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The generic rational differential equation $dw/dz = P_n(z, w)/Q_n(z, w)$ on $\mathbb{CP}^2$ carries no interesting transverse structure

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Abstract. Let Γ be a non-solvable pseudogroup of holomorphic transformations in one variable fixing zero. Then for any z_0 sufficiently near zero and outside some real analytic set containing zero and depending on Γ, for any germ of biholomorphism ϕ defined at z_0 with $\phi(z_0) - z_0$ sufficiently small, there exists a sequence $\gamma_n \in \Gamma$ which tends to ϕ uniformly on some neighborhood of zero. If we let Γ be the holonomy pseudogroup of a compact leaf, we find that holomorphic codimension-1 foliations with non-solvable holonomy admit no transverse geometric structure in addition to the conformal one. This applies, in particular, to the singular foliation induced on $\mathbb{CP}^2$ by the differential equation $dw/dz = P_n(z, w)/Q_n(z, w)$ where P_n and Q_n are the generic polynomials of degree n, hence the title of this paper.

1. *Introduction*

This paper is related to our previous works [**BLL1, BLL2**]. Recall the following classical setting: let $\mathcal{F}$ be a dimension p, codimension q topological foliation on a topological manifold M; let x be any point of M, $\mathcal{F}_x$ be the leaf of $\mathcal{F}$ at x and $T(\alpha_1, \ldots, \alpha_q)$ be any immersed q-disk transverse to $\mathcal{F}$ and based at x (that is, $T(0, \ldots, 0) = x$). Then for any loop $c(t)$ based at x and remaining constantly on $\mathcal{F}_x$, we can define a Poincaré 'first-return' map h_c from a certain neighborhood D_c of x in T to its image $h_c(D_c)$. Precisely, if $x' \in T$ is close enough to x (in some convenient foliated tubular neighborhood of the image of c) one can lift the closed path $c(t)$ to a possibly non-closed path $c_\varepsilon(t)$ which starts at x' and remains along the x'-leaf. Then by definition $h_c(c_\varepsilon(0)) = c_\varepsilon(1)$. Details can be found in [**Go**].

It so happens that the map h_c depends only on the homotopy class of c in $\pi_1(\mathcal{F}_x)$; so the set of all h_c with $c \in \pi_1(\mathcal{F}_x)$ is a pseudogroup Γ of homeomorphisms of T fixing the origin. It depends neither on x nor T in an essential way (pseudogroups attached

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to the same leaf are conjugate). It encloses much important information about $\mathcal{F}$. If $\mathcal{F}$ is endowed with any additional transverse structure (differentiability, invariant measure, Finsler metric and so forth) then this shows on Γ (typically, a subclass of transversals T with 'good' parametrizations will exist and Γ will consist of local automorphisms of this subclass).

Consider the case where $\mathcal{F}$ is transversally holomorphic with complex codimension-1: a 'good' parametrization of T will then realize Γ as a pseudogroup of holomorphic transformations of $\mathbb{C}$ fixing zero. This is already an important and difficult case, so we will, from now on, restrict ourselves to it. It appears that the pseudogroups under consideration form two distinct classes with very different dynamics, according to whether they are solvable or not. The notion of solvability for general pseudogroups is in fact rather tricky, but in our particular context Γ can be made into a group in a natural way: to each element of Γ is attached a germ of an invertible holomorphic function fixing zero, and the set of those germs is a subgroup Γ_0 of $\mathrm{Diff}(\mathbb{C}, 0)$ (the group of all germs of convergent formal series of the form $f(z) = az + o(z)$). We can state that Γ is a *realization* of Γ_0; we pass from Γ_0 to Γ by specifying a definition domain for each germ. We can state that Γ is solvable at zero if Γ_0 is solvable; experience has shown that this definition is consistent.

It has been shown by various authors that a generic solvable Γ is conjugate to some subgroup of the two-parameter group of transformations $az(1 + bz)^{-1}$; consult [**EISV**]. Non-solvable pseudogroups are much more chaotic, as will soon become clear. Before stating our theorem, recall that of Shcherbakov.

THEOREM. [**Sh**] *If Γ is not solvable, then there exists an open dense subset Ω of a neighborhood of 0 in $\mathbb{C}$, consisting of finitely many connected components Ω_i, such that the orbit of each $z \in \Omega_i$ is dense in Ω_i.*

Recently, Nakaï improved [**Na**] this theorem showing that the complementary Σ of Ω is a real analytic subset of $\mathbb{C}$ and that somehow it depends only on Γ_0: precisely, to any Γ_0 is attached some Σ_0 ($\Sigma_0 = \{0\}$ for generic Γ_0), and Σ is contained in Σ_0 whatever the chosen realization of Γ_0.

It is natural to ask whether non-solvable pseudogroups may preserve some geometric structure on Ω. Our next result answers this question negatively (Γ, Ω, etc. are as before).

THEOREM 1. *If Γ is non-solvable then, for any $z_0, z_1 \in \Omega_i$, for any local biholomorphism ϕ with $\phi(z_0) = z_1$, there exists a sequence $\gamma_n \in \Gamma$ which tends to ϕ uniformly on some neighborhood of z_0.*

This implies that Γ can preserve no 'special' collection of local holomorphic coordinates. In particular, G does not preserve any geometric structure in the sense of Lie, Cartan, Ehresmann or Gromov (see the remark after Proposition 9). To explain our next result (which is the main step in the proof of Theorem 1) we observe the following. Let X be a germ of a holomorphic vector field defined on some domain D. Let U be a compact subset of D and τ a positive real number. If τ is small enough (depending on U), the solutions $z(t)$ of the real differential equation $\dot{z} = X, z(0) = \alpha$ are defined for all $(\alpha, t) \in U \times [0, \tau]$. Then we can define a function $\phi(z)$ sending α to $z(\tau)$.

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Definition 2. We say that X is *approached by* Γ *on* D if for any U and τ as before, there exists a sequence γ_n in Γ that converges to ϕ uniformly on U.

The following result is both folklore and straightforward.

PROPOSITION 3. *Let D be a given domain. The set $\mathfrak{g}(D)$ of all fields approached by Γ on D is a real Lie algebra w.r.t. the usual Lie bracket of vector fields, and it is closed w.r.t. the topology of uniform convergence on compact subsets of D.*

With this, we can easily prove Theorem 1 using the following result of Nakaï.

THEOREM. [**Na**] *Let Γ be a non-solvable subgroup of* $\mathrm{Diff}(\mathbb{C}, 0)$. *Then, for any z_0 sufficiently small and outside some real analytic set containing zero and depending on Γ, there are two transverse holomorphic germs of vector fields X, Y approached by Γ on some neighborhood of z_0.*

Roughly speaking, Nakaï's theorem means that $\mathfrak{g}(D)$ is transitive on D for D small enough. It is, in fact, infinite dimensional (Nakaï did not mention this in his paper but clearly knew it); Theorem 1 implies that $\mathfrak{g}$, in fact, contains any holomorphic field which is defined on all of D.

Finally, let us give an application of our theorem to foliation theory. Consider foliations $\mathcal{F}$ of $\mathbb{CP}^2$ which are compactifications of those defined on $\mathbb{C}^2$ by some rational differential equation $dw/dz = P_n(z, w)/Q_n(z, w)$ with P_n and Q_n two polynomials of degree $n \geq 2$. Recall that, according to a theorem of Singer (see [**CS**]), this equation can be integrated in the sense of Liouville iff $\mathcal{F}$ is transversely affine on a Zariski open set—this just means that in the vicinity of any regular point there exists a local chart where the equation reads as $d\xi = 0$, two such charts being linked by an *affine* transition function $\xi' = a\xi + b$ $(a \neq 0)$. We can define transversely projective foliations in a similar vein, by simply replacing the affinities by homographies.

Now, let us combine our result with the generic non-solvability of $\mathcal{F}$.

THEOREM 4. *For each integer $n \geq 2$, there is a Zariski open subset O of the set of couples of degree n polynomials such that if $(P_n, Q_n) \in O$, the foliation $\mathcal{F}$ induced by the equation $dw/dz = P_n(z, w)/Q_n(z, w)$ admits no transverse structure except the given conformal one.*

In [**CS**], Camacho and Scardua have suggested a geometrico-differential approach to reveal a hierarchy of integrabilities between the best possible (Liouville) and the worst (strictly no closed pseudogroup stabilizes the equation under consideration). Our result implies that there is no step between the two extremes.

Recently, Malgrange defined in [**Ma**] the notion of Galois groupoid of a complex foliation. As a consequence of our result, the Galois groupoid of the differential equation above is generically trivial (i.e. maximal).

Our main goal now is to prove Theorem 1; this is done in the next section. The proof has Proposition 3 as its main tool, so we also give a proof of this, although it is a well-known result. We do not prove Theorem 4 because (a) only a small part of it is not in our Theorem 1 and (b) the main task in the proof—to show that the line at infinity has

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non-solvable holonomy for a Zariski-dense set of couples (P, Q)—is a direct application of the ideas contained in [**II**].

2. *Development*

2.1. *Notation and tools.* From now on, Γ is a fixed non-solvable pseudogroup; Ω is as in Shcherbakov's theorem; z_0 and z_1 are in the same connected component of Ω; X and Y are as in Nakaï's theorem, as stated earlier and ϕ is any germ of a biholomorphism with $\phi(z_0) = z_1$. Finally, for any holomorphic vector field Z and real number t, we let ϕ_Z^t denote the t-time flow of Z whenever it makes sense.

We will need the following.

Fact 5. On their common domain of definition, X and Y span an infinite dimensional real Lie algebra.

Proof. In Nakaï's construction it appears that X and Y are defined on a sector at zero, with asymptotic development

$$X = (az^p + o(z^p))\frac{\partial}{\partial z}, \quad Y = (bz^q + o(z^q))\frac{\partial}{\partial z} \quad (ab \neq 0,\ p > q > 1).$$

So, by letting $X_1 = [X, Y]$ and defining $X_{n+1} = [X, X_n]$ inductively we obtain

$$X_n = (a^n b(q-p)(q-1)(p+q-2)\dots((n-2)p+q-n+1)z^{np+q-n} + o(z^{np+q-n}))\frac{\partial}{\partial z}.$$

We can see that these fields are all non-zero and have a contact with the null vector field at zero that is strictly increasing with n. So they must all be linearly independent over $\mathbb{C}$, *a fortiori* over $\mathbb{R}$. As they belong to the Lie algebra spanned by X and Y, the proof is done. □

We now let D denote some domain containing z_0 with the following properties:

(1) X and Y both are approached by Γ on D;
(2) D is a topological square with two arcs of X-orbits and two arcs of Y-orbits as edges; and
(3) X and Y are transverse on all of D and its boundary.

When talking about pseudoflows of non-complete vector fields, we have to be cautious about definition domains: for the sake of precision, we consider from now on (without changing the notation) that X and Y are only defined on D.

We let G denote the set of all germs of the form

$$\psi(z) = \phi_X^{t_0} \circ \phi_Y^{t_1} \circ \dots \circ \phi_X^{t_{2n}} \circ \phi_Y^{t_{2n+1}}(z) \tag{1}$$

where each t_i is small enough for ϕ to be defined. We note that G is a pseudogroup with respect to the composition of germs. Of course, G is not a Lie group, because X and Y are not complete on D. It is not even a local Lie group, because if it were its Lie algebra $\mathfrak{g}$ would be a finite-dimensional real Lie algebra containing X and Y, contradicting fact 5. Our first aim is to show that G acts transitively on frame bundles J^k of any order $k \geq 0$ (see §2.2 and Proposition 9). This will need some machinery. Let G_0 be the subset of all germs in G which fix z_0 (that is, $\psi(z_0) = z_0$). Note that G_0 is a subgroup of $\mathrm{Diff}(\mathbb{C}, z_0)$; furthermore it is a *connected* one.

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LEMMA 6. *G_0 is a pathwise-connected subgroup of* Diff$(\mathbb{C}, z_0)$.

Proof. Because of our choice of D, if G contains the germ ψ of formula (1), then it also contains any of the germs

$$\psi_t(z) = \phi_X^{t.t_0} \circ \phi_Y^{t.t_1} \circ \cdots \circ \phi_X^{t.t_{2n}} \circ \phi_Y^{t.t_{2n+1}}(z)$$

with $t \in [0, 1]$. As ψ_t is a path from the identity to ψ in G, we see that G is a connected space. As it is also a principal G_0-bundle on the topological disk D, this implies that G_0 is connected, since it is the fiber of a connected bundle over a simply connected space. □

Connected groups are easier to cope with than pseudogroups, so we will rather work on G_0. As connected Lie groups are again easier to study than general connected groups, we shall see G_0 as a projective limit of Lie groups. For this, consider J^k, the bundle of k-jets of invertible holomorphic functions on D. Recall J^k is the set of such formal Taylor expansions as

$$a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots + a_k\varepsilon^k + o(\varepsilon^k), \quad a_0 \in D,\ a_1 \neq 0.$$

So, J^0 is just D and J^1 is a $\mathbb{C}^*$-bundle on D, in fact it is equivalent to the tangent bundle of D minus the zero section. We further define for any integers $l > k$ a projection

$$\pi_l^k : J^l \to J^k, \quad (a_0, \ldots, a_k, \ldots, a_l) \to (a_0, \ldots, a_k)$$

and this makes J^l into a fiber bundle over J^k. For formal purposes we choose as a base point in J^k the jet I_k equal to $z = z_0 + \varepsilon + o(\varepsilon^k)$, and this is compatible with each of the projections π_l^k. Finally let D_k denote the fiber of (π_l^0) at z_0: if we write for a moment $\varepsilon = z - z_0$, the law of composition of Taylor polynomials endows D_k with the structure of a connected solvable complex Lie group of complex dimension k; for $l > k$, D_l is an extension of D_k, and J^l is a D_l/D_k bundle over J^k (for $k = 0$, this implies that J^l is a D_l-principal bundle over J^0).

There is for any k a natural pseudoaction of G on J^k by the composition of Taylor expansions. Namely, if $g \in G$ is defined at a_0, we set

$$g(a_0 + \cdots + a_k\varepsilon^k) = g(a_0) + \sum_{j=1}^{k} \left(\sum_{i=1}^{k} \left(\frac{g^{(i)}(a_0)}{i!} \sum_{\mathcal{P}_j^i} (a_{p_1} \ldots a_{p_i}) \right) \varepsilon^j \right) \tag{2}$$

the third sum being taken on the following set of i-uples,

$$\mathcal{P}_i^j = \{(p_1, \ldots, p_i) \in \mathbb{N}^i : p_1 \geq 1, \ldots, p_i \geq 1, p_1 + \cdots + p_i = j\}.$$

Recalling the definitions of G_0 and D_k, we can see that for any k there is a natural action of the former on the latter; this action is by left translations, so there is a morphism ϕ_k from G_0 to D_k. But G_0 is connected and D_k is a Lie group; so by Von Neumann's theorem, the image $G_k = \phi_k(G_0)$ of G_0 must be a (real) Lie subgroup of D_k.

2.2. *Transitivity of G on J^k.* We now study the action of G on J^k. As usual, we state that two jets ϕ and ψ in J^k are in the same G-orbit if there exists $g \in G$ defined at the base point of ϕ and such that $g \circ \phi = \psi$. This is an equivalence relation. First, check the regularity of the G-orbits.

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LEMMA 7. *For any integer k, the orbits of G in J^k are the leaves of a (real) analytic foliation $\mathcal{F}_k$.*

Proof. Indeed since G_k is a Lie subgroup of D_k, the G_k-orbits in D_k are the leaves of an analytic foliation $\mathcal{F}_k^0$ of D_k whose tangent bundle is spanned by the Lie algebra of G_k. Now, for any z_1 in D there is a g in G such that $g(z_0) = z_1$, and this implies that the G-orbits intersect the fiber of π_k^0 at z_1 along an analytic foliation (namely, $g^*(\mathcal{F}_k^0)$). Moreover, this foliation depends analytically on z_1: to see this last fact, recall that g can be made to depend analytically on a finite number of parameters $t_0, \ldots, t_{2n+1}$ in formula (1). So, the G-orbits intersect fibers of π_k^0 along a real analytic foliation $\mathcal{F}_k^1$. We have also shown that $\mathcal{F}_k^1$ is invariant by the pseudoflows of X and Y. Finally, since X and Y vanish nowhere on D they are transverse to the fibers of π_k^0, but the leaves of $\mathcal{F}_k^1$ are along these fibers, so that $\mathcal{F}_k^1$ is transverse to both X and Y.

Consider now the real vector bundle on J^k which is spanned by X, Y and the tangent bundle of $\mathcal{F}_k^1$: it has the same dimension $2 + \dim(\mathcal{F}_k^1)$ everywhere because X and Y are transverse to $\mathcal{F}_k^1$; it is analytic because it is spanned by analytic fields; it is integrable because X and Y preserve $\mathcal{F}_k^1$. Therefore, it is tangent to an analytic foliation $\mathcal{F}_k$. □

Next, compute the differential of $g \in G$ when seen as a diffeomorphism on J^k w.r.t. the canonical coordinates $(a_0, \ldots, a_k)$. Differentiating in (2), we see that the Jacobian matrix of g at $(a_0, \ldots, a_k)$ is superior triangular with each diagonal term equal to $g'(a_0)$, and the general term $\alpha_j^i (i > j)$:

$$\begin{cases} \alpha_j^i = \displaystyle\sum_{i=1}^{n} \frac{g^{(i)}(a_0)}{i!} \sum_{\mathcal{P}_j^i} \frac{\partial}{\partial a_i}(a_{p_1} \times \cdots \times a_{p_i}) \quad (i > j > 1) \\ \alpha_j^1 = \dfrac{g^{(i+1)}(a_0)}{i!} \displaystyle\sum_{\mathcal{P}_j^i} a_{p_1} \times \cdots \times a_{p_i} \end{cases}$$

so that if g fixes the point $(a_0, \ldots, a_k)$ of J^k, then its linear part at this point is

$$\mathrm{d}g\left(\sum_{i=0}^{k} \alpha_i \frac{\partial}{\partial a_i}\right) = \sum_{i=0}^{k} \alpha_i \frac{\partial}{\partial a_i} + \frac{g^{k+1}(a_0)}{k!} \alpha_0 \frac{\partial}{\partial a_k}. \tag{3}$$

The following lemma is just elementary linear algebra applied to formula (3).

LEMMA 8. *Suppose that $\phi \in G$ fixes the point $(a_0, \ldots, a_k)$ of J^k. Further assume that ϕ preserves some real vector subspace V of the tangent space at $(a_0, \ldots, a_k)$ to J^k and that V surjects on $T_{a_0}D$ through the projection π_k^0. Then at least one of the two following conditions holds:*

(1) *V contains the complex line spanned by $\partial/\partial a_k$, or*

(2) *$\phi^{k+1}(a_0) = 0$ (so that $d\phi(a_0, \ldots, a_k)$ is in fact the identity; cf. formula (3)).*

Proof. The hypothesis implies that V must contain vectors of the form

$$A = k!\frac{\partial}{\partial a_0} + \sum_{i=1}^{k} \alpha_i \frac{\partial}{\partial a_i}, \quad B = ik!\frac{\partial}{\partial a_0} + \sum_{i=1}^{k} \beta_i \frac{\partial}{\partial a_i}.$$

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Since it is ϕ-invariant, it must also contain $d\phi(A) - A$ and $d\phi(B) - B$ which, according to formula (2), are respectively equal to $\phi^{k+1}(a_0)\,\partial/\partial a_k$ and $i\phi^{k+1}(a_0)\,\partial/\partial a_k$. So if $\phi^{k+1}(a_0)$ is non-zero, then V must contain the complex line spanned by these two fields, as stated. □

Now recall a well-known theorem of Kobayashi.

THEOREM. [**Ko**] *Let H be a transitive connected group of C^∞ transformations on some manifold W. If for any $w \in W$ and $h \in H$ the condition $h(w) = w$ implies $dh(w) =$identity, then H is a Lie group of dimension* $\dim(W)$.

We are ready to show the following.

PROPOSITION 9. *For any k and any jet ϕ in J^k there is a g in G which is defined at the base point a_0 of ϕ and has ϕ as its jet at a_0. In other words, G acts transitively on J^k for any k*†.

Proof. We show by reduction to the absurd that $\dim(\mathcal{F}^k) = \dim(J^k)$, which is just another statement of Proposition 9. It is already known that G acts transitively on J^0. So if G does not act transitively on any J^k as well, there would exist a smaller integer $k > 0$ such that G does not act transitively on J^k. Now, consider any jet ϕ in J^k and any $g \in G$ stabilizing it: $g \circ \phi = \phi$. Since G is not transitive on J^k, $\mathcal{F}_k$ has more than one leaf; this means that the differential dg of g at ϕ must leave invariant a non-trivial real vector subspace V of the tangent space to J^k at ϕ (namely, the tangent plane to the leaf of $\mathcal{F}_k$ at ϕ). This V surjects on $T_{z_0}D$ through π_k^0 because G is transitive on D. On the other hand, it cannot contain the real line spanned by $\partial/\partial a_k$ because then it would have to be all of the tangent space to J^k at I_k and this would imply transitivity of G on J^k, contradicting the hypothesis. Therefore, applying Lemma 8 we have that the differential of g at ϕ is the identity. Now, apply Kobayashi's theorem with $H = G_0$ and W the G_0-orbit of I_k in J^k. Our previous results then imply that G_0 is a Lie group, which is false by Fact 5. □

Remark. Ehresmann has formalized a general definition of geometric structure (the ideas were already contained in the works of Lie and Cartan) including tensors, connexions and any known differential structure (see also [**Gr**]). In particular, such structure is revealed in some high-order jet bundle (also called a frame bundle) J^k by some function which should be invariant under the corresponding isometric pseudo-group. Proposition 9 exactly states that this is not the case for G.

2.3. *Proof of Theorem 1 and Proposition 3.*

Proof of Proposition 3. For Proposition 3 to be true, $\mathfrak{g}(D)$ has to be:

(1) real-homogeneous;
(2) closed for uniform convergence on compact subsets;
(3) stable w.r.t. sum of vectors; and
(4) stable w.r.t. Lie brackets.

† *Remark.* We could have done with only claiming and showing transitivity on J^3; but this would not have made the proof simpler.

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Point (1) is a trivial consequence of Definition 2, through the well-known formula $\phi^t_{kZ} = \phi^{kt}_Z$. Point (2) is again obvious: let Z_n be a sequence in $\mathfrak{g}(D)$ converging to some Z; let U be a compact subset of D and $t \in \mathbb{R}$ a time such that the germ $\psi(z) = \phi^t_Z(z)$ is well defined on all of U. Then for n large enough, this is also true of $\psi_n(z) = \phi^t_{Z_n}(z)$ because D is a neighborhood of U, U is compact and Z_n tends to Z. Let γ_n be in Γ, with $|\psi_n(z) - \gamma_n(z)| < 1/n$ uniformly on U (this is possible since Z_n is in $\mathfrak{g}(D)$). Then the sequence γ_n tends to ψ uniformly on U. As this applies to any U and t, this means (by Definition 2) that Z is in $\mathfrak{g}(D)$. Point (3) comes from point (4) and the formula

$$\lim_{n\to+\infty} (\phi^{t/n}_Z \phi^{t/n}_T)^n = \phi^t_{Z+T}.$$

Point (4) then comes from point (2), point (3) and the formula

$$\lim_{n\to+\infty} n((\phi^{1/n}_Z)_* T - T) = [Z, T].$$

This ends the proof. □

COROLLARY 10. *For any hyperbolic germ ϕ_0 in* Diff$(\mathbb{C}, z_0)$ *there is a sequence γ_n in Γ which tends to ϕ_0 on some neighborhood of z_0.*

Proof. According to Proposition 9, G_0 contains a germ of the form $f(z) = z_0 + 2(z - z_0) + o(z - z_0)$. Such a germ is linearizable in a neighborhood of z_0 by Königs' theorem. So, letting ω denote a suitable local coordinate, we can write $f(\omega) = 2\omega$ and assume f is defined at least on the disk $\mathbb{D} = \{|\omega| < 1\}$. Since ϕ_0 is hyperbolic, it is locally linearizable, so (after possibly having to rescale $\mathbb{D}$) we can further assume that ϕ_0 is the time-one flow of some vector field Z in $\mathfrak{g}(\mathbb{D})$. Now write $Z = \sum_{i=1}^{\infty} (a_n + ib_n)\omega^n \, \partial/\partial\omega$ with each a_i and each b_i real. Using Proposition 3, we only have to show that all the fields $\omega^n \, \partial/\partial\omega$ and $i\omega^n \, \partial/\partial\omega$ with $n \geq 1$ are in $\mathfrak{g}(\mathbb{D})$.

For any $\lambda \in \mathbb{C}^*$ with $|\lambda| \leq 1$, choose a germ of the form $g_\lambda(\omega) = \lambda\omega + o(\omega)$: we easily see that $f^n \circ g \circ f^{-n}$ tends uniformly to $\lambda\omega$ on $\mathbb{D}$ as n tends to $+\infty$. This means that $\mathfrak{g}(\mathbb{D})$ contains the fields $\omega \, \partial/\partial\omega$ and $i\omega \, \partial/\partial\omega$ by Definition 2. Now, let $p > 1$ be an integer and $h_p(\omega) = \omega + \omega^p + o(\omega^p)$: then, the field

$$T = (1-p)^{-1} \left(h_{p*} \left(\omega \frac{\partial}{\partial\omega} \right) - \omega \frac{\partial}{\partial\omega} \right) = (\omega^p + o(\omega^p)) \frac{\partial}{\partial\omega}$$

is in $\mathfrak{g}(\mathbb{D}')$ for some disk $\mathbb{D}'$ possibly smaller than $\mathbb{D}$. But then, as a straightforward computation shows, the sequence $2^{(p-1)n} f^n_*(T)$ tends uniformly to $\omega^p \, \partial/\partial\omega$ on $\mathbb{D}$, so this field must be in $\mathfrak{g}(\mathbb{D})$ after all. Finally, replacing h_p by $k_p(\omega) = \omega + i\omega^p + o(\omega^p)$ and using the same argument, we obtain that $i\omega^p \, \partial/\partial\omega$ is also in $\mathfrak{g}(\mathbb{D})$. This ends the proof. □

Proof of Theorem 1. First check that we can assume $z_1 \in D$. Indeed by Shcherbakov's theorem, the Γ-orbit of z_1 is dense in the z_1-connected component of Ω, so there is a point of this orbit (say $z_2 = \gamma(z_1)$) which is in D. If Theorem 1 holds true under the additional assumption $z_1 \in D$, then since $\gamma \circ \phi(z_0) = z_2$ is in D there is a sequence γ_n converging to $\gamma \circ \phi$ on some neighborhood of z_0. Now $\gamma^{-1}\gamma_n$ tends to ϕ on the same neighborhood.

From now on, assume that z_1 is in D. Then by Proposition 9 there is a $g \in G$ with $g(z_1) = z_0$ and (say) $g'(z_1) = 2\phi'(z_0)$. Then, $g \circ \phi$ fixes z_0 and is hyperbolic. So, by Corollary 10, $g \circ \phi$ is in G. Since g and $g \circ \phi$ are in G, ϕ must be in G. □

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