{\mathcal{O}(-1)}(\overline{1})$ in $H^1(\mathbb{P}_X^{r-1}, \mathbf{I}, \mathcal{O}(-1))$.

Now, we can choose $\chi_{e_1,\dots,e_r} := e_1^\vee \wedge \dots \wedge e_r^\vee$ as a generator of $\det((\mathcal{O}_X^r)^\vee)$. Then $p^*\chi_{e_1,\dots,e_r}$ generates $p^*\det((\mathcal{O}_X^r)^\vee)$ and we can use this to define a new symmetric morphism $\tilde{e}_1 \otimes \chi_{e_1,\dots,e_r}$:

$$\mathcal{O}(-1) \xrightarrow{\tilde{e}_1 \otimes \chi_{e_1,\dots,e_r}} p^*\det((\mathcal{O}_X^r)^\vee).$$

Localizing at the generic point of $\mathbb{P}_X^{r-1}$ as above, we can see $\tilde{e}_1 \otimes \chi_{e_1,\dots,e_r}$ as an element in $C^0(\mathbb{P}_X^{r-1}, \mathbf{W}, p^*\det((\mathcal{O}_X^r)^\vee) \otimes \mathcal{O}(-1))$, and we can consider its exterior product with $d_0^{\mathcal{O}(-1)}(\tilde{e}_2) \cdot \dots \cdot d_0^{\mathcal{O}(-1)}(\tilde{e}_r)$.

It follows from [8, Corollary 4.8] that

$$(\tilde{e}_1 \otimes \chi_{e_1,\dots,e_r}) \star (d_0^{\mathcal{O}(-1)}(\tilde{e}_2) \cdot \dots \cdot d_0^{\mathcal{O}(-1)}(\tilde{e}_r))$$

is a cycle on $\mathbb{P}_X^{r-1} \times \mathbb{P}_X^{r-1}$ and we can pull-back along the diagonal to obtain an element $\xi_{e_1,\dots,e_r}$ in $H^{r-1}(\mathbb{P}_X^{r-1}, \mathbf{I}^{r-1}, \det((\mathcal{O}_X^r)^\vee)(-r))$ (compare with [2, Corollary 6.13]). Now $\omega_{\mathbb{P}_X^{r-1}/X} = \det(\Omega_{\mathbb{P}_X^{n-1}/X}) = \det((\mathcal{O}_X^n)^\vee)(-r)$ and therefore $\xi_{e_1,\dots,e_r}$ can be seen as a class in the group $H^{r-1}(\mathbb{P}_X^{r-1}, \mathbf{I}^{r-1}, \omega_{\mathbb{P}_X^{r-1}/X})$.

Definition 6.1 We call $\xi_{e_1,\dots,e_r}$ the *orientation class* of $\mathbb{P}_X^{r-1}$.

It turns out that $\xi_{e_1,\dots,e_r}$ does not depend on the choice of a basis, as we will see later (see Theorem 8.1). When r is odd, we will also see that $\xi_{e_1,\dots,e_r}$ is the Euler class of some bundle and as such is also defined when $\mathcal{E}$ is not free.

We first deal with the properties of $\xi_{e_1,\dots,e_r}$ under the push-forward homomorphism

$$p_* : H^{r-1}(\mathbb{P}_X^{r-1}, \mathbf{I}^{r-1}, \omega_{\mathbb{P}_X^{r-1}/X}) \to H^0(X, \mathbf{W}).$$

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Let $v_1,\dots,v_r$ be a basis of $\mathcal{O}_X^r$. Let $s_{v_1,\dots,v_r} : X \to \mathbb{P}_X^{r-1}$ be the immersion given by the identification of the vanishing locus $V(\tilde{v}_2,\dots,\tilde{v}_r)$ with X. Observe that $\xi_{v_1,\dots,v_r}$ is precisely supported on $X = V(\tilde{v}_2,\dots,\tilde{v}_r)$, and we can therefore consider it as an element of $H_X^{r-1}(\mathbb{P}_X^{r-1},\mathbf{I}^{r-1},\omega_{\mathbb{P}_X^{r-1}/X})$.

Lemma 6.2 *The equality*

$$(s_{v_1,\dots,v_r})_*(1) = \xi_{v_1,\dots,v_r}$$

holds in $H_X^{r-1}(\mathbb{P}_X^{r-1},\mathbf{I}^{r-1},\omega_{\mathbb{P}_X^{r-1}/X})$.

Proof: Let $D = D_+(\tilde{v}_1)$. For any $2 \le i \le r$, let $\tilde{V}_i = \tilde{v}_i/\tilde{v}_1$ and consider the Koszul complex $Kos(\tilde{V}_2,\dots,\tilde{V}_r)$ on D generated by the regular sequence $\tilde{V}_2,\dots,\tilde{V}_r$. This complex is naturally isomorphic to its dual ([5, Definition 4.1]), i.e. we have a symmetric isomorphism:

$$\rho : Kos(\tilde{V}_2,\dots,\tilde{V}_n) \to \mathrm{Hom}_{\mathcal{O}_D}(Kos(\tilde{V}_2,\dots,\tilde{V}_n),\mathcal{O}_D).$$

On D, $\xi_{v_1,\dots,v_r}$ is represented by the symmetric isomorphism

$$(\tilde{v}_1 \otimes \chi_{v_1,\dots,v_r}) \cdot d_0^{\mathcal{O}(-1)}(\tilde{v}_2) \cdot \dots \cdot d_0^{\mathcal{O}(-1)}(\tilde{v}_r)$$

since $\tilde{v}_1 \otimes \chi_{v_1,\dots,v_r}$ is an isomorphism. Choosing $1/\tilde{v}_1$ as generator of $\mathcal{O}(-1)$, we see that the restriction of $d_0^{\mathcal{O}(-1)}(\tilde{v}_i)$ to D is precisely $d_0(\tilde{V}_i)$ for any $2 \le i \le r$ and the restriction of $\tilde{v}_1 \otimes \chi_{v_1,\dots,v_r}$ to D is the symmetric morphism $\mathcal{O}_D \to \det(A^{r-1})^\vee$ given by $1 \mapsto \tilde{V}_2 \wedge \dots \wedge \tilde{V}_r$. Therefore $(\xi_{v_1,\dots,v_r})_{|D}$ coincides with the symmetric form ρ on $Kos(\tilde{V}_2,\dots,\tilde{V}_r)$ (use [5, Remark 4.2]). The latter is $(s_{v_1,\dots,v_r|_D})_*(1)$ by [13, §2.4]. To conclude, observe that since $s_{v_1,\dots,v_r}(X) \subset D$, the localization homomorphism

$$H_X^{r-1}(\mathbb{P}_X^{r-1},\mathbf{I}^{r-1},\omega_{\mathbb{P}_X^{r-1}/X}) \to H_X^{r-1}(D,\mathbf{I}^{r-1},\omega_{D/X})$$

is an isomorphism by [1, Corollary 2.3]. □

Proposition 6.3 *Let* $v_r,\dots,v_r$ *be any basis of* $\mathcal{O}_X^r$. *Then* $p_*(\xi_{v_r,\dots,v_r}) = 1$ *in* $H^0(X,\mathbf{W})$.

Proof: This is clear since $ps_{v_1,\dots,v_r} = Id$. □

We can identify further the class $\xi_{v_1,\dots,v_r}$. Recall from Section 1 that we have an exact sequence of sheaves on $\mathbb{P}_X^{r-1}$

$$0 \longrightarrow \mathcal{G} \longrightarrow p^*(\mathcal{O}_X^r)^\vee \longrightarrow \mathcal{O}(1) \longrightarrow 0.$$

and that the determinant of $\mathcal{G}$ is $\det(\mathcal{O}_X^r)^\vee(-1)$. If r is odd, this determinant is the same as $\omega_{\mathbb{P}_X^{r-1}/X}$ (modulo 2). So we can see the Euler class $c(G^\vee)$ as an element of $H^{r-1}(\mathbb{P}_X^{r-1},\mathbf{I}^{r-1},\omega_{\mathbb{P}_X^{r-1}/X})$.

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Proposition 6.4 *Suppose that r is odd. Then we have $c(G^\vee) = \xi_{v_1,\dots,v_r}$ for any basis $v_1,\dots,v_r$ of $\mathcal{O}_X^r$.*

Proof: Consider the exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow p^*(\mathcal{O}_X^r)^\vee \longrightarrow \mathcal{O}(1) \longrightarrow 0.$$

Choose a basis $v_1,\dots,v_r$ of $\mathcal{O}_X^r$. Restricting this sequence to $D_+(\tilde{v}_1)$ and choosing $\tilde{v}_1$ as a generator of $\mathcal{O}(1)$, we see that $\mathcal{G}$ is the free $\mathcal{O}_{D_+(\tilde{v}_1)}$-module generated by the elements $(\tilde{v}_2/\tilde{v}_1,-1,0,\dots,0),\dots,(\tilde{v}_r/\tilde{v}_1,0,\dots,0,-1)$.

The projection of $p^*(\mathcal{O}_X^r)^\vee$ onto the first factor yields a morphism of sheaves $s : \mathcal{G} \to \mathcal{O}_{\mathbb{P}_X^{r-1}}$ and a direct computation shows that we have an exact sequence of sheaves

$$\mathcal{G} \xrightarrow{s} \mathcal{O}_{\mathbb{P}_X^{r-1}} \longrightarrow (s_{v_1,\dots,v_r})_*\mathcal{O}_{\mathrm{Spec}(A)} \longrightarrow 0.$$

Restricting this sequence to $D_+(\tilde{v}_1)$ and using the above basis of $\mathcal{G}_{|_{D_+(\tilde{v}_1)}}$, we see that $s_{|_{D_+(\tilde{v}_1)}}$ is given by the regular sequence $\tilde{v}_2/\tilde{v}_1,\dots,\tilde{v}_r/\tilde{v}_1$.

We can consider the Koszul complex $K(s)$ associated to s, together with its natural symmetric isomorphism ([5, Remark 4.2])

$$\rho : K(s) \to T^{r-1}\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}_X^{r-1}}}(K(s),\det(\mathcal{G})).$$

Observe that $K(s)$ is supported on $s_{v_1,\dots,v_r}(X)$. Localizing at the generic point of $s_{v_1,\dots,v_r}(X)$, we get an element in $H^{r-1}_{s_{v_1,\dots,v_r}(X)}(\mathbb{P}_X^{r-1},\mathbf{I}^{r-1},\det(\mathcal{G}))$ which represents the Euler class $c(G^\vee)$ by [9, Théorème 14.3.1]. Restricting to $D_+(\tilde{v}_1)$, we check as in the proof of Lemma 6.2 that it coincides with $\xi_{v_1,\dots,v_r}$. □

Remark 6.5 In particular, we see that if r is odd then $\xi_{v_1,\dots,v_r}$ is independent of the basis.

7. $\mathbb{P}(E)$ and $\mathbb{P}(E \oplus \mathbf{1}_X)$

In this section, we prove some technical results comparing the cohomology of $\mathbb{P}(E)$ and $\mathbb{P}(E \oplus \mathbf{1}_X)$.

Let X be a scheme, and let E be a rank r vector bundle over X. As described in Section 1, we have a closed embedding $s : X \to \mathbb{P}(E \oplus \mathbf{1}_X)$ and its open complement $\eta : U \to \mathbb{P}(E \oplus \mathbf{1}_X)$ is a vector bundle over $\mathbb{P}(E)$ with projection $q : U \to \mathbb{P}(E)$. The closed embedding s induces an isomorphism

$$s_* : H^{i-r}(X,\mathbf{I}^{j-r}) \to H^i_X(\mathbb{P}(E \oplus \mathbf{1}_X),\mathbf{I}^j,\omega_{\mathbb{P}(E\oplus\mathbf{1}_X)/X})$$

and the projection q an isomorphism

$$q^* : H^i(\mathbb{P}(E),\mathbf{I}^j,\omega_{\mathbb{P}(E)/X} \otimes \mathcal{O}(-1)) \to H^i(U,\mathbf{I}^j,\omega_{U/X}).$$

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Lemma 7.1 *The long exact sequence of localization yields short (split) exact sequences*

$$0 \to H^{i-r}(X,\mathbf{I}^{j-r}) \xrightarrow{s_*} H^i(\mathbb{P}(E\oplus \mathbf{1}_X),\mathbf{I}^j,\omega_{\mathbb{P}(E\oplus\mathbf{1}_X)/X}) \to H^i(\mathbb{P}(E),\mathbf{I}^j,\omega_{\mathbb{P}(E)/X}(-1)) \to 0.$$

Proof: It suffices to observe that p_* is a left inverse of s_*. □

The same result holds twisted by a line bundle L over X. Suppose next that the cohomology of $\mathbb{P}(E\oplus\mathbf{1}_X)$ is twisted by $\omega_{\mathbb{P}(E\oplus\mathbf{1}_X)/X}(1)$ instead of $\omega_{\mathbb{P}(E\oplus\mathbf{1}_X)/X}$.

The complement of $\mathbb{P}(E)$ in $\mathbb{P}(E\oplus\mathbf{1}_X)$ is E and the embedding $s: X \to \mathbb{P}(E\oplus\mathbf{1}_X)$ factorizes through E to give an embedding $z: X \to E$ (which is the zero section of the vector bundle E). By flat excision, the morphism $\tilde{\iota}: E \to \mathbb{P}(E\oplus\mathbf{1}_X)$ induces an isomorphism on the cohomology groups with support on X. We therefore get an isomorphism (trivializing $\mathcal{O}(-1)$ in an obvious way)

$$(\iota^*)^{-1}z_*: H^{i-r}(X,\mathbf{I}^{j-r}) \to H^i_X(\mathbb{P}(E\oplus\mathbf{1}_X),\mathbf{I}^j,\omega_{\mathbb{P}(E\oplus\mathbf{1}_X)/X}(1)).$$

As in the previous case, the projection $q: U \to \mathbb{P}(E)$ induces an isomorphism

$$q^*: H^i(\mathbb{P}(E),\mathbf{I}^j,\omega_{\mathbb{P}(E)/X}) \to H^i(U,\mathbf{I}^j,\omega_{U/X}(1)).$$

The aim of the next lemma is to understand the connecting homomorphism δ in the long exact sequence of localization with coefficients in $\omega_{\mathbb{P}(E\oplus\mathbf{1}_X)/X}(-1)$ associated to the open embedding $U \subset \mathbb{P}(E\oplus\mathbf{1}_X)$.

Lemma 7.2 *Let* $\epsilon: H^{i-r+1}(X,\mathbf{I}^{j-r+1}) \to H^{i-r+1}(X,\mathbf{I}^{j-r})$ *be the homomorphism induced by the morphism of sheaves* $\mathbf{I}^{j-r+1} \to \mathbf{I}^{j-r}$. *Then the following diagram commutes*

$$\begin{array}{ccc} H^i(\mathbb{P}(E),\mathbf{I}^j,\omega_{\mathbb{P}(E)/X}) & \xrightarrow{\tilde{\iota}^*\delta q^*} & H^{i+1}_X(E,\mathbf{I}^j,\omega_{E/X}) \\ \downarrow{\scriptstyle p_*} & & \uparrow{\scriptstyle z_*} \\ H^{i-r+1}(X,\mathbf{I}^{j-r+1}) & \xrightarrow[\epsilon]{} & H^{i-r+1}(X,\mathbf{I}^{j-r}). \end{array}$$

Proof: Recall from Section 1 that the blow-up of $\mathbb{P}(E\oplus\mathbf{1}_X)$ along X can be identified with the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}(E)}(1)\oplus\mathbf{1}_{\mathbb{P}(E)})$. Hence we have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{s} & \mathbb{P}(E\oplus\mathbf{1}_X) & \xleftarrow{\eta} & U \\ \uparrow{\scriptstyle p} & & \uparrow{\scriptstyle \tilde{p}} & \swarrow{\scriptstyle \tilde{\eta}} & \downarrow{\scriptstyle q} \\ \mathbb{P}(E) & \longrightarrow & Bl_X(\mathbb{P}(E\oplus\mathbf{1}_X)) & \longrightarrow & \mathbb{P}(E) \end{array}$$

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satisfying Hypothesis 1.2 in [4, §1]. Observe that the composition of the two morphisms in the bottom line is the identity, because of the identification of the exceptional fibre $\mathbb{P}(E)$ with $\mathbb{P}(\mathbf{1}_{\mathbb{P}(E)})$ in $Bl_X(\mathbb{P}(E\oplus\mathbf{1}_X)) = \mathbb{P}(\mathcal{O}_{\mathbb{P}(E)}(1)\oplus\mathbf{1}_{\mathbb{P}(E)})$.

In such a situation, Balmer and Calmès describe in [4, Theorem 1.4 (B)] the connecting homomorphism

$$\delta : W^i(U,\omega_{U/X}(1)) \to W_X^{i+1}(\mathbb{P}(E\oplus\mathbf{1}_X),\omega_{\mathbb{P}(E\oplus\mathbf{1}_X)/X}(1))$$

appearing in the long exact sequence of localization for Witt groups as $s_* p_*(q^*)^{-1}$ (see [3] for more information on Witt groups). Surprisingly enough, their argument basically boils down to understanding the case of a blow-up along a closed subscheme of codimension 1 (!). As we want to adapt their proof to $\mathbf{I}^j$-cohomology, we briefly recall the setting of [4, Lemma 4.2]. Let B be a smooth scheme over a scheme X and let E be a smooth principal divisor in B with open complement U. Denote by $\tilde{\imath} : E \to B$ the closed embedding and by $\tilde{v} : U \to B$ the open embedding. Let $\mathcal{O}(E)$ be the line bundle associated to E. Then the composite

$$W^i(B,\omega_{B/X}\otimes\mathcal{O}(E)) \xrightarrow{\tilde{v}^*} W^i(U,\omega_{B/U}) \xrightarrow{\partial} W_E^{i+1}(B,\omega_{B/X})$$

is equal to $\tilde{\imath}_*\tilde{\imath}^*$.

In the same setting, we would like to compute the composite

$$H^i(B,\mathbf{I}^j,\omega_{B/X}\otimes\mathcal{O}(E)) \xrightarrow{\tilde{v}^*} H^i(U,\mathbf{I}^j,\omega_{B/U}) \xrightarrow{\partial} H_E^{i+1}(B,\mathbf{I}^j,\omega_{B/X}).$$

By definition of $\mathcal{O}(E)$, we have an exact sequence of sheaves of $\mathcal{O}_B$-modules

$$0 \longrightarrow \mathcal{O}(E)^\vee \xrightarrow{s} \mathcal{O}_B \longrightarrow \tilde{\imath}_*\mathcal{O}_E \longrightarrow 0.$$

The morphism s can be seen as a symmetric morphism for the duality $\mathcal{O}(E)^\vee$

$$s : \mathcal{O}(E)^\vee \to \mathrm{Hom}_{\mathcal{O}_B}(\mathcal{O}(E)^\vee,\mathcal{O}(E)^\vee)$$

which becomes an isomorphism on U (for the duality $\mathcal{O}_U$ since we can trivialize $\mathcal{O}_E^\vee$ on U). It follows that s defines a cycle in $C^0(B,\mathbf{W},\mathcal{O}(E)^\vee)$.

Let $\alpha \in H^i(B,\mathbf{I}^j,\omega_{B/X}\otimes\mathcal{O}(E))$ and $\overline{\alpha} \in C^i(B,\mathbf{I}^j,\omega_{B/X}\otimes\mathcal{O}(E))$ be a lift of α. Then $\alpha \star s \in C^i(B,\mathbf{I}^j,\omega_{B/X})$ represents the class of $\tilde{v}^*(\alpha)$ in $H^i(U,\mathbf{I}^j,\omega_{B/U})$ and it follows from the Leibnitz formula for $\mathbf{I}^j$-cohomology [8, Corollary 4.8] that (up to signs)

$$\partial(\alpha\star s) = (-1)^i\alpha\cdot d^0(s) = d^0(s)\cdot\alpha.$$

Thus $\partial v^*(\alpha) = d^0(s)\cdot\alpha$.

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We now compute the composite

$$H^i(B,\mathbf{I}^j,\omega_{B/X}\otimes\mathcal{O}(E)) \xrightarrow{\tilde{\iota}^*} H^i(E,\mathbf{I}^j,\omega_{E/X}) \xrightarrow{\tilde{\iota}_*} H^{i+1}_E(B,\mathbf{I}^{j+1},\omega_{B/X}).$$

The projection formula yields $\tilde{\iota}_*\tilde{\iota}^*(\alpha) = \tilde{\iota}_*(1)\cdot\alpha$. By definition of the push-forward map for $\mathbf{I}^j$-cohomology, we see that the class of $d^0(s)$ in $H^1(B,\mathbf{I},\mathcal{O}(E)^\vee)$ is precisely $\tilde{\iota}_*(1)$, and it follows therefore that $\epsilon\tilde{\iota}_*(1) = d^0(s)$ where

$$\epsilon : H^1(B,\mathbf{I},\mathcal{O}(E)^\vee) \to H^1(B,\mathbf{W},\mathcal{O}(E)^\vee).$$

Thus $\epsilon\tilde{\iota}_*\tilde{\iota}^* = \partial v^*$ in this situation. We can now argue as Balmer and Calmès ([4, proof of the Main Lemma 3.5, proof of Theorem 5.1]) to get a commutative diagram:

$$\begin{array}{ccc} H^i(\mathbb{P}(E),\mathbf{I}^j,\omega_{\mathbb{P}(E)/X}) & \xrightarrow{\delta q^*} & H^{i+1}_X(\mathbb{P}(E\oplus\mathbf{1}_X),\mathbf{I}^j,\omega_{\mathbb{P}(E\oplus\mathbf{1}_X)/X}(1)) \\ \downarrow{\scriptstyle p_*} & & \uparrow{\scriptstyle s_*} \\ H^{i-r+1}(X,\mathbf{I}^{j-r+1}) & \xrightarrow[\epsilon]{} & H^{i-r+1}(X,\mathbf{I}^{j-r}). \end{array}$$

This proves the result. □

Corollary 7.3 *The following diagram commutes:*

$$\begin{array}{ccc} H^i(\mathbb{P}^{r-1}_X,\mathbf{I}^j,\omega_{\mathbb{P}^{r-1}_X/X}) & \xrightarrow{\tilde{\iota}^*\delta q^*} & H^{i+1}_X(\mathbb{A}^{r-1}_X,\mathbf{I}^j,\omega_{\mathbb{A}^{r-1}_X/X}) \\ \uparrow{\scriptstyle \xi_{e_2,\ldots,e_r}} & & \uparrow{\scriptstyle z_*} \\ H^{i-r+2}(X,\mathbf{I}^{j-r+2}) & \xrightarrow[\epsilon]{} & H^{i-r+2}(X,\mathbf{I}^{j-r+1}). \end{array}$$

Proof: This follows immediately from Lemma 7.2 with $E = \mathcal{O}^r_X$ and from the fact that in this situation the multiplication by $\xi_{e_2,\ldots,e_r}$ is a section of p_* by Proposition 6.3. □

Pursuing our study of the localization sequence, we switch to the study of the homomorphism

$$H^{i+1}_X(\mathbb{P}(E\oplus\mathbf{1}_X),\mathbf{I}^j,\omega_{\mathbb{P}(E\oplus\mathbf{1}_X)/X}(1)) \to H^{i+1}(\mathbb{P}(E\oplus\mathbf{1}_X),\mathbf{I}^j,\omega_{\mathbb{P}(E\oplus\mathbf{1}_X)/X}(1))$$

when $\mathcal{E} = \mathcal{O}^{r-1}_X$. The next result is a computation of it, after identification of the left-hand term by flat excision (and trivialization of $\mathcal{O}(-1)$).

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Lemma 7.4 *The following diagram commutes:*

$$\begin{array}{ccc} H^{i+1}_X(\mathbb{A}^{r-1}_X,\mathbf{I}^j,\omega_{\mathbb{A}^{r-1}_X/X}) & \xrightarrow{(\tilde{\imath}^*)^{-1}} & H^{i+1}(\mathbb{P}^{r-1}_X,\mathbf{I}^j,\omega_{\mathbb{P}^{r-1}_X/X}(1)) \\ \uparrow{\scriptstyle z_*} & & \uparrow{\scriptstyle \mu^{r-1}_{(p')^*\det(\mathcal{O}^r_X)^\vee}} \\ H^{i-r+2}(X,\mathbf{I}^{j-r+1}) & \xrightarrow[\pi]{} & H^{i-r+2}(X,\bar{\mathbf{I}}^{j-r+1}). \end{array}$$

Proof: Let $\alpha \in H^{i-r+2}(X,\mathbf{I}^{j-r+1})$. By definition,

$$\mu^{r-1}_{(p')^*\det(\mathcal{O}^{r-1}_X)^\vee}\pi(\alpha) = c(\mathcal{O}(1))^{r-2}\partial_{\det(\mathcal{O}^{r-1}_X)^\vee(-1)}(p')^*\pi(\alpha).$$

Since $(p')^*$ and π commute, we can use Lemma 3.1 to get

$$\partial_{\det(\mathcal{O}^{r-1}_X)^\vee(-1)}(p')^*\pi(\alpha) = c(\det(\mathcal{O}^{r-1}_X)\otimes\mathcal{O}(1))\cdot(p')^*(\alpha).$$

So it suffices to compute $c(\mathcal{O}(1))^{r-2}c(\det(\mathcal{O}^{r-1}_X)\otimes\mathcal{O}(1))\cdot(p')^*(\alpha)$ to prove the result. Using the projection formula, we see that $z_*(\alpha) = z_*(1)\cdot(p'')^*(\alpha)$ and we are then reduced to prove the result for $\alpha = 1$.

Recall from Section 6 that $c(\mathcal{O}(1))$ is represented by the class of $d_0^{\mathcal{O}(-1)}\tilde{e}_i$ for any $i = 1,\dots,r$ and that $c(\det(\mathcal{O}^{r-1}_X)\otimes\mathcal{O}(1))$ is represented by the class of $d_0^{\det(E)^\vee(-1)}(\tilde{e}_2\cdot\chi_{e_1,\dots,e_r})$.

It follows that $c(\mathcal{O}(1))^{r-2}c(\det(\mathcal{O}^{r-1}_X)\otimes\mathcal{O}(1))$ is represented by the class of

$$d_0^{\mathcal{O}(-1)}\tilde{e}_r\cdot\ldots\cdot d_0^{\mathcal{O}(-1)}\tilde{e}_3\cdot d_0^{\det(\mathcal{O}^{r-1}_X)^\vee(-1)}(\tilde{e}_2\cdot\chi_{e_1,\dots,e_r})$$

which is supported on $s(X)$. Restricting to E, we see that it also represents $(\tilde{\imath}^*)^{-1}z_*(1)$ by [13, §2.4]. □

We now turn to the homomorphism

$$H^{i+1}(\mathbb{P}(E\oplus\mathbf{1}_X),\mathbf{I}^j,\omega_{\mathbb{P}(E\oplus\mathbf{1}_X)/X}(1)) \to H^{i+1}(U,\mathbf{I}^j,\omega_{U/X}(1))$$

where we identify the right-hand term with $H^{i+1}(\mathbb{P}(E),\mathbf{I}^j,\omega_{\mathbb{P}(E)/X})$ by homotopy invariance.

Lemma 7.5 *Suppose that $\mathcal{E} = \mathcal{O}^{r-1}_X$. Then the following diagram commutes:*

$$\begin{array}{ccc} H^{i+1}(\mathbb{P}(E\oplus\mathbf{1}_X),\mathbf{I}^j,\omega_{\mathbb{P}(E\oplus\mathbf{1}_X)/X}(1)) & \xrightarrow{(q^*)^{-1}\eta^*} & H^{i+1}(\mathbb{P}(E),\mathbf{I}^j,\omega_{\mathbb{P}(E)/X}) \\ \uparrow{\scriptstyle \mu^{r-1}_{\det(E\oplus\mathbf{1}_X)^\vee}} & & \uparrow{\scriptstyle \xi_{e_2,\dots,e_n}} \\ H^{i-r+2}(X,\bar{\mathbf{I}}^{j-r+1}) & \xrightarrow[\partial]{} & H^{i-r+1}(X,\mathbf{I}^{j-r+2}). \end{array}$$

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Proof: Let $\alpha \in H^{i-n+2}(X,\bar{\mathbf{I}}^{j-r+1})$. Using Lemma 5.4, we see that

$$(q^*)^{-1}\eta^*\mu^{r-1}_{\det(\mathcal{O}_X^{r-1}\oplus\mathbf{1}_X)^\vee}(\alpha) = \mu^{r-1}_{\det(\mathcal{O}_X^{r-1})^\vee}(\alpha).$$

The lemma follows then from the explicit description of $\xi_{e_2,\dots,e_n}$ of Section 6 and Proposition 2.1. □

8. The split case

In this section, A is a smooth k-algebra and $X = \mathrm{Spec}(A)$. We always suppose that $r \geq 2$.

If $\beta \in H^i(\mathbb{P}^{r-1}_X,\mathbf{I}^j,\mathcal{L})$ and $\alpha \in H^r(X,\mathbf{I}^s,\mathcal{N})$, we denote by $\alpha\cdot\beta$ the class of $p^*(\alpha)\cdot\beta$ in $H^{i+r}(\mathbb{P}^{r-1}_X,\mathbf{I}^{j+s},p^*\mathcal{N}\otimes\mathcal{L})$. In the statement of the following proposition, we consider the reduced cohomology groups of Definition 5.9. Observe that the pull-back p^* and the right multiplication by the orientation class induce homomorphisms

$$p^* : H^i(X,\mathbf{I}^j,\mathcal{L}) \to \tilde{H}^i(\mathbb{P}^{r-1}_X,\mathbf{I}^j,p^*\mathcal{L})$$

and

$$\xi_{e_1,\dots,e_r} : H^i(X,\mathbf{I}^j,\mathcal{L}) \to \tilde{H}^{i+r-1}(\mathbb{P}^{r-1}_X,\mathbf{I}^{j+r-1},p^*\mathcal{L}\otimes\omega_{\mathbb{P}^{r-1}_X/X})$$

by composing with the projection of the cohomology groups to the reduced cohomology groups.

Theorem 8.1 *Let $e_1,\dots,e_r$ be a basis of A^r, and let L be a line bundle over X. If r is even, then*

$$p^*\oplus\xi_{e_1,\dots,e_r} : H^i(X,\mathbf{I}^j,\mathcal{L})\oplus H^{i-r+1}(X,\mathbf{I}^{j-r+1},\mathcal{L}) \to \tilde{H}^i(\mathbb{P}^{r-1}_X,\mathbf{I}^j,p^*\mathcal{L})$$

is an isomorphism and $\tilde{H}^i(\mathbb{P}^{r-1}_X,\mathbf{I}^j,p^*\mathcal{L}\otimes\mathcal{O}(-1)) = 0$ for any $i,j\in\mathbb{Z}$. If r is odd, then

$$p^* : H^i(X,\mathbf{I}^j,\mathcal{L}) \to \tilde{H}^i(\mathbb{P}^{r-1}_X,\mathbf{I}^j,p^*\mathcal{L})$$

and

$$\xi_{e_1,\dots,e_r} : H^{i-r+1}(X,\mathbf{I}^{j-r+1},\mathcal{L}) \to \tilde{H}^i(\mathbb{P}^{r-1}_X,\mathbf{I}^j,p^*\mathcal{L}\otimes\mathcal{O}(-1))$$

are isomorphisms for any $i,j\in\mathbb{Z}$. In both cases, if $v_1,\dots,v_r$ is any other basis of A^r, then we have $\xi_{e_1,\dots,e_r} = \xi_{v_1,\dots,v_r}$.

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Proof: We prove the result for a trivial invertible $\mathcal{O}_X$-module $\mathcal{L}$, the general case being strictly the same. If $e_1,\dots,e_r$ is a basis of A^r, we put $\mathcal{E} = A^{r-1}$ with basis $e_2,\dots,e_r$. We then have $\mathbb{P}^{r-1}_X = \mathbb{P}(E \oplus \mathbf{1}_X)$ and we use the notations of Section 1 in the rest of the proof.

We start with the computation of the reduced cohomology of $\mathbb{P}^1_X$, i.e. we assume that $r = 2$. If e_1,e_2 is a basis of A^2, then we can identify $X = \mathrm{Spec}(A)$ with the zero locus of the section $\tilde{e}_2$. This yields an embedding $s_{e_1,e_2} : X \to \mathbb{P}^1_X$ and split exact sequences (Lemma 7.1):

$$0 \longrightarrow H^{i-1}(X,\mathbf{I}^{j-1}) \xrightarrow{(s_{e_1,e_2})_*} H^i(\mathbb{P}^1_X,\mathbf{I}^j,\omega_{\mathbb{P}^1_X/X}) \longrightarrow H^i(\mathbb{A}^1,\mathbf{I}^j,\omega_{\mathbb{A}^1_X/X}) \longrightarrow 0$$

Now $(s_{e_1,e_2})_*(1) = \xi_{e_1,e_2}$ by Lemma 6.2 and in general $(s_{e_1,e_2})_*(\alpha) = p^*(\alpha)\cdot\xi_{e_1,e_2}$ by the projection formula. The retraction p_* and homotopy invariance give

$$H^i(\mathbb{P}^1_X,\mathbf{I}^j,\omega_{\mathbb{P}^1_X/X}) \simeq [H^{i-1}(X,\mathbf{I}^{j-1})]\cdot\xi_{e_1,e_2} \oplus [H^i(X,\mathbf{I}^j,\det(A^2)^\vee)]\cdot 1.$$

Since $\omega_{\mathbb{P}^1_X/X}$ is a square in $\mathrm{Pic}(\mathbb{P}^1_X)$, we see that in that case the ordinary cohomology coincides with the reduced cohomology and this case is settled.

We now have to compute $H^i(\mathbb{P}^1_X,\mathbf{I}^j,\omega_{\mathbb{P}^1_X/X}(-1))$. We again use the long exact sequence associated to the embedding $s_{e_1,e_2} : X \to \mathbb{P}^1_X$. Using Corollary 7.3, Lemmas 7.4 and 7.5 as well as the 5-lemma, we see that

$$\mu^1 : H^{i-1}(X,\bar{\mathbf{I}}^{j-1}) \to H^i(\mathbb{P}^1_X,\mathbf{I}^j,\omega_{\mathbb{P}^1_X/X}(-1))$$

is an isomorphism. Therefore the reduced group is trivial and the case $r = 2$ is settled.

Suppose by induction that the theorem is true for $r-1$. As usual, we set $\mathcal{E} = A^{r-1}$ and we therefore have split exact sequences by Lemma 7.1 :

$$0 \to H^{i-r}(X,\mathbf{I}^{j-r}) \xrightarrow{s_*} H^i(\mathbb{P}(E\oplus\mathbf{1}_X),\mathbf{I}^j,\omega_{\mathbb{P}(E\oplus\mathbf{1}_X)/X}) \to H^i(\mathbb{P}(E),\mathbf{I}^j,\omega_{\mathbb{P}(E)/X}(-1)) \to 0$$

Lemma 5.4 shows that these sequences induce split exact sequences

$$0 \to H^{i-r}(X,\mathbf{I}^{j-r}) \xrightarrow{s_*} \tilde{H}^i(\mathbb{P}(E\oplus\mathbf{1}_X),\mathbf{I}^j,\omega_{\mathbb{P}(E\oplus\mathbf{1}_X)/X}) \to \tilde{H}^i(\mathbb{P}(E),\mathbf{I}^j,\omega_{\mathbb{P}(E)/X}(-1)) \to 0$$

and therefore Lemma 6.2 yields the result for $\tilde{H}^i(\mathbb{P}^{r-1}_X,\mathbf{I}^j,\omega_{\mathbb{P}^{r-1}_X/X})$. We now deal with $H^i(\mathbb{P}^{r-1}_X,\mathbf{I}^j,\omega_{\mathbb{P}^{r-1}_X/X}(-1))$. We have a commutative diagram with exact lines (use Lemmas 7.4 and 7.5)

$$\begin{array}{ccccc}
H^{i-r+1}(X,\mathbf{I}^{j-r+1}) & \longrightarrow & H^i(\mathbb{P}^{r-1}_X,\mathbf{I}^j,\omega_{\mathbb{P}^{r-1}_X/X}(-1)) & \longrightarrow & H^i(\mathbb{P}^{r-2},\mathbf{I}^j,\omega_{\mathbb{P}^{r-2}/X}) \\
\| & & \uparrow{\scriptstyle \mu^{r-1}_{\det((A^r)^\vee)}} & & \uparrow{\scriptstyle \xi_{e_2,\dots,e_r}} \\
H^{i-r+1}(X,\mathbf{I}^{j-r+1}) & \xrightarrow[\pi]{} & H^{i-r+1}(X,\bar{\mathbf{I}}^{j-r+1}) & \xrightarrow[\partial]{} & H^{i-r+1}(X,\mathbf{I}^{j-r+2}).
\end{array}$$

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Once again, Lemma 5.4 shows that the same diagram holds for reduced groups. By induction, Lemmas 7.3 and $H^i(\mathbb{P}^{r-1}_X, \mathbf{I}^j, \omega_{\mathbb{P}^{r-1}_X/X}(-1))$ has the desired form.

To conclude, we still have to prove that $\xi_{v_1,\dots,v_r} = \xi_{e_1,\dots,e_r}$ for any basis $v_1,\dots,v_r$ of A^r. If r is odd, this is a straightforward consequence of Proposition 6.4.

Suppose that r is even. Then $\omega_{\mathbb{P}^{r-1}_X/X} = p^*\det((A^r)^\vee)$ modulo 2 and we can see $\xi_{v_1,\dots,v_r}$ as a class in $H^{r-1}(\mathbb{P}^{r-1}_X, \mathbf{I}^{r-1}, p^*\det((A^r)^\vee))$. We just computed that this group is equal to the direct sum of

$$[H^{r-1}(X, \mathbf{I}^{r-1}, \det((A^r)^\vee)) \cdot 1] \oplus [H^0(X, \mathbf{W}) \cdot \xi_{e_1,\dots,e_r}]$$

and a sum of $\mu^m(H^{r-1-m}(X, \bar{\mathbf{I}}^{r-1-m}))$ for even $m \leq r-2$. Therefore

$$\xi_{v_1,\dots,v_r} = p^*(\beta) + \sum \mu^m \alpha_m + \alpha \cdot \xi_{e_1,\dots,e_r}$$

for some $\alpha_m \in H^{r-1-m}(X, \bar{\mathbf{I}}^{r-1-m})$, some $\beta \in H^{r-1}(X, \mathbf{I}^{r-1}, \det((A^r)^\vee)$ and some $\alpha \in H^0(X, \mathbf{W})$. Applying p_*, we get $\alpha = 1$ by Proposition 6.3. Using the projection π, we see that $\alpha_m = 0$ for any m (use Lemma 5.2). It remains to show that $\beta = 0$. But both $\xi_{e_1,\dots,e_r}$ and $\xi_{v_1,\dots,v_r}$ are represented by a cycle supported on a rational point (over X) and there exists a $\mathbb{A}^{r-1}$ not containing their supports. Restricting to this affine plane and using homotopy invariance, we get $\beta = 0$. □

9. The projective bundle theorem

If E is a bundle of odd rank over a smooth scheme X, we have the following theorem. Recall from Section 1.1 that $\omega_{\mathbb{P}(E)/X} = p^*\det(\mathcal{E})^\vee(-1)$ in $Pic(\mathbb{P}(E))/2$.

Theorem 9.1 *Let X be a smooth scheme and let E be a vector bundle over X of odd rank r. Let $p : \mathbb{P}(E) \to X$ be the projective bundle associated to E and*

$$0 \longrightarrow \mathcal{G} \longrightarrow p^*\mathcal{E}^\vee \longrightarrow \mathcal{O}(1) \longrightarrow 0$$

the canonical sequence. Then

$$p^* : H^i(X, \mathbf{I}^j, \mathcal{L}) \to \tilde{H}^i(\mathbb{P}(E), \mathbf{I}^j, p^*\mathcal{L})$$

and

$$c(G^\vee) : H^{i+1-r}(X, \mathbf{I}^{j+1-r}, \mathcal{L}) \to \tilde{H}^i(\mathbb{P}(E), \mathbf{I}^j, p^*\mathcal{L} \otimes \omega_{\mathbb{P}(E)/X})$$

are isomorphisms for any $i, j \in \mathbb{Z}$ and any line bundle L over X.

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Proof: We consider the set $\mathcal{S}$ of open affine subschemes of X on which the restriction of E is free. For any $n \in \mathbb{N}$, we let $\mathcal{S}_n$ be the set of open subschemes of X which are covered by at most n elements in $\mathcal{S}$. We prove that the theorem is true on any element of $\mathcal{S}_n$ by induction on n.

If $n = 1$, then the result follows from Proposition 6.4 and Theorem 8.1. Suppose now that the theorem holds for any element of $\mathcal{S}_{n-1}$ and let $Y \in \mathcal{S}_n$. By definition, $Y = Z \cup T$ with $Z \in \mathcal{S}_{n-1}$ and $T \in \mathcal{S}_1$. Since X is separated, it follows that $Z \cap T \in \mathcal{S}_{n-1}$. Using the Mayer-Vietoris sequence, we see that the theorem holds on Y. It follows that the theorem holds for any $n \in \mathbb{N}$ and any element of $\mathcal{S}_n$, hence also for X since it is noetherian and thus quasi-compact. □

If E is of even rank, then things are more complicated. However, there is a simple case.

Theorem 9.2 *Let E be a bundle of even rank r on a smooth scheme X. Then $\tilde{H}^i(\mathbb{P}(E), \mathbf{I}^j, p^*\mathcal{L} \otimes \mathcal{O}(-1)) = 0$ for any $i, j \in \mathbb{Z}$ and any line bundle L over X.*

Proof: If X is affine and E is free, then the result follows from Theorem 8.1. We can use the same argument as in the proof of Theorem 9.1 to conclude. □

Now comes the painful computation of the untwisted cohomology of $\mathbb{P}(E)$ when E is of even rank. To start with, we prove a lemma showing that the push-forward induces a well-defined operation on reduced groups.

Lemma 9.3 *The homomorphism p_* induces a homomorphism*

$$p_* : \tilde{H}^i(\mathbb{P}(E), \mathbf{I}^j, \omega_{\mathbb{P}(E)/X}) \to H^{i+1-r}(X, \mathbf{I}^{j-r+1}).$$

Proof: By definition, it suffices to prove that $p_* \Theta_{\text{even}} = 0$. Let m be an even integer such that $m \leq r-1$ and $\alpha \in H^{i-m}(X, \bar{\mathbf{I}}^{j-m})$. Then α is supported on a finite number of codimension $i-m$ points in X. Let Y denote the union of the closure of these points. We can then see α as an element in $H^{i-m}_Y(X, \bar{\mathbf{I}}^{j-m})$. Now remark that $\mu^m(\alpha)$ is supported on $p^{-1}(Y) \subset \mathbb{P}(E)$ and then by definition of the transfer map we have $p_* \mu^m_{\det(E)^\vee}(\alpha) \in H^{i-r+1}_Y(X, \mathbf{I}^{j-r+1})$. Since m and r are even, we get $m \leq r-2$ and $i-r+1 < i-m$. This implies that $H^{i-r+1}_Y(X, \mathbf{I}^{j-r+1}) = 0$ and the result is proved. □

Theorem 9.4 *Let X be a smooth scheme and let E be a vector bundle over X of even rank r. Let $p : \mathbb{P}(E) \to X$ be the projective bundle associated to E. Then we have a long exact sequence*

$$\cdots \to H^{i-r}(X, \mathbf{I}^{j-r+1}) \xrightarrow{\epsilon c(E)} H^i(X, \mathbf{I}^j, \det(\mathcal{E})^\vee) \xrightarrow{p^*} \tilde{H}^i(\mathbb{P}(E), \mathbf{I}^j, \omega_{\mathbb{P}(E)/X}) \xrightarrow{p_*} H^{i-r+1}(X, \mathbf{I}^{j-r+1}) \to \cdots$$

The sequence splits if and only if $\epsilon c(E) = 0$.

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Proof: In this proof, we denote by ω the invertible sheaf $\omega_{\mathbb{P}(E)/X}$ and by ω' the invertible sheaf $\omega_{\mathbb{P}(E\oplus\mathbf{1}_X)/X}$. We consider the long exact sequence of localization with coefficients in $\omega'(1)$ (which is equivalent to $(p')^*\det(\mathcal{E})^\vee$ modulo 2) associated to the closed embedding $s : X \to \mathbb{P}(E \oplus \mathbf{1}_X)$ with open complement $\eta : U \to \mathbb{P}(E \oplus \mathbf{1}_X)$.

The first step of the proof is to show that this localization sequence can be replaced by an exact sequence easier to manage. More precisely, we will construct surjective homomorphisms f_i, g_i, h_i such that the following diagram commutes

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
H^i(\mathbb{P}(E\oplus\mathbf{1}_X),\mathbf{I}^j,\omega'(1)) & \xrightarrow{f_i} & H^i(X,\mathbf{I}^j,\det(\mathcal{E})^\vee)\oplus H^{i-r}(X,\bar{\mathbf{I}}^{j-r}) \\
\downarrow{\scriptstyle (q^*)^{-1}\eta} & & \downarrow{\scriptstyle (p^*\ \ \mu^r)} \\
H^i(\mathbb{P}(E),\mathbf{I}^j,\omega) & \xrightarrow{g_i} & \tilde{H}^i(\mathbb{P}(E),\mathbf{I}^j,\omega) \\
\downarrow{\scriptstyle \delta q^*} & & \downarrow{\scriptstyle \epsilon p_*} \\
H^i_X(\mathbb{P}(E\oplus\mathbf{1}_X),\mathbf{I}^j,\omega'(1)) & \xrightarrow{h_i} & H^{i-r+1}(X,\mathbf{I}^{j-r}) \\
\downarrow{\scriptstyle ext} & & \downarrow{\scriptstyle \binom{c(E)}{\pi}} \\
H^i(\mathbb{P}(E\oplus\mathbf{1}_X),\mathbf{I}^j,\omega'(1)) & \xrightarrow{f_{i+1}} & H^{i+1}(X,\mathbf{I}^j,\det(\mathcal{E})^\vee)\oplus H^{i+1-r}(X,\bar{\mathbf{I}}^{j-r}) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

and show that the kernel is an exact complex. It will follow that the sequence on the right, that we denote by $\mathcal{C}$, is indeed a long exact sequence.

We start with the definitions of f_i and g_i. Let m be an even integer. Then Lemma 5.4 yields a commutative diagram

$$\begin{array}{ccc}
H^i(\mathbb{P}(E\oplus\mathbf{1}_X),\mathbf{I}^j,\det(\mathcal{E})^\vee(-m)) & \xrightarrow{\eta^*} & H^i(U,\mathbf{I}^j,q^*(\det(\mathcal{E})^\vee(-m))) \\
\uparrow{\scriptstyle \mu^m_{\det(\mathcal{E}\oplus\mathbf{1}_X)^\vee}} & & \uparrow{\scriptstyle q^*} \\
H^{i-m}(X,\bar{\mathbf{I}}^{j-m}) & \xrightarrow[\mu^m_{\det(\mathcal{E})^\vee}]{} & H^i(\mathbb{P}(E),\mathbf{I}^j,\det(\mathcal{E})^\vee(-m)).
\end{array}$$

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It follows that we have a commutative diagram with (split) injective horizontal maps

$$\begin{array}{ccc}
\bigoplus_{\substack{2\le m\le r-1\\ m \text{ even}}} H^{i-r}(X,\bar{\mathbf{I}}^{j-m}) & \xrightarrow{\sum_m \mu^m_{\det(E\oplus 1)^\vee}} & H^i(\mathbb{P}(E\oplus \mathbf{1}_X),\mathbf{I}^j,\omega'(1)) \\
\| & & \downarrow{\scriptstyle (q^*)^{-1}\eta^*} \\
\bigoplus_{\substack{2\le m\le r-1\\ m \text{ even}}} H^{i-r}(X,\bar{\mathbf{I}}^{j-m}) & \xrightarrow[\sum_m \mu^m_{\det(E)^\vee}]{} & H^i(\mathbb{P}(E),\mathbf{I}^j,\omega).
\end{array}$$

By definition, the cokernel of the bottom arrow is the reduced group $\tilde{H}^i(\mathbb{P}(E),\mathbf{I}^j,\omega)$ and we denote by

$$g_i : H^i(\mathbb{P}(E),\mathbf{I}^j,\omega) \to \tilde{H}^i(\mathbb{P}(E),\mathbf{I}^j,\omega)$$

the quotient map.

Now $\tilde{H}^i(\mathbb{P}(E\oplus \mathbf{1}_X),\mathbf{I}^j,\omega'(1))$ is the cokernel of the (split injective) map

$$\sum_m \mu^m : \bigoplus_{\substack{2\le m\le r\\ m \text{ even}}} H^{i-r}(X,\bar{\mathbf{I}}^{j-r}) \to H^i(\mathbb{P}(E\oplus \mathbf{1}_X),\mathbf{I}^j,\omega'(1)).$$

In the above diagram, we've considered even integers $m \le r-1$. Since r is even, we see that the cokernel of the top map is the direct sum

$$\tilde{H}^i(\mathbb{P}(E\oplus \mathbf{1}_X),\mathbf{I}^j,\omega'(1)) \oplus H^{i-r}(X,\bar{\mathbf{I}}^{j-r})$$

and we denote by f_i' the quotient map.

Therefore, the above diagram induces a commutative diagram

$$\begin{array}{ccc}
H^i(\mathbb{P}(E\oplus \mathbf{1}_X),\mathbf{I}^j,\omega'(1)) & \xrightarrow{f_i'} & \tilde{H}^i(\mathbb{P}(E\oplus \mathbf{1}_X),\mathbf{I}^j,\omega'(1)) \oplus H^{i-r}(X,\bar{\mathbf{I}}^{j-r}) \\
\downarrow{\scriptstyle (q^*)^{-1}\eta^*} & & \downarrow \\
H^i(\mathbb{P}(E),\mathbf{I}^j,\omega) & \xrightarrow[g_i]{} & \tilde{H}^i(\mathbb{P}(E),\mathbf{I}^j,\omega)
\end{array}$$

with surjective horizontal maps. We can further identify the left-hand vertical homomorphism using Lemma 5.4 and we obtain a commutative diagram

$$\begin{array}{ccc}
H^i(\mathbb{P}(E\oplus \mathbf{1}_X),\mathbf{I}^j,\omega'(1)) & \xrightarrow{f_i'} & \tilde{H}^i(\mathbb{P}(E\oplus \mathbf{1}_X),\mathbf{I}^j,\omega'(1)) \oplus H^{i-r}(X,\bar{\mathbf{I}}^{j-r}) \\
\downarrow{\scriptstyle (q^*)^{-1}\eta^*} & & \downarrow{\scriptstyle ((q^*)^{-1}\eta^* \ \ \mu^r)} \\
H^i(\mathbb{P}(E),\mathbf{I}^j,\omega) & \xrightarrow[g_i]{} & \tilde{H}^i(\mathbb{P}(E),\mathbf{I}^j,\omega).
\end{array}$$

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Moreover, the vertical arrows are surjective and have the same kernels. Now Theorem 9.1 proves that

$$(p')^* : H^i(X,\mathbf{I}^j,\det(\mathcal{E})^\vee) \to \tilde{H}^i(\mathbb{P}(E\oplus \mathbf{1}_X),\mathbf{I}^j,\omega'(1))$$

is an isomorphism. We define f_i as the composite $\begin{pmatrix} ((p')^*)^{-1} & 0 \\ 0 & Id \end{pmatrix} f_i'$. It follows that we obtain a commutative diagram

$$\begin{array}{ccc} H^i(\mathbb{P}(E\oplus \mathbf{1}_X),\mathbf{I}^j,\omega'(1)) & \xrightarrow{f_i} & H^i(X,\mathbf{I}^j,\det(\mathcal{E})^\vee)\oplus H^{i-r}(X,\bar{\mathbf{I}}^{j-r}) \\ \downarrow{\scriptstyle (q^*)^{-1}\eta^*} & & \downarrow{\scriptstyle (p^*\ \ \mu^r)} \\ H^i(\mathbb{P}(E),\mathbf{I}^j,\omega) & \xrightarrow[g_i]{} & \tilde{H}^i(\mathbb{P}(E),\mathbf{I}^j,\omega) \end{array}$$

as required.

We now define h_i as the composite

$$H_X^{i+1}(\mathbb{P}(E\oplus \mathbf{1}_X),\mathbf{I}^j,\omega'(1)) \xrightarrow{\tilde{\imath}^*} H_X^{i+1}(E,\mathbf{I}^j,\omega_{E/X}) \xrightarrow{(z_*)^{-1}} H^{i+1-r}(X,\mathbf{I}^{j-r})$$

and we observe that h_i is an isomorphism. It follows from Lemmas 7.2 and 9.3 that the diagram

$$\begin{array}{ccc} H^i(\mathbb{P}(E),\mathbf{I}^j,\omega) & \xrightarrow{g_i} & \tilde{H}^i(\mathbb{P}(E),\mathbf{I}^j,\omega) \\ \downarrow{\scriptstyle \delta q^*} & & \downarrow{\scriptstyle \epsilon p_*} \\ H_X^i(\mathbb{P}(E\oplus \mathbf{1}_X),\mathbf{I}^j,\omega'(1)) & \xrightarrow{h_i} & H^{i-r+1}(X,\mathbf{I}^{j-r}) \end{array}$$

commutes.

We are thus left to prove that the following diagram commutes

$$\begin{array}{ccc} H_X^i(\mathbb{P}(E\oplus \mathbf{1}_X),\mathbf{I}^j,\omega'(1)) & \xrightarrow{h_i} & H^{i-r+1}(X,\mathbf{I}^{j-r}) \\ \downarrow{\scriptstyle ext} & & \downarrow{\scriptstyle \binom{c(E)}{\pi}} \\ H^i(\mathbb{P}(E\oplus \mathbf{1}_X),\mathbf{I}^j,\omega'(1)) & \xrightarrow{f_{i+1}} & H^{i+1}(X,\mathbf{I}^j,\det(\mathcal{E})^\vee)\oplus H^{i+1-r}(X,\bar{\mathbf{I}}^{j-r}). \end{array}$$

To achieve this, we observe first that the following diagram commutes because of the definition of the Euler class of E and the naturality of the extension of support

$$\begin{array}{ccccc} H_X^{i+1}(\mathbb{P}(E\oplus \mathbf{1}_X),\mathbf{I}^j,\omega'(1)) & \xrightarrow{\tilde{\imath}^*} & H_X^{i+1}(E,\mathbf{I}^j,\omega_{E/X}) & \xleftarrow{z_*} & H^{i+1-r}(X,\mathbf{I}^{j-r}) \\ \downarrow{\scriptstyle ext} & & \downarrow{\scriptstyle ext} & \swarrow{\scriptstyle (p'')^*c(E)} & \\ H^{i+1}(\mathbb{P}(E\oplus \mathbf{1}_X),\mathbf{I}^j,\omega'(1)) & \xrightarrow[\tilde{\imath}^*]{} & H^{i+1}(E,\mathbf{I}^j,\omega_{E/X}). & & \end{array}$$

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Together with Theorem 9.1, this shows that

$$ext(\tilde{\imath}^*)^{-1}z_*(\alpha) = (p')^*c(E)(\alpha) + \sum_{m \text{ even}} \mu^m_{\det(\mathcal{E})^\vee}(\alpha_{i+1-m})$$

in $H^{i+1}(\mathbb{P}(E\oplus \mathbf{1}_X),\mathbf{I}^j,\omega'(1))$. Applying $\pi_{\det(\mathcal{E})^\vee}$ and using Lemma 5.2, we obtain that $\pi_{\det(\mathcal{E})^\vee}(ext)(\tilde{\imath}^*)^{-1}z_*(\alpha)$ equals to

$$(p')^*\overline{c}_r(E)(\pi(\alpha)) + \sum_{m \text{ even}}^{2\leq m\leq r} [\overline{c}(\mathcal{O}(1))^m(\alpha_{i+1-m}) + \overline{c}(\mathcal{O}(1))^{m-1}(\pi\partial_{\det(E)^\vee}(\alpha_{i+1-m}))].$$

On the other hand, the commutative diagram

$$\begin{array}{ccc} H^{i+1-r}(X,\mathbf{I}^{j-r}) & \xrightarrow{ext(\tilde{\imath}^*)^{-1}z_*} & H^{i+1}(\mathbb{P}(E\oplus\mathbf{1}_X),\mathbf{I}^j,\omega_{\mathbb{P}(E\oplus\mathbf{1}_X)/X}(1)) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_{\det(\mathcal{E})^\vee}} \\ H^{i+1-r}(X,\overline{\mathbf{I}}^{j-r}) & \xrightarrow[ext(\tilde{\imath}^*)^{-1}z_*]{} & H^{i+1}(\mathbb{P}(E\oplus\mathbf{1}_X),\overline{\mathbf{I}}^j) \end{array}$$

and Lemma 4.4 yield $ext(\tilde{\imath}^*)^{-1}z_*(\pi(\alpha)) = \overline{c}(G^\vee_{E\oplus\mathbf{1}_X})(\pi(\alpha))$.

But $\overline{c}(G^\vee_{E\oplus\mathbf{1}_X}) = \sum_{m=0}^r \overline{c}_{r-m}(E)\overline{c}(\mathcal{O}(1))^m$ and we deduce from the commutative diagram above that

$$ext(\tilde{\imath}^*)^{-1}z_*(\alpha) = (p')^*c(E)(\alpha) + \sum_{m \text{ even}} \mu^m_{\det(\mathcal{E})^\vee}(\overline{c}_{r-m}(E)(\pi(\alpha))).$$

It follows that the diagram

$$\begin{array}{ccc} H^i_X(\mathbb{P}(E\oplus\mathbf{1}_X),\mathbf{I}^j,\omega'(1)) & \xrightarrow{h_i} & H^{i-r+1}(X,\mathbf{I}^{j-r}) \\ \downarrow{\scriptstyle ext} & & \downarrow{\scriptstyle \binom{c(E)}{\pi}} \\ H^i(\mathbb{P}(E\oplus\mathbf{1}_X),\mathbf{I}^j,\omega'(1)) & \xrightarrow{f_{i+1}} & H^{i+1}(X,\mathbf{I}^j,\det(\mathcal{E})^\vee)\oplus H^{i+1-r}(X,\overline{\mathbf{I}}^{j-r}). \end{array}$$

commutes. Observe finally that f_i and g_i have the same kernel and that h_i is an isomorphism, it follows therefore that $\mathcal{C}$ is exact.

The second step of the proof is to replace $\mathcal{C}$ by the exact sequence of the statement of the theorem.

Denote temporarily by $\mathcal{C}'$ the sequence

$$\cdots \to H^{i-r}(X,\mathbf{I}^{j-r+1}) \xrightarrow{\xi c(E)} H^i(X,\mathbf{I}^j,\det(\mathcal{E})^\vee) \xrightarrow{p^*} \tilde{H}^i(\mathbb{P}(E),\mathbf{I}^j,\omega_{\mathbb{P}(E)/X}) \xrightarrow{p_*} H^{i-r+1}(X,\mathbf{I}^{j-r+1}) \to \cdots$$

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and suppose for a while that $\mathcal{C}'$ is a complex. We define a map $f : \mathcal{C}' \to \mathcal{C}$ by

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
H^{i-r}(X,\mathbf{I}^{j-r+1}) & \xrightarrow{\epsilon} & H^{i-r}(X,\mathbf{I}^{j-r}) \\
{\scriptstyle \epsilon c(E)}\downarrow & & \downarrow{\scriptstyle \binom{c(E)}{\pi}} \\
H^{i}(X,\mathbf{I}^{j},\det(\mathcal{E})^\vee) & \xrightarrow{\binom{1}{0}} & H^{i}(X,\mathbf{I}^{j},\det(\mathcal{E})^\vee) \oplus H^{i-r}(X,\bar{\mathbf{I}}^{j-r}) \\
{\scriptstyle p^*}\downarrow & & \downarrow{\scriptstyle (p^* \ \ \mu^r)} \\
\tilde{H}^{i}(\mathbb{P}(E),\mathbf{I}^{j},\omega_{\mathbb{P}(E)/X}) & = & \tilde{H}^{i}(\mathbb{P}(E),\mathbf{I}^{j},\omega_{\mathbb{P}(E)/X}) \\
{\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle \epsilon p_*} \\
H^{i-r+1}(X,\mathbf{I}^{j-r+1}) & \xrightarrow{\epsilon} & H^{i-r+1}(X,\mathbf{I}^{j-r}) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

and we check that this is a morphism of complexes.

It is easy to see that the cone $C(f)$ of f is homotopic to the long exact sequence

$$\cdots \to H^{i-r}(X,\mathbf{I}^{j-r+1}) \xrightarrow{\epsilon} H^{i-r}(X,\mathbf{I}^{j-r}) \xrightarrow{\pi} H^{i-r}(X,\bar{\mathbf{I}}^{j-r}) \xrightarrow{\partial} H^{i-r+1}(X,\mathbf{I}^{j-r+1}) \to \cdots$$

Since $\mathcal{C}$ and $C(f)$ are exact, it follows that $\mathcal{C}'$ is also exact.

We are thus reduced to prove that $\mathcal{C}'$ is a complex. The arguments of Lemma 9.3 show that $p_* p^* = 0$, and we have $\epsilon c(E) p_* = c(E)\epsilon p_* = 0$ since $\mathcal{C}$ is exact. To prove that $p^*\epsilon c(E) = \epsilon c(p^*E) p^* = 0$, we observe that the exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow p^*\mathcal{E} \longrightarrow \mathcal{G}_E^\vee \longrightarrow 0$$

and [9, Proposition 13.3.2] give $c(p^*E) = c(G_E^\vee)c(\mathcal{O}(-1))$. Up to sign, the latter is $c(\mathcal{O}(-1))c(G_E^\vee)$ which becomes 0 after applying ϵ because of Lemma 3.1. □

To close this section, we exhibit an example of a vector bundle of even rank over a smooth affine $\mathbb{R}$-scheme X such that $\epsilon c(E)$ is non trivial.

Example 9.5 Consider the real 2-sphere $S^2 = \mathrm{Spec}(\mathbb{R}[x,y,z]/x^2 + y^2 + z^2 - 1)$. Then $H^2(S^2,\mathbf{I}^2) = \mathbb{Z}$ by [9, Théorème 16.3.8]. The exact sequence of sheaves

$$0 \longrightarrow \mathbf{I}^2 \longrightarrow \mathbf{I} \longrightarrow \bar{\mathbf{I}} \longrightarrow 0$$

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yields an exact sequence

$$H^1(S^2,\bar{\mathbf{I}}) \longrightarrow H^2(S^2,\mathbf{I}^2) \xrightarrow{\epsilon} H^2(S^2,\mathbf{I}) \longrightarrow H^2(S^2,\bar{\mathbf{I}}) .$$

But $H^1(S^2,\bar{\mathbf{I}})$ is a 2-torsion group and $H^2(S^2,\bar{\mathbf{I}}) = 0$. Therefore ϵ is an isomorphism. If P denotes the (algebraic) tangent bundle to S^2, then $c(P) \neq 0$ in $H^2(S^2,\mathbf{I}^2)$ by [9, Corollaire 15.3.2, Théorème 16.3.11, Théorème 16.4.4] (in fact it is not hard to compute that $c(P) = \pm 2$ depending on the chosen orientation). Therefore $\epsilon c(P) \neq 0$.

10. Integral Stiefel-Whitney classes

The following theorem is a generalization of Lemma 3.1:

Theorem 10.1 *Let X be a smooth scheme and let E be a vector bundle of odd rank r over X. Let*

$$\partial_{\det(\mathcal{E})^\vee} : H^{r-1}(X,\bar{\mathbf{I}}^{r-1}) \to H^r(X,\mathbf{I}^r,\det(\mathcal{E})^\vee)$$

be the connecting homomorphism. Then $c(E) = \partial_{\det(\mathcal{E})^\vee}(\overline{c}_{r-1}(E))$.

Proof: Because of the exact sequence of sheaves on $\mathbb{P}(E)$:

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow p^*\mathcal{E} \longrightarrow \mathcal{G}_E^\vee \longrightarrow 0$$

we have $p^*c(E) = c(p^*E) = c(G_E^\vee)c(\mathcal{O}(-1))$ by [9, Proposition 13.3.2]. Since E is of odd rank, $c(G_E^\vee) \in H^{r-1}(\mathbb{P}(E),\mathbf{I}^{r-1},\det(\mathcal{E})^\vee(-1))$. Lemma 3.1 gives a commutative diagram

$$\begin{array}{ccc} H^{r-1}(\mathbb{P}(E),\mathbf{I}^{r-1},\det(\mathcal{E})^\vee(-1)) & \xrightarrow{c(\mathcal{O}(-1))} & H^r(\mathbb{P}(E),\mathbf{I}^r,\det(\mathcal{E})^\vee) \\ \downarrow{\scriptstyle \pi_{\det(\mathcal{E})^\vee(-1)}} & \nearrow{\scriptstyle \partial_{\det(\mathcal{E})^\vee}} & \\ H^{r-1}(\mathbb{P}(E),\bar{\mathbf{I}}^{r-1}) & & \end{array}$$

computing the multiplication by $c(\mathcal{O}(-1))$. Since $\pi_{\det(\mathcal{E})^\vee(-1)}c(G_E^\vee) = \overline{c}(G_E^\vee)$, we get $c(G_E^\vee)c(\mathcal{O}(-1)) = \partial_{\det(\mathcal{E})^\vee}\overline{c}(G_E^\vee)$ and therefore $p^*c(E) = \partial_{\det(\mathcal{E})^\vee}\overline{c}(G_E^\vee)$. This shows that $\epsilon_{p^*\det(\mathcal{E})^\vee}p^*c(E) = 0$ and then $p^*\epsilon_{\det(\mathcal{E})^\vee}c(E) = 0$.

Theorem 9.1 implies that $\epsilon_{\det(\mathcal{E})^\vee}c(E) = 0$. Therefore there exists $\alpha \in H^{r-1}(X,\bar{\mathbf{I}}^{r-1})$ such that $\partial_{\det(\mathcal{E})^\vee}(\alpha) = c(E)$. It remains to show that α can be chosen to be $\overline{c}_{r-1}(E)$.

We have an exact sequence:

$$H^{r-1}(\mathbb{P}(E),\mathbf{I}^{r-1},p^*\det(\mathcal{E})^\vee) \xrightarrow{\pi_{\det(\mathcal{E})^\vee}} H^{r-1}(\mathbb{P}(E),\bar{\mathbf{I}}^{r-1}) \xrightarrow{\partial_{\det(\mathcal{E})^\vee}} H^r(\mathbb{P}(E),\mathbf{I}^r,p^*\det(\mathcal{E})^\vee).$$

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Since $\partial_{\det(\mathcal{E})^\vee}\overline{c}(G^\vee) = p^*c(E) = \partial_{\det(\mathcal{E})^\vee}(p^*\alpha)$, we have $p^*\alpha - \overline{c}(G_E^\vee) = \pi_{\det(\mathcal{E})^\vee}(\beta)$ for some $\beta \in H^{r-1}(\mathbb{P}(E),\mathbf{I}^{r-1},p^*\det(\mathcal{E})^\vee)$.

By Theorem 9.1, we can write

$$\beta = p^*\gamma + \sum_{2\leq m\leq r-1}^{r \text{ even}} \mu^m_{\det(\mathcal{E})^\vee}(\alpha_{r-1-m})$$

for some $\gamma \in H^{r-1}(X,\mathbf{I}^{r-1},\det(\mathcal{E})^\vee)$ and $\alpha_{r-1-m} \in H^{r-1-m}(X,\overline{\mathbf{I}}^{r-1-m})$. Using Proposition 5.7, we obtain

$$\pi_{\det(\mathcal{E})^\vee}(\beta) = p^*\pi_{\det(\mathcal{E})^\vee}(\gamma) + \sum_{2\leq m\leq r-1}^{r \text{ even}} (\xi^m\alpha_{r-1-m} + \xi^{m-1}\pi_{\det(\mathcal{E})^\vee}\partial_{\det(\mathcal{E})^\vee}\alpha_{r-1-m}),$$

and Lemma 4.3 yields

$$\overline{c}(G_E^\vee) = \sum_{m=0}^{r-1} \xi^m\overline{c}_{r-1-m}(E).$$

The equality $p^*\alpha - \overline{c}(G_E^\vee) = \pi_{\det(\mathcal{E})^\vee}(\beta)$ and Theorem 4.1 imply thus that we have $\alpha_{r-1-m} = \overline{c}_{r-1-m}(E)$ for any $2 \leq m \leq r-1$ such that m is even. Therefore

$$p^*\alpha - p^*\overline{c}_{r-1}(E) = p^*\pi_{\det(\mathcal{E})^\vee}(\gamma) + \sum_{2\leq m\leq r-1}^{r \text{ even}} \xi^{m-1}\pi_{\det(\mathcal{E})^\vee}\partial_{\det(\mathcal{E})^\vee}\alpha_{r-1-m}.$$

Since p^* is injective by Theorem 4.1, it follows then that $\alpha - \overline{c}_{r-1}(E) = \pi_{\det(\mathcal{E})^\vee}(\delta)$ for some $\delta \in H^{r-1}(X,\mathbf{I}^{r-1},\det(\mathcal{E})^\vee)$.

Therefore we can choose $\alpha = \overline{c}_{r-1}(E)$ and the theorem is proved. □

This theorem is the motivation of the next definition:

Definition 10.2 Let X be a smooth scheme and let E be a vector bundle of odd rank r over X. For any odd m, we define the *m-th integral Stiefel-Whitney class $c_m(E)$ of E* by $c_m(E) = \partial_{\det(\mathcal{E})^\vee}\overline{c}_{m-1}(E)$.

The first expected property of these classes should be that they generalize the Stiefel-Whitney classes of Section 4. Indeed, it is the case:

Proposition 10.3 *For any odd m, we have $\pi_{\det(\mathcal{E})^\vee}c_m(E) = \overline{c}_m(E)$.*

Proof: Theorem 10.1 proves that $\partial_{\det(\mathcal{E})^\vee}(\overline{c}_{r-1}(E)) = c(E)$. Since we also have $\partial_{\det(\mathcal{E})^\vee}(\overline{c}(G_E^\vee)) = c(E)$, we see that $\overline{c}(G_E^\vee) - \overline{c}_{r-1}(E)$ is in the kernel of $\partial_{\det(\mathcal{E})^\vee}$. By exactness of the sequence linking the different powers of the fundamental ideals, there exists $\beta \in H^{r-1}(\mathbb{P}(E),\mathbf{I}^{r-1},\det(\mathcal{E})^\vee)$ such that $\overline{c}(G_E^\vee) - \overline{c}_{r-1}(E) = \pi_{\det(\mathcal{E})^\vee}(\beta)$.

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By Theorem 9.1, we can write

$$\beta = p^*\gamma + \sum_{2\leq m\leq r-1} \mu^m_{\det(\mathcal{E})^\vee}\alpha_{r-1-m}$$

and Proposition 5.7 yields

$$\pi_{\det(\mathcal{E})^\vee}(\beta) = p^*\pi_{\det(\mathcal{E})^\vee}(\gamma) + \sum_{2\leq m\leq r-1} (\xi^m\alpha_{r-1-m} + \xi^{m-1}\pi_{\det(\mathcal{E})^\vee}\partial_{\det(\mathcal{E})^\vee}\alpha_{r-1-m}).$$

On the other hand,

$$\pi_{\det(\mathcal{E})^\vee}(\beta) = \overline{c}(G_E^\vee) - \overline{c}_{r-1}(E) = \sum_{m=1}^{r-1} \xi^m\overline{c}_{r-1-m}(E)$$

by Lemma 4.3. Comparing (and using Theorem 4.1), we find $\alpha_m = \overline{c}_{r-1-m}(E)$ and $\overline{c}_{r-m}(E) = \pi_{\det(\mathcal{E})^\vee}\partial_{\det(\mathcal{E})^\vee}(\overline{c}_{r-1-m}(E))$. But $\partial_{\det(\mathcal{E})^\vee}(\overline{c}_{r-1-m}(E)) = c_{r-m}(E)$ by definition and the proposition is proved. □

For any line bundle L over X, the multiplication by the integral Stiefel-Whitney classes defines homomorphisms $c_m(E) : H^i(X,\mathbf{I}^j,\mathcal{L}) \to H^{i+m}(X,\mathbf{I}^{j+m},\mathcal{L}\otimes \det(\mathcal{E})^\vee)$ satisfying the following functorial properties:

Proposition 10.4 *Let X be a smooth scheme and let E be a vector bundle of odd rank r over X. Then*

1. *If $f : X \to Y$ is a flat morphism, then $f^*c_m(E) = c_m(f^*E)\circ f^*$.*
2. *If $f : X \to Y$ is a proper morphism, then $f_*(c_m(f^*E)) = c_m(E)\circ f_*$.*
3. *$c_1(E) = c(\det(E))$ and $c_r(E) = c(E)$.*
4. *If $[E] = [E']$ in $K_0(X)$, then $c_m(E) = c_m(E')$ for any odd m.*

Proof: The first two assertions are straightforward consequences of [11, Theorem 3.2] and Proposition 2.1. The third assertion is a reformulation of Lemma 3.1 and Theorem 10.1. To prove the last assertion, observe that since $[E] = [E']$ in $K_0(X)$, we have $\overline{c}_{m-1}(E) = \overline{c}_{m-1}(E')$ for any m because of Whitney formula. Using Theorem 10.1, we get

$$c_m(E) = \partial_{\det(\mathcal{E})^\vee}(\overline{c}_{m-1}(E)) = \partial_{\det(\mathcal{E}')^\vee}(\overline{c}_{m-1}(E')) = c_m(E').$$

□

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Remark 10.5 Recall that the Steenrod operation Sq^2, as defined by Brosnan [7] and Voevodsky [26], satisfies the following "quadratic" description (on smooth schemes over a field of characteristic different from 2) due to Totaro ([24, Theorem 1.1]):

$$\begin{array}{ccc} H^i(X,\bar{\mathbf{I}}^i) & \xrightarrow{\partial} & H^{i+1}(X,\mathbf{I}^{i+1}) \\ & \searrow^{Sq^2} & \downarrow \pi \\ & & H^{i+1}(X,\bar{\mathbf{I}}^{i+1}). \end{array}$$

If E is a vector bundle of odd rank over X with trivial determinant, Proposition 10.3 gives in particular the formula $Sq^2(\bar{c}_{m-1}(E)) = \bar{c}_m(E)$.

11. The projective bundle theorem for Milnor-Witt sheaves

Let X be a smooth connected scheme of dimension d and let $x \in X^{(i)}$ for some $i \in \mathbb{N}$. For any $j \in \mathbb{N}$, let $K^{\boldsymbol{M}}_{j-i}(k(x))$ be the $(j-i)$-th Milnor K-theory group as defined in [18, §1], with the convention that $K^{\boldsymbol{M}}_{j-i}(k(x)) = 0$ if $j - i < 0$. For any $y \in X^{(i+1)}$, there is a residue homomorphism

$$d_i : K^{\boldsymbol{M}}_{j-i}(k(x)) \to K^{\boldsymbol{M}}_{j-i-1}(k(y))$$

defined for instance in [18, §2]. We thus get a sequence of abelian groups $C(X,\mathbf{K}^{\boldsymbol{M}}_j)$

$$K^{\boldsymbol{M}}_j(k(X)) \xrightarrow{d_0} \bigoplus_{x\in X^{(1)}} K^{\boldsymbol{M}}_{j-1}(k(x)) \xrightarrow{d_1} \cdots \longrightarrow \bigoplus_{x\in X^{(d)}} K^{\boldsymbol{M}}_{j-d}(k(x))$$

which turns out to be a complex by [16]. As in the case of $\mathbf{I}^j$-cohomology, we can associate a sheaf $\mathbf{K}^{\boldsymbol{M}}_j$ to the presheaf

$$U \mapsto \ker(K^{\boldsymbol{M}}_j(k(U)) \to \bigoplus_{x\in U^{(1)}} K^{\boldsymbol{M}}_{j-1}(k(x)))$$

and the above complex is a flasque resolution of this sheaf ([23, §6]). We denote by $H^i(X,\mathbf{K}^{\boldsymbol{M}}_j)$ the cohomology groups of the above complex (which agree with the cohomology groups of the corresponding sheaf). These cohomology groups satisfy good functorial properties as summarized in [23]. In particular, homotopy invariance holds and there are proper push-forwards. This allows to define Euler classes of vector bundles using the usual procedure (see Section 3).

Let $p : \mathbb{P}(E) \to X$ be a projective bundle. To avoid mixing up the notations, we denote by $e(\mathcal{O}(1))$ the Euler class of $\mathcal{O}(1)$ in $H^1(\mathbb{P}(E),\mathbf{K}^{\boldsymbol{M}}_1)$. In this setup, the projective bundle formula holds as recalled by the next result.

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Theorem 11.1 *Let X be a smooth scheme and let E be a vector bundle of rank r over X. Then*

$$H^i(\mathbb{P}(E),\mathbf{K}^M_j) = \bigoplus_{m=0}^{r-1} H^{i-m}(\mathbb{P}(E),\mathbf{K}^M_{j-m})\cdot e(\mathcal{O}(1))^m$$

for any $i,j \in \mathbb{N}$.

Proof: We can mimic the proof of Theorem 4.1 which goes through. □

In the same situation as above, there are homomorphisms

$$s_{j-i} : K^M_{j-i}(k(x)) \to \overline{I}^{j-i}(k(x))$$

for any $i,j \in \mathbb{N}$ and any $x \in X^{(i)}$ ([18, §4]). These maps induce a morphism of complexes $C(X,\mathbf{K}^M_j) \to C(X,\bar{\mathbf{I}}^j)$ ([9, Théorème 10.2.6]) and thus a morphism of sheaves $s : \mathbf{K}^M_j \to \bar{\mathbf{I}}^j$. We also denote by $s : H^i(X,\mathbf{K}^M_j) \to H^i(X,\bar{\mathbf{I}}^j)$ the induced homomorphism on cohomology groups. We state now an obvious compatibility result between the Euler classes in both $\mathbf{K}^M_j$ and $\bar{\mathbf{I}}^j$ theories.

Lemma 11.2 *Let E be a rank d vector bundle over X. Then the following diagram commutes*

$$\begin{array}{ccc} H^i(X,\mathbf{K}^M_j) & \xrightarrow{e(E)} & H^{i+d}(X,\mathbf{K}^M_{j+d}) \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} \\ H^i(X,\bar{\mathbf{I}}^j) & \xrightarrow[\bar{c}(E)]{} & H^{i+d}(X,\bar{\mathbf{I}}^{j+d}) \end{array}$$

for any $i,j \in \mathbb{N}$.

Proof: This follows from the compatibility results between the pull-back maps ([9, Proposition 10.4.1]) and push-forward maps ([9, Lemme 10.4.4]). □

We can now define for any $j \in \mathbb{Z}$ the Milnor-Witt K-theory sheaf $\mathbf{K}^{MW}_{j,\mathcal{L}}$ twisted by a line bundle L over X as the fibre product of sheaves

$$\begin{array}{ccc} \mathbf{K}^{MW}_{j,\mathcal{L}} & \longrightarrow & \mathbf{I}^j_{\mathcal{L}} \\ \downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{L}}} \\ \mathbf{K}^M_j & \xrightarrow[s]{} & \bar{\mathbf{I}}^j. \end{array}$$

This definition agrees with the definition given in [20] by [19, Théorème 5.3]. This sheaf admits a flasque resolution by a complex of sheaves $C(X,\mathbf{K}^{MW}_j,\mathcal{L})$ of the

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form

$$K_j^{MW}(k(X),\omega_{x_0}^{\mathcal{L}}) \xrightarrow{d_0^{\mathcal{L}}} \bigoplus_{x\in X^{(1)}} K_{j-1}^{MW}(k(x),\omega_x^{\mathcal{L}}) \xrightarrow{d_1^{\mathcal{L}}} \cdots \longrightarrow \bigoplus_{x\in X^{(d)}} K_{j-d}^{MW}(k(x),\omega_x^{\mathcal{L}}).$$

where $K_{j-i}^{MW}(k(x),\omega_x)$ is the $(j-i)$-th Milnor-Witt group of $k(x)$ twisted by ω_x as defined in [20, Remark 3.21]. The cohomology groups of $\mathbf{K}_{j,\mathcal{L}}^{MW}$ satisfy the same functorial properties as the cohomology groups of $\mathbf{I}_{\mathcal{L}}^j$ (as demonstrated in [8]).

The fibre product

$$\begin{array}{ccc} \mathbf{K}_{j,\mathcal{L}}^{MW} & \longrightarrow & \mathbf{I}_{\mathcal{L}}^j \\ \downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{L}}} \\ \mathbf{K}_j^M & \xrightarrow[s]{} & \bar{\mathbf{I}}^j. \end{array}$$

shows that there is an exact sequence of sheaves

$$0 \longrightarrow \mathbf{I}_{\mathcal{L}}^{j+1} \xrightarrow{\epsilon'_{\mathcal{L}}} \mathbf{K}_{j,\mathcal{L}}^{MW} \xrightarrow{\theta_{\mathcal{L}}} \mathbf{K}_j^M \longrightarrow 0$$

for any line bundle L over X and any $j \in \mathbb{Z}$. We denote the connecting homomorphism by $\delta_{\mathcal{L}} : H^i(X,\mathbf{K}_j^M) \to H^{i+1}(X,\mathbf{I}^{j+1},\mathcal{L})$. The commutative diagram of sheaves with exact lines

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{I}_{\mathcal{L}}^{j+1} & \xrightarrow{\epsilon'_{\mathcal{L}}} & \mathbf{K}_{j,\mathcal{L}}^{MW} & \xrightarrow{\theta_{\mathcal{L}}} & \mathbf{K}_j^M & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle s} & & \\ 0 & \longrightarrow & \mathbf{I}_{\mathcal{L}}^{j+1} & \xrightarrow[\epsilon_{\mathcal{L}}]{} & \mathbf{I}_{\mathcal{L}}^j & \xrightarrow[\pi_{\mathcal{L}}]{} & \bar{\mathbf{I}}^j & \longrightarrow & 0 \end{array}$$

yields a proof of the next lemma.

Lemma 11.3 *The following diagram commutes*

$$\begin{array}{ccc} H^i(X,\mathbf{K}_j^M) & \xrightarrow{\delta_{\mathcal{L}}} & H^{i+1}(X,\mathbf{I}^{j+1},\mathcal{L}) \\ \downarrow{\scriptstyle s} & & \| \\ H^i(X,\bar{\mathbf{I}}^j) & \xrightarrow[\partial_{\mathcal{L}}]{} & H^{i+1}(X,\mathbf{I}^{j+1},\mathcal{L}) \end{array}$$

for any line bundle L *over* X *and any* $i,j \in \mathbb{N}$.

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Corollary 11.4 *Let E be a rank r vector bundle over X. Then the following diagram commutes*

$$\begin{array}{ccc} H^i(X,\mathbf{K}^M_j) & \xrightarrow{\delta_{\mathcal{L}}} & H^{i+1}(X,\mathbf{I}^{j+1},\mathcal{L}) \\ \downarrow{\scriptstyle e(E)} & & \downarrow{\scriptstyle c(E)} \\ H^{i+r}(X,\mathbf{K}^M_{j+r}) & \xrightarrow[\delta_{\mathcal{L}\otimes\det(\mathcal{E})^\vee}]{} & H^{i+1+r}(X,\mathbf{I}^{j+1+r},\mathcal{L}\otimes\det(\mathcal{E})^\vee) \end{array}$$

for any line bundle L over X and any $i,j\in\mathbb{N}$.

Proof: In view of Lemmas 11.2 and 11.3, it suffices to show that the diagram

$$\begin{array}{ccc} H^i(X,\bar{\mathbf{I}}^j) & \xrightarrow{\partial_{\mathcal{L}}} & H^{i+1}(X,\mathbf{I}^{j+1},\mathcal{L}) \\ \downarrow{\scriptstyle \bar{c}(E)} & & \downarrow{\scriptstyle c(E)} \\ H^{i+r}(X,\bar{\mathbf{I}}^{j+r}) & \xrightarrow[\partial_{\mathcal{L}\otimes\det(\mathcal{E})^\vee}]{} & H^{i+1+r}(X,\mathbf{I}^{j+1+r},\mathcal{L}\otimes\det(\mathcal{E})^\vee) \end{array}$$

commutes. This is Lemma 3.2. □

Corollary 11.5 *The following diagram commutes*

$$\begin{array}{ccc} H^{i-m}(X,\mathbf{K}^M_{j-m}) & \xrightarrow{s} & H^{i-m}(X,\bar{\mathbf{I}}^{j-m}) \\ \downarrow{\scriptstyle e(\mathcal{O}(1))^m p^*} & & \downarrow{\scriptstyle \mu_{\mathcal{L}}^{m+1}} \\ H^i(\mathbb{P}^n_X,\mathbf{K}^M_j) & \xrightarrow[\delta_{\mathcal{L}(-m-1)}]{} & H^{i+1}(\mathbb{P}^n_X,\mathbf{I}^{j+1},\mathcal{L}(-m-1)) \end{array}$$

for any $m\in\mathbb{N}$, any line bundle L over X and any $i,j\in\mathbb{N}$.

Proof: The diagram

$$\begin{array}{ccccc} H^{i-m}(X,\mathbf{K}^M_{j-m}) & \xrightarrow{s} & H^{i-m}(X,\bar{\mathbf{I}}^{j-m}) & & \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} & & \\ H^{i-m}(\mathbb{P}^n_X,\mathbf{K}^M_{j-m}) & \xrightarrow{s} & H^{i-m}(\mathbb{P}^n_X,\bar{\mathbf{I}}^{j-m}) & \xrightarrow{\partial_{\mathcal{L}(-1)}} & H^{i-m+1}(\mathbb{P}^n_X,\mathbf{I}^{j-m+1},\mathcal{L}(-1)) \\ \downarrow{\scriptstyle e(\mathcal{O}(1))^m} & & \downarrow{\scriptstyle \bar{c}(\mathcal{O}(1))^m} & & \downarrow{\scriptstyle c(\mathcal{O}(1))^m} \\ H^i(\mathbb{P}^n_X,\mathbf{K}^M_j) & \xrightarrow[s]{} & H^i(\mathbb{P}^n_X,\bar{\mathbf{I}}^j) & \xrightarrow[\partial_{\mathcal{L}(-m-1)}]{} & H^{i+1}(\mathbb{P}^n_X,\mathbf{I}^{j+1},\mathcal{L}(-m-1)) \end{array}$$

commutes by Lemmas 3.2 and 11.2. The bottom composite is equal to $\delta_{\mathcal{L}(-m-1)}$ by Lemma 11.3. □

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We know restrict to the special case $X = \mathrm{Spec}(k)$. The reader can refer to Remark 11.9 for more explanations on why we restrict to this case. In order to compute the $\mathbf{K}_j^{MW}$-cohomology of $\mathbb{P}_k^n$, we will have to understand the homomorphisms

$$\delta : H^i(\mathbb{P}_k^n, \mathbf{K}_j^M) \to H^{i+1}(\mathbb{P}_k^n, \mathbf{I}^{j+1})$$

and

$$\delta_{\mathcal{O}(-1)} : H^i(\mathbb{P}_k^n, \mathbf{K}_j^M) \to H^{i+1}(\mathbb{P}_k^n, \mathbf{I}^{j+1}, \mathcal{O}(-1))$$

for any $i, j \in \mathbb{N}$.

Notation We denote by A_j^i the kernel of δ and by B_{j+1}^{i+1} its cokernel. Accordingly, we denote by $A_j^i(\mathcal{O}(-1))$ the kernel of $\delta_{\mathcal{O}(-1)}$ and by $B_{j+1}^{i+1}(\mathcal{O}(-1))$ its cokernel.

Using the long exact sequence of cohomology groups associated to the exact sequences

$$0 \longrightarrow \mathbf{I}^{j+1} \longrightarrow \mathbf{K}_j^{MW} \longrightarrow \mathbf{K}_j^M \longrightarrow 0$$

and

$$0 \longrightarrow \mathbf{I}_{\mathcal{O}(-1)}^{j+1} \longrightarrow \mathbf{K}_{j,\mathcal{O}(-1)}^{MW} \longrightarrow \mathbf{K}_j^M \longrightarrow 0,$$

we get short exact sequences

$$0 \longrightarrow B_{j+1}^i \longrightarrow H^i(\mathbb{P}_k^n, \mathbf{K}_j^{MW}) \longrightarrow A_j^i \longrightarrow 0 \tag{2}$$

and

$$0 \to B_{j+1}^i(\mathcal{O}(-1)) \to H^i(\mathbb{P}_k^n, \mathbf{K}_j^{MW}, \mathcal{O}(-1)) \to A_j^i(\mathcal{O}(-1)) \to 0. \tag{3}$$

Proposition 11.6 *We have*

$$A_j^i = \begin{cases} K_{j-i}^M(k) & \text{if } i \text{ is even.} \\ 2K_{j-i}^M(k) & \text{if } i \text{ is odd.} \end{cases}$$

$$A_j^i(\mathcal{O}(-1)) = \begin{cases} 2K_{j-i}^M(k) & \text{if } i \text{ is even.} \\ K_{j-i}^M(k) & \text{if } i \text{ is odd.} \end{cases}$$

$$B_{j+1}^i = \begin{cases} I^{j+1}(k) & \text{if } i = 0. \\ I^{j+1-n}(k) & \text{if } i = n \text{ and } n \text{ odd.} \\ 0 & \text{else.} \end{cases}$$

$$B_{j+1}^i(\mathcal{O}(-1)) = \begin{cases} I^{j+1-n}(k) & \text{if } i = n \text{ and } n \text{ even.} \\ 0 & \text{else.} \end{cases}$$

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Proof: By definition of A^i_j and B^{i+1}_{j+1}, we have an exact sequence

$$0 \longrightarrow A^i_j \longrightarrow H^i(\mathbb{P}^n_k, \mathbf{K}^M_j) \xrightarrow{\delta} H^{i+1}(\mathbb{P}^n_k, \mathbf{I}^{j+1}) \longrightarrow B^{i+1}_{j+1} \longrightarrow 0.$$

The projective bundle theorem for $\mathbf{K}^M_j$-cohomology shows that

$$H^i(\mathbb{P}^n_k, \mathbf{K}^M_j) = \bigoplus_{m=0}^{n} H^{i-m}(k, \mathbf{K}^M_{j-m}) \cdot e(\mathcal{O}(1))^m$$

and therefore $H^i(\mathbb{P}^n_k, \mathbf{K}^M_j) = K^M_{j-i}(k) \cdot e(\mathcal{O}(1))^i$.

Suppose first that n is even. Then Theorem 8.1 shows that

$$H^{i+1}(\mathbb{P}^n_k, \mathbf{I}^{j+1}) = H^{i+1}(k, \mathbf{I}^{j+1}) \oplus \bigoplus_{2 \leq m \leq n}^{m \text{ even}} H^{i+1-m}(k, \bar{\mathbf{I}}^{j+1-m})$$

and therefore

$$H^{i+1}(\mathbb{P}^n_k, \mathbf{I}^{j+1}) = \begin{cases} I^{j+1}(k) & \text{if } i = -1. \\ \overline{I}^{j-i}(k) & \text{if } i \geq 0 \text{ is odd.} \\ 0 & \text{else.} \end{cases}$$

Thus the only possible case where δ is possibly non-trivial is when i is odd. Corollary 11.5 shows that we have a commutative diagram

$$\begin{array}{ccc} K^M_{j-i}(k) & \xrightarrow{s} & \overline{I}^{j-i}(k) \\ \downarrow{\scriptstyle e(\mathcal{O}(1))^i} & & \downarrow{\scriptstyle \mu^{i+1}} \\ H^i(\mathbb{P}^n_k, \mathbf{K}^M_j) & \xrightarrow[\delta]{} & H^{i+1}(\mathbb{P}^n_k, \mathbf{I}^{j+1}). \end{array}$$

It follows that $\delta : H^i(\mathbb{P}^n_k, \mathbf{K}^M_j) \to H^{i+1}(\mathbb{P}^n_k, \mathbf{I}^{j+1})$ is surjective when n is even and i is odd.

Suppose now that n is odd. In that case,

$$H^{i+1}(\mathbb{P}^n_k, \mathbf{I}^{j+1}) = H^{i+1}(k, \mathbf{I}^{j+1}) \oplus \bigoplus_{2 \leq m \leq n}^{m \text{ even}} H^{i+1-m}(k, \bar{\mathbf{I}}^{j+1-m}) \oplus H^{i+1-n}(k, \mathbf{I}^{j+1-n})$$

and the only difference with the above case is when $i = n - 1$. In this situation, we claim that

$$\delta : H^{n-1}(\mathbb{P}^n_k, \mathbf{K}^M_j) \to H^n(\mathbb{P}^n_k, \mathbf{I}^{j+1})$$

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is trivial. Now Theorem 8.1 and Proposition 6.3 show that the push-forward map (which is defined since $\omega_{\mathbb{P}^n_k/k}$ is trivial)

$$p_* : H^n(\mathbb{P}^n_k, \mathbf{I}^{j+1}) \to H^0(k, \mathbf{I}^{j+1-n})$$

is an isomorphism. The map $\mathbf{I}^{j+1} \to \mathbf{K}^{MW}_j$ in the exact sequence of sheaves

$$0 \longrightarrow \mathbf{I}^{j+1} \longrightarrow \mathbf{K}^{MW}_j \longrightarrow \mathbf{K}^M_j \longrightarrow 0$$

gives a commutative diagram

$$\begin{array}{ccc} H^n(\mathbb{P}^n_k, \mathbf{I}^{j+1}) & \longrightarrow & H^n(\mathbb{P}^n_k, \mathbf{K}^{MW}_j) \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\ H^0(k, \mathbf{I}^{j+1-n}) & \longrightarrow & H^0(k, \mathbf{K}^{MW}_j). \end{array}$$

The bottom map is injective and the left p_* is an isomorphism. It follows that $H^n(\mathbb{P}^n_k, \mathbf{I}^{j+1}) \to H^n(\mathbb{P}^n_k, \mathbf{K}^{MW}_j)$ is injective and therefore δ is trivial.

The computation of the twisted case follows exactly the same arguments as developed above. □

We can now explicit and prove the computation of $H^i(\mathbb{P}^n, \mathbf{K}^{MW}_j)$.

Theorem 11.7 *We have*

$$H^i(\mathbb{P}^n_k, \mathbf{K}^{MW}_j) = \begin{cases} K^{MW}_j(k) & \text{if } i = 0. \\ K^M_{j-i}(k) & \text{if } i \text{ is even and } i \neq 0. \\ 2K^M_{j-i}(k) & \text{if } i \text{ is odd and } i \neq n. \\ K^{MW}_{j-n}(k) & \text{if } i = n \text{ and } n \text{ is odd.} \end{cases}$$

$$H^i(\mathbb{P}^n_k, \mathbf{K}^{MW}_j, \mathcal{O}(-1)) = \begin{cases} 2K^M_{j-i}(k) & \text{if } i \text{ is even and } i \neq n. \\ K^M_{j-i}(k) & \text{if } i \text{ is odd.} \\ K^{MW}_{j-n}(k) & \text{if } i = n \text{ and } n \text{ is even.} \end{cases}$$

Proof: Proposition 11.6 above shows that the only time when the exact sequences (2) and (3) above could be non-trivial extensions are exactly when Milnor-Witt K-theory groups appear in the statement of the theorem, the other cases being obvious.

Suppose first that $i = 0$ and the twist is trivial. In that case, the pull-back map induces a commutative diagram (essentially by the definition of the pull-back map

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for Milnor-Witt K-theory, see for instance [9, Corollaire 10.4.2])

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathbb{P}^n,\mathbf{I}^{j+1}) & \longrightarrow & H^0(\mathbb{P}^n,\mathbf{K}_j^{MW}) & \longrightarrow & H^0(\mathbb{P}^n,\mathbf{K}_j^M) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle p^*} & & \uparrow{\scriptstyle p^*} & & \uparrow{\scriptstyle p^*} & & \\ 0 & \longrightarrow & H^0(k,\mathbf{I}^{j+1}) & \longrightarrow & H^0(k,\mathbf{K}_j^{MW}) & \longrightarrow & H^0(k,\mathbf{K}_j^M) & \longrightarrow & 0 \end{array}$$

The bottom sequence is exact. The left-hand and right-hand vertical maps are isomorphisms, and it follows that the top sequence is also exact. The five lemma allows to conclude the the middle vertical map is an isomorphism.

Using the push-forward maps p_* and essentially the same arguments as above yield the remaining cases $i = n$ and n is odd, and $i = n$ and n is even. □

Recall now that the i-th Chow-Witt group of a smooth scheme X twisted by a line bundle L is by definition $H^i(X,\mathbf{K}_i^{MW},\mathcal{L})$ ([9, Définition 10.2.14]). In the following statement, we use the identifications $K_0^{MW}(k) = GW(k)$ (the Grothendieck-Witt group of k) and $K_0^M(k) = \mathbb{Z}$. We also write $2\mathbb{Z}$ instead of $\mathbb{Z}$ to stress the different natures of these groups in $\widetilde{CH}^i(\mathbb{P}_k^n)$ and $\widetilde{CH}^i(\mathbb{P}_k^n,\mathcal{O}(-1))$.

Corollary 11.8 *We have*

$$\widetilde{CH}^i(\mathbb{P}_k^n) = \begin{cases} GW(k) & \text{if } i = 0 \text{ or } i = n \text{ with } n \text{ odd.} \\ \mathbb{Z} & \text{if } i \text{ is even and } i \neq 0. \\ 2\mathbb{Z} & \text{if } i \text{ is odd and } i \neq n. \end{cases}$$

$$\widetilde{CH}^i(\mathbb{P}_k^n,\mathcal{O}(-1)) = \begin{cases} 2\mathbb{Z} & \text{if } i \text{ is even and } i \neq n. \\ \mathbb{Z} & \text{if } i \text{ is odd.} \\ GW(k) & \text{if } i = n \text{ and } n \text{ is even.} \end{cases}$$

Remark 11.9 The computation of the total $\mathbf{K}_j^{MW}$-cohomology of $\mathbb{P}_X^n$ for a general base scheme X is much more complicated than for $\mathrm{Spec}(k)$. The reason is that the map

$$\delta : H^i(\mathbb{P}_X^n,\mathbf{K}_j^M) \to H^{i+1}(\mathbb{P}_X^n,\mathbf{I}^{j+1})$$

is more complicated to study (the same holds for its twisted analogue). In general, its kernel and cokernel can be expressed in terms of some cohomology groups of the base with coefficients in either $\mathbf{K}_j^M$ or $2\mathbf{K}_j^M$. This allows to gather some information on the total $\mathbf{K}_j^{MW}$-cohomology of $\mathbb{P}_X^n$. Unfortunately the answer seems to be up to extensions only, i.e. the answer presents itself under the form of short exact sequences that don't split in general. The situation is even worse in the case of $\mathbb{P}(E)$ for a general vector bundle E.

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