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The strong sweeping out property for convolution powers

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Abstract. Pointwise convergence a.e. of $\mu^k f$ has been studied by Bellow, Jones and Rosenblatt. We answer two of their questions: the strong sweeping out property holds (in this case) iff the measure μ has unbounded angular ratio and the stability is equivalent to the bounded angular ratio.

Let (X, Σ, m) denote a non-atomic Lebesgue probability space, and $\tau : X \to X$ an invertible, measure preserving transformation. If μ is a probability measure on $\mathbb{Z}$, $f \in L^1(X)$, define $(\mu f)(x) = \sum_{k=-\infty}^{\infty} \mu(k) f(\tau^k x)$. The sequence of measures (ν_n) is said to have the *strong sweeping out property*, if, given $\epsilon > 0$, there is a set $B \in \Sigma$ with $m(B) < \epsilon$ such that

$$\limsup \nu_n \chi_B(x) = 1 \text{ a.e.} \quad \text{and} \quad \liminf \nu_n \chi_B(x) = 0 \text{ a.e.}$$

(see [**AB**] for a further discussion). There is also a continuous variant: if $(U_t)_{t\in\mathbb{R}}$ is a measurable, measure preserving flow on X (see [**K**, p. 10] for technical details) and μ is a probability measure on $\mathbb{R}$, one can define $(\mu f)(x) = \int_{\mathbb{R}} f(U_t x)\, d\mu(t)$ and there is a corresponding notion of the strong sweeping out property.

In [**AB**, Theorem 2.2] a general *transfer principle* was proved, saying that (given a sequence (ν_n)) if this property is satisfied for one flow, then it holds for any aperiodic, measurable, measure preserving flow. Several sufficient conditions for the strong sweeping out property have been given in [**AB**]. In Theorem 1 we want to single out the general approach which is used more or less implicitly in [**AB**], namely that of using several simple spaces X and special sets to test for the strong sweeping out property.

THEOREM 1. *Assume that (ν_n) is a sequence of probability measures on $\mathbb{R}$ and for all $\epsilon, M > 0$ there is a probability space (X, Σ, m) with a measurable, measure preserving flow $(U_t)_{t\in\mathbb{R}}$ such that $m(x : \sup_n \nu_n \chi_A(x) > 1 - \epsilon) > M \cdot m(A)$ for some set $A \in \Sigma$. Then (ν_n) has the strong sweeping out property.*

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Remark 1. A corresponding result holds for measures ν_n on $\mathbb{Z}$ and invertible, measure preserving transformations τ.

Our principal aim is the case of convolution powers, i.e. $\nu_n = \mu^n$ for $n \in \mathbb{N}$, where μ is a probability measure on $\mathbb{Z}$ (respectively $\mathbb{R}$), $\mu \neq \delta_0$ (i.e. not concentrated in 0). A number of convergence results for $\mu^n f$ have been shown in [**BJR**], see also [**B**] for a survey of results. $\hat{\mu}(t) = \sum \mu(\{k\}) e^{-2\pi i k t}$ $(t \in \mathbb{R})$ denotes the Fourier transform of μ (similarly in the continuous case). It was already conjectured in [**BJR**] that with regard to almost everywhere convergence the behaviour depends essentially on properties of the *angular ratio* of $\hat{\mu}$:

$$\limsup_{t \to 0} \frac{|\hat{\mu}(t) - 1|}{1 - |\hat{\mu}(t)|} = \infty. \tag{UR}$$

THEOREM 2. *Suppose that μ is a probability measure on $\mathbb{Z}$ ($\mu \neq \delta_k$, i.e. not concentrated in a single point) and that $\hat{\mu}$ has unbounded angular ratio (i.e. property (UR) holds). Then (μ^n) has the strong sweeping out property.*

Remark 2. A corresponding result holds for probability measures on $\mathbb{R}$ and measurable, measure preserving flows.

Under a somewhat stronger condition on μ, Theorem 2 was proved in [**AB**, Theorem 6.1] (the case where μ has finite second moment was settled earlier in [**BJR**]). The condition of [**AB**] was characterized in [**L2**] by the behaviour of the partial sums for the expectations of μ compared to those of the second moment and those for $\sum \mu(\{k\})$. Furthermore [**L2**, Example 3], a construction was given for measures μ not satisfying this stronger condition, but still having unbounded angular ratio (in addition, they have expectation zero and for given $p < 2$ one can find such a μ with finite moment of order p).

Note that a converse has been proved in [**BJR**, Theorem 1.6]: if μ has bounded angular ratio and it is strictly aperiodic [**BJR**, Definition 1.5], then $\mu^n f$ converges pointwise a.e. for $f \in L^p(X)$, $p > 1$. (The assumption that μ is strictly aperiodic is not needed in full strength for convergence itself, but rather for their description of the limit; the only case that causes troubles arises if μ is supported by some arithmetic progression $k + l\mathbb{Z}$ that does not contain zero. Then the full sequence shows a cyclic behaviour, but it can be split into finitely many convergent subsequences.) See also [**B**] and [**BC**] for results on the case $p = 1$. In §3 of [**BJR**] the relation to the condition of *stability* is explained (the strong sweeping out property is clearly incompatible with stability and pointwise convergence a.e. on L^2 implies stability). Combined with our Theorem 2, it follows that stable measures are also characterized by bounded angular ratio, i.e.

$$\limsup_{t \to 0} \frac{|\hat{\mu}(t) - 1|}{1 - |\hat{\mu}(t)|} < \infty.$$

Our proof of Theorem 2 uses Theorem 1. In the proofs of Theorems 1.1, 1.10 and 6.1 of [**AB**], finite products of $[0, 1]$ with translations by irrationals and products of subintervals have been used for testing (see also [**L1**]). In our case, it is even enough to consider $[0, 1]$ with various translations and subintervals.

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Proof of Theorem 1. This results from [**AB**, Theorem 2.2]. Given $0 < \epsilon < 1$, $M > 0$ and $N_0 \in \mathbb{N}$, choose (X, Σ, m, U_t) and $A \in \Sigma$ satisfying the condition of our Theorem 1 with M replaced by $M + N_0/(1-\epsilon)$. Since (U_t) is measure preserving, we have

$$m(x : \nu_n \chi_A(x) > 1 - \epsilon) < \frac{m(A)}{1-\epsilon},$$

hence

$$m\Big(x : \sup_{n \geq N_0} \nu_n \chi_A(x) > 1 - \epsilon\Big) > M \cdot m(A).$$

Then there exists $N \in \mathbb{N}$ such that

$$F = \Big\{x \in X : \max_{N_0 \leq n \leq N} \nu_n \chi_A(x) > 1 - \epsilon\Big\}$$

satisfies

$$m(F) > M \cdot m(A).$$

Since this depends only on the class of $\nu_n \chi_A$ with respect to a.e. equivalence, we may choose a pointwise defined version as follows: $\nu_n \chi_A(x) = \int \chi_A(U_t x)\, d\nu_n(t)$ if the integrand is ν_n-measurable and $\nu_n \chi_A(x) = 0$ otherwise. These versions are used in the definition of F. For $x \in X$ we consider $A(x) = \{r \in \mathbb{R} : U_r x \in A\}$ and $F(x) = \{r \in \mathbb{R} : U_r x \in F\}$.

By the individual ergodic theorem, the set $A(x)$ has a density a.e. and furthermore $d(A(x)) = E(A, \Sigma_U)(x)$ gives the conditional expectation with respect to the σ-algebra Σ_U of U-invariant sets (see [**K**, §1.2]).

Since $m(F) > M \cdot m(A)$, we obtain

$$m(x : E(F, \Sigma_U)(x) > M \cdot E(A, \Sigma_U)(x)) > 0.$$

Hence, we can find $x \in X$ such that $A(x)$ and $F(x)$ have a density (in particular, they are Lebesgue measurable in $\mathbb{R}$) and $d(F(x)) > M \cdot d(A(x))$. If $r \in F(x)$, then for some n we have $\nu_n \chi_A(U_r x) > 1 - \epsilon$. By our definition, it follows that

$$\nu_n \chi_A(U_r x) = \int \chi_A(U_{r'} U_r x)\, d\nu_n(r') = \int \chi_{A(x)}(r + r')\, d\nu_n(r').$$

We adopt the same pointwise definition for $\nu_n \chi_{A(x)}$ as above and obtain $\nu_n \chi_A(U_r x) = \nu_n \chi_{A(x)}(r)$. The same conclusion holds if $\nu_n \chi_{A(x)}(r) > 1 - \epsilon$.

Thus

$$F(x) = \Big\{r : \max_{N_0 \leq n \leq N} \nu_n \chi_{A(x)}(r) > 1 - \epsilon\Big\}.$$

It follows that $E = A(x)$ satisfies the condition of [**AB**, Theorem 2.2] which completes the proof. (In fact, [**AB**] makes the assumption that the measures ν_n are concentrated on a countable subset of $\mathbb{R}$, but this is not used in the proof of Theorem 2.2.) □

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Proof of Theorem 2. We use Theorem 1 (respectively its discrete variant). Assume that $0 < \epsilon < 1$ and $M \geq 1$ are given. For $t > 0$, we put

$$\rho = \left|1 - \frac{\hat{\mu}(t)}{|\hat{\mu}(t)|}\right| \quad \text{and} \quad L = \frac{|\hat{\mu}(t) - 1|}{1 - |\hat{\mu}(t)|}.$$

If t is sufficiently close to zero, both quantities are defined, ρ gets small and, by assumption, L can be made large. Hence, we can find $t > 0$ such that $0 < \rho < 1/M$ and $L > 32\pi M^2/\epsilon$. We consider $X = [0, 1[\cong \mathbb{R}/\mathbb{Z}$ with normalized Lebesgue measure m and the transformation $\tau(x) = x + t \pmod 1$.

Put $k = [2\pi/\rho] + 1$ (where $[\alpha]$ denotes the integer part of α), $|\hat{\nu}_k(t)| = 1 - r$ and $\hat{\mu}(t)/|\hat{\mu}(t)| = e^{2\pi i\xi}$ with $0 < \xi < 1$. Finally, set $\delta = \max(\rho/8, \sqrt{r/(8\epsilon)})$ and $A =]-2\delta, 2\delta[$.

We claim:

(i) $\delta < 1/(4M)$;

(ii) $\nu_j \chi_A(x) > 1 - \epsilon$ for $|x - j\xi| < \delta$, $j = 1, \ldots, k$.

Since $\rho = 2\sin(\pi\xi)$, we have $1/k < \rho/(2\pi) < \xi < \rho/4 \leq 2\delta$ and it follows from (ii) that $\sup \nu_j \chi_A(x) > 1 - \epsilon$ for all $x \in X$; since $m(A) = 4\delta$, (i) shows that the condition of Theorem 1 will be fulfilled.

To prove (i), first note that

$$\rho \geq |1 - \hat{\mu}(t)| - \left|\hat{\mu}(t) - \frac{\hat{\mu}(t)}{|\hat{\mu}(t)|}\right| = |1 - \hat{\mu}(t)|\left(1 - \frac{1}{L}\right) = (1 - |\hat{\mu}(t)|)(L - 1).$$

Consequently $\rho/L \geq \frac{1}{2}(1 - |\hat{\mu}(t)|)$ and $|\hat{\mu}(t)| \geq 1 - 2\rho/L$. (I thank A. Bellow for pointing out an error in the first version of this computation.) Hence, $1 - r = |\hat{\mu}(t)|^k \geq (1 - 2\rho/L)^k$. This gives

$$-r \geq \ln(1 - r) \geq k \ln\left(1 - 2\frac{\rho}{L}\right) \geq -4k\frac{\rho}{L} \quad \left(\text{since } \frac{\rho}{L} < \frac{1}{4}\right).$$

From $k < 4\pi/\rho$, we obtain $r < 16\pi/L$. Thus $\sqrt{r/(8\epsilon)} < \sqrt{2\pi/(L\epsilon)} < 1/(4M)$. Since $\rho < 1/M$, claim (i) follows.

Next, we prove (ii): as in [**L2**], we write $\langle\alpha\rangle$ for the number in $]-\frac{1}{2}, \frac{1}{2}]$ satisfying $\alpha - \langle\alpha\rangle \in \mathbb{Z}$. By elementary estimates (see [**L2**, proof of Lemma 2]), we have

$$|\hat{\nu}_j(t)| = e^{-2\pi ij\xi}\hat{\nu}_j(t) = \sum_s \nu_j(s)\cos(2\pi(st + j\xi)) \leq 1 - 8\sum_s \nu_j(s)\langle st + j\xi\rangle^2.$$

Hence for $j \leq k$, we obtain

$$\sum_s \nu_j(s)\langle st + j\xi\rangle^2 \leq \frac{r}{8}.$$

This gives for $|x - j\xi| < \delta$,

$$\nu_j(s : |\langle st + x\rangle| < 2\delta) > 1 - \frac{r}{8\delta^2}$$

(by (i), we have $\delta < \frac{1}{4}$). Since $\delta \geq \sqrt{r/(8\epsilon)}$, we conclude that $\nu_j \chi_A(x) > 1 - \epsilon$. □

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Note added in Proof. After finishing the paper, I have received a preprint by M. Akcoglu, A. Bellow, R. L. Jones and J. M. Rosenblatt 'Divergence of Convolution Powers' which also contains a proof for the result described in Theorem 2.

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